考試科目	線性代數	所別	應用數學系	考試時間	5月16日 星期六	第一節

1. Prove or give a counterexample to the following statements:
   (a) (10%) Let A be an $n \times n$ complex matrix. If A is invertible, then A is diagonalizable.
   (b) (10%) Let A be an $n \times m$ matrices, for some $n \neq m$. It is IMPOSSIBLE to find an $m \times n$ matrix B, such that $AB = I_n$ and $BA = I_m$.
   (c) (10%) Let $T: V \to W$ be a linear transformation, where V and W are vector spaces such that $\dim V = \dim W = n$. If T is not a zero transformation, then for each non-zero $n \times n$ matrix A, there is a basis β for V and a basis γ for W, such that $[T]_\beta^\gamma = A$. ($[T]_\beta^\gamma$ is the matrix representation of T with respect to β and γ.)
   (d) (10%) Let A and B be two $n \times n$ similar matrices. Then the two matrices have the same trace, determinant, and rank.

2. (15%) Let V be a vector space over a field $\mathbb{F}$. Let $\mathcal{L}(\mathbb{F}, V)$ be the collection of all linear transformations from $\mathbb{F}$ to V. Show that $\mathcal{L}(\mathbb{F}, V)$ is isomorphic to V by the definition of isomorphic vector spaces. (NO credit will be given if you did not prove it by definition.)

3. (15%) Let $f(t) = t^4 + 3t^3$ be the characteristic polynomial of a square matrix A. Let $B = A^4 + 3A^3 + A - 6I$. Show that B is an invertible matrix.

4. (15%) Let V be the vector space of continuous complex-valued functions defined on $[0, 2\pi]$. Let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ be the function defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\, dt.$$

   (a) Show that $\langle \cdot, \cdot \rangle$ is an inner product on V.
   (b) Let $S = \{e^{int} \mid n \text{ is an integer}\}$. Show that S is an orthonormal set on V with respect to the inner product $\langle \cdot, \cdot \rangle$.

5. (15%) Let $T: V \to V$ be a linear operator on a finite-dimensional vector space V, and let $\mathbf{v}$ be an arbitrary non-zero vector in V. If $W = \operatorname{span}(\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \ldots, T^n\mathbf{v}, \ldots\})$, prove the following statements.
   (a) W is a T-invariant subspace of V. (That is, $T(W) \subset W$.)
   (b) There is a positive integer k, such that $\beta = \{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \ldots, T^k\mathbf{v}\}$ is a basis for W.

備考	試題隨卷繳交

考試科目	分析概論	所別	應用數學系	考試時間	5月16日 星期六	第2節

1. (20%) Let X be an open subset of $\mathbb{R}^n$ and λ_n be the Lebesgue measure on $\mathbb{R}^n$. Show that $L^2(X)$ is a Hilbert space and find the dual space $L^2(X)^*$ of $L^2(X)$.

2. (20%) Let $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ be the Laplace operator on $\mathbb{R}^2$. Express Δ in terms of the polar coordinates (r, θ) and identify all harmonic functions which depend only on r.

3. (20%) Let $F : [a, b] \to \mathbb{R}$ be a function. Show that F is an indefinite integral if and only if F is absolutely continuous.

4. (20%) Let X be a metric space. Show that X is separable if and only if X is of second countable.

5. (20%) Let $\{f_n\}$ be a sequence of continuous real-valued functions defined on $\mathbb{R}^n$ such that $|f_n| \le 2^{-n}$ on $\mathbb{R}^n$, for all $n \ge 1$. Show that the series $\sum_{n=1}^{\infty} f_n$ is a well-defined continuous function on $\mathbb{R}^n$.

備考	試題隨卷繳交