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Proof by model: a new knowledge-based reachability analysis methodology for Petri net

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To solve the state explosion problem in the reachability analysis of Petri nets, Chao recently broke the NP(nondeterministic polynomial time)-complete barrier by developing the first closed-form solution of the number of Control Related States for the kth-order system. In this paper, we propose a new proof methodology known as proof by model, which is based on the validated information of the reverse net, to simplify and accelerate the construction of the closed-form solution for Petri nets. Here, we apply this methodology to the proof procedure of *Top-Right* systems with one non-sharing resource placed in the top position of the right-side process. The core theoretical and data basis are that any forbidden (resp. live) state in a Petri net is non-reachable (resp. live) in its reverse net; and the validated information of the *Bottom-Right* system, the reverse net of *Top-Right*.

Keywords: control systems; discrete event systems; flexible manufacturing systems; petri nets.

1. Introduction

Petri nets (PNs) have been widely applied for modelling and analysing flexible manufacturing systems or resource allocation systems (Ezpeleta *et al.*, 1995; Chao, 2005, 2006, 2011a,b,c, 2012; Lee *et al.*, 2005; Uzam & Zhou, 2006; Shih & Chao, 2010; Zimmermann, 2015) Reachability analysis (Ichikawa *et al.*, 1985; Hiraishi & Ichikawa, 1988; Lee *et al.*, 1990; Ferrarini, 1994; Kostin, 2003; Mizuno *et al.*, 2007; Miyamoto & Horiguchi, 2011) can be used to verify system properties, such as liveness, boundedness and reversibility. However, the large number of states generated (called the state explosion problem) is the persistent problem of using PNs to model various systems. Lee *et al.* (1990) have shown that the reachability problem (i.e. whether a marking is reachable) is NP-complete for even a live and safe Free Choice net. It is of theoretical interest and significance to find the exact number $\breve{R}$ of the reachable states of the research target PNs, because previous approaches have only found bounds (e.g. TimeNet tool; Zimmermann 2015). Another challenge of reachability problems is to know how to narrow the computation time to obtain reachable states and other information, which is an NP-complete problem,

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within a reasonable waiting time for a large PN. To the best of our knowledge, there is nothing in the literature that addresses such an issue. This problem is highly difficult, even for a marked graph. We (Liang & Chao, 2012; Chao, 2014) have successfully solved the problem by the construction of a closed-form solution for particular PNs.

Chao (2014) defined the *k*th-order system (defined in Definition 1) which is the simplest class of S^3PR (Systems of Simple Sequential Processes with Resources); by applying the concept of the complete reachability graph, split the reachability graph of the control net into *reachable, forbidden, deadlock non-reachable* and *non-reachable+empty-siphon states* (below, we call all of the different types of states Control Related States); integrating graph theory and combinatorial mathematics, pioneered the first closed-form solution to compute the number of Control Related States for a *k*th-order system. Notably, this solution reduces the computation time for the exponential increase ($O(2^k)$) of a *k*th-order system's Control Related States to intra-seconds. We have also extended and applied Chao's (2014) key methodology in enumerating the number of Control Related States of *Top-Right* (Chao & Yu, 2013) and *Bottom-Right* (Chao & Yu, 2014) by the viewpoint of letting the left process be the master control process.

Chao (2015) showed that it needs an additional 10 controllers for the deadlocks prevention policy of a fifth-order system. Due to the contributions of the closed-form solution listed above, we Chao & Yu (2015b,c) propose a new concept, the moment to launch resource allocation (MLR), to launch a partial deadlock avoidance/prevention policy for a real-time and large system to save the cost of deadlock prevention policy for reducing both the number of controllers and their allocation time. Presently, we can use the future deadlock ratio of the current state (i.e. the number of deadlock states/the number of reachable states), which can be derived in real-time by closed-form formulae, as the indicator to launch resource allocation.

However, the main problem is that without a knowledge-based relationship between PNs, both for the construction of a closed-form solution and for structural analysis-based deadlock prevention policy by siphon computation, N different structure nets need N times the independent analysis efforts. This is an important research issue for real-time, dynamic resource allocation systems because the new allocated resource creates a new net structure, new reachable states and also the new deadlock states derived from the new reachable states. Besides, the innovation of robot systems, Internet of Things and cloud computing system will let N be a very large number; gradually, even an unlimited number. However, few studies have been conducted on knowledge-based analysis for PNs. Furthermore, we also found that the complicated proof procedures by siphon concept are barriers to comprehend the whole methodology in *Bottom-Right* (Chao & Yu, 2014). We need a more brief and theoretical proof procedure to simplify the construction of closed-form formulae for more complicated systems.

To solve the problems listed above, we propose a new proof methodology called 'proof by model'. Chao (2014) showed the relationship of forbidden and non-reachable states between a PN and its reverse net in Lemmas 1 and 2. Based on Lemmas 1 and 2, in this paper we first prove that a reverse state of a live state in a PN is also a live state and that the number of live states in a PN and its reverse net are the same in Theorem 2. Here, we construct the knowledge-based analysis methodology for the construction of a closed-form solution of PNs presently based on Lemmas 1 and 2 and Theorem 2 and validated closed-form solution information of its reverse net. According to this methodology, to construct the closed-form solution, we can directly omit the effort of the computation for live states based on Theorem 2. We also show how to apply this methodology to the theoretical proof procedure marked by 'proof by model'. Here, we do not redo the whole construction effort of the closed-form formulae of the *Top-Right* system purely according to structure analysis by the siphon concept again.

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The approach is explained as follows. Let $\breve{R}$, L and F be the number of reachable, live and forbidden states of a kth-order system, respectively. Chao (2014) proved that the total number of states in a kth-order system is 3^k; the number of non-reachable states is $3^k - \breve{R}$; $F = \breve{R} - L$. For *Bottom-Right* (denoted as B in this section), we have proven that the number of reachable states in $B = 2\breve{R} + \Theta$, where Θ is the number of non-reachable markings in a kth-order system that are reachable states in B; the number of live states is $2L + A + C$, where A (resp. C) is the number of non-reachable (resp. forbidden) markings in a kth-order system that are live states in B. In Theorem 2, we prove that a PN N and its reverse net N^r have the same number of live states and that the live states in N are exactly the reverse states of live states in N^r. Hence, applying 'proof by model' with the given and validated closed-form solution information of N^r to compute the Control Related States of N can allow us to only focus on reachable and deadlock states due to $F = \breve{R} - L$, where L is known and validated.

To identify N and N^r clearly, we will investigate the proof procedure of the *Top-Right* kth-order system as the case study. In Appendix C, we apply our methodology to a *Top-Left* k-net system, which is more complicated and has a different net structure to the *Top-Right* system.

The rest of the paper is organized as follows. Section 2 presents the definition of a variant kth-order system and the closed-form solution of kth-order system's Control Related States. In Section 3, we show the known and validated characteristics of the *Bottom-Right* kth-order system proven in Chao & Yu (2014). Based on the results obtained and the methodology applied in Sections 2 and 3, we list the proof procedures of the closed-form solution of Control Related States in a *Top-Right* system mainly by 'proof by model' in Section 4; partial regular proof procedures are listed in Appendix B. Finally, Section 5 presents the paper's conclusions. Appendix A presents the preliminaries concerning PNs, which is optional for experts in PNs. In Appendix C, we apply our methodology to a *Top-Left* k-net.

2. The closed-form solution of Control Related States of kth-order systems

Here, we redefine the kth-order system (Chao, 2014) with one non-sharing resource place, in which each resource place carries only one token for different structure systems.

DEFINITION 1 A variant kth-order system is a subclass of S^3PR, with k resource places $r_1, r_2, \ldots, r_k$ shared between two processes N_1 and N_2 and one non-sharing resource place $r'_{\text{gen}}(= r^*)$ used by an operation place p^* in p_2.

(1) $M_0(r^*) = 1$ and $\forall r \in P_R, M_0(r) = 1$.

(2) N_1 (resp. N_2) uses $r_1, r_2, \ldots, r_k$ (resp. $r_k, r_{k-1}, \ldots, r_2, r_1$) in that order.

(3) $M_0(p_0) = k, M_0(p'_0) = k + 1$, where p_0 and p'_0 are the idle places in the processes N_1 and N_2, respectively.

(4) Holder places of r_j in N_1 and N_2 are denoted as p_j and p'_j, respectively.

(5) The compound circuit containing $r_i, r_{i+1}, \ldots, r_{j-1}, r_j$ is called the $(r_i - r_j)$-region.

(6) If r'_{gen} does not exist, then it is called a kth-order system. The location of r'_{gen} is between r_{gen} and $r_{\text{gen}+1}$.

(7) There are three possibilities for the token initially at r_i to sit at: $p_i(N_1), p'_i(N_2)$ and r_i. The corresponding token or r_i state is denoted by 1, -1 and 0, respectively.

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(8) x^y means r_{gen} is at x state ($x = 1, 0, -1$) and r'_{gen} is at y state ($y = 0, -1$), where subscript 'gen' is the location of a non-sharing resource being used by an operation place p^*. The system is denoted as a Top-Right kth-order system when gen $= 1$; Bottom-Right kth-order system when gen $= k - 1$.

Examples are shown in Figs. 1, 2, 3(a) and 4(a).

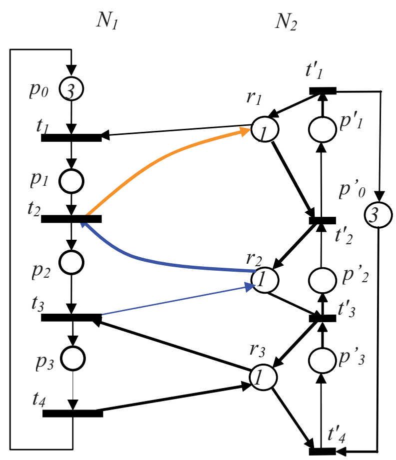

FIG. 1. Third-order system.

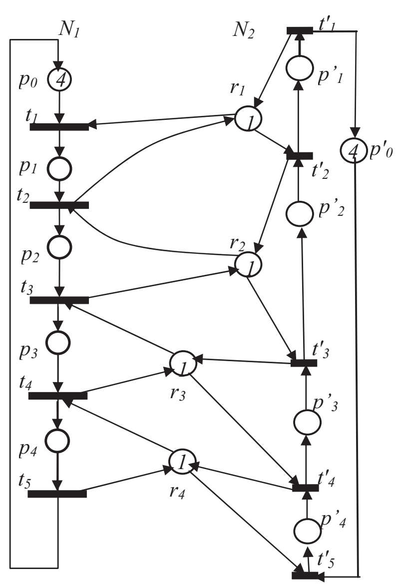

FIG. 2. Fourth-order system.

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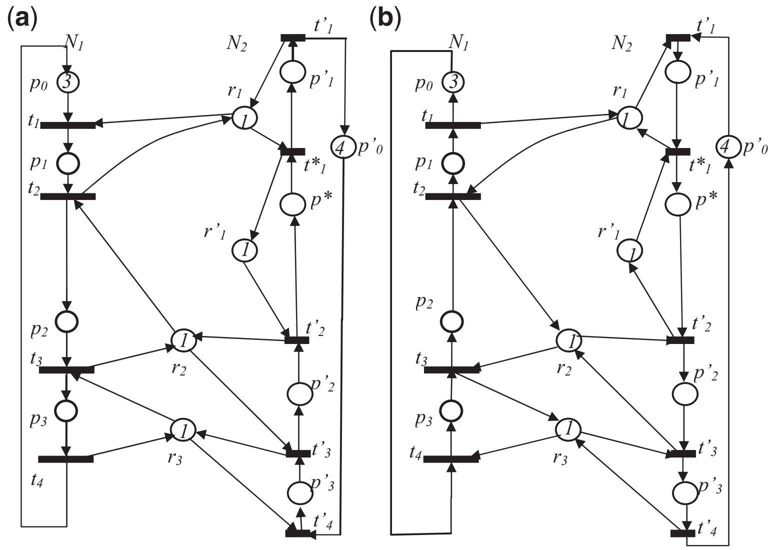

FIG. 3. Third-order *Top-Right* system (a) N and (b) reverse N^r.

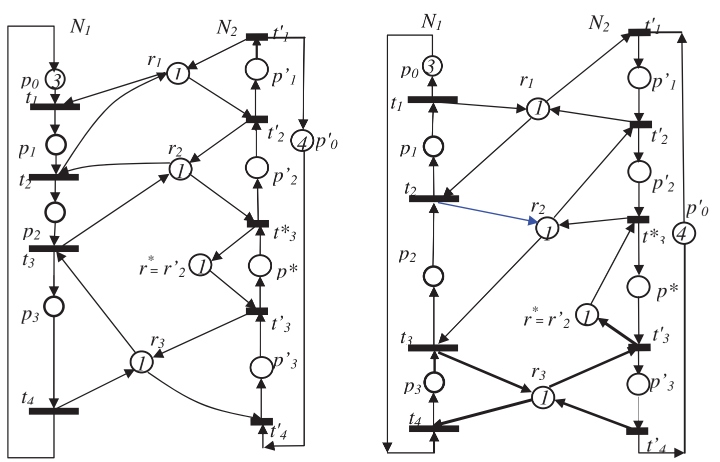

FIG. 4. Third-order *Bottom-Right* system (a) N and (b) reverse N^r.

2.1. *The classification of Control Related States*

DEFINITION 2 (Chao, 2014) $s = (x_1x_2 \ldots x_k), x_i = 1$, 0 or $-1, i = 1$ to k, is a state for a kth-order system N, x_i is the token initially at r_i to sit at: $p_i(N_1), r_i$ or $p'_i(N_2)$, respectively. $(x_ix_{i+1} \ldots x_qx_{q+1}), k \geq i \geq 1, k \geq q \geq i \geq 1$ (embedded in s) is a substate of s.

By Definitions 1 and 2, we transform the notation of states from the viewpoint of the token distribution between the 'place's into the viewpoint of 'resource's. In (7) of Definition 1, we define r_i state denoted by 1(token at p_i), -1(token at p'_i) and 0 (token at r_i). By this state notation, we not only shorten the

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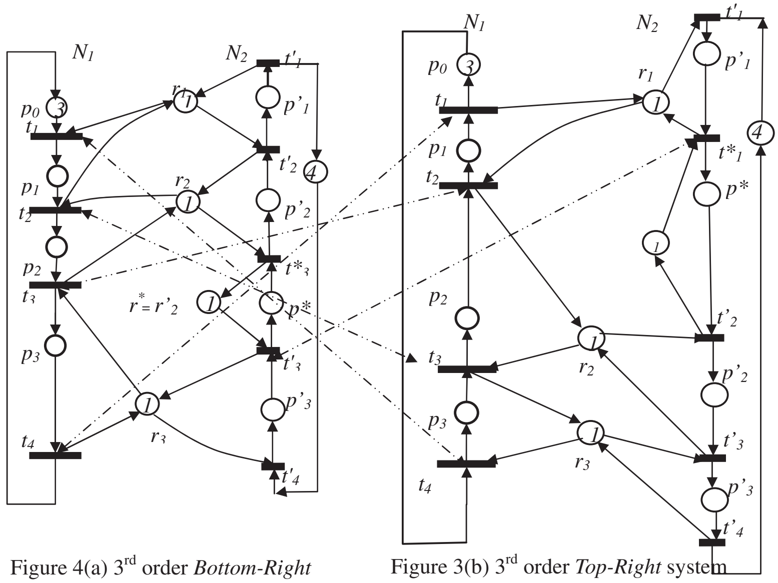

Figure 4(a) 3rd order *Bottom-Right* system *N* (Chao and Yu 2014).

Figure 3(b) 3rd order *Top-Right* system reverse N^r (Chao and Yu 2013).

FIG. 5. The mapping diagram of *Bottom-Right* and reverse net of *Top-Right*.

representation of states in INA (Integrated Net Analyser, 1992), but also it is easy to link it to the figures shown in this paper. For example, a state $(1 - 1 - 1)$ in Fig. 1 can clearly show that operation places p_1 in left process, p'_2 and p'_3in right process contain the tokens and is a deadlock state due to the empty siphon in a third-order system, while in INA using the tokens distribution of 11 operation/resource places to show a state with the number of tokens in $p_1 = 1, p'_2 = 1$ and $p'_3 = 1$ and is hardly associated with a deadlock state.

Let N be a PN and N^r be the reverse net of N. N^r is the net where all of the input arcs in N are reversed to output arcs; output arcs are reversed to input arcs. The net in Fig. 3(b) (resp. 4(b)) is the reverse net of 3(a) (resp. 4(a)). Rebuilding the index number of transitions (t_1, t_2, t_3, t_4) as (t_4, t_3, t_2, t_1), $(t'_1, t'_2, t^*_3, t'_3, t'_4)$ as $(t'_4, t'_3, t'_2, t^*_1, t'_1)$ and the index number of resources (r_1, r_2, r_3) as (r_3, r_2, r_1) in Fig. 4(a), we can find that *Bottom-Right* and the reverse net of *Top-Right* are the same structure nets, as shown in Fig. 5. That is, the reverse net N^r of *Top-Right* (Fig. 3(b)) is *Bottom-Right* (Fig. 4(a)); also, the reverse net N^r of *Bottom-Right* (Fig. 4(b)) is *Top-Right* (Fig. 3(a)). In addition, a reverse state of state (abc) in N is (cba) in N^r.

By enumerating the token flow of each resource place, Chao (2014) proposed the concept of a complete reachability graph (Fig. 6), which lists all states and all paths from which any state can be reachable for all states in a kth-order system. Letting $(0_1 \ldots 0_k)$ be the initial state, based on a complete reachability graph of a kth-order system, we can say that a state is a *reachable state* if there is a directed path from the initial state $(0_1 \ldots 0_k)$; a *live state* if there is a directed path from a state to the initial state; a *deadlock state* is a state that has no output arc; a *forbidden state* is a state that has no directed path to the initial state but has a directed path to a deadlock state; *non-reachable states* are the states that have no directed path from the initial state; *non-reachable + empty-siphon states* are states that are non-reachable from the initial state in both N and the reverse net of N.

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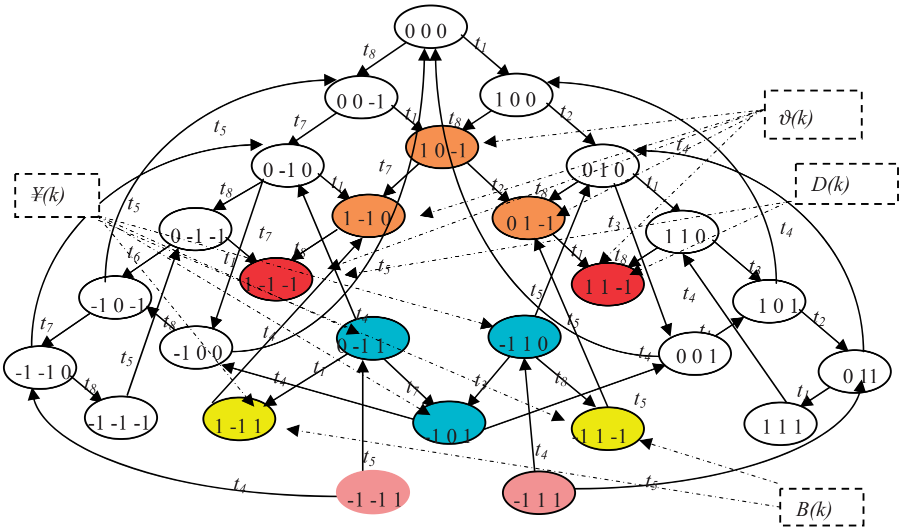

FIG. 6. Complete reachability graph of a third-order system (Fig. 1).

According to graph theory, Chao (2014) found the important Lemmas 1 and 2 and Theorem 1.

LEMMA 1 (Chao, 2014) Any forbidden state in N is non-reachable in N^r.

LEMMA 2 (Chao, 2014) Any non-reachable state in N is a forbidden one or a non-reachable one in N^r.

THEOREM 1 (Chao, 2014) $\vartheta(k) = ¥(k) - B(k)$, where $\vartheta(k)$, $¥(k)$ and $B(k)$ are the number of forbidden, non-reachable and non-reachable+empty-siphon states in a kth-order system, respectively.

Extending Lemmas 1 and 2, we have

THEOREM 2 Any reverse state of a live state in N is a live state in N^r, and the number of live states in N is equal to the number of live states in N^r.

Proof. Assume that the reverse state s_{L}^r of a live state s_{L} in N is not a live state in N^r, being perhaps a forbidden state or non-reachable state instead. This assumption means that s_{L}^r is a forbidden or a non-reachable state in N^r but s_{L} is a live state in N, which violates Lemma 1 or Lemma 2. Hence, s_{L}^r must be a live state in N^r. Assume that the number of live states in N and N^r is not equal. This means that there is a state s_{L} with its reverse state s_{L}^r being not a live state in both N and N^r, which also violates Lemma 1 or Lemma 2. Hence, the number of live states in N is equal to the number of live states in N^r. □

In Fig. 6, there is a directed path from the initial state (0 0 0) to the deadlock state $(1\ {-1}\ {-1})$ in N: $(0\ 0\ 0) \rightarrow (0\ 0\ {-1}) \rightarrow (1\ 0\ {-1}) \rightarrow (1\ {-1}\ 0) \rightarrow (1\ {-1}\ {-1})$. In N^r, we can find that there is a path from the $({-1}\ {-1}\ 1)$ state to the initial state (0 0 0): $({-1}\ {-1}\ 1) \rightarrow (0\ {-1}\ 1) \rightarrow ({-1}\ 0\ 1) \rightarrow ({-1}\ 0\ 0) \rightarrow (0\ 0\ 0)$, where $({-1}\ {-1}\ 1)$ is the reverse state of $(1\ {-1}\ {-1})$. Hence, we have Theorem 3.

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THEOREM 3 A state s_R in N is a reachable state if and only if the reverse state of s_R (maybe a non-reachable state) is reachable to the initial state and contains no forbidden substate in N^r.

Proof. A reachable state s_R in N is a state that is reachable from the initial state through a directed path σ. Reversing all of the input arcs of σ, we can find that there is also a directed path σ' from the reverse state of s_R to the initial state in N^r. A state s'_R that belongs to N^r and contains a forbidden substate will be reachable to a deadlock state but not the initial state so that the reverse state of s'_R will be a non-reachable state in N. Hence, s_R, the reverse state of s'_R which contains a forbidden substate, is not a reachable state in N. □

Below, we list the important properties of Control Related States in a kth-order system (Chao, 2014).

For the third-order system, there are three types of unmarked (resp. non-reachable) siphon states: $(1\ -1\ x)$, $(x\ 1\ -1)$ and $(1\ 0\ -1)$ [resp. $(-1\ 1\ x)$, $(x\ -1\ 1)$ and $(-1\ 0\ 1)$], where $x = -1,\ 0,\ 1$.

By Definition 2, we have some characteristics of non-reachable and forbidden states of a kth-order system.

A substate of $(-1\ x\ x \ldots x\ 1)(x = 1,\ 0,\ -1)$ *corresponds to a non-reachable state (Chao, 2014).*

A substate of $(1\ x\ x \ldots x\ -1)(x = 1,\ 0,\ -1)$ *corresponds to a forbidden or a non-reachable state (Chao, 2014).*

State $s = (x\ x \ldots x\ 1\ x\ x \ldots x\ -1\ x\ x \ldots x\ 1\ x\ x \ldots x\ -1\ x\ x \ldots x)$ *cannot be a reachable state. This means that a reachable state cannot have two substates of* $(1\ x\ x \ldots x\ -1)$ *(Chao, 2014).*

If $s = (x_1 x_2 \ldots x_{i-1} 1_i x_{i+1} x_{i+2} \ldots x_k)$ *does not carry a substate of* $(1_g x_{g+1} x_{g+2} \ldots x_k)$, $g > i$, *then* s *with* $x_m = 0$ *or* $1, m = 1$ *to* $i - 1$ *and* $x_j = 0$ *or* $-1, j = i + 1$ *to* k *are the only reachable states (Chao, 2014).*

2.2. *Computation of the number of reachable states*

By enumerating the token distribution of a kth-order system, Chao (2014) has proven:

LEMMA 3 (Chao, 2014)

(1) s is a live state if and only if $s = \{(y_1 \ldots y_k) | y_i = -1 \text{ or } 0\}$, or $s = \{(x_1 \ldots x_k) | x_i = 1 \text{ or } 0\}$.

(2) The set of live states $L_k = \{(x_1 \ldots x_k) | x_i = 1 \text{ or } 0\} \cup \{(y_1 \ldots y_k) | y_i = -1 \text{ or } 0\} = L_a \cup L_b$.

(3) The total number of live states is $2^{k+1} - 1$.

THEOREM 4 (Chao, 2014)

(1) The possible reachable states are $s = \{(x_1 x_2 \ldots x_j y_{j+1} \ldots y_k) | 0 \leq j \leq k\} = \{(x_1 \ldots x_j 1 y_{j+2} \ldots y_k) | 1 \leq j \leq k\} \cup \{(y_1 \ldots y_k)\}$, where $x_i = 1$ or $0 (i = 1$ to $j)$ and $y_p = 0$ or $-1 (p = j + 2$ to $k) = L_c \cup L_d$.

(2) The total number of reachable states is $(k + 2)2^{(k-1)}$.

COROLLARY 1 (Chao, 2014)

(1) The number of forbidden states $\vartheta(k) = (k - 2)2^{(k-1)} + 1$.

(2) The number of non-reachable states $¥(k) = 3^k - (k + 2)2^{(k-1)}$.

(3) The number of non-reachable+empty-siphon states $B(k) = 3^k - k2^k - 1$.

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THEOREM 5 (Chao, 2014) In a kth-order system, a deadlock state has the pattern: $(1_1 1_2 \ldots 1_m - 1_{m+1} - 1_{m+2} \ldots - 1_k)$, $1 \leq m < k$. The total number of deadlock states $D(k) = k - 1$.

To sum up, the total number of each type of Control Related States in a kth-order system in Chao (2014) is shown below.

The total number of states is 3^k.

The total number of live states $L(k) = 2^{k+1} - 1$.

The total number of reachable states $R(k) = (k+2)2^{(k-1)}$.

The number of forbidden states $\vartheta(k) = R(k) - L(k) = (k-2)2^{(k-1)} + 1$.

The number of non-reachable states $¥(k) = 3^k - R(k) = 3^k - (k+2)2^{(k-1)}$.

The number of non-reachable + empty-siphon states $B(k) = ¥(k) - \vartheta(k) = 3^k - k2^k - 1$.

The total number of deadlock states $D(k) = k - 1$.

3. Methodology to enumerate the Control Related States of a Bottom-Right kth-order system

We first define *the equivalent* (defined in Definition 3) of a net. By this instrument, we can analyse the effect of a non-sharing resource in a kth-order system.

DEFINITION 3 (Chao & Yu, 2014) The equivalent $N^e = (P^e \cup P^e_{\mathrm{R}}, T^e, F^e, W^e)$ of a net $N = (P \cup P_{\mathrm{R}}, T, F, W)$ (P_{NR} is the set of non-sharing places) is defined as

(1) ${P^e}_{\mathrm{R}} = P_{\mathrm{R}} \backslash P_{\mathrm{NR}}$;

(2) $P^e = P \backslash \bigcup_{r \in P_{\mathrm{NR}}} H(r)$;

(3) $T^e = T \backslash \bigcup_{r \in P_{\mathrm{NR}}} r^{\bullet}$;

(4) $F^e = (F \bigcup_{r \in P_{\mathrm{NR}}} ({}^{\bullet}r, r^{\bullet^{\bullet}}) \cup ({}^{\bullet}(r^{\bullet}), {}^{\bullet}r) \backslash \bigcup_{r \in P_{\mathrm{NR}}} [(H(r), H(r)^{\bullet});$

$\cup ({}^{\bullet}H(r), H(r)) \cup ({}^{\bullet}r, r) \cup (r, r^{\bullet}) \cup (r^{\bullet}, r^{\bullet^{\bullet}}) \cup ({}^{\bullet}(r^{\bullet}), r^{\bullet})]$

(5) $W^e : F^e \rightarrow Z$.

We say that the net in Fig. 1 is the equivalent of the net in Figs. 3(a) and 4(a) because the net is exactly the same as the net except that the net has one non-sharing resource place r^*.

DEFINITION 4 (Chao & Yu, 2014) The reverse net of N^e is denoted as N^{er}.

In this article, we denote N^e as a kth-order system and N as a variant kth-order system that contains a non-sharing resource (for example, *Bottom-Right*). Let state s in N^e $be(x_1 x_2 \ldots x_{k-1} x_k)$. By Definition 1, the state of *Top-Right* will be $(x_1^y \ldots x_{k-1} x_k)$; *Bottom-Right* will be $(x_1 \ldots x_{k-1}^y x_k)$, where $y = 0$ or -1. According to the reverse net concept in Section 2, the state $(x_1^y \ldots x_{k-1} x_k)$ in *Top-Right* and state $(x_k \ldots x_2^y x_1)$ in *Bottom-Right* are the reverse states of each other, where $y = 0$ or -1.

For every reachable (resp. live) state $s(x\,x \ldots x\,x)$ in N^e (a kth-order system), both states $(x\,x\,x \ldots x^0\,x)$ and $(x\,x\,x \ldots x^{-1}\,x)$ in *Bottom-Right* are reachable (resp. live) states. We (Chao & Yu, 2014) have shown that, in N, the number of reachable states $(R') > 2R$ and the number of live states $(L') > 2L$.

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Because of a non-sharing resource, we have shown the following: (1) markings that are non-reachable in N^e may become reachable in N(the number of which is denoted as Θ); (2) forbidden markings in N^e may be live in N (the number of which is denoted as $C(k)$); and (3) non-reachable markings in N^e may be live in N(the number of which is denoted as $A(k)$). Thus, we have:

$$R' = 2R + \Theta, \tag{1}$$

$$L' = 2L + A(k) + C(k). \tag{2}$$

3.1. *The characteristics of a Bottom-Right kth-order system*

Let N^e be a kth-order system and N^{B} be *a Bottom-Right* kth-order system in this section. Here, we list the important characteristics of *Bottom-Right* in Chao & Yu (2014).

For the *Bottom-Right* third-order system, there are three types of unmarked (resp. non-reachable) siphon states: $(1\ -1^{-1}\ x)$, $(x\ 1^{-1}\ -1)$ and $(1\ 0^{-1}\ -1)$ [resp. $(-1\ 1^{-1}\ x)$, $(x\ -1^{-1}\ 1)$ and $(-1\ 0^{-1}\ 1)$], where $x = -1,\ 0,\ 1$.

LEMMA 4 A substate of $(-1\ x\ x\ldots x^{-1}\ 1)$ $(x = 1,\ 0,\ -1)$ corresponds to a non-reachable state.

COROLLARY 2 A substate of $(1\ x\ x\ldots x^{-1}\ -1)(x = 1,\ 0,\ -1)$ corresponds to a forbidden or non-reachable state.

LEMMA 5 Both $s = (1\ 0_2\ 0_3\ 0_4\ldots 0^{-1}_{k-1}\ 0_k)$ and $s' = (-1\ 0_2\ 0_3\ 0_4\ldots 0^{-1}_{k-1}\ 0_k)$ correspond to two legal markings M.

LEMMA 6 Let $s = (x_1x_2x_3x_4\ldots -1_i0_{i+1}\ldots 0^0_{k-1}1_k)$, where $i = 1$ to $k-1$; $x_j = 0, j = i+1$ to $k-1$; $x_n = 0$ or $1, 0 \le n < i-1$, be such that only the bottom $r_i - r_k$ siphon in N^{er} is unmarked.

(1) M is non-reachable in N^e.

(2) $M^* = M + r^*$ is reachable in N^{B}.

(3) The total number of such M^* is $2^{k-1} - 1$.

THEOREM 6 The total number of reachable states in Bottom-Right is $2R + 2^{k-1} - 1 = 2(k+2)2^{(k-1)} + 2^{(k-1)} - 1 = (2k+5)2^{(k-1)} - 1$.

By Lemma 13 in Chao & Yu (2014), $s = (x_1x_2\ldots 1_j\ldots 0_{k-2}0^0_{k-1} - 1_k)$ is a live state in *Bottom-Right*, where $x_i = 0$ or $1, i = 1$ to $j-1$; s is a non-reachable state where $x_i = -1, i = 1$ to $j-1$. The total number of possible live states is $2^{(j-1)}$.

THEOREM 7 The total number of forbidden markings in N^e that may be live in N^{B} is $C_{\mathrm{B}}(k) = 2^{k-1} - 1$.

By Lemma 14 in Chao & Yu (2014), $s = (x_1x_2\ldots - 1_{k-2}0^0_{k-1}1_k)$ is a non-reachable state in *Bottom-Right*, where $x_i = -1, i = 1$ to $k-3$; s is a live state where $x_i = 0, i = 1$ to $k-3$; s is a forbidden state where $x_i = 1, i = 1$ to $k-3$. The total number of possible live states is 1.

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By Lemma 15 in Chao & Yu (2014), $s = (x_1x_2 \ldots -1_j \ldots 0^0_{k-1}1_k)$ is a live state in *Bottom-Right*, where $x_i = 0, i = 1$ to $j-1$; a non-reachable state where $x_i = -1, i = 1$ to $j-1$; a forbidden state where $x_i = 1, i = 1$ to $j-1$. The total number of possible live states is 1.

THEOREM 8 The total number of non-reachable markings in N^e that may be live in N^{B} is $A_B(k) = k - 1$.

THEOREM 9 The total number of live states in Bottom-Right is $18 \times 2^{k-2} + k - 4$.

4. Computation of Control Related States of a Top-Right kth-order system

Let N^e be a kth-order system and N^{T} be a *Top-Right* kth-order system in this section.

OBSERVATION 1 (Chao & Yu, 2013)

(1) Any unmarked siphon state carries a substate $(1^{-1}00 \ldots 0-1)$.

(2) Any non-reachable state carries a substate $(-1^{-1}00 \ldots 01)$, where the number '0' goes from 0 to $k-2$.

Note that the $(1^000 \ldots 0-1)$ obtained by replacing 1^{-1} with 1^0 is not an unmarked siphon state because r_2 is not used by any process and t_1^* is potentially firable in Fig. 3(a).

For the third-order system, there are three types of unmarked (resp. non-reachable) siphon states: $(1^{-1} -1\, x)$, $(x\ 1\ -1)$ and $(1^{-1}\ 0\ -1)$ [resp. $(-1^{-1}\ 1\ x)$, $(x\ -1\ 1)$ and $(-1^{-1}\ 0\ 1)$], where $x = -1,\ 0,\ 1$.

LEMMA 7 (Chao & Yu, 2013) A substate of $(-1^{-1}\ x\ x \ldots x\ 1)(x = 1, 0, -1)$ corresponds to a non-reachable state, where the number l of x's goes from 0 to $k-2$; $l = 0$ to $k-2$.

Proof by model. According to Corollary 2, a substate of $(1\,x\,x \ldots x^{-1}\ -1)(x = 1,\ 0,\ -1)$ corresponds to a forbidden or non-reachable state of the *Bottom-Right* system. Hence, the reverse substate $(-1^{-1}\,x\,x \ldots x\,1)$ in *Top-Right* is a non-reachable state according to Lemmas 1 and 2. □

COROLLARY 3 (Chao & Yu, 2013) A substate of $(1^{-1}\ x\ x \ldots x\ -1)(x = 1,\ 0,\ -1)$ corresponds to a forbidden or non-reachable state, where the number l of x's goes from 0 to $k-2$; $l = 0$ to $k-2$.

Proof by model. According to Lemma 4, a substate of $(-1\ x\ x \ldots x^{-1}\ 1)(x = 1,\ 0,\ -1)$ corresponds to a non-reachable state in *Bottom-Right*. Hence, the reverse substate $(1^{-1}\ x\ x \ldots x\ -1)$ in *Top-Right* corresponds to a forbidden or non-reachable state according to Lemmas 1 and 2. □

LEMMA 8 (Chao & Yu, 2013) Let M be a reachable marking in N^e; then, both $M^* = M + r^*$ and $M' = M + p^*$ are reachable in N^{T}.

Proof. There are no unmarked siphons in N^{er} because M is reachable in N^e. There are also no unmarked siphons under both M' and M^* in N^{T}. Hence, they are both reachable in N^{T}. □

The following lemma helps to prove in the sequel that some states are legal.

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LEMMA 9 (Chao & Yu, 2013) Both $s = (1^{-1}\ 0_2\ 0_3\ 0_4 \ldots 0_{j-1}\ 0_j)$ and $s' = (-1^0\ 1_2\ 0_3\ 0_4 \ldots 0_{j-1}\ 0_j)$ correspond to two legal markings M.

Proof. Let $\sigma = t_2 t_3 \ldots t_{n-1} t_n t_1^*$. Then, $M[\sigma > M_0$; hence, M is a *legal marking* because M does not necessarily evolve to a deadlock state. □

Markings that are non-reachable in N^e may become reachable in N^{T}.

LEMMA 10 (Chao & Yu, 2013) Let M be such that only the top $r_1 - r_2$ region in N^{er} is unmarked.

(1) M is non-reachable in N^e.

(2) $M^* = M + r^*$ is reachable in N^{T}.

Proof. (1) By *Lemma* 1, M is non-reachable in N^e.

(2) Under M^*, there are no unmarked siphons in N^{T}; hence, $M^* = M + r^*$ is reachable in N^{T}. □

In general, we have

LEMMA 11 (Chao & Yu, 2013) Let $s = (-1^0 0_2 0_3 0_4 \ldots 0_{j-1} 1_j x_{j+1} x_{j+2} \ldots x_k)$ be such that only the top $r_1 - r_j$ siphon in N^{er} is unmarked.

(1) M is non-reachable in N^e.

(2) $M^* = M + r^*$ is reachable in N^{T}.

(3) The total number of such M^* is $R(k - j)$.

Proof by model. Let $s' = (x_k \ldots x_{j+2}\ x_{j+1}\ 1_j\ 0_{j-1} \ldots 0_4\ 0_3\ 0_2^0 - 1)$ be the reverse state of s. By Lemma 13 in Chao & Yu (2013) , s' is a live state, where $x_i = 0$ or $1, i = j + 1$ to k; by Theorem 3, s is a reachable state *if and only if* s' is reachable to the initial state $(0_k \ldots 0_{j+2}\ 0_{j+1}\ 0_j\ 0_{j-1} \ldots 0_4\ 0_3\ 0_2^0\ 0)$. Hence, the total number of such M^* is dependent on the number of possibilities such that the $(r_k - r_{j+1})$ region can be reachable to $(0_k \ldots 0_{j+2}\ 0_{j+1})$, which equals $R(k - j)$. □

THEOREM 10 (Chao & Yu, 2013) The total number of reachable states in N^{T} is $R'(k) = 2R(k) + \Theta(k-2)$, where $\Theta(k - 2) = \sum_{j=2\ to\ k} R(k - j)$.

Proof. There are two cases:

(1) M is reachable in N^e.

By Lemma 8, both $M^* = M + r^*$ and $M' = M + p^*$ are reachable in N^{T}. Hence, there are $2R$ such states because there are R reachable states in N^{T}.

(2) M is non-reachable in N^e.

By Lemma 11, there are $\Theta(k - 2) = \sum_{j=2\ to\ k} R(k - j)$ states that are non-reachable in N^e but are reachable inN^{T}.

Combining (1)–(2), we have $R'(k) = 2R(k) + \Theta(k - 2)$. □

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COROLLARY 4 (Chao & Yu, 2013) $R'(k) = 2(k+2)2^{(k-1)} + \Theta(k-2) = (5k+7)2^{k-2}$.

Proof.

$$\begin{aligned} R'(k) &= 2(k+2)2^{(k-1)} + \Theta(k-2) = 2R(k) - (R(k) - 3R(k-1)) \\ &= R(k) + 3R(k-1) = (k+2)2^{k-1} + 3(k+1)2^{k-2} = (5k+7)2^{k-2}. \end{aligned}$$ □

According to Theorems 2 and 9, we can derive the number of live states of *Top-Right* system as $18 \times 2^{k-2} + k - 4$. For the integrity of proof procedure and to validate Theorem 2, we list the Lemmas B3–B6 and Theorems B1–B3 in Appendix B to show how to enumerate the number of live states of *Top-Right* system.

THEOREM 11 (Chao & Yu, 2013) $\vartheta'(k) = (5k-11)2^{k-2} - (k-4)$.

Proof.

$$\begin{aligned} \vartheta'(k) &= R'(k) - L'(k) \\ &= (5k+7)2^{k-2} - (18 \times 2^{k-2} + k - 4) \\ &= (5k-11)2^{k-2} - (k-4). \end{aligned}$$ □

THEOREM 12 (Chao & Yu, 2013) $¥'(k) = 2 \times 3^k - (5k+7)2^{k-2}$

Proof.

$$\begin{aligned} ¥'(k) &= 2 \times 3^k - R'(k) \\ &= 2 \times 3^k - (5k+7)2^{k-2}. \end{aligned}$$ □

THEOREM 13 (Chao & Yu, 2013) Denote $D'(k)$ as the total number of deadlock states in N^{T}, where $D'(k) = D(k) + D(k-1) = 2k - 3$.

Proof. p^* is a trap of strict minimal siphon (SMS) s^*, which contains a non-sharing resource r^* (Fig. 3(a)); the deadlock pattern of N^{T} must include two situations: $M(p^*) = 0$(s^* is not an empty siphon) and $M(p^*) = 1$(s^* may be an empty siphon).

(1) $M(p^*) = 1$: The deadlock pattern of $N^{\mathrm{T}}(1_1^{-1}1_2 \ldots 1_i \ldots 1_j - 1_{j+1} \ldots - 1_k), 1 \leq j \leq k-1$ is reachable. In this situation, the number of deadlock states is $D(k)$, determined by N^e.

(2) $M(p^*) = 0 : M(p_1) = 1$ (trap of s^*), t_1^* cannot be enabled; $M(p_2) = 1$ (siphon of s^*), t_2' cannot be enabled. Because p_2 is the trap of the next SMS $\{p_3, r_2, r_3, p_2'\}$, the deadlock condition of subnet $(x_2 \ldots x_k)$ must be met. In this case, the total number of deadlock states is $D(k-1)$, which is determined by a $(k-1)$th-order system.

Hence, $D'(k) = D(k) + D(k-1) = k - 1 + k - 2 = 2k - 3$. □

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The formulae of Control Related States listed above are consistent with the reachability analysis using the INA (Starke 1992) tool.

Application: In Appendix C, we extend our methodology to a *Top-Left* k-net system, where any resource place is shared between μ processes (called k-net; *Top-Left* k-net is a k-net with a *Top* non-sharing resource place in the left process). The total number of live states is $Ł_k = 2^k + (\mu)^k - 2[Ł'_k = 2Ł_k + (\mu)^{k-1} - 1 + (\mu - 1)(k-1)$ for the *Top-Left* k-net]. The total number of reachable states can be similarly analysed as $Ŗ_k = 2^k + (\mu - 1)2^{(k-1)}(1 - x^k)/(1 - x), x = \mu/2[Ŗ'_k = 2Ŗ_k + ((\mu)^{k-1} - 1)$ for the *Top-Left* k-net]. $\vartheta_k = Ł_k - Ŗ_k$. See Appendix C for an explanation.

5. Conclusions

Based on Lemmas 1 and 2, we first derive Theorems 2 and 3 to prove that a reverse state of a live state in a PN is also a live state in its reverse net; a reverse state of a reachable state in a PN will contain no forbidden sub-states. Due to the contributions of Lemmas 1 and 2, Theorems 1–3, we propose a new knowledge-based analysis concept, 'proof by model', for the construction of a closed-form solution of a PN based on the validated information of its reverse net. This concept is especially significant for the oncoming so-called Industry 4.0 intelligent manufacturing era, because when a resource is dynamically allocated, we should not re-analyse the whole system by siphon computation for a new deadlock avoidance/prevention policy of a new PN model, but rather reuse the validated information to construct the policy. The 'proof by model' based on the reverse net concept is our first step towards knowledge-based analysis of the PN reachability problem.

Here, we demonstrate how to apply 'proof by model' to the proof procedures of closed-form formulae construction for a *Top-Right* kth-order system with validated information from a *Bottom-Right* system, which is the reverse net of *Top-Right*. Some regular proof procedures by siphon concept are shown in Appendix B for comparison. Applying the 'proof by model' concept, the analysis effort can be reduced to focus only on the computation of the number of reachable and deadlock states only because according to Theorem 2 both *Top-Right* and *Bottom-Right* systems have the same number of live states, which is the validated information of *Bottom-Right* system. Hence, Lemmas B3–B6 and Theorems B1–B3 that are applied for enumerating the number of live states of *Top-Right* are redundant, but we show them here for the integrity of the proof procedure and the validation of Theorem 2.

According to the knowledge-based analysis concept, we can also construct the knowledge of a validated sub-states information system, by which more complicated PNs can be constructed. Moreover, many future research works can be extended from this concept, such as the effects of adding non-sharing resources, processes or tokens into a PN such that with the new elements listed above, the system could possibly be a 'self-learning' knowledge-based reachability analysis system of PN.

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A. Preliminaries about Petri nets

A Petri net is a four-tuple $N = (P, T, F, W)$, where P is the set of *places*, T is the set of *transitions*, $F \subseteq (P \times T) \cup (T \times P)$ is called *flow relation* of the net, which is represented by arcs with arrows from places to transitions or vice versa, and W: $F \rightarrow$ Z (the set of nonnegative integers) is a mapping that assigns a weight to an arc. M_0: $P \rightarrow Z$ is the *initial marking* assigned to each place $p \in P, M_0(p)$ tokens. (N, M_0) is called a marked net or a net system. In the special case where W maps onto $\{0, 1\}$, the PN is said to be *ordinary* (otherwise, *general*). $N' = (P', T', F', W')$ is called a subnet of N where $P' \subseteq P, T' \subseteq T, F' = F \cap ((P' \times T') \cup (T' \times P')$ and W: $F' \rightarrow$ Z.

The set of input (resp. output) transitions of a place p is denoted by ${}^{\bullet}p$ (resp. $p^{\bullet}$). Similarly, the set of input (resp. output) places of a transition t is denoted by ${}^{\bullet}t$ (resp. $t^{\bullet}$). Finally, an ordinary PN such that (s.t.) $\forall t \in T, |t^{\bullet}| = |{}^{\bullet}t| = 1$ is called a *State Machine* (*SM*). It is called a *Marked Graph* if $\forall p \in {}^{\bullet}P, |p^{\bullet}| = |{}^{\bullet}p| = 1$. A PN is strongly connected if $\forall x, x' \in (P \cup T)$ such that $x \neq x'$ and there is a direct path from x to x'. A node x in $N = (P, T, F, W)$ is either $p \in P$ or $t \in T$. An elementary direct path Γ in N is a graphical object containing a sequence of nodes such that there is an arc between each two successive nodes in the sequence with the notation: $\Gamma = [n_1 n_2 \ldots n_k], k \geq 1$, where $n_i \neq n_j$ for $i \neq j$. N^r is the reverse net of N obtained by reversing the direction of all arcs in N with the initial marking unchanged. A is the incidence matrix of a net with m places and n transitions: $A = [a_{ij}]$; a matrix of integers and its typical entry are given by $a_{ij} = a_{ij}^{+}$—a_{ij}^{-}, where $a_{ij}^{-} = W(i, j)$ is the weight of the arc from transition i to its output place j and $a_{ij}^{+} = W(j, i)$ is the weight of the arc to transition i from its input place j. $A^r = -A$, where A^r is the incidence matrix of the reverse net N^r of N.

Given a marking M, a transition t is *enabled* if $\forall p \in {}^{\bullet}t, M(p) \geq W(p, t)$; this is denoted by $M[t >$. Firing an enabled transition t results in a new marking M_1, which is obtained by removing $W(p, t)$ tokens from each place $p \in {}^{\bullet}t$ and placing $W(t, p')$ tokens into each place $p' \in t^{\bullet}$, moving the system state from M_0 to M_1. Repeating this process, the state reaches M' by firing a sequence $\sigma = t_1 t_2 \ldots t_k$ of transitions. M' is said to be reachable from M_0; i.e. $M_0[\sigma > M'$. M_0 is reached in N^r by firing a sequence $\sigma^r = t_k t_{k-1} \ldots t_2 t_1$ of transitions from M'; i.e. $M'[\sigma^r > M_0$ in N^r and $M_0 = M' + A^r \bullet x(\sigma^r)$, where $x(\sigma^r)$ is the firing vector to reach M_0 from M'. $R(N, M_0)$ is the set of markings reachable from M_0. A forbidden (resp. live) marking or state is one that is (resp. not), or necessarily evolves into, a deadlock marking.

A transition $t \in T$ is live at M_0 if $\forall M \in R(N, M_0), \exists M' \in R(N, M), t$ is enabled at M'. A PN is *live* at M_0 *if* $\forall t \in T, t$ is live at M_0. A PN is said to be *deadlock-free* if at least one transition is enabled at every reachable marking.

For a Petri net (N, M_0), a non-empty subset S (resp. τ) of places is called a *siphon* (resp. *trap*) if ${}^{\bullet}S \subseteq S^{\bullet}$ (resp. $\tau^{\bullet} \subseteq^{\bullet} \tau$), i.e. every transition having an output (resp. input) place in S has an input (resp. output) place in S (resp. τ). A siphon is a set of places where tokens can continuously flow out such that $M_0(S) = \sum_{p \in S} M_0(p) = 0$, where S is called an *empty siphon* or *unmarked siphon* at M_0; all output transitions of S are permanently dead. A *minimal siphon* does not contain a siphon as a proper subset. It is called a *strict minimal siphon* (*SMS*), denoted by S, if it does not contain a trap.

An integer vector Y (with components $Y(p), p \in P$), denoted by $Y = \sum Y(p)p$, is called a P-invariant if $Y \neq 0$ and $Y^{\mathrm{T}} \bullet A = 0$, where A is the incidence matrix. $\|Y\| = \{p \in P | Y(p) \neq 0\}$ is the *support* of Y. A *minimal* P-invariant does not contain another P-invariant as its proper subset. If a siphon $S \subset \|Y\|$, then $[S] = \|Y\| \backslash S$ is called the *complementary siphon* of S and $S \cup [S]$ is the *support* of a P-invariant.

DEFINITION A1 (Ezpeleta *et al.*, 1995) A simple sequential process (S^2P) is a net $N = (P \cup \{p^0\}, T, F)$ where (1) $P \neq \emptyset$ $p^0 \notin P$ (p^0 is called the process idle or initial or final operation place), (2) N is a strongly connected SM and (3) every circuit of N contains the place p^0.

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Transitions in $p^{0\bullet}$ and ${}^\bullet p^0$ are called *source* and *sink transitions*, respectively.

Definition A2 (Ezpeleta *et al.*, 1995) A simple sequential process with resources (S^2PR), also called a working process, is a net $N = (P \cup \{p^0\} \cup P_R, T, F)$ such that (1) the subnet generated by $X = P \cup \{p^0\} \cup T$ is an S^2P; (2) $P_R \neq \emptyset$ and $P \cup \{p^0\} \cap P_R = \emptyset$; (3) $\forall p \in P, \forall t \in {}^\bullet p, \forall t' \in p^\bullet, \exists r_p \in P_R, {}^\bullet t \cap P_R = t'^\bullet \cap P_R = \{r_p\}$; (4) the two following statements are verified: $\forall r \in P_R$: (a) ${}^{\bullet\bullet}r \cap P = r^{\bullet\bullet} \cap P \neq \emptyset$ and (b) ${}^\bullet r \cap r^\bullet = \emptyset$; and (5) ${}^{\bullet\bullet}(p^0) \cap P_R = (p^0)^{\bullet\bullet} \cap P_R = \emptyset$. $\forall p \in P$, where p is called an operation place. $\forall r \in P_R$, where r is called a resource place. $H(r) = {}^{\bullet\bullet}r \cap P$ denotes the set of holders of r (i.e. operation places that use r). Any resource r is associated with a minimal P-invariant whose support is denoted by $\rho(r) = \{r\} \cup H(r)$.

Definition A3 (Ezpeleta *et al.*, 1995) A system of S^2PR (S^3PR) is defined recursively as follows: (1) An S^2PR is defined as an S^3 PR and (2) Let $N_i = (P_i \cup P_i^0 \cup P_{Ri}, T_i, F_i), i \in \{1, 2\}$ be two S^3 PR such that $(P_1 \cup P_1^0) \cap (P_2 \cup P_2^0) = \emptyset$. $P_{R1} \cap P_{R2} = P_C (\neq \emptyset)$ and $T_1 \cap T_2 = \emptyset$. The net $N = (P \cup P^0 \cup P_R, T, F)$ resulting from the composition of N_1 and N_2 *via* P_C (denoted by $N_1 o N_2$) is defined as follows: (1) $P = P_1 \cup P_2$; (2) $P^0 = P_1^0 \cup P_2^0$; (3) $P_R = P_{R1} \cup P_{R2}$; (4) $T = T_1 \cup T_2$; and (5) $F = F_1 \cup F_2$ is also an S^3 PR.

B. Regular proof procedure of *Top-Right*

Lemma B1 A substate of $(-1^{-1}xx\ldots x1)(x = 1, 0, -1)$ corresponds to a non-reachable state, where the number l of x's goes from 0 to $k-2$; $l = 0$ to $k-2$.

Proof. This is proven by induction. The lemma holds for the case of $l = 0$ because $(-1^{-1}1)$ is a non-reachable state, as discussed above. Now, assuming that the lemma holds for $l = 0$ to $i-1$, we need to prove that it also holds for $l = i+1$. There are three possible values of the last x in the substate:

(1) $x = -1$: Then, we have the substate of $(x1) = (-11)$, which corresponds to a non-reachable state.
(2) $x = 1$: The problem is reduced to the substate of $(-1^{-1}xx\ldots x1)$ with $l = i$, which has been assumed to correspond to a non-reachable state.
(3) $x = 0$: Then, we consider the penultimate x. The arguments repeat and, eventually, the substate becomes $(-1^{-1}00\ldots 01)$, which is a non-reachable state according to Observation 1(2). □

Lemma B2 Let $s = (-1^0 0_2 0_3 0_4 \ldots 0_{j-1} 1_j x_{j+1} x_{j+2} \ldots x_k)$ be such that only the top $r_1 - r_j$ siphon in N^{er} is unmarked.

(1) M is non-reachable in N^e.
(2) $M^* = M + r^*$ is reachable in N^T.
(3) The total number of such M^* is $R(k-j)$.

Proof. The proofs of (1) and (2) are similar to that of Lemma 10. (3) $s = (x_{j+1} x_{j+2} \ldots x_k)$ is a substate of M for the $(r_{j+1} - r_k)$ subnet. If there are no unmarked siphons in the reverse of containing the $(r_{j+1} - r_k)$ subnet, so neither will be the reverse of the $(r_{j+1} - r_k)$ subnet. Thus, any unmarked siphon in N^{er} must include r_1 and r_2, which is impossible for the same reason as that held in (1). Thus, the total number of M^* is the same as the number of reachable states in $(r_{j+1} - r_k)$, which equals $R(k-j)$. □

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LEMMA B3 (Chao & Yu, 2013) Let M be a live marking in N^e; then, both $M^* = M + r^*$ and $M' = M + p^*$ are live in N^{T}.

Proof. There are no unmarked siphons in N^e because M is live in N^e. There are also no unmarked siphons under both M' and M^* in N^{T}. Hence, they are both live in N^{T}. □

The number of markings for $C(k)$ and $A(k)$ of *Top-Right* is computed by the following lemma.

LEMMA B4 (Chao & Yu, 2013) Let $s = (1^0 0 - 1_3 x_4 \ldots x_{k-2} x_{k-1} x_k)$ correspond to marking M such that there are unmarked siphons in only the top $r_1 - r_3$ region in N^e.

(1) If $M(p'_4) = 1(x_4 = -1)$, then $M' = M + r^*$ is a forbidden marking (necessarily evolving into an unmarked state) in N^{T}. M' is a non-live marking in N^{T}.

(2) If $M(r_4) = 1(x_4 = 0)$, then no SMS is unmarked under $M' = M + r^*$ in N^{T}. M' may be a live marking in N^{T}.

(3) If $M(p_4) = 1(x_4 = 1)$, then $M' = M + r^*$ is a non-reachable state in N^{T}.

(4) The total number of possible live markings under M is 1.

Proof by model. By Lemma 14 in (Chao & Yu, 2014), $s_{\mathrm{B}} = (x_k x_{k-1} \ldots - 1_3 0_2^0 1_1)$ is a non-reachable state in *Bottom-Right*, where $x_i = -1, i = 4$ to k; a live state where $x_i = 0, i = 4$ to k; a forbidden state where $x_i = 1, i = 4$ to k. The total number of possible live states is 1. The reverse state of $s_{\mathrm{B}} = (x_k x_{k-1} \ldots - 1_3 0_2^0 1_1)$ is $s_{\mathrm{T}} = (1^0 0 - 1_3 x_4 \ldots x_{k-2} x_{k-1} x_k)$. By Theorem 2, we have the total number of possible live markings under M being 1, where $x_i = 0, i = 4$ to k. □

Proof (by siphon concept). (1) Let $t'_2 \in r^{*\bullet}$. Fire t'_2 at M to reach a new state $s' = (1^{-1} 0_2 0_3 - 1_4 x_4 \ldots x_{k-2} x_{k-1} x_k)$, which corresponds to an unmarked siphon state and is forbidden. (2) Fire t_1^* at M' again to reach a new state $s'' = (1^{-1} 0 0_3 0_4 x_5 \ldots x_{k-2} x_{k-1} x_k)$, which corresponds to a legal marking if $x_5 = x_6 = \ldots = x_{k-2} = x_{k-1} = x_k = 0$ based on Lemma 9. Hence, M' may be a live marking in N. (3) $(-1_3 1_4)$ is a substate of an unmarked siphon in N^{er}. Hence, $M' = M + r^*$ is a non-reachable state in N^{T}. (4) This follows from parts of (1)–(3) of this lemma. □

REMARK OF THE PROOF OF **Lemma B4**: (1) By Lemma 2, when $x_4 = -1$ in s_{B} (a non-reachable state), s_{T} will be a forbidden state in *Top-Right* because s_{T} is a reachable state. (2) When $x_4 = 1$ in s_{T}, by Lemma 1, s_{T} will be non-reachable because the reverse state s_{B} is a forbidden state.

LEMMA B5 (Chao & Yu, 2013) Let $s = (1^0\, 0\, 0_3\, 0_4 \ldots 0_{j-1}\, - 1_j\, x_{j+1}\, x_{j+2} \ldots x_k)$ correspond to marking M such that there are unmarked siphons in only the top $r_1 - r_j$ siphon in N^e.

(1) If $M(p'_{j+1}) = 1(x_{j+1} = -1)$, then $M' = M + r^*$ is a forbidden marking (necessarily evolving into an unmarked state) in N. M' is a non-live marking in N^{T}.

(2) If $M(r_{j+1}) = 1(x_{j+1} = 0)$, then no SMS is unmarked under $M' = M + r^*$ in N^{T}. M' may be a live marking in N^{T}.

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(3) If $M(p_{j+1}) = 1(x_{j+1} = 1)$, then $M' = M + r^*$ is a non-reachable state in N^{T}.

(4) The total number of possible live markings under M is 1^{k-j}.

Proof by model. The proof is similar to that for Lemma B4. □

Proof (by siphon concept). (1) Let $t_2' \in r^{*\bullet}$. Fire $t_j' t_{j-1}' \ldots t_3' t_2'$ at M to reach a new state $s' = (1^{-1}\ 0\ 0_3\ 0_4 \ldots 0_j\ -1_{j+1}\ x_{j+2} \ldots x_k)$, which corresponds to an unmarked siphon state and is forbidden. (2) Fire at M' again to reach a new state $s'' = (1^{-1}\ 0\ 0_3\ 0_4 \ldots 0_j\ 0_{j+1}\ x_{j+2} \ldots x_k)$, which corresponds to a legal marking if $x_{j+2} = x_{j+3} = \ldots = x_{k-2} = x_{k-1} = x_k = 0$ based on Lemma 9. (3) $(-1_j 1_{j+1})$ is a substate of an unmarked siphon in N^{er}. Hence, $M' = M + r^*$ is a non-reachable state in N^{T}. (4) Based on parts of (1)–(3) of this lemma, for M' to be a live marking in N^{T}, $x_{j+1} = x_{j+2} = \ldots = x_k = 0$. Hence, the total number of possible live markings under M is 1^{k-j}. □

THEOREM B1 (Chao & Yu, 2013) The total number of forbidden markings in N^{T} that may be live in N^e is $C_T(k) = k - 1$.

Proof. By summing 1^{k-j} (Lemma B5) from $j = 2$ to k, we have $C_T(k) = 1+1+\cdots+1 = \sum_{j=2\ to\ k} 1^{j-2} = k - 1$. □

LEMMA B6 (Chao & Yu, 2013) Let $s = (-1^0\ 0_2\ 0_3\ 0_4 \ldots 0_{j-1}\ 1_j\ x_{j+1}\ x_{j+2} \ldots x_k)$ correspond to marking M such that there are unmarked siphons in only the top $r_1 - r_j$ siphon in N^{er}.

(1) If $M(p'_{j+1}) = 1(x_{j+1} = -1)$, then $M' = M + r^*$ is a non-live marking in N^{T}.

(2) If $M(r_{j+1}) = 1(x_{j+1} = 0)$, then no SMS is unmarked under $M' = M + r^*$ in N^{T}. M' is a legal marking in N^{T}.

(3) If $M(p_{j+1}) = 1(x_{j+1} = 1)$, then $M' = M + r^*$ is an unmarked state in N^r. M' is a legal marking in N^{T}.

(4) The total number of possible live markings under M is 2^{k-j}.

Proof by model. By Lemma 13 in Chao & Yu (2014), $s_{\mathrm{B}} = (x_k\ x_{k-1} \ldots 1_j \ldots 0_3\ 0_2^0\ -1_1)$ is a live state in *Bottom-Right*, where $x_i = 0$ or $1, i = j - 1$ to k; a non-reachable state where $x_i = -1, i = j - 1$ to k; the total number of possible live states is $2^{(k-j)}$. The reverse state of $s_{\mathrm{B}} = (x_k\ x_{k-1} \ldots 1_j \ldots 0_3\ 0_2^0\ -1_1)$ is $s_{\mathrm{T}} = (-1^0\ 0_2\ 0_3 \ldots 1_j \ldots x_{k-2}\ x_{k-1}\ x_k)$. By Theorem 2, we have the total number of possible live markings under M being $2^{(k-j)}$, where $x_i = 0$ or $1, i = j - 1$ to k. □

Proof (by siphon concept). (1) $(1_j - 1_{j+1})$ is a substate of an unmarked siphon in N^{T}. Hence, $M' = M + r^*$ is a non-live state in N^{T}.

(2) This corresponds to a legal marking if $x_{j+2} = x_{j+3} = \ldots = x_{k-2} = x_{k-1} = x_k = 0$ based on Lemma 9.

(3) $M = (-1^0 0 0_3 0_4 \ldots 0_{j-1} 1_j 1_{j+1} x_{j+2} \ldots x_k)$. $M'' = M + r^*$ has no unmarked siphons in N^r just as $M' = M + r^*$. Hence, $M\prime = M + r^*$ is a live state in N^{T}.

(4) Based on parts of (1)–(3) of this lemma, for M' to be a live marking in N^{T}, $x_{j+1} = x_{j+2} = \ldots = x_k = 0$ or 1. Hence, the total number of possible live markings under M is 2^{k-j}. □

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THEOREM B2 (Chao & Yu, 2013) The total number of non-reachable markings in N^e that may be live in N^{T} is $A_{\mathrm{T}}(k) = 2^{k-1} - 1$.

Proof. By summing 2^{k-j} from $j = 2$ to k, we have $A_{\mathrm{T}}(k) = 1 + 2 + 2^2 + \ldots + 2^{k-2} = \sum_{j=2 \text{ to } k} 2^{j-2} = 2^{k-1} - 1$. □

By Lemma 1, the forbidden markings in *Top-Right* are non-reachable markings in *Bottom-Right*. Hence, $C_{\mathrm{T}}(k)$ of *Top-Right*=$A_{\mathrm{B}}(k)$ of *Bottom-Right*; $A_{\mathrm{T}}(k)$ of *Top-Right*=$C_{\mathrm{B}}(k)$ of *Bottom-Right.*

THEOREM B3 (Chao & Yu, 2013) $L'(k) = 18 \times 2^{k-2} + k - 4$.

Proof.

$$\begin{aligned} L'(k) &= 2L(k) + A_{\mathrm{T}}(k) + C_{\mathrm{T}}(k) = 2((2^{k+1}) - 1) + 2^{k-1} - 1 + (k-1) \\ &= 2^{k+2} + 2^{k-1} + k - 4 \\ &= 16 \times 2^{k-2} + 2 \times 2^{k-2} + k - 4 \\ &= 18 \times 2^{k-2} + k - 4. \end{aligned}$$

□

C. Applying to k-net and Top-Left k-net

In k-net, *Top-Left* k-net and *Bottom-Left* k-net, let y_i^j denote the ith token state at Process $j (> 1)$. $y_i^j = -1$ means the ith token is at operation place p_i of Process j and not at operation place p_i of other processes. Hence, $y_i^2 + y_i^3 + \cdots + y_i^\mu = y_i = -1$ with $(\mu - 1)$ possibilities; i.e. exactly one of $y_i^2, y_i^3, \ldots, y_i^\mu$ equals -1; the rest are 0. $y_i^j = 0$ means that the ith token is at resource place r_i. Thus, $y_i \leq 0$.

Chao (2014) constructed the formulae of $Ł_k$ and $Ŗ_k$ for the k-net in Theorems C1 and C2, as extracted, respectively, below:

THEOREM C1 (Chao, 2014) For a k-net with μ processes, the total number of live states is $Ł_k = 2^k + (\mu)^k - 1$.

THEOREM C2 (Chao, 2014) For a k-net with μ processes, the total number of reachable states is $Ŗ_k = 2^k + (\mu - 1)y(1 - x^k)/(1 - x)$, where $x = \mu/2$ and $y = 2^{(k-1)}$.

Rebuilding the index number of transitions $(t_5^1, t_4^1, t_3^*, t_3^1, t_2^1, t_1^1)$ as $(t_1^1, t_1^*, t_2^1, t_3^1, t_4^1, t_5^1)$, etc., and the index number of resources (r_1, r_2, r_3, r_4) as (r_4, r_3, r_2, r_1) in Fig. A.1.(b), we can find that the *Bottom-Left* k-net (Chao & Yu, 2015a) is the reverse net of the *Top-Left* k-net, as shown in Fig. A.1.

Here, we extend to construct the formulae of $Ł'_k$ and $Ŗ'_k$ for the *Top-Left* k-net based on these results. The presence of the non-sharing resource place increases the number of states by a factor of 2. Based on Theorem B3, we can extend to $Ł'_k = 2Ł_k + A'(k) + C'(k)$, where $A'(k)$ and $C'(k)$ are as defined below:

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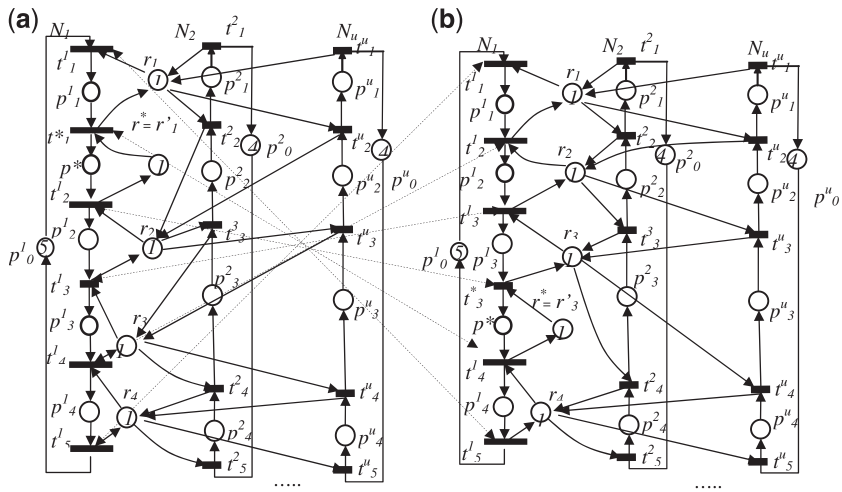

FIG. A.1. (a) Fourth *Top-Left* k-net system. (b) Fourth *Bottom-Left* k-net system.

THEOREM C3 For a k-net with μ processes,

(1) the total number of forbidden markings in the k-net that may be live in the Top-Left k-net is $C'(k) = (\mu)^{k-1} - 1$.

(2) the total number of non-reachable markings in the k-net that may be live in the Top-Left k-net is $A'(k) = (\mu - 1)(k - 1)$.

Proof. There are $(\mu - 1)$ possible top (resp. but non) empty siphons in the *Top-Left* k-net (resp. k-net) containing $r_{1,} r_2$ and r^*.

(1) $s = (1_1^0\, x_2\, x_3\, x_4 \ldots x_i \ldots x_{k-1}\, x_k) x_i = 0$, or $y_i^2, \ldots$ or y_i^{μ}, we have the total number is $(u^{(i)})$. Because $2 \leq i \leq k$, and we have to exclude substate $(0_2 0_3 \ldots 0_k)$. Hence, $C'(k) = [(\mu)^{k-1} - 1]$.

(2) For each such state, there are $(k - 1)$ states that may be live. Hence, $A'(k) = (\mu - 1)(k - 1)$. □

THEOREM C4 For a Top-Left k-net with μ processes, the total number of live markings $Ł'_k = 2Ł_k + (\mu)^{k-1} - 1 + (\mu - 1)(k - 1)$.

Proof.

$$
\begin{aligned}
Ł'_k &= 2Ł_k + A'(k) + C'(k) \\
&= 2Ł_k + (\mu)^{k-1} - 1 + (\mu - 1)(k - 1).
\end{aligned}
$$
□

We have revised the number of $C'(k)$ and $A'(k)$ of *Bottom-Left* k-net (Chao & Yu, 2014) in Chao & Yu (2015a) due to the inconsistent analysis from the viewpoint of *Bottom-Right*: (1) the total number of forbidden markings in the k-net that may be live in the *Bottom-Left* k-net is $C'(k) = (\mu - 1)(k - 1)$;

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(2) the total number of non-reachable markings in the k-net that may be live in the *Bottom-Left* k-net is $A'(k) = (\mu)^{k-1} - 1$. Based on Lemma 1, the forbidden markings in the *Top-Left* k-net are non-reachable markings in the *Bottom-Left* k-net. Hence, $C'(k)$ of the *Top-Left* k-net=$A'(k)$ of the *Bottom-Left* k-net; $A'(k)$ of the *Top-Left* k-net=C$'(k)$ of the *Bottom-Left* k-net.

THEOREM C5 For a Top-Left k-net with μ processes, the total number of reachable markings $Ŗ'_k = 2Ŗ_k + ((\mu)^{k-1} - 1)$.

Proof. Let $s = (x_1^0 \dots 0 \dots 1_j \dots x_k) 2 \leq j \leq k$ be the states pattern of the reachable states of which are non-reachable markings in the k-net but reachable markings in *Top-Left* k-net. The condition are: (1) $x_m = 0, y_m^2, y_m^3, \dots, y_m^\mu, j+1 \leq m \leq k$ and (2) $x_m = y_m^2, y_m^3, \dots y_m^\mu, m = 1$. The total number of such states is $(\mu - 1)((\mu)^{k-2} + (\mu)^{k-3} + \cdots + 0) = (\mu - 1)((\mu)^{k-1} - 1)/(\mu - 1) = ((\mu)^{k-1} - 1)$. Hence, $Ŗ'_k = 2Ŗ_k + ((\mu)^{k-1} - 1)$. □

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