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Global cluster synchronization in nonlinearly coupled community networks with heterogeneous coupling delays

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HIGHLIGHTS

- New theory on cluster synchronization of nonlinearly coupled networks with delays.
- The framework accommodates a large class of network systems.
- Our approach releases common constraints in previous synchronization theory.
- Delay-independent and delay-dependent criteria for cluster synchronization given.
- Networks exhibit new synchronization scenarios under the synchronization criteria.

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ABSTRACT

This investigation establishes the global cluster synchronization of complex networks with a community structure based on an iterative approach. The units comprising the network are described by differential equations, and can be non-autonomous and involve time delays. In addition, units in the different communities can be governed by different equations. The coupling configuration of the network is rather general. The coupling terms can be non-diffusive, nonlinear, asymmetric, and with heterogeneous coupling delays. Based on this approach, both delay-dependent and delay-independent criteria for global cluster synchronization are derived. We implement the present approach for a nonlinearly coupled neural network with heterogeneous coupling delays. Two numerical examples are given to show that neural networks can behave in a variety of new collective ways under the synchronization criteria. These examples also demonstrate that neural networks remain synchronized in spite of coupling delays between neurons across different communities; however, they may lose synchrony if the coupling delays between the neurons within the same community are too large, such that the synchronization criteria are violated.

1. Introduction

Over the past few decades, complex networks of coupled dynamical systems have been widely exploited to model many real-world complex systems in the sciences, engineering, society, and so on (see Boccaletti, Latora, Moreno, Chavez, & Hwang, 2006; Strogatz, 2001; Wang & Chen, 2003). For instance, the celebrated Hopfield neural network, serving as content-addressable memory systems, has provided a model for understanding biologically inspired architectures for information and image processing (Hopfield, 1982). Synchronization, an important and inherent phenomenon in a wide range of real systems including biological and physical systems, has attracted considerable attention from researchers (Glass, 2001; Pikovsky, Rosenblum, & Kurths, 2001). Inspired by the pioneering work of Pecora and Carroll (1990), the synchronization of complex networks has received increasing attention in a wide range of research areas such as engineering, biology, and physics. More specifically, in recent years the synchronization of coupled neural networks has been extensively investigated by virtue of its wide potential application in many fields including secret communication (Sheikhan, Shahnazi, & Garoucy, 2013; Xia & Cao, 2008), pattern recognition (Haken, 2005; Hoppensteadt & Izhikevich, 2000), and parallel image processing (Bräunl, Feyrer, Rapf, & Reinhardt, 2013; Krinsky, Biktashev, & Efimov, 1991).

Complex networks of coupled systems can behave collectively in many ways, and this behavior is possibly distinct from their behavior in isolation. This collective behavior is determined by the dynamics of the individual nodes, coupling configuration between the nodes, and coupling time delays. Among these factors, delays

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are ubiquitous in many natural and artificial systems, and there have been various attempts to incorporate them into network modeling (Crook, Ermentrout, Vanier, & Bower, 1997; Yu, Cao, & Chen, 2008). Indeed, coupling delays are known to be capable of modifying the collective dynamics of coupled systems.

Various synchronization phenomena that appear in a wide range of real systems have been investigated and several different synchronization models and scenarios have been reported in the literature. These synchronization protocols include complete or identical synchronization (Li & Yang, 2015; Lu, Chen, & Chen, 2006; Wu, 2005; Yu, Cao, & Lu, 2007), phase synchronization (Rosenblum, Pikovsky, & Kurths, 1996), lag synchronization (Rosenblum, Pikovsky, & Kurths, 1997), partial synchronization (Vreeswijk, 1996), generalized synchronization (Margheri & Martins, 2010), almost synchronization (Femat & Solis-Perales, 1999), and cluster synchronization (Belykh, Belykh, Hasler, & Nevidin, 2003; Belykh, Belykh, Nevidin, & Hasler, 2003; Qin & Chen, 2004). Among these synchronization protocols, the complete synchronization of a network with identical nodes means that all nodes eventually approach to uniform dynamical behavior. This is the simplest form of synchronization, and has been intensively studied. Cluster synchronization is the phenomenon that divides the nodes in a network into several groups, known as clusters or communities, such that all nodes in the same community attain complete synchronization. However, the dynamics of nodes in different communities do not tend to coincide. The cluster synchronization of complex networks has attracted increasing research interest in recent years because of its applications in areas including the biological sciences (Kaneko, 1994; Zemanová, Zhou, & Kurths, 2006), neurological sciences (Schnitzler & Gross, 2005), engineering control (Passino, 2002), ecological sciences (Montbró, Kurths, & Blasius, 2004), communication engineering (Kouomou & Woafo, 2003; Rulkov, 1996), and distributed computation (Hwang, Tan, & Chen, 2004).

The coupled systems for which the investigations of cluster synchronization have been conducted include the ones composed of identical subsystems without delays (Belykh, Belykh, Hasler, & Nevidin, 2003; Ma, Liu, & Zhang, 2006; Qin & Chen, 2004; Wu & Chen, 2009; Zhang, Ma, & Chen, 2014; Zhang, Ma, & Zhang, 2013) or with delays (Cao & Li, 2009; Song & Zhao, 2014). Among them, the approach proposed by Ma et al. (2006) involved construction of a coupling scheme to stabilize arbitrarily selected cluster synchronization patterns for coupled identical chaotic networks. Wu and Chen (2009) studied the cluster synchronization of linearly and symmetrically coupled ordinary differential equations based on the geometrical analysis of the synchronization manifold. Zhang, Ma, and Zhang (2013) decomposed the coupling matrix and employed the Lyapunov function method to investigate the cluster synchronization in networks with asymmetric negative couplings. By introducing competitive inter-cluster edges and assigning edge weights to mimic more realistic networks, Zhang, Ma, and Chen (2014) developed a modified small-world networks, and showed that the new model with inter-cluster co-competition balance admits robustness of cluster synchronous patterns. The cluster synchronization of Hopfield-type neural networks has been investigated in Zhang, Ma, and Chen (2014) and Zhang, Ma, and Zhang (2013). Cao and Li (2009) and Song and Zhao (2014) used the linear matrix inequality technique to investigate delayed hybrid neural networks with linear and nonlinear couplings, respectively. Qin and Chen (2004) investigated the cluster synchronization of coupled Josephson equations by constructing different coupling schemes. Cluster synchronization in three-dimensional lattices of diffusively coupled oscillators was studied in Belykh, Belykh, Hasler, and Nevidin (2003). In reality, networks may exhibit community structure, and nodes in different communities have different functions (Girvan & Newman, 2002; Newman, 2003). Recently, the cluster synchronization of networks with nonidentical nodes was investigated. The cluster synchronization of coupled nonidentical systems was studied by Lu, Liu, and Chen (2010a) who investigated the relation between cluster synchronization and un-weighted graph topology. The same authors also examined local cluster synchronization in general bi-directed networks of coupled nonidentical maps (Lu, Liu, & Chen, 2010b). The persistence of cluster synchronization manifolds in lattices of nonidentical chaotic oscillators was studied in Belykh, Belykh, Nevidin, and Hasler (2003).

Existing reports in literature contain a number of investigations on the cluster synchronization of coupled systems in which either linear couplings without coupling delays (Belykh, Belykh, Hasler, & Nevidin, 2003; Lu et al., 2010a; Ma et al., 2006; Qin & Chen, 2004; Wu & Chen, 2009; Zhang, Ma, & Chen, 2014; Zhang, Ma, & Zhang, 2013), or linear couplings with coupling delays are considered (Cao & Li, 2009). In addition, systems with coupling delays that have been considered for addressing cluster synchronization problems commonly exhibit homogeneous coupling delays, which means that all coupling delays are identical.

In the literature, there exist much fewer studies on the cluster synchronization for nonlinearly coupled systems than those for linearly coupled systems. By analyzing the stability of cluster synchronization manifolds, Belykh and Hasler (2011) and Juang and Liang (2014) investigated the local cluster synchronization in networks of nonlinearly coupled neurons without delays. A few investigations established the cluster synchronization of nonlinearly coupled systems with coupling delays. Song and Zhao (2014) considered nonlinearly coupled identical systems with homogeneous discrete time-varying delays and homogeneous distributed time-varying delays, and derived delay-dependent criteria for cluster synchronization based on the Lyapunov function method and linear matrix inequality technique. In real-world networks, all the units in real systems may not transmit information at the same rate (Jalan & Singh, 2014). In addition, the transmission delays may depend on both the transmission speed and locations of interacting nodes (Crook et al., 1997; Faye & Faugeras, 2010). For instance, Crook et al. (1997) considered the axonal delay which is larger if the distance between interacting neurons is larger or conduction velocity value is smaller, and showed that large delays can result in a loss of synchrony. Thus, it is more realistic to incorporate the heterogeneity in coupling delays into the system model. We note that Jalan and Singh (2014) studied the phase synchronized clusters in networks of coupled identical maps in the presence of heterogeneous coupling delays, and investigated the impact of heterogeneity in delays on the phenomenon of cluster synchronization. To the best of our knowledge, a new approach to establishing the cluster synchronization of nonlinearly coupled systems with coupling delays is still in demand, particularly for coupled nonidentical systems or coupled systems with heterogeneous coupling delays.

Let us consider the following network system:

$$\dot{\mathbf{x}}_i(t) = \tilde{\mathbf{F}}_i(\mathbf{x}_i^t, t) + \sum_{j\in\mathcal{N}} \tilde{\omega}_{ij}(t)\tilde{\mathbf{G}}_{ij}(\mathbf{x}_j^t), \quad i \in \mathcal{N},\ t \geq t_0, \tag{1}$$

where $\mathcal{N} := \{1, \dots, N\}$, $\mathbf{x}_i(t) = (x_{i,1}(t), \dots, x_{i,K}(t))^T \in \mathbb{R}^K$ represents the state of the ith node at time t, $\mathbf{x}_i^t \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^K)$, with $\tau_M \geq 0$, is defined as $\mathbf{x}_i^t(\theta) = \mathbf{x}_i(t+\theta)$ for $\theta \in [-\tau_M, 0]$, and $\tilde{\mathbf{F}}_i$ is a smooth function, describing the intrinsic dynamics of the ith node. Each $\tilde{\omega}_{ij}(t)$, which denotes the coupling coefficient from the jth node to the ith node, is a bounded function of t, and its corresponding coupling function $\tilde{\mathbf{G}}_{ij}$ is assumed to be smooth. The matrix $\mathbf{W}(t) := [\tilde{\omega}_{ij}(t)]_{1\leq i,j\leq N}$ refers to the connection matrix for the system (1). Notably, the form of (1) covers the ordinary differential equation case when $\tau_M = 0$. In this investigation, we consider system (1) to be divided into m communities, and the intrinsic dynamics ($\tilde{\mathbf{F}}_i$) of all individual nodes in the same community to be identical. Without loss of generality, we set

$$\mathcal{M} := \{1, \ldots, m\},$$
$$\tilde{\mathcal{N}}_r := \left\{1 + \sum_{s=0}^{r-1} N_s, \ldots, \sum_{s=0}^{r} N_s\right\}, \quad r = 1, \ldots, m,$$

where $m \in \mathbb{N} - \{0\}$, $N_0 := 0$, and $N_r \in \mathbb{N}$ for $r \in \mathcal{M}$. Herein, $\mathcal{M}$ is the index set of communities in system (1), N_r is the number of nodes in the rth community, and $\tilde{\mathcal{N}}_r$ collects the indices of nodes in the rth community. Notably, $\sum_{s\in\mathcal{M}} N_s = N$ and $\bigcup_{s\in\mathcal{M}} \tilde{\mathcal{N}}_s = \mathcal{N}$. For later use, we relabel the indices in $\tilde{\mathcal{N}}_r$, $r \in \mathcal{M}$, such that these indices are numbered from 1 to N_r, and then collect the relabeled indices in the index set:

$$\mathcal{N}_r = \{1, \ldots, N_r\}. \tag{2}$$

Accordingly, we can denote by (r, p) the two dimensional index for the pth node in the rth community, for $r \in \mathcal{M}$ and $p \in \mathcal{N}_r$. Notably, there exists a bijective map $\mathcal{J} : \mathcal{N} \to \{(s, q) : s \in \mathcal{M}, p \in \mathcal{N}_s\}$, defined by

$$\mathcal{J}(i) = (r, p), \tag{3}$$

satisfying $i = \sum_{s=0}^{r-1} N_s + p$, where $i \in \tilde{\mathcal{N}}_r$, $r \in \mathcal{M}$, and $p \in \mathcal{N}_r$. Map $\mathcal{J}$ can depict the community structure of the system (1); more precisely, the ith node in the system (1) can be regarded as the pth node in the rth community if $\mathcal{J}(i) = (r, p)$. In this paper, we investigate the cluster synchronization of system (1) by assuming that the intrinsic dynamics of all nodes and the connections between nodes are merely determined by the communities to which these nodes belong. Namely, functions $\tilde{\mathbf{F}}_i$ and $\tilde{\mathbf{G}}_{ij}$ of system (1) satisfy

$$\tilde{\mathbf{F}}_i = \mathbf{F}_r = (F_{r,1}, \ldots, F_{r,K})^T \quad \text{and}$$
$$\tilde{\mathbf{G}}_{ij}(\mathbf{x}_j^t) = \mathbf{G}_{rs}(\mathbf{x}_j(t - \tau_{rs}(t))), \tag{4}$$

if $\mathcal{J}(i) = (r, p)$ and $\mathcal{J}(j) = (s, q)$. Herein, $\mathbf{G}_{rs} = (G_{rs,1}, \ldots, G_{rs,K})^T$ satisfies

$$G_{rs,k}(\mathbf{x}_j(t - \tau_{rs}(t))) = G_{rs,k}(x_{j,k}(t - \tau_{rs}(t))), \tag{5}$$

where $G_{rs,k}$ is a non-decreasing and differentiable function, and $\tau_{rs}(t)$ is a continuous function, with $0 \le \tau_{rs}(t) \le \tau_M$. We note that function $\mathbf{F}_r$ represents the intrinsic dynamics of each node in the rth community, $\mathbf{G}_{rs}$ stands for the coupling function corresponding to the connection from nodes in the sth community to nodes in the rth community, and $\tau_{rs}(t)$ is the time-dependent transmission delay corresponding to function $\mathbf{G}_{rs}$. For later use, we set

$$\bar{\tau}_r := \sup\{\tau_{rr}(t) : t \ge t_0\}, \quad r = 1, \ldots, m. \tag{6}$$

Notably, system (1) is a nonlinearly coupled system if some $G_{rs,k}$ is a nonlinear function. In addition, system (1) is said to include homogeneous coupling delays if all $\tau_{rs}(t)$ are identical; otherwise, it has heterogeneous coupling delays. Existing attempts to address the cluster synchronization problem for systems of neural networks and neuronal networks in the literature largely admit the form of (1) or forms that are similar; for example, see Lu et al. (2010a), Ma et al. (2006), Wu and Chen (2009), Zhang, Ma, and Chen (2014), and Zhang, Ma, and Zhang (2013). The system (1) is said to attain *global cluster synchronization* if

$$x_{i,k}(t) - x_{j,k}(t) \to 0, \quad \text{as } t \to \infty, \text{ for all } k \in \mathcal{K}, \tag{7}$$

if $i, j \in \tilde{\mathcal{N}}_r$ for some $r \in \mathcal{M}$, for every solution $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))^T$ of (1), where $\mathcal{K} := \{1, \ldots, K\}$ and $\mathbf{x}_j(t) = (x_{j,1}(t), \ldots, x_{j,K}(t))^T$, for $j \in \mathcal{N}$. To the best of our knowledge, the global cluster synchronization of nonlinearly coupled nonidentical systems in the form of (1) satisfying (4) and (5) is not yet established, even if coupling delays $\tau_{rs}(t)$ are identical. In this paper, we aim to establish the global cluster synchronization of community network (1) which satisfies (4) and (5), and is with nonlinear coupling functions and heterogeneous coupling delays.

We note that the use of control schemes, including pinning control and adaptive control, to establish the cluster synchronization of networks also presents synchronization problems. These investigations establish the cluster synchronization of controlled networks by adding controllers to the network. This work can be found in Liu and Chen (2011), Wang, Fu, and Li (2009), and Wu, Zhou, and Chen (2009) for the linearly coupling case, and in Wang, Feng, Xu, and Zhao (2012) and Wang and Cao (2013) for the nonlinear coupling case. We note that Wang and Cao (2013) and Wang et al. (2012) investigated the cluster synchronization of nonlinearly coupled nonidentical systems with homogeneous time-varying delay and without delay, respectively, by using pinning control.

The remainder of this paper is organized as follows. We establish the global cluster synchronization of the system (1) in Section 2. Then, we illustrate the implementation of our approach by studying the global cluster synchronization of a nonlinearly coupled neural networks, possibly with heterogeneous coupling delays, in Section 3. The paper is concluded in Section 4.

2. Cluster synchronization of system (1)

In this section, we establish the global cluster synchronization of system (1), which satisfies (4) and (5). We manipulate our synchronization approach by first relabeling the variables and parameters of system (1) based on the community structure of the system, and introduce three basic assumptions imposed upon the system in Section 2.1. We then derive delay-dependent and delay-independent criteria for the global cluster synchronization of system (1) in Section 2.2.

2.1. Preliminaries

Recall that the community structure of system (1) can be depicted through the map $\mathcal{J}$, defined in (3). With the help of the map $\mathcal{J}$, we rewrite the connection matrix $\mathbf{W}(t) = [\tilde{\omega}_{ij}(t)]_{1\le i,j\le N}$ in the following block form:

$$\mathbf{W}(t) = [W_{rs}(t)]_{1\le r,s\le m} = \begin{pmatrix} W_{11}(t) & \cdots & W_{1m}(t) \\ \vdots & \ddots & \vdots \\ W_{m1}(t) & \cdots & W_{mm}(t) \end{pmatrix}, \tag{8}$$

with

$$W_{rs}(t) = [\omega_{rs}^{(pq)}(t)]_{1\le p\le N_r, 1\le q\le N_s} = \begin{pmatrix} \omega_{rs}^{(11)}(t) & \cdots & \omega_{rs}^{(1N_s)}(t) \\ \vdots & \ddots & \vdots \\ \omega_{rs}^{(N_r1)}(t) & \cdots & \omega_{rs}^{(N_rN_s)}(t) \end{pmatrix},$$

where

$$\omega_{rs}^{(pq)}(t) := \tilde{\omega}_{ij}(t) \quad \text{if } \mathcal{J}(i) = (r, p) \text{ and } \mathcal{J}(j) = (s, q). \tag{9}$$

Recall that $\tilde{\omega}_{ij}(t)$ refers to the coupling coefficient from the jth node to the ith node. By (9), $\omega_{rs}^{(pq)}(t)$ is the exact coupling coefficient from the qth node in the sth community to the pth node in the rth community. Accordingly, $W_{rs}(t)$ can be regarded as the sub-connection matrix corresponding to the connections from the nodes in the sth community to nodes in the rth community. We note that the connection from the qth node in the sth community to the pth node in the rth community is said to be excitatory (resp., inhibitory) at time t if $\omega_{rs}^{(pq)}(t) > 0$ (resp., $\omega_{rs}^{(pq)}(t) < 0$).

Assume that $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))^T$ is an arbitrary solution of system (1), where $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t))^T$, $i = 1, \ldots, N$, and

$(\mathbf{x}_1^t, \ldots, \mathbf{x}_N^t)^T$ is the corresponding evolution of system (1), defined by $\mathbf{x}_i^t(\theta) = \mathbf{x}_i(t+\theta)$ for $\theta \in [-\tau_M, 0]$, $i = 1, \ldots, N$. Through map $\mathcal{J}$, we relabel $\mathbf{x}_i(t)$ and $\mathbf{x}_i^t(t)$, $i = 1, \ldots, N$, as follows:

$$\mathbf{x}_{r,p}(t) = (x_{r,p,1}(t), \ldots, x_{r,p,K}(t))^T := \mathbf{x}_i(t) \quad \text{and} \quad \mathbf{x}_{r,p}^t := \mathbf{x}_i^t \tag{10}$$

if $\mathcal{J}(i) = (r, p)$. With the help of (2)–(5), (9), and (10), system (1) can be rewritten in the following component form:

$$\dot{x}_{r,p,k}(t) = F_{r,k}(\mathbf{x}_{r,p}^t, t) + \sum_{s\in\mathcal{M}} \sum_{q\in\mathcal{N}_s} \omega_{rs}^{(pq)}(t) \times G_{rs,k}(x_{s,q,k}(t - \tau_{rs}(t))), \tag{11}$$

for all $(r, p, k) \in \mathcal{A}_x$ and $t \geq t_0$, where

$$\mathcal{A}_x := \{(s, q, l) : s \in \mathcal{M}, q \in \mathcal{N}_s, l \in \mathcal{K}\}. \tag{12}$$

In this paper, the index (r, p) corresponds to the pth node in the rth community, and (r, p, k) corresponds to the kth variable of the pth node in the rth community. For instance, $\mathbf{x}_{r,p}$ represents the state of the pth node in the rth community, and $x_{r,p,k}$ is the state of the kth variable of the pth node in the rth community. Moreover, we denote by $\mathbf{X}(t)$, or $(\mathbf{x}_{r,p}(t))$, or $(x_{r,p,k}(t))$ the solution of system (11) at time t, evolved from an arbitrary initial condition, and denote by $\mathbf{X}^t$, or $(\mathbf{x}_{r,p}^t)$, or $(x_{r,p,k}^t)$ the corresponding evolution of system (11), as follows:

$$\begin{aligned} \mathbf{X}(t) &= (\mathbf{x}_{r,p}(t)) = (x_{r,p,k}(t)) \\ &:= (\mathbf{x}_{1,1}(t), \ldots, \mathbf{x}_{1,N_1}(t), \ldots, \mathbf{x}_{r,p}(t), \\ &\quad \ldots, \mathbf{x}_{m,1}(t), \ldots, \mathbf{x}_{m,N_m}(t))^T, \\ \mathbf{X}^t &= (\mathbf{x}_{r,p}^t) = (x_{r,p,k}^t) \\ &:= (\mathbf{x}_{1,1}^t, \ldots, \mathbf{x}_{1,N_1}^t, \ldots, \mathbf{x}_{r,p}^t, \ldots, \mathbf{x}_{m,1}^t, \\ &\quad \ldots, \mathbf{x}_{m,N_m}^t)^T, \end{aligned} \tag{13}$$

where $\mathbf{x}_{r,p}(t) = (x_{r,p,1}(t), \ldots, x_{r,p,K}(t))^T \in \mathbb{R}^K$ and $\mathbf{x}_{r,p}^t = (x_{r,p,1}^t, \ldots, x_{r,p,K}^t)^T \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^K)$, for $(r, p) \in \{(s, q) : s \in \mathcal{M}, q \in \mathcal{N}_s\}$. Recall that $\mathbf{X}^t(\theta) = \mathbf{X}(t+\theta)$, $\mathbf{x}_{r,p}^t(\theta) = \mathbf{x}_{r,p}(t+\theta)$, and $x_{r,p,k}^t(\theta) = x_{r,p,k}(t+\theta)$ for $\theta \in [-\tau_M, 0]$.

According to the definition in (7), system (11), i.e. system (1), attains *global cluster synchronization* if

$$x_{r,p,k}(t) - x_{r,p+1,k}(t) \to 0, \quad \text{as } t \to \infty, \text{ for all } (r, p, k) \in \mathcal{A}_z, \tag{14}$$

for every solution $(x_{r,p,k}(t))$ of system (11), where $\mathcal{A}_z$ is defined as

$$\mathcal{A}_z := \{(s, q, l) : s \in \mathcal{M}, q \in \mathcal{N}_s - \{N_s\}, l \in \mathcal{K}\}. \tag{15}$$

Next, let us introduce three assumptions imposed on system (11). The first assumption requires that each matrix $W_{rs}(t)$, defined in (8), has equal row-sums, as follows:

Assumption (I): For all $r, s \in \mathcal{M}$ and $t \geq t_0$, there exists some $\kappa_{rs}(t)$ such that

$$\sum_{1\leq q\leq N_s} \omega_{rs}^{(pq)}(t) = \kappa_{rs}(t), \quad \text{for all } p = 1, \ldots, N_r.$$

Diffusive couplings have been largely considered in neural network systems in the literature, see Belykh, Belykh, Hasler, and Nevidin (2003), Li and Yang (2015), Ma et al. (2006), Qin and Chen (2004) and Wang et al. (2009). We note that system (11) is diffusively coupled if $\kappa_{rs}(t) = 0$ for all $r, s \in \mathcal{M}$ and $t \geq t_0$, cf. Cao and Li (2009) and Song and Zhao (2014). Define the cluster synchronous manifold of system (11):

$$\mathcal{S} := \{(\mathbf{x}_1, \ldots, \mathbf{x}_N) : \mathbf{x}_i = \mathbf{x}_j \in \mathbb{R}^K \text{ if } i, j \in \tilde{\mathcal{N}}_r \text{ for some } r \in \mathcal{M}\}. \tag{16}$$

The invariance of $\mathcal{S}$ under the flow generated by system (11) is a prerequisite to the cluster synchronization of system (11), and it can be guaranteed by assumption (I), cf. Lu et al. (2010a). The second assumption is associated with the dissipative property of system (11) as follows:

Assumption (D): All solutions of (11) eventually enter and then remain in some compact set $\mathcal{Q} := \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_m$, where

$$\mathcal{Q}_r := Q_r \times \cdots \times Q_r \subset \mathbb{R}^{N_r K},$$
$$Q_r := [\check{\varrho}_{r,1}, \hat{\varrho}_{r,1}] \times \cdots \times [\check{\varrho}_{r,K}, \hat{\varrho}_{r,K}] \subset \mathbb{R}^K, \quad r \in \mathcal{M}.$$

Notably, under assumption (D), an arbitrary solution $\mathbf{X}(t) = (\mathbf{x}_{r,p}(t)) = (x_{r,p,k}(t))$ of system (11) exists on $[t_0, \infty)$, and eventually enters and then remains in $\mathcal{Q}$, where each $\mathbf{x}_{r,p}(t)$ (resp., $x_{r,p,k}(t)$) eventually enters and then remains in Q_r (resp., $[\check{\varrho}_{r,k}, \hat{\varrho}_{r,k}]$). Throughout this paper, we denote by $\mathbf{\Phi}$, or $(\Phi_{r,p})$, or $(\phi_{r,p,k})$ an arbitrary function in $\mathcal{C}([-\tau_M, 0]; \mathbb{R}^N)$. Herein,

$$\begin{aligned} \mathbf{\Phi} &= (\Phi_{r,p}) = (\phi_{r,p,k}) \\ &:= (\Phi_{1,1}, \ldots, \Phi_{1,N_1}, \ldots, \Phi_{r,p}, \ldots, \Phi_{m,1}, \ldots, \Phi_{m,N_m})^T, \end{aligned} \tag{17}$$

where $\Phi_{r,p} := (\phi_{r,p,1}, \ldots, \phi_{r,p,K})^T$ for $(r, p) \in \{(s, q) : s \in \mathcal{M}, q \in \mathcal{N}_s\}$, and each $\phi_{r,p,k} \in \mathcal{C}([-\tau_M, 0]; \mathbb{R})$.

With $\mathcal{Q}$ defined in assumption (D), we define the following set in $\mathcal{C}([-\tau_M, 0]; \mathbb{R}^N)$:

$$\mathcal{C}_{\mathcal{Q}} := \{\mathbf{\Phi} = (\Phi_{r,p}) : \Phi_{r,p} \in \mathcal{C}_{Q_r}, (r, p) \in \{(s, q) : s \in \mathcal{M}, q \in \mathcal{N}_s\}\}, \tag{18}$$

where

$$\mathcal{C}_{Q_r} := \{(\phi_1, \ldots, \phi_K)^T : \phi_k \in \mathcal{C}([-\tau_M, 0]; [\check{\varrho}_{r,k}, \hat{\varrho}_{r,k}]), k \in \mathcal{K}\}.$$

The last assumption is related to the argument structure of function $F_{r,k}$ in system (11), cf. (4). For each $(r, k) \in \{(s, l) : s \in \mathcal{M}, l \in \mathcal{K}\}$, we decompose $F_{r,k}(\Phi, t) - F_{r,k}(\Psi, t)$ as follows:

$$F_{r,k}(\Phi, t) - F_{r,k}(\Psi, t) = h_{r,k}(\phi_k(0), \psi_k(0), t) + w_{r,k}(\Phi, \Psi, t), \tag{19}$$

where $\Phi = (\phi_1, \ldots, \phi_K)^T$, $\Psi = (\psi_1, \ldots, \psi_K)^T \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^K)$. Such a decomposition in (19) is always achievable because a trivial decomposition is attained when setting $h_{r,k} \equiv 0$. Application of this argument certainly requires a nontrivial decomposition where the terms involving $\phi_k(0)$, $\psi_k(0)$ are collected in $h_{r,k}$, and the others in $w_{r,k}$. We illustrate the nontrivial decomposition for system (51) in Section 3. Now, let us introduce the following assumption imposed on functions $h_{r,k}$ and $w_{r,k}$.

Assumption (F): For each $(r, k) \in \{(s, l) : s \in \mathcal{M}, l \in \mathcal{K}\}$, there exist $\check{\mu}_{r,k}, \hat{\mu}_{r,k} \in \mathbb{R}$, $\rho_{r,k}^w \geq 0$, and $\bar{\mu}_{r,k}^{(l)}, \bar{\beta}_{r,k}^{(l)} \geq 0$, for $l \in \mathcal{K} - \{k\}$, such that for all $\Phi = (\phi_1, \ldots, \phi_K)^T$, $\Psi = (\psi_1, \ldots, \psi_K)^T \in \mathcal{C}_{Q_r}$, and $t \geq t_0$, the following properties hold:

$$\text{(F-i): } \begin{cases} \check{\mu}_{r,k} \leq \dfrac{h_{r,k}(\phi_k(0), \psi_k(0), t)}{\phi_k(0) - \psi_k(0)} \leq \hat{\mu}_{r,k} & \text{if } \phi_k(0) - \psi_k(0) \neq 0, \\ h_{r,k}(\phi_k(0), \psi_k(0), t) = 0, & \text{if } \phi_k(0) - \psi_k(0) = 0, \end{cases}$$

(F-ii): $|w_{r,k}(\Phi, \Psi, t)| \leq \rho_{r,k}^w$, and there exist $\tau_{r,k}^{(l)} = \tau_{r,k}^{(l)}(\Phi, \Psi, t) \in [0, \tau_M]$, such that

$$|w_{r,k}(\Phi, \Psi, t)| \leq \sum_{l\in\mathcal{K}-\{k\}} \bar{\mu}_{r,k}^{(l)} |\phi_l(0) - \psi_l(0)| + \bar{\beta}_{r,k}^{(l)} |\phi_l(-\tau_{r,k}^{(l)}) - \psi_l(-\tau_{r,k}^{(l)})|.$$

Remark 1. (i) Recall the notations in (13) and (17), and $\mathcal{C}_{\mathcal{Q}}$ in (18). In this paper, $\mathbf{\Phi} = (\Phi_{r,p}) = (\phi_{r,p,k}) \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^N)$ plays the role of an arbitrary evolution $\mathbf{X}^t = (\mathbf{x}_{r,p}^t) = (x_{r,p,k}^t)$ of system (11) at an arbitrary fixed time t, where each $\Phi_{r,p}$ (resp., $\phi_{r,p,k}$) plays the role of $\mathbf{x}_{r,p}^t$ (resp., $x_{r,p,k}^t$). Thus, it is not difficult to observe that under assumption (D), an arbitrary evolution $\mathbf{X}^t = (\mathbf{x}_{r,p}^t)$ of system (11) exists on $[t_0, \infty)$, and eventually enters and then remains in $\mathcal{C}_{\mathcal{Q}}$, where each $\mathbf{x}_{r,p}^t$ eventually enters and then remains in $\mathcal{C}_{Q_r}$. (ii) In Section 2.2, we shall see in (20) and (21) that Φ and Ψ in (19) play the roles of an arbitrary pair of $\mathbf{x}_{r,p}$ and $\mathbf{x}_{r,p+1}$ of an arbitrary evolution $(\mathbf{x}_{r,p}(t))$ of system (11). Recall that under assumption (D), each $\mathbf{x}_{r,p}^t$ eventually enters and then remains in $\mathcal{C}_{Q_r}$. This explains why assumption (F) considers $\Phi, \Psi \in \mathcal{C}_{Q_r}$.

2.2. Synchronization criteria

First, let us show the main principle of our approach to establish the global cluster synchronization of system (11). We suppose that $\mathbf{X}(t) = (\mathbf{x}_{r,p}(t)) = (x_{r,p,k}(t))$ is an arbitrary solution of system (11), and $\mathbf{X}^t = (\mathbf{x}^t_{r,p}) = (x^t_{r,p,k})$ is the corresponding evolution. Set

$$\begin{aligned}\mathbf{Z}(t) &= (\mathbf{z}_{r,p}(t)) = (z_{r,p,k}(t)) \\ &:= (\mathbf{z}_{1,1}(t), \dots, \mathbf{z}_{1,N_1-1}(t), \dots, \mathbf{z}_{r,p}(t), \dots, \mathbf{z}_{m,1}(t), \\ &\quad \dots, \mathbf{z}_{m,N_m-1}(t))^T,\end{aligned}$$

where $\mathbf{z}_{r,p} = (z_{r,p,1}(t), \dots, z_{r,p,K}(t))^T := \mathbf{x}_{r,p}(t) - \mathbf{x}_{r,p+1}(t)$, for $(r,p) \in \{(s,q) : s \in \mathcal{M}, q \in \mathcal{N}_s - \{N_s\}\}$. Then, $\mathbf{Z}(t)$ satisfies the following difference–differential system associated with system (11):

$$\dot{z}_{r,p,k}(t) = H_{r,p,k}(\mathbf{X}^t, t), \quad (r,p,k) \in \mathcal{A}_z,\ t \geq t_0, \tag{20}$$

where $z_{r,p,k}(t) := x_{r,p,k}(t) - x_{r,p+1,k}(t)$, $\mathcal{A}_z$ is defined in (15), and

$$\begin{aligned}H_{r,p,k}(\mathbf{\Phi}, t) &:= F_{r,k}(\Phi_{r,p}, t) - F_{r,k}(\Phi_{r,p+1}, t) \\ &\quad + \sum_{s\in\mathcal{M}} \sum_{q\in\mathcal{N}_s} [\omega^{(pq)}_{rs}(t) - \omega^{((p+1)q)}_{rs}(t)] \\ &\quad \times G_{rs,k}(\phi_{s,q,k}(-\tau_{rs}(t))),\end{aligned} \tag{21}$$

for $\mathbf{\Phi} = (\Phi_{r,p}) = (\phi_{r,p,k})$ as defined in (17). Herein, the roles of $\mathbf{\Phi}$, $\Phi_{r,p}$, and $\phi_{r,p,k}$ are discussed in Remark 1. In the following, we recompose the terms $H_{r,p,k}$ defined (21), such that system (20) can be recast into the form

$$\begin{aligned}\dot{z}_{r,p,k}(t) &= h_{r,p,k}(x_{r,p,k}(t), x_{r,p+1,k}(t), t) \\ &\quad + \tilde{h}_{r,p,k}(x^t_{r,p,k}, x^t_{r,p+1,k}, t) + w_{r,p,k}(t),\end{aligned} \tag{22}$$

for $(r,p,k) \in \mathcal{A}_z$ and $t \geq t_0$, where $w_{r,p,k}(t) = w_{r,p,k}(\mathbf{X}^t, t)$. The precise formulations and properties of $h_{r,p,k}$, $\tilde{h}_{r,p,k}$, and $w_{r,p,k}$ are presented in Propositions 1 and 2, respectively. Each component in (22) takes the form:

$$\dot{z}(t) = h(x(t), y(t), t) + \tilde{h}(x_t, y_t, t) + w(t), \quad t \geq t_0 \tag{23}$$

where $z(t) = x(t) - y(t)$. In the Appendix, we introduce (23) precisely, and show that $z(t)$ converges to an interval $[-\bar{\nu}, \bar{\nu}]$ as $t \to \infty$ under some conditions; moreover, $[-\bar{\nu}, \bar{\nu}]$ can be estimated. Accordingly, we capture the convergent property to each $z_{r,p,k}(t)$ in (22). By an iterative argument, as performed in Proposition 3 and Theorem 1, we further verify that each $z_{r,p,k}(t)$ approaches zero; hence, system (11) attains global cluster synchronization.

To recompose the terms $H_{r,p,k}$ defined by (21), for each $r \in \mathcal{M}$, we define the following two matrices:

$$L_r := \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} \in \mathbb{R}^{(N_r-1)\times N_r},$$

$$R_r := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{N_r\times(N_r-1)}$$

where N_r is the number of nodes in the rth community of system (11).

Lemma 1. *Assume that assumption* (I) *holds. Then, for any* $r, s \in \mathcal{M}$ *and* $t \geq t_0$,

$$L_r W_{rs}(t) \Xi = \bar{W}_{rs}(t) L_s \Xi,$$

for $\Xi = (\xi_1, \dots, \xi_{N_s})^T \in \mathbb{R}^{N_s}$, *where* $W_{rs}(t)$ *is defined by* (8), *and*

$$\begin{aligned}\bar{W}_{rs}(t) &= [\bar{\omega}^{(pq)}_{rs}(t)]_{1\leq p\leq N_r-1, 1\leq q\leq N_s-1} \\ &= \begin{pmatrix} \bar{\omega}^{(11)}_{rs}(t) & \cdots & \bar{\omega}^{(1(N_s-1))}_{rs}(t) \\ \vdots & \ddots & \vdots \\ \bar{\omega}^{((N_r-1)1)}_{rs}(t) & \cdots & \bar{\omega}^{((N_r-1)(N_s-1))}_{rs}(t) \end{pmatrix} \\ &:= L_r W_{rs}(t) R_s.\end{aligned}$$

Proof. For $s \in \mathcal{M}$,

$$R_s L_s = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -1 \\ 0 & \cdots & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{N_s\times N_s}.$$

Thus, for $r, s \in \mathcal{M}$ and $t \geq t_0$,

$$L_r W_{rs}(t)(I_{N_s} - R_s L_s)\Xi = L_r W_{rs}(t)(\xi_{N_s}, \dots, \xi_{N_s})^T,$$

for $\Xi = (\xi_1, \dots, \xi_{N_s})^T \in \mathbb{R}^{N_s}$, where I_{N_s} is the identity matrix of size N_s, and $(\xi_{N_s}, \dots, \xi_{N_s})^T \in \mathbb{R}^{N_s}$. Accordingly, we obtain

$$\begin{aligned}L_r W_{rs}(t)(I_{N_s} - R_s L_s)\Xi &= \kappa_{rs}(t)\xi_{N_s} L_r(1, \dots, 1)^T \\ &= (0, \dots, 0)^T \in \mathbb{R}^{N_r-1},\end{aligned}$$

where $(1, \dots, 1)^T \in \mathbb{R}^{N_r}$, recalling assumption (I). Hence, we verify the assertion. □

For later use, with $\bar{\omega}^{(pq)}_{rs}(t)$ defined in Lemma 1, we set

$$\check{\omega}^{(pq)}_{rs} := \inf\{\bar{\omega}^{(pq)}_{rs}(t) : t \geq t_0\}, \tag{24}$$

$$\hat{\omega}^{(pq)}_{rs} := \sup\{\bar{\omega}^{(pq)}_{rs}(t) : t \geq t_0\}, \tag{25}$$

$$|\omega|^{(pq)}_{rs} := \sup\{|\bar{\omega}^{(pq)}_{rs}(t)| : t \geq t_0\}. \tag{26}$$

Based on (19) and Lemma 1, we decompose function $H_{r,p,k}$, defined in (21), into three parts, as seen in the following proposition.

Proposition 1. *Consider system* (11) *which satisfies assumptions* (I) *and* (F). *Then, functions* $H_{r,p,k}$, $(r,p,k) \in \mathcal{A}_z$, *can be decomposed as*

$$\begin{aligned}H_{r,p,k}(\mathbf{\Phi}, t) &= h_{r,p,k}(\phi_{r,p,k}(0), \phi_{r,p+1,k}(0), t) \\ &\quad + \tilde{h}_{r,p,k}(\phi_{r,p,k}, \phi_{r,p+1,k}, t) + w_{r,p,k}(\mathbf{\Phi}, t),\end{aligned}$$

where $h_{r,p,k} = h_{r,p,k}(\phi_{r,p,k}(0), \phi_{r,p+1,k}(0), t)$, $\tilde{h}_{r,p,k} = \tilde{h}_{r,p,k}(\phi_{r,p,k}, \phi_{r,p+1,k}, t)$, *and* $w_{r,p,k} = w_{r,p,k}(\mathbf{\Phi}, t)$ *are defined by*

$$\begin{aligned}h_{r,p,k} &:= h_{r,k}(\phi_{r,p,k}(0), \phi_{r,p+1,k}(0), t), \\ \tilde{h}_{r,p,k} &:= \bar{\omega}^{(pp)}_{rr}(t)[G_{rr,k}(\phi_{r,p,k}(-\tau_{rr}(t))) \\ &\quad - G_{rr,k}(\phi_{r,p+1,k}(-\tau_{rr}(t)))], \\ w_{r,p,k} &:= w_{r,k}(\Phi_{r,p}, \Phi_{r,p+1}, t) + \sum_{(s,q)\in\mathcal{B}-\{(r,p)\}} \bar{\omega}^{(pq)}_{rs}(t) \\ &\quad \times [G_{rs,k}(\phi_{s,q,k}(-\tau_{rs}(t))) - G_{rs,k}(\phi_{s,q+1,k}(-\tau_{rs}(t)))].\end{aligned}$$

Herein, $\mathcal{B} := \{(s,q) : s \in \mathcal{M}, q \in \mathcal{N}_s - \{N_s\}\}$; $G_{rs,k}$ *is defined in* (5), $\mathbf{\Phi} = (\Phi_{r,p}) = (\phi_{r,p,k})$ *in* (17), *functions* $h_{r,k}$ *and* $w_{r,k}$ *in* (19), *and* $\bar{\omega}^{(pq)}_{rs}(t)$ *in* Lemma 1.

Proof. By Lemma 1, for $r, s \in \mathcal{M}$ and $p \in \mathcal{N}_r - \{N_r\}$,

$$\sum_{q\in\mathcal{N}_s}[\omega_{rs}^{(pq)}(t) - \omega_{rs}^{((p+1)q)}(t)]\xi_q = \sum_{q\in\mathcal{N}_s-\{N_s\}} \bar{\omega}_{rs}^{(pq)}(t)(\xi_q - \xi_{q+1}), \tag{27}$$

for $\xi_q \in \mathbb{R}$, $q \in \mathcal{N}_s$. As seen from (19), we can rewrite the part $F_{r,k}(\Phi_{r,p}, t) - F_{r,k}(\Phi_{r,p+1}, t)$ of $H_{r,p,k}(\mathbf{\Phi}, t)$ in (21), as follows:

$$\begin{aligned} &F_{r,k}(\Phi_{r,p}, t) - F_{r,k}(\Phi_{r,p+1}, t) \\ &\quad = h_{r,k}(\phi_{r,p,k}(0), \phi_{r,p+1,k}(0), t) + w_{r,k}(\Phi_{r,p}, \Phi_{r,p+1}, t). \end{aligned} \tag{28}$$

On the other hand, applying (27) rewrites the remaining part of $H_{r,p,k}(\mathbf{\Phi}, t)$ in (21), as follows:

$$\begin{aligned} &\sum_{s\in\mathcal{M}}\sum_{q\in\mathcal{N}_s}[\omega_{rs}^{(pq)}(t) - \omega_{rs}^{((p+1)q)}(t)]G_{rs,k}(\phi_{s,q,k}(-\tau_{rs}(t))) \\ &\quad = \sum_{s\in\mathcal{M}}\sum_{q\in\mathcal{N}_s-\{N_s\}} \{\bar{\omega}_{rs}^{(pq)}(t)[G_{rs,k}(\phi_{s,q,k}(-\tau_{rs}(t))) \\ &\qquad - G_{rs,k}(\phi_{s,q+1,k}(-\tau_{rs}(t)))]\}. \end{aligned} \tag{29}$$

The decomposition of $H_{r,p,k}$ then follows directly from (21), (28), and (29). □

Based on Proposition 1, system (20) can be recast into system (22). The following proposition draws the properties associated with functions $h_{r,p,k}$, $\tilde{h}_{r,p,k}$, and $w_{r,p,k}$, defined in Proposition 1. We note that in this proposition, $\check{\mu}_{r,k}$, $\hat{\mu}_{r,k}$, $\bar{\mu}_{r,k}^{(l)}$, $\bar{\beta}_{r,k}^{(l)}$, $\rho_{r,k}^{w}$, and $\tau_{r,k}^{(l)}$ are defined in assumption (F), $\mathcal{A}_z$ in (15), $\mathbf{\Phi} = (\phi_{r,p,k})$ in (17), $\mathcal{C}_{\mathcal{Q}}$ in (18), $\check{\omega}_{rs}^{(pq)}$, $\hat{\omega}_{rs}^{(pq)}$, and $|\omega|_{rs}^{(pq)}$ in (24)–(26), and $\mathcal{B}$ in Proposition 1.

Proposition 2. *Consider system (11) which satisfies assumptions (I), (D), and (F). Then, for all $(r, p, k) \in \mathcal{A}_z$, $\mathbf{\Phi} = (\phi_{r,p,k}) \in \mathcal{C}_{\mathcal{Q}}$, and $t \geq t_0$, functions $h_{r,p,k} = h_{r,p,k}(\phi_{r,p,k}(0), \phi_{r,p+1,k}(0), t)$, $\tilde{h}_{r,p,k} = \tilde{h}_{r,p,k}(\phi_{r,p,k}, \phi_{r,p+1,k}, t)$, and $w_{r,p,k}(\mathbf{\Phi}, t)$, defined in Proposition 1, satisfy the following properties:*

(i) $\begin{cases} \check{\mu}_{r,p,k} \leq h_{r,p,k}/\Delta\phi_{r,p,k}(0) \leq \hat{\mu}_{r,p,k} & \text{if } \Delta\phi_{r,p,k}(0) \neq 0, \\ h_{r,p,k} = 0 & \text{if } \Delta\phi_{r,p,k}(0) = 0, \end{cases}$

(ii) $|\tilde{h}_{r,p,k}(\phi_{r,p,k}, \phi_{r,p+1,k}, t)| \leq \rho_{r,p,k}^{h}$, *and*

$$\begin{cases} \check{\beta}_{r,p,k} \leq \dfrac{\tilde{h}_{r,p,k}}{\Delta\phi_{r,p,k}(-\tau_{rr}(t))} \leq \hat{\beta}_{r,p,k} \\ \quad \text{if } \Delta\phi_{r,p,k}(-\tau_{rr}(t)) \neq 0, \\ \tilde{h}_{r,p,k} = 0 \quad \text{if } \Delta\phi_{r,p,k}(-\tau_{rr}(t)) = 0, \end{cases}$$

(iii) $|w_{r,p,k}(\mathbf{\Phi}, t)| \leq \rho_{r,p,k}^{w}$, *and*

$$\begin{aligned} &|w_{r,p,k}(\mathbf{\Phi}, t)| \\ &\quad \leq \sum_{(s,q,l)\in\mathcal{A}_z-\{(r,p,k)\}} [\bar{\mu}_{r,p,k}^{(s,q,l)}|\phi_{s,q,l}(0) - \phi_{s,q+1,l}(0)| \\ &\qquad + \bar{\beta}_{r,p,k}^{(s,q,l)}|\phi_{s,q,l}(-\tau_{r,p,k}^{(s,q,l)}) - \phi_{s,q+1,l}(-\tau_{r,p,k}^{(s,q,l)})|], \end{aligned}$$

where $\Delta\phi_{r,p,k}(\theta) := \phi_{r,p,k}(\theta) - \phi_{r,p+1,k}(\theta)$, for all $\theta \in [-\tau_M, 0]$,

$$\check{\mu}_{r,p,k} := \check{\mu}_{r,k}, \qquad \hat{\mu}_{r,p,k} := \hat{\mu}_{r,k},$$

$$\check{\beta}_{r,p,k} := \begin{cases} \check{\omega}_{rr}^{(pp)}\check{L}_{rr,k} & \text{if } \check{\omega}_{rr}^{(pp)} \geq 0, \\ \check{\omega}_{rr}^{(pp)}\hat{L}_{rr,k} & \text{if } \check{\omega}_{rr}^{(pp)} < 0, \end{cases}$$

$$\hat{\beta}_{r,p,k} := \begin{cases} \hat{\omega}_{rr}^{(pp)}\hat{L}_{rr,k} & \text{if } \hat{\omega}_{rr}^{(pp)} \geq 0, \\ \hat{\omega}_{rr}^{(pp)}\check{L}_{rr,k} & \text{if } \hat{\omega}_{rr}^{(pp)} < 0, \end{cases}$$

$$\bar{\mu}_{r,p,k}^{(s,q,l)} := \begin{cases} \bar{\mu}_{r,k}^{(l)} & \text{if } (s,q) = (r,p), l \neq k, \\ 0 & \text{otherwise}, \end{cases}$$

$$\bar{\beta}_{r,p,k}^{(s,q,l)} := \begin{cases} \bar{\beta}_{r,k}^{(l)} & \text{if } (s,q) = (r,p), l \neq k, \\ |\omega|_{rs}^{(pq)}\hat{L}_{rs,k} & \text{if } (s,q) \neq (r,p), l = k, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tau_{r,p,k}^{(s,q,l)} := \begin{cases} \tau_{r,k}^{(l)} & \text{if } (s,q) = (r,p), l \neq k, \\ \tau_{rs}(t) & \text{if } (s,q) \neq (r,p), l = k, \\ 0 & \text{otherwise}, \end{cases}$$

and $\rho_{r,p,k}^{h}$ and $\rho_{r,p,k}^{w}$ are arbitrary quantities satisfying

$$\begin{aligned} &\rho_{r,p,k}^{h} \geq 2|\omega|_{rr}^{(pp)}\bar{G}_{rr,k}, \ \rho_{r,p,k}^{w} \\ &\quad \geq \rho_{r,k}^{w} + 2\left(\sum_{(s,q)\in\mathcal{B}-\{(r,p)\}} |\omega|_{rs}^{(pq)}\bar{G}_{rs,k}\right). \end{aligned}$$

Herein,

$$\check{L}_{rs,k} := \min\{(G_{rs,k})'(\xi) : \xi \in [\check{\varrho}_{s,k}, \hat{\varrho}_{s,k}]\} \geq 0, \tag{30}$$

$$\hat{L}_{rs,k} := \max\{(G_{rs,k})'(\xi) : \xi \in [\check{\varrho}_{s,k}, \hat{\varrho}_{s,k}]\} \geq 0, \tag{31}$$

$$\bar{G}_{rs,k} := \max\{|G_{rs,k}(\xi)| : \xi \in [\check{\varrho}_{s,k}, \hat{\varrho}_{s,k}]\} \geq 0, \tag{32}$$

where $G_{rs,k}$ is introduced in (5), and $[\check{\varrho}_{s,k}, \hat{\varrho}_{s,k}]$ is defined in assumption (D).

Proof. Notably, $\mathbf{\Phi} = (\Phi_{r,p}) = (\phi_{r,p,k}) \in \mathcal{C}_{\mathcal{Q}}$ implies that $\Phi_{r,p} \in \mathcal{C}_{Q_r}$ for all $(r, p) \in \{(s, q) : s \in \mathcal{M}, q \in \mathcal{N}_s\}$, and $\phi_{r,p,k}(\theta) \in [\check{\varrho}_{r,k}, \hat{\varrho}_{r,k}]$ for all $\theta \in [-\tau_M, 0]$ and $(r, p, k) \in \mathcal{A}_z$. Thus,

$$\begin{aligned} &|\bar{\omega}_{rs}^{(pq)}(t)[G_{rs,k}(\phi_{s,q,k}(-\tau_{rs}(t))) - G_{rs,k}(\phi_{s,q+1,k}(-\tau_{rs}(t)))]| \\ &\quad \leq 2|\omega|_{rs}^{(pq)}\bar{G}_{rs,k}, \end{aligned} \tag{33}$$

for all $(r, p, k) \in \mathcal{A}_z$ and $(s, q) \in \mathcal{B}$, recalling (26) and (32). With the help of (33), the boundedness of $\tilde{h}_{r,p,k}$ and $w_{r,p,k}$ in assertions (ii) and (iii) can be verified through assumption (F) and the definitions of $\tilde{h}_{r,p,k}$ and $w_{r,p,k}$ in Proposition 1. On the other hand, via the mean-value theorem, $\tilde{h}_{r,p,k} = \tilde{h}_{r,p,k}(\phi_{r,p,k}, \phi_{r,p+1,k}, t)$ and $w_{r,p,k} = w_{r,p,k}(\mathbf{\Phi}, t)$ can be expressed as

$$\begin{aligned} \tilde{h}_{r,p,k} &= \bar{\omega}_{rr}^{(pp)}(t)(G_{rr,k})'(\sigma_{r,p}^{\tau}(t)) \\ &\quad \times [\phi_{r,p,k}(-\tau_{rr}(t)) - \phi_{r,p+1,k}(-\tau_{rr}(t))] \end{aligned} \tag{34}$$

$$\begin{aligned} w_{r,p,k} &= w_{r,k}(\Phi_{r,p}, \Phi_{r,p+1}, t) \\ &\quad + \sum_{(s,q)\in\mathcal{B}-\{(r,p)\}} \bar{\omega}_{rs}^{(pq)}(t)(G_{rs,k})'(\sigma_{s,q}^{\tau}(t)) \\ &\quad \times [\phi_{s,q,k}(-\tau_{rs}(t)) - \phi_{s,q+1,k}(-\tau_{rs}(t))], \end{aligned} \tag{35}$$

for some $\sigma_{s,q}^{\tau}(t)$ between $\phi_{s,q,k}(-\tau_{rs}(t))$ and $\phi_{s,q+1,k}(-\tau_{rs}(t))$, $(s, q) \in \mathcal{B}$. Notably, for all $(s, q) \in \mathcal{B}$, $\sigma_{s,q}^{\tau}(t) \in [\check{\varrho}_{s,k}, \hat{\varrho}_{s,k}]$ and

$$0 \leq \check{L}_{rs,k} \leq (G_{rs,k})'(\sigma_{s,q}^{\tau}(t)) \leq \hat{L}_{rs,k}, \tag{36}$$

recalling (30) and (31). Based on assumption (F), (24)–(26), and (36), the remaining assertions follow directly from (34), (35), and the definition of $h_{r,p,k}$ in Proposition 1. □

The terms $\check{\mu}_{r,p,k}$, $\hat{\mu}_{r,p,k}$, $\check{\beta}_{r,p,k}$, $\hat{\beta}_{r,p,k}$, $\rho_{r,p,k}^{h}$, $\rho_{r,p,k}^{w}$, $\bar{\mu}_{r,p,k}^{(s,q,l)}$, and $\bar{\beta}_{r,p,k}^{(s,q,l)}$ in Proposition 2 exist if system (11) satisfies assumptions (I), (D), and (F), functions $G_{rs,k}$ and $G'_{rs,k}$ are continuous and bounded on $[\check{\varrho}_{s,k}, \hat{\varrho}_{s,k}]$, and all $\bar{\omega}_{rs}^{(pq)}(t)$ are bounded functions of t. With these terms and $\bar{\tau}_r$ defined in (6), we define the following quantities:

$$\begin{aligned} \eta_{r,p,k} &:= -\hat{\mu}_{r,p,k} - \hat{\beta}_{r,p,k} + \bar{\beta}_{r,p,k}\bar{\tau}_r \\ &\quad \times \bar{\tau}_r(\check{\mu}_{r,p,k} + \hat{\mu}_{r,p,k} + \check{\beta}_{r,p,k} + \hat{\beta}_{r,p,k}), \end{aligned} \tag{37}$$

$$\tilde{\eta}_{r,p,k} := -\hat{\mu}_{r,p,k} - \hat{\beta}_{r,p,k}, \tag{38}$$

$$\bar{L}_{r,p,k}^{(s,q,l)} := \bar{\mu}_{r,p,k}^{(s,q,l)} + \bar{\beta}_{r,p,k}^{(s,q,l)}, \tag{39}$$

where

$$\bar{\beta}_{r,p,k} := \max\{|\check{\beta}_{r,p,k}|, |\hat{\beta}_{r,p,k}|\}.$$

We further introduce the following conditions for the cluster synchronization of system (11):

Condition (S1): $\hat{\mu}_{r,p,k} + \hat{\beta}_{r,p,k} < 0$ and

$$\bar{\beta}_{r,p,k}\bar{\tau}_r < \frac{3\rho^h_{r,p,k}(\hat{\mu}_{r,p,k} + \hat{\beta}_{r,p,k})}{(\hat{\mu}_{r,p,k} + \check{\mu}_{r,p,k} + \hat{\beta}_{r,p,k} + \check{\beta}_{r,p,k})(3\rho^h_{r,p,k} + \rho^w_{r,p,k})},$$

for all $(r, p, k) \in \mathcal{A}_z$.

Condition (S2): $\bar{\beta}_{r,p,k} < -\hat{\mu}_{r,p,k}/[1 + \rho^w_{r,p,k}/\rho^h_{r,p,k}]$, for all $(r, p, k) \in \mathcal{A}_z$.

Recall from (6) that $\bar{\tau}_r \in [0, \tau_M]$ is a bound of the time-dependent transmission delay $\tau_{rr}(t)$. Thus, condition (S1) and quantity $\eta_{r,p,k}$ are delay-dependent because they evolve $\bar{\tau}_r$, whereas condition (S2) and quantities $\tilde{\eta}_{r,p,k}$ and $\bar{L}^{(s,q,l)}_{r,p,k}$ are delay-independent. We note that $\eta_{r,p,k} > 0$ (resp., $\tilde{\eta}_{r,p,k} > 0$) for all $(r, p, k) \in \mathcal{A}_z$, under condition (S1) (resp., (S2)). In the remainder of this section, with condition (S1) (resp., (S2)) and quantities $\eta_{r,p,k}$ (resp., $\tilde{\eta}_{r,p,k}$) and $\bar{L}^{(s,q,l)}_{r,p,k}$, we derive a delay-dependent (resp., delay-independent) criterion for the cluster synchronization of system (11). We illustrate how $\check{\mu}_{r,p,k}$, $\hat{\mu}_{r,p,k}$, $\check{\beta}_{r,p,k}$, $\hat{\beta}_{r,p,k}$, $\rho^h_{r,p,k}$, $\rho^w_{r,p,k}$, $\bar{\mu}^{(s,q,l)}_{r,p,k}$, and $\bar{\beta}^{(s,q,l)}_{r,p,k}$ in Proposition 2; hence condition (S1) and $\eta_{r,p,k}$ and $\bar{L}^{(s,q,l)}_{r,p,k}$, are determined. We then derive a delay-dependent criterion for the cluster synchronization of system (51), a nonlinearly coupled neural network with delays, in Section 3.

Recall that $\mathbf{X}(t) = (\mathbf{x}_{r,p}(t)) = (x_{r,p,k}(t))$ is an arbitrary solution of system (11), and $\mathbf{X}^t = (\mathbf{x}^t_{r,p}) = (x^t_{r,p,k})$ is the corresponding evolution. Then, $\mathbf{Z}(t) = (\mathbf{z}_{r,p}(t)) = (z_{r,p,k}(t))$, where $z_{r,p,k}(t) := x_{r,p,k}(t) - x_{r,p+1,k}(t)$ for $(r, p, k) \in \mathcal{A}_z$, satisfies system (20). Based on Proposition 1, system (20) is recast to system (22). Let us relabel the three-dimensional indices in system (22) to one-dimensional indices, through the bijective mapping ℓ: $\mathcal{A}_z \to \{1, \dots, (N - m)K\}$, defined by

$$\ell(r, p, k) = k + (p - 1)K + \sum_{s\in\mathcal{M},\ s<r} (N_s - 1)K. \tag{40}$$

The labeling $\ell(r, p, k)$ corresponds to the sequence (r, p, k) by considering the order of r, p, and k in succession; more precisely,

$$\ell(r_1, p_1, k_1) < \ell(r_2, p_2, k_2) \quad \text{if } r_1 < r_2,$$
$$\ell(r, p_1, k_1) < \ell(r, p_2, k_2) \quad \text{if } p_1 < p_2,$$
$$\ell(r, p, k_1) < \ell(r, p, k_2) \quad \text{if } k_1 < k_2.$$

For later use, we define

$$\check{\mathcal{L}}_{r,p,k} := \{(s, q, l) \in \mathcal{A}_z : \ell(s, q, l) < \ell(r, p, k)\}, \tag{41}$$
$$\hat{\mathcal{L}}_{r,p,k} := \{(s, q, l) \in \mathcal{A}_z : \ell(s, q, l) > \ell(r, p, k)\}. \tag{42}$$

By setting $z_{\ell(r,p,k)} := z_{r,p,k}$, $x_{\ell(r,p,k)} := x_{r,p,k}$, $x^t_{\ell(r,p,k)} := x^t_{r,p,k}$, $h_{\ell(r,p,k)} := h_{r,p,k}$, $\tilde{h}_{\ell(r,p,k)} := \tilde{h}_{r,p,k}$, and $w_{\ell(r,p,k)} := w_{r,p,k}$, we can rewrite system (22) as follows:

$$\begin{aligned}\dot{z}_{\ell(r,p,k)}(t) &= h_{\ell(r,p,k)}(x_{\ell(r,p,k)}(t), x_{\ell(r,p+1,k)}(t), t) \\ &\quad + \tilde{h}_{\ell(r,p,k)}(x^t_{\ell(r,p,k)}, x^t_{\ell(r,p+1,k)}, t) + w_{\ell(r,p,k)}(t),\end{aligned} \tag{43}$$

for $(r, p, k) \in \mathcal{A}_z$ and $t \geq t_0$. Under assumption (D), every component in system (43) takes the form (23) (restated as (91)). In the Appendix, we summarize the convergent property of (91) in Proposition A.1 under conditions (A1) and (H_0). By Proposition 2, under assumptions (I), (D), and (F), every component in (43) satisfies condition (H_0), with $\check{\mu} = \check{\mu}_{r,p,k}$, $\hat{\mu} = \hat{\mu}_{r,p,k}$, $\check{\beta} = \check{\beta}_{r,p,k}$, $\hat{\beta} = \hat{\beta}_{r,p,k}$, $\rho^h = \rho^h_{r,p,k}$, $\bar{\tau} = \bar{\tau}_r$. In addition, every component in (43) satisfies condition (A1) under condition (S1). Consequently, by Proposition A.1, there exist $(N - m)K$ intervals $[-\bar{\nu}_{\ell(r,p,k)}, \bar{\nu}_{\ell(r,p,k)}]$ to which $z_{\ell(r,p,k)}(t)$ converges as $t \to \infty$, where $(r, p, k) \in \mathcal{A}_z$; moreover,

$$0 \leq \bar{\nu}_{\ell(r,p,k)} \leq |w_{\ell(r,p,k)}|^{\max}(\infty)/\eta_{r,p,k}, \tag{44}$$

where $\eta_{r,p,k}$ is defined in (37). Herein, $|w_{\ell(r,p,k)}|^{\max}(\infty) := \lim_{T\to\infty} |w_{\ell(r,p,k)}|^{\max}(T)$, where $|w_{\ell(r,p,k)}|^{\max}(T) := \sup\{|w_{\ell(r,p,k)}(t)| : t \geq T\}$, for $T \geq t_0$.

The following proposition shows that $\bar{\nu}_{\ell(r,p,k)}$, defined in (44), can further be estimated iteratively.

Proposition 3. *Assume that assumptions* (I), (D), *and* (F), *and condition* (S1) *hold. Then, for each* $(r, p, k) \in \mathcal{A}_z$, *there exists a sequence* $\{\nu^{(n)}_{\ell(r,p,k)}\}_{n=0}^{\infty}$ *which satisfies*

$$0 \leq \bar{\nu}_{\ell(r,p,k)} \leq \nu^{(n)}_{\ell(r,p,k)} \tag{45}$$

where

$$\nu^{(n)}_{\ell(r,p,k)} := \left[\sum_{(s,q,l)\in\check{\mathcal{L}}_{r,p,k}} \bar{L}^{(s,q,l)}_{r,p,k}\nu^{(n)}_{\ell(s,q,l)} + \sum_{(s,q,l)\in\hat{\mathcal{L}}_{r,p,k}} \bar{L}^{(s,q,l)}_{r,p,k}\nu^{(n-1)}_{\ell(s,q,l)}\right]\Big/\eta_{r,p,k},$$

for $n \geq 1$, *and* $0 \leq \bar{\nu}_{\ell(r,p,k)} \leq \nu^{(0)}_{\ell(r,p,k)} := \rho^w_{r,p,k}/\eta_{r,p,k}$, *where* $\eta_{r,p,k}$ *is defined in* (37), $\bar{L}^{(s,q,l)}_{r,p,k}$ *in* (39), $\check{\mathcal{L}}_{r,p,k}$ *in* (41), *and* $\hat{\mathcal{L}}_{r,p,k}$ *in* (42).

Proof. We prove the proposition by induction. First, we consider $n = 0$ and $(r, p, k) \in \mathcal{A}_z$. By assumption (D), there exists a $t_1 > t_0$ such that $\mathbf{X}^t = (\mathbf{x}^t_{r,p}) \in \mathcal{C}_{\mathcal{Q}}$ and each $\mathbf{x}^t_{r,p} \in \mathcal{C}_{Q_r}$ for all $t \geq t_1$. According to property (iii) of Proposition 2, $|w_{\ell(r,p,k)}(t)| = |w_{r,p,k}(\mathbf{X}^t, t)| \leq \rho^w_{r,p,k}$ for all $(r, p, k) \in \mathcal{A}_z$ and $t \geq t_1$. It follows that $|w_{\ell(r,p,k)}|^{\max}(\infty) \leq \rho^w_{r,p,k}$, which yields $0 \leq \bar{\nu}_{\ell(r,p,k)} \leq \rho^w_{r,p,k}/\eta_{r,p,k} = \nu^{(0)}_{\ell(r,p,k)}$, for each $(r, p, k) \in \mathcal{A}_z$, recalling (44).

Next, we assume that (45) holds and $\nu^{(n)}_{\ell(s,q,l)}$ is defined, and hence $z_{\ell(s,q,l)}(t)$ converges to the interval $[-\nu^{(n)}_{\ell(s,q,l)}, \nu^{(n)}_{\ell(s,q,l)}]$ as $t \to \infty$, for $n \in \{0, 1, \dots, n_0 - 1\}$, $(s, q, l) \in \mathcal{A}_z$ and $n = n_0$, $(s, q, l) \in \check{\mathcal{L}}_{r,p,k}$, for some $(r, p, k) \in \mathcal{A}_z$ and $n_0 \geq 1$. Recall that $\mathbf{X}^t = (x_{r,p,k}(t)) \in \mathcal{C}_{\mathcal{Q}}$ for all $t \geq t_1$, and $x^t_{s,q,l}(\theta) - x^t_{s,q+1,l}(\theta) = x_{s,q,l}(t + \theta) - x_{s,q+1,l}(t + \theta) = z_{s,q,l}(t + \theta) = z_{\ell(s,q,l)}(t + \theta)$ for all $\theta \in [-\tau_M, 0]$. By assumption (D) and property (iii) of Proposition 2,

$$\begin{aligned}|w_{\ell(r,p,k)}(t)| &= |w_{r,p,k}(\mathbf{X}^t, t)| \\ &\leq \sum_{(s,q,l)\in\mathcal{A}_z-\{(r,p,k)\}} [\bar{\mu}^{(s,q,l)}_{r,p,k}|x^t_{s,q,l}(0) - x^t_{s,q+1,l}(0)| \\ &\quad + \bar{\beta}^{(s,q,l)}_{r,p,k}|x^t_{s,q,l}(-\tau^{(s,q,l)}_{r,p,k}) - x^t_{s,q+1,l}(-\tau^{(s,q,l)}_{r,p,k})|] \\ &= \sum_{(s,q,l)\in\mathcal{A}_z-\{(r,p,k)\}} [\bar{\mu}^{(s,q,l)}_{r,p,k}|z_{\ell(s,q,l)}(t)| \\ &\quad + \bar{\beta}^{(s,q,l)}_{r,p,k}|z_{\ell(s,q,l)}(t - \tau^{(s,q,l)}_{r,p,k})|],\end{aligned}$$

for all $t \geq t_1$. Then, we obtain

$$|w_{\ell(r,p,k)}|^{\max}(\infty) \leq \sum_{(s,q,l)\in\check{\mathcal{L}}_{r,p,k}} \bar{L}^{(s,q,l)}_{r,p,k}\nu^{(n_0)}_{\ell(s,q,l)} + \sum_{(s,q,l)\in\hat{\mathcal{L}}_{r,p,k}} \bar{L}^{(s,q,l)}_{r,p,k}\nu^{(n_0-1)}_{\ell(s,q,l)},$$

which yields $0 \leq \bar{\nu}_{\ell(r,p,k)}(t) \leq |w_{\ell(r,p,k)}|^{\max}(\infty)/\eta_{\ell(r,p,k)} \leq \nu^{(n_0)}_{\ell(r,p,k)}$, recalling (44). Hence, we complete the proof. □

We observe that $\{(\nu_1^{(n)}, \dots, \nu_{(N-m)K}^{(n)})^T\}_{n=0}^{\infty}$, with $\nu_\ell^{(n)} = \nu_{\ell(r,p,k)}^{(n)}$, $\ell = 1, \dots, (N-m)K$, defined in Proposition 3, is exactly the Gauss–Seidel iteration for solving the linear system:

$$\mathbf{M}\mathbf{v} = \mathbf{0}, \tag{46}$$

where

$$\mathbf{M} := D_{\mathbf{M}} - L_{\mathbf{M}} - U_{\mathbf{M}} = [m_{\imath\jmath}]_{1 \le \imath,\jmath \le (N-m)K}, \tag{47}$$

with

$$m_{\imath\jmath} := \begin{cases} \eta_{r,p,k} & \text{if } \imath = \jmath = \ell(r,p,q), \\ -\bar{L}_{r,p,k}^{(s,q,l)} & \text{if } \imath = \ell(r,p,k), \jmath = \ell(s,q,l), \text{ and } \imath \ne \jmath. \end{cases}$$

Herein, $D_{\mathbf{M}}$, $-L_{\mathbf{M}}$, and $-U_{\mathbf{M}}$ represent the diagonal, strictly lower-triangular, and strictly upper-triangular parts of $\mathbf{M}$, respectively, and $\eta_{r,p,k}$ and $\bar{L}_{r,p,k}^{(s,q,l)}$ are defined in (37) and (39), respectively. Notice that $\mathbf{M}$ is regarded as delay-dependent, since the diagonal entries $\eta_{r,p,k}$ are delay-dependent, as indicated in (37). The following theorem transforms the problem of global cluster synchronization of system (11) into solving linear system (46).

Theorem 1. *Consider system (11) which satisfies assumptions* (I), (D), *and* (F) *and condition* (S1). *Then, the system attains global cluster synchronization if the Gauss–Seidel iteration for the linear system (46) converges to zero, the unique solution of (46), or equivalently,*

$$\lambda_{\text{syn}} := \max_{1 \le \sigma \le (N-m)K} \{|\lambda_\sigma| : \lambda_\sigma : \text{eigenvalue of } (D_{\mathbf{M}} - L_{\mathbf{M}})^{-1} U_{\mathbf{M}}\} < 1,$$

where matrices $D_{\mathbf{M}}$, $L_{\mathbf{M}}$, and $U_{\mathbf{M}}$ are defined in (47).

Proof. Recall that $\mathbf{X}(t) = (x_{r,p,k}(t))$ is an arbitrary solution of system (11), and $z_{\ell(r,p,k)}(t) = x_{r,p,k}(t) - x_{r,p+1,k}(t)$ converges to $[-\bar{\nu}_{\ell(r,p,k)}, \bar{\nu}_{\ell(r,p,k)}]$ for all $(r,p,k) \in \mathcal{A}_z$. By Proposition 3, $0 \le \bar{\nu}_{\ell(r,p,k)} \le \nu_{\ell(r,p,k)}^{(n)}$ for all $n \in \mathbb{N}$ and $(r,p,k) \in \mathcal{A}_z$. Accordingly, if $\{(\nu_1^{(n)}, \dots, \nu_{(N-m)K}^{(n)})^T\}_{n=0}^{\infty}$, the Gauss–Seidel iteration for the linear system (46), converges to zero, system (11) achieves global cluster synchronization. Notice that condition (S1) implies that all diagonal entries of $\mathbf{M}$ (i.e., $\eta_{r,p,k}$, for $(r,p,k) \in \mathcal{A}_z$) are positive. Accordingly, $(D_{\mathbf{M}} - L_{\mathbf{M}})^{-1}$ and λ_{syn} exist; moreover, the Gauss–Seidel iteration for solving linear system (46) converges to zero if and only if $\lambda_{\text{syn}} < 1$. Hence, we complete the proof. □

The criterion in Theorem 1 is delay-dependent because condition (S1) and matrix $\mathbf{M}$ are delay-dependent. The application of arguments parallel to those of Proposition 3 and Theorem 1, but using Proposition A.2 instead of Proposition A.1 enables us to derive a delay-independent criterion for the cluster synchronization of system (11). We obtain such a criterion by defining a delay-independent matrix $\tilde{\mathbf{M}}$ that plays the role of $\mathbf{M}$ in Theorem 1, as follows:

$$\tilde{\mathbf{M}} := D_{\tilde{\mathbf{M}}} - L_{\tilde{\mathbf{M}}} - U_{\tilde{\mathbf{M}}} = [\tilde{m}_{\imath\jmath}]_{1 \le \imath,\jmath \le (N-m)K}, \tag{48}$$

with

$$\tilde{m}_{\imath\jmath} := \begin{cases} \tilde{\eta}_{r,p,k} & \text{if } \imath = \jmath = \ell(r,p,q), \\ -\bar{L}_{r,p,k}^{(s,q,l)} & \text{if } \imath = \ell(r,p,k), \jmath = \ell(s,q,l), \text{ and } \imath \ne \jmath, \end{cases}$$

where $D_{\tilde{\mathbf{M}}}$, $-L_{\tilde{\mathbf{M}}}$, and $-U_{\tilde{\mathbf{M}}}$ are diagonal, strictly lower-triangular, and strictly upper-triangular parts of $\tilde{\mathbf{M}}$, respectively. Furthermore, $\tilde{\eta}_{r,p,k}$ and $\bar{L}_{r,p,k}^{(s,q,l)}$ are defined in (38) and (39), respectively.

Theorem 2. *Consider system (11) which satisfies assumptions* (I), (D), *and* (F) *and condition* (S2). *Then, the system attains global cluster synchronization if the Gauss–Seidel iteration for the linear system:*

$$\tilde{\mathbf{M}}\mathbf{v} = 0, \tag{49}$$

converges to zero, the unique solution of (49), or equivalently,

$$\tilde{\lambda}_{\text{syn}} := \max_{1 \le \sigma \le (N-m)K} \{|\lambda_\sigma| : \lambda_\sigma : \text{eigenvalue of } (D_{\tilde{\mathbf{M}}} - L_{\tilde{\mathbf{M}}})^{-1} U_{\tilde{\mathbf{M}}}\} < 1,$$

where $\tilde{\mathbf{M}}$, $D_{\tilde{\mathbf{M}}}$, $-L_{\tilde{\mathbf{M}}}$, and $-U_{\tilde{\mathbf{M}}}$ are defined in (48).

Let us discuss the conditions in Theorems 1 and 2. In Theorem 1, condition (S1) implies that each $\hat{\mu}_{r,p,k} + \hat{\beta}_{r,p,k} < 0$, and each $\bar{\tau}_r$ is sufficiently small, such that all diagonal entries of matrix $\mathbf{M}$ (i.e. $\eta_{r,p,k} := -\hat{\mu}_{r,p,k} - \hat{\beta}_{r,p,k} + \bar{\beta}_{r,p,k}\bar{\tau}_r(\check{\mu}_{r,p,k} + \hat{\mu}_{r,p,k} + \check{\beta}_{r,p,k} + \hat{\beta}_{r,p,k})$, $(r,p,k) \in \mathcal{A}_z$) are positive. Recall from (6) that $\bar{\tau}_r \ge 0$ is the bound of $\tau_{rr}(t)$, where $\tau_{rr}(t)$ refers to the coupling delays between neurons within the rth community. In general, positive and sufficiently large diagonal entries of matrix $\mathbf{M}$ promote the convergence of the Gauss–Seidel iteration for linear system (46). Thus, the criterion in Theorem 1 prefers a negative value for $\hat{\mu}_{r,p,k} + \hat{\beta}_{r,p,k}$ with a large magnitude and small $\bar{\tau}_r$, such that each $\eta_{r,p,k}$ is positive and has a sufficiently large magnitude. Moreover, the criterion in Theorem 1 requires that coupling delays between neurons within the same community are small enough. Similarly, we can observe that the criterion in Theorem 2 is independent of delays, and prefers negative $\hat{\mu}_{r,p,k}$ with a large magnitude, and $\check{\beta}_{r,p,k}$ and $\hat{\beta}_{r,p,k}$ with a small magnitude. In Section 3, we provide the precise synchronization criterion established in Theorem 1, when considering system (51).

In the following remark, let us compare our assumption imposed on the connection matrix $\mathbf{W}(t) = [\tilde{\omega}_{ij}(t)]_{1 \le i,j \le N} = [W_{rs}(t)]_{1 \le r,s \le m}$, cf. (8), with those in the existing related work.

Remark 2. Time-independent connection matrices have commonly been considered by researchers carrying out work related to ours. Let us denote by $\mathbf{W} = [\tilde{\omega}_{ij}]_{1 \le i,j \le N} = [W_{rs}]_{1 \le r,s \le m}$ the connection matrices considered therein. Cao and Li (2009) and Song and Zhao (2014) considered $\mathbf{W}$ that satisfies that all row sums are zero, all off-diagonal entries are non-negative, and all rows in W_{rs}, $r \ne s$, are the same. Lu et al. (2010a) considered that $\mathbf{W}$ satisfies the common intercluster coupling condition, under which every nonzero matrix W_{rs} cannot have zero rows. The approach pursued by Zhang, Ma, and Zhang (2013) requires $\tilde{\omega}_{ij} + \tilde{\omega}_{ji} \ge 0$ for all i, j, with i, j in different index sets of communities. Ma et al. (2006), Wu and Chen (2009), and Zhang, Ma, and Chen (2014) considered $\mathbf{W}$ to be symmetric. Recall that our approach considers time-dependent connection matrix $\mathbf{W}(t)$, and assumption (I) to be the unique requirement imposed upon $\mathbf{W}(t)$. Assumption (I), under which the cluster synchronous manifold is positively invariant, is a prerequisite to the cluster synchronization problem, and is also required in all the studies mentioned above.

3. Implementation of approach

The proposed synchronization framework developed in Section 2 accommodates a large class of network systems in the form of (1). As mentioned in the Introduction, the cluster synchronization of Hopfield-type neural networks has been investigated in Zhang, Ma, and Chen (2014) and Zhang, Ma, and Zhang (2013). In this section, to demonstrate the present approach, we implement our approach to a Hopfield-type neural network consisting of two communities of neurons. These two communities contain two and three identical neurons, respectively. We consider the first two neurons in the network as belonging to the first community, and the remaining neurons to belong to the second community; moreover, the dynamics of each neuron in the rth community, $r = 1, 2$, is governed by

$$\dot{\mathbf{x}}(t) = \mathbf{F}_r(\mathbf{x}^t) := \begin{pmatrix} -x_1(t) + a_r g(x_2(t - \tau_r^I)) \\ -x_2(t) + a_r g(x_1(t - \tau_r^I)) \end{pmatrix}, \quad t \ge 0, \tag{50}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t))^T$, $\mathbf{x}^t \in \mathcal{C}([-\tau_r^I, 0]; \mathbb{R}^2)$ is defined as $\mathbf{x}^t(\theta) = \mathbf{x}(t+\theta)$ for $\theta \in [-\tau_r^I, 0]$, $a_r > 0$, $g(\xi) := \tanh(\xi)$, and $\tau_r^I \geq 0$ represents the internal delay.

Denote by $\tilde{\mathcal{N}}_1 = \{1, 2\}$ (resp., $\tilde{\mathcal{N}}_2 = \{3, 4, 5\}$) the neuron indices in the first (resp., second) community. Then, we consider the following nonlinearly coupled neural networks with coupling delays:

$$\dot{\mathbf{x}}_i(t) = \tilde{\mathbf{F}}_i(\mathbf{x}_i(t-\tau_i^I)) + \sum_{j\in\mathcal{N}} \tilde{\omega}_{ij}\tilde{\mathbf{G}}_{ij}(\mathbf{x}_j(t-\tau_{ij})),$$
$$i \in \mathcal{N},\ t \geq 0, \tag{51}$$

with

$$\tilde{\mathbf{F}}_i(\mathbf{x}_i(t-\tau_i^I)) := \mathbf{F}_r(\mathbf{x}_i(t-\tau_r^I)) \quad \text{if } i \in \tilde{\mathcal{N}}_r,$$
$$\tilde{\mathbf{G}}_{ij}(\mathbf{x}_j(t-\tau_{ij})) = \mathbf{G}_{rs}(\mathbf{x}_j(t-\tau_{rs})) := \begin{pmatrix} g(x_{j,1}(t-\tau_{rs})) \\ 0 \end{pmatrix} \quad \text{if } i \in \tilde{\mathcal{N}}_r,\ j \in \tilde{\mathcal{N}}_s,$$

where $\mathbf{x}_i(t) = (x_{i,1}(t), x_{i,2}(t))^T$, $\mathcal{N} = \tilde{\mathcal{N}}_1 \bigcup \tilde{\mathcal{N}}_2$, $\mathbf{F}_r$ is defined in (50), and $\tau_{rs} \geq 0$, $r, s \in \{1, 2\}$, refers to the transmission delay corresponding to the connection from neurons in the sth community to neurons in the rth community. Obviously, system (51) is in the form of (1) which satisfies (4) and (5), with $t_0 = 0$, $m = 2$, $N_1 = 2$, $N_2 = 3$, $K = 2$, $\tilde{\omega}_{ij}(t) \equiv \tilde{\omega}_{ij}$, $\tau_{rs}(t) \equiv \tau_{rs}$, and $\tau_M = \tau_M^* := \max\{\tau_1^I, \tau_2^I, \tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}\}$. Moreover, functions $\mathbf{F}_r = (F_{r,1}, F_{r,2})^T$ and $\mathbf{G}_{rs} = (G_{rs,1}, G_{rs,2})^T$ in (4) and (5) are now

$$F_{r,1}(\Phi, t) = -\phi_1(0) + a_r g(\phi_2(-\tau_r^I)), \tag{52}$$

$$F_{r,2}(\Phi, t) = -\phi_2(0) + a_r g(\phi_1(-\tau_r^I)), \tag{53}$$

$$G_{rs,1}(\xi) = g(\xi),\ G_{rs,2}(\xi) = 0, \tag{54}$$

for $\Phi = (\phi_1, \phi_2)^T \in \mathcal{C}([-\tau_M^*, 0]; \mathbb{R}^2)$, $\xi \in \mathbb{R}$, and $t \geq 0$. Recalling (2) and (3), we obtain the index sets $\mathcal{N}_1 = \{1, 2\}$ and $\mathcal{N}_2 = \{1, 2, 3\}$, and map $\mathcal{J}$, which depicts the community structure, and now satisfies

$$\mathcal{J}(1) = (1, 1),\ \mathcal{J}(2) = (1, 2),\ \mathcal{J}(3) = (2, 1),$$
$$\mathcal{J}(4) = (2, 2),\ \mathcal{J}(5) = (2, 3). \tag{55}$$

Thus, we read the ith neuron in (51) as the pth node in the rth community, and relabel $\mathbf{x}_i = (x_{i,1}, x_{i,2})^T =: \mathbf{x}_{r,p} = (x_{r,p,1}, x_{r,p,2})^T$, if $\mathcal{J}(i) = (r, p)$; more precisely,

$$x_{1,k} = x_{1,1,k}, \qquad x_{2,k} = x_{1,2,k}, \qquad x_{3,k} = x_{2,1,k},$$
$$x_{4,k} = x_{2,2,k}, \qquad x_{5,k} = x_{2,3,k},$$

for $k = 1, 2$. As seen from (7), system (51) achieves global cluster synchronization if

$$Err(t) := \sqrt{\sum_{k\in\{1,2\}} \sum_{i\in\{1,3,4\}} [x_{i,k}(t) - x_{i+1,k}(t)]^2} \to 0, \quad \text{as } t \to \infty,$$

for every solution $(\mathbf{x}_1(t), \ldots, \mathbf{x}_5(t))^T$ of system (51), where $\mathbf{x}_i(t) = (x_{i,1}(t), x_{i,2}(t))^T$, $i = 1, \ldots, 5$. Herein, function $Err(t)$ is referred to as the synchronization error for the corresponding solution.

Next, we need to derive synchronization criteria for system (51) based on Theorem 1. We obtain these synchronization criteria precisely and clearly by further assuming that the connection matrix $\mathbf{W} := [\tilde{\omega}_{ij}]_{1\leq i,j\leq 5}$ of system (51) is

$$\mathbf{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \tag{56}$$

with

$$W_{11} = \begin{pmatrix} 0 & \beta_1 \\ \beta_1 & 0 \end{pmatrix}, \qquad W_{12} = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_1 & 0 \end{pmatrix},$$

$$W_{21} = \begin{pmatrix} -\gamma_2 & 0 \\ 0 & -\gamma_2 \\ -\gamma_2 & 0 \end{pmatrix}, \qquad W_{22} = \begin{pmatrix} 0 & \beta_2 & 0 \\ 0 & 0 & \beta_2 \\ \beta_2 & 0 & 0 \end{pmatrix},$$

where $\beta_r > 0$ and $\gamma_r \geq 0$, for $r = 1, 2$. Apparently, matrix $\mathbf{W}$ satisfies assumption (I). We derive the delay-dependent synchronization criteria for system (51), which satisfies (56), in the following theorem and corollary. To this end, we define

$$\mathbf{M}^* := \begin{pmatrix} \eta_1^* & -a_1 & -\gamma_1 & 0 & 0 & 0 \\ -a_1 & 1 & 0 & 0 & 0 & 0 \\ -\gamma_2 & 0 & 1 & -a_2 & -\beta_2 & 0 \\ 0 & 0 & -a_2 & 1 & 0 & 0 \\ -\gamma_2 & 0 & -\beta_2 & 0 & \eta_2^* & -a_2 \\ 0 & 0 & 0 & 0 & -a_2 & 1 \end{pmatrix}, \tag{57}$$

$$\tau_1^* := \frac{3g(\rho_{1,1}^*)(1+\beta_1 g'(\rho_{1,1}^*))}{[2+(1+g'(\rho_{1,1}^*))\beta_1][a_1+3\beta_1 g(\rho_{1,1}^*)+\gamma_1 g(\rho_{2,1}^*)]}, \tag{58}$$

$$\tau_2^* := \frac{3g(\rho_{2,1}^*)(1+\beta_2 g'(\rho_{2,1}^*))}{[2+(1+g'(\rho_{2,1}^*))\beta_2][a_2+4\beta_2 g(\rho_{2,1}^*)+\gamma_2 g(\rho_{1,1}^*)]}, \tag{59}$$

where

$$\rho_{r,k}^* := \begin{cases} a_r + \beta_r + \gamma_r & \text{if } k = 1, \\ a_r & \text{if } k = 2, \end{cases} \tag{60}$$

$$\eta_r^* := 1 + \beta_r g'(\rho_{r,1}^*) - \beta_r \tau_{rr}[2 + (1 + g'(\rho_{r,1}^*))\beta_r], \tag{61}$$

for $r = 1, 2$.

Theorem 3. *System (51) attains global cluster synchronization in spite of delays τ_1^I, τ_2^I, τ_{12}, and τ_{21} if τ_{11} and τ_{22} are sufficiently small such that*

$$\tau_{11} < \tau_1^* \quad \text{and} \quad \tau_{22} < \tau_2^*, \tag{62}$$

and the Gauss–Seidel iteration for the linear system:

$$\mathbf{M}^*\mathbf{v} = 0 \tag{63}$$

converges to zero, the unique solution of (63), or equivalently,

$$\lambda^* := \max_{1\leq\sigma\leq 6}\{|\lambda_\sigma| : \lambda_\sigma : \text{eigenvalue of } (D_{\mathbf{M}^*} - L_{\mathbf{M}^*})^{-1} U_{\mathbf{M}^*}\} < 1. \tag{64}$$

Herein, $\mathbf{M}^*$ and τ_r^*, $r = 1, 2$, are defined in (57)–(59), and $D_{\mathbf{M}^*}$, $-L_{\mathbf{M}^*}$, and $-U_{\mathbf{M}^*}$ are the diagonal, strictly lower-triangular, and strictly upper-triangular parts of $\mathbf{M}^*$, respectively.

Proof. Let us first verify that system (51) satisfies assumptions (I), (D), and (F). Notably, system (51) satisfies assumption (I), with $\kappa_{11}(t) \equiv \beta_1$, $\kappa_{12}(t) \equiv \gamma_1$, $\kappa_{21}(t) \equiv -\gamma_2$, and $\kappa_{22}(t) \equiv \beta_2$. From (6), we obtain

$$\bar{\tau}_1 = \tau_{11} \quad \text{and} \quad \bar{\tau}_2 = \tau_{22}. \tag{65}$$

Notice that $|g(\xi)| < 1$ for $\xi \in \mathbb{R}$. As seen from (51) and (56), it is not difficult to verify that system (51) satisfies assumption (D), with

$$[\check{\varrho}_{r,k}, \hat{\varrho}_{r,k}] = [-\rho_{r,k}^*, \rho_{r,k}^*], \quad r, k \in \{1, 2\}, \tag{66}$$

where $\rho_{r,k}^*$ is defined in (60). Next, let us establish assumption (F) for system (51). As seen from (19), (52), and (53), the functions $F_{r,k}$ now satisfy

$$F_{r,k}(\Phi, t) - F_{r,k}(\Psi, t) = h_{r,k}(\phi_k(0), \psi_k(0), t) + w_{r,k}(\Phi, \Psi, t), \tag{67}$$

for $r, k \in \{1, 2\}$, $\Phi = (\phi_1, \phi_2)^T$, $\Psi = (\psi_1, \psi_2)^T \in \mathcal{C}([-\tau_M^*, 0]; \mathbb{R}^2)$, where

$$h_{r,1}(\phi_1(0), \psi_1(0), t) = -[\phi_1(0) - \psi_1(0)],$$

$$h_{r,2}(\phi_2(0), \psi_2(0), t) = -[\phi_2(0) - \psi_2(0)],$$
$$w_{r,1}(\Phi, \Psi, t) = a_r[g(\phi_2(-\tau_r^I)) - g(\psi_2(-\tau_r^I))],$$
$$w_{r,2}(\Phi, \Psi, t) = a_r[g(\phi_1(-\tau_r^I)) - g(\psi_1(-\tau_r^I))].$$

By the mean-value theorem, $w_{r,k}$, $r, k \in \{1, 2\}$, can be expressed as

$$w_{r,1}(\Phi, \Psi, t) = a_r g'(\xi_{r,2})[\phi_2(-\tau_r^I) - \psi_2(-\tau_r^I)], \tag{68}$$
$$w_{r,2}(\Phi, \Psi, t) = a_r g'(\xi_{r,1})[\phi_1(-\tau_r^I) - \psi_1(-\tau_r^I)], \tag{69}$$

where $\xi_{r,k}$ is some number between $\phi_k(-\tau_r^I)$ and $\psi_k(-\tau_r^I)$. Recall that $|g(\xi)| < 1$ and $0 < g'(\xi) \le 1$ for all $\xi \in \mathbb{R}$. By (67)–(69), we can verify that system (51) satisfies assumption (F), with

$$\begin{aligned} &\check{\mu}_{r,k} = \hat{\mu}_{r,k} = -1, \qquad \rho_{r,1}^w = \rho_{r,2}^w = 2a_r, \qquad \bar{\mu}_{r,k}^{(l)} = 0, \\ &\bar{\beta}_{r,k}^{(l)} = a_r, \qquad \tau_{r,k}^{(l)} = \tau_r^I, \end{aligned} \tag{70}$$

for $r, k, l \in \{1, 2\}$, $l \neq k$. Directed computations yield that for matrices W_{rs} in (56), the associated matrices $\bar{W}_{rs}(t)$, $r, s \in \{1, 2\}$, defined in Lemma 1, are now

$$\bar{W}_{11}(t) \equiv \begin{pmatrix} -\beta_1 \end{pmatrix}, \qquad \bar{W}_{12}(t) \equiv \begin{pmatrix} \gamma_1 & 0 \end{pmatrix},$$
$$\bar{W}_{21}(t) \equiv \begin{pmatrix} -\gamma_2 \\ \gamma_2 \end{pmatrix}, \qquad \bar{W}_{22}(t) \equiv \begin{pmatrix} 0 & \beta_2 \\ -\beta_2 & -\beta_2 \end{pmatrix}.$$

Subsequently, we obtain from (24)–(26) that

$$\begin{cases} -\check{\omega}_{11}^{(11)} = -\hat{\omega}_{11}^{(11)} = |\omega|_{11}^{(11)} = \beta_1, \\ \check{\omega}_{12}^{(11)} = \hat{\omega}_{12}^{(11)} = |\omega|_{12}^{(11)} = \gamma_1, \\ \check{\omega}_{12}^{(12)} = \hat{\omega}_{12}^{(12)} = |\omega|_{12}^{(12)} = \check{\omega}_{22}^{(11)} = \hat{\omega}_{22}^{(11)} = |\omega|_{22}^{(11)} = 0, \\ -\check{\omega}_{21}^{(11)} = -\hat{\omega}_{21}^{(11)} = |\omega|_{21}^{(11)} = \check{\omega}_{21}^{(21)} = \hat{\omega}_{21}^{(21)} = |\omega|_{21}^{(2,1)} = \gamma_2, \\ \check{\omega}_{22}^{(12)} = \hat{\omega}_{22}^{(12)} = |\omega|_{22}^{(12)} = -\check{\omega}_{22}^{(21)} = -\hat{\omega}_{22}^{(21)} = |\omega|_{22}^{(21)} = \beta_2, \\ -\check{\omega}_{22}^{(22)} = -\hat{\omega}_{22}^{(22)} = |\omega|_{22}^{(22)} = \beta_2. \end{cases}$$

From (54) and (66), the quantities $\check{L}_{rs,k}$, $\hat{L}_{rs,k}$, and $\bar{G}_{rs,k}$, defined in (30)–(32), become

$$\check{L}_{rs,1} = \min\{g'(\xi) : \xi \in [-\rho_{s,1}^*, \rho_{s,1}^*]\} = g'(\rho_{s,1}^*), \tag{71}$$
$$\hat{L}_{rs,1} = \max\{g'(\xi) : \xi \in [-\rho_{s,1}^*, \rho_{s,1}^*]\} = 1, \tag{72}$$
$$\bar{G}_{rs,1} = \max\{|g(\xi)| : \xi \in [-\rho_{s,1}^*, \rho_{s,1}^*]\} = g(\rho_{s,1}^*), \tag{73}$$
$$\check{L}_{rs,2} = \hat{L}_{rs,2} = \bar{G}_{rs,2} = 0, \tag{74}$$

for $r, s \in \{1, 2\}$. We note that $\mathcal{A}_z$, defined in (15), is now

$$\begin{aligned} \mathcal{A}_z = \mathcal{A}_z^* := \{&(1, 1, 1), (1, 1, 2), (2, 1, 1), (2, 1, 2), \\ &(2, 2, 1), (2, 2, 2)\}. \end{aligned}$$

From (70)–(74), the quantities $\check{\mu}_{r,p,k}$, $\hat{\mu}_{r,p,k}$, $\check{\beta}_{r,p,k}$, $\hat{\beta}_{r,p,k}$, $\rho_{r,p,k}^h$, $\rho_{r,p,k}^w$, $\bar{\mu}_{r,p,k}^{(s,q,l)}$, and $\bar{\beta}_{r,p,k}^{(s,q,l)}$, defined in Proposition 2, can be chosen as follows:

$$\check{\mu}_{r,p,k} = \hat{\mu}_{r,p,k} = -1, \quad \text{for all } (r, p, k) \in \mathcal{A}_z^*,$$

$$\check{\beta}_{r,p,k} = \begin{cases} -\beta_r & \text{if } (r, p, k) \in \{(1, 1, 1), (2, 2, 1)\}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\hat{\beta}_{r,p,k} = \begin{cases} -\beta_r g'(\rho_{r,1}^*) & \text{if } (r, p, k) \in \{(1, 1, 1), (2, 2, 1)\}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\rho_{r,p,k}^w = \begin{cases} \rho_*^w & \text{if } (r, p, k) = (1, 1, 1), \\ 2a_1 & \text{if } (r, p, k) = (1, 1, 2), \\ \rho_{**}^w & \text{if } (r, p, k) \in \{(2, 1, 1), (2, 2, 1)\}, \\ 2a_2 & \text{if } (r, p, k) \in \{(2, 1, 2), (2, 2, 2)\}, \end{cases}$$

$$\bar{\mu}_{r,p,k}^{(s,q,l)} = 0, \quad \text{for all } (r, p, k) \in \mathcal{A}_z^*,$$

$$\bar{\beta}_{r,p,k}^{(s,q,l)} = \begin{cases} a_1 & \text{if } (r, p) = (s, q) = (1, 1), \\ & (k, l) \in \{(1, 2), (2, 1)\}, \\ a_2 & \text{if } (r, p) = (s, q) \in \{(2, 1), (2, 2)\}, \\ & (k, l) \in \{(1, 2), (2, 1)\}, \\ \gamma_1 & \text{if } (r, s) = (1, 2), (p, q) = (1, 1), \\ & k = l = 1, \\ \gamma_2 & \text{if } (r, s) = (2, 1), (p, q) \in \{(1, 1), (2, 1)\}, \\ & k = l = 1, \\ \beta_2 & \text{if } (r, s) = (2, 2), (p, q) \in \{(1, 2), (2, 1)\}, \\ & k = l = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau_{r,p,k}^{(s,q,l)} = \begin{cases} \tau_1^I & \text{if } (r, p) = (s, q) = (1, 1), \\ & (k, l) \in \{(1, 2), (2, 1)\}, \\ \tau_2^I & \text{if } (r, p) = (s, q) \in \{(2, 1), (2, 2)\}, \\ & (k, l) \in \{(1, 2), (2, 1)\}, \\ \tau_{12} & \text{if } (r, s) = (1, 2), (p, q) \in \{(1, 1), (1, 2)\}, \\ & k = l \in \{1, 2\}, \\ \tau_{21} & \text{if } (r, s) = (2, 1), (p, q) \in \{(1, 1), (2, 1)\}, \\ & k = l \in \{1, 2\}, \\ \tau_{22} & \text{if } (r, s) = (2, 2), (p, q) \in \{(1, 2), (2, 1)\}, \\ & k = l \in \{1, 2\}, \\ 0 & \text{otherwise}, \end{cases}$$

$\rho_{r,p,k}^h = 2\beta_r g(\rho_{r,1}^*)$ if $(r, p, k) \in \{(1, 1, 1), (2, 2, 1)\}$, and $\rho_{r,p,k}^h$ is an arbitrary positive number if $(r, p, k) \in \mathcal{A}_z^* - \{(1, 1, 1), (2, 2, 1)\}$, where $\rho_*^w := 2[a_1 + \gamma_1 g(\rho_{2,1}^*)]$, $\rho_{**}^w := 2[a_2 + \beta_2 g(\rho_{2,1}^*) + \gamma_2 g(\rho_{1,1}^*)]$. By these quantities and (65), $\eta_{r,p,k}$ and $\bar{L}_{r,p,k}^{(s,q,l)}$, defined in (37) and (39), are determined as follows:

$$\eta_{r,p,k} = \begin{cases} \eta_1^* & \text{if } (r, p, k) = (1, 1, 1), \\ \eta_2^* & \text{if } (r, p, k) = (2, 2, 1), \\ 1 & \text{if } (r, p, k) \in \{(1, 1, 2), (2, 1, 1), (2, 1, 2), (2, 2, 2)\} \end{cases} \tag{75}$$

and

$$\bar{L}_{r,p,k}^{(s,q,l)} = \bar{\beta}_{r,p,k}^{(s,q,l)}, \tag{76}$$

for all $(r, p, k), (s, q, l) \in \mathcal{A}_z^*$, with $(r, p, k) \neq (s, q, l)$, where η_r^*, $r = 1, 2$, are defined in (61). By (40), mapping ℓ satisfies

$$\begin{aligned} &\ell(1, 1, 1) = 1, \ell(1, 1, 2) = 2, \ell(2, 1, 1) = 3, \\ &\ell(2, 1, 2) = 4, \ell(2, 2, 1) = 5, \ell(2, 2, 2) = 6. \end{aligned} \tag{77}$$

With (75)–(77), we obtain that matrix $\mathbf{M}$, defined in (47), is as follows:

$$\begin{pmatrix} \eta_{1,1,1} & -\bar{L}_{1,1,1}^{(1,1,2)} & -\bar{L}_{1,1,1}^{(2,1,1)} & -\bar{L}_{1,1,1}^{(2,1,2)} & -\bar{L}_{1,1,1}^{(2,2,1)} & -\bar{L}_{1,1,1}^{(2,2,2)} \\ -\bar{L}_{1,1,2}^{(1,1,1)} & \eta_{1,1,2} & -\bar{L}_{1,1,2}^{(2,1,1)} & -\bar{L}_{1,1,2}^{(2,1,2)} & -\bar{L}_{1,1,2}^{(2,2,1)} & -\bar{L}_{1,1,2}^{(2,2,2)} \\ -\bar{L}_{2,1,1}^{(1,1,1)} & -\bar{L}_{2,1,1}^{(1,1,2)} & \eta_{2,1,1} & -\bar{L}_{2,1,1}^{(2,1,2)} & -\bar{L}_{2,1,1}^{(2,2,1)} & -\bar{L}_{2,1,1}^{(2,2,2)} \\ -\bar{L}_{2,1,2}^{(1,1,1)} & -\bar{L}_{2,1,2}^{(1,1,2)} & -\bar{L}_{2,1,2}^{(2,1,1)} & \eta_{2,1,2} & -\bar{L}_{2,1,2}^{(2,2,1)} & -\bar{L}_{2,1,2}^{(2,2,2)} \\ -\bar{L}_{2,2,1}^{(1,1,1)} & -\bar{L}_{2,2,1}^{(1,1,2)} & -\bar{L}_{2,2,1}^{(2,1,1)} & -\bar{L}_{2,2,1}^{(2,1,2)} & \eta_{2,2,1} & -\bar{L}_{2,2,1}^{(2,2,2)} \\ -\bar{L}_{2,2,2}^{(1,1,1)} & -\bar{L}_{2,2,2}^{(1,1,2)} & -\bar{L}_{2,2,2}^{(2,1,1)} & -\bar{L}_{2,2,2}^{(2,1,2)} & -\bar{L}_{2,2,2}^{(2,2,1)} & \eta_{2,2,2} \end{pmatrix},$$

which is exactly $\mathbf{M}^*$. It is not difficult to verify that system (51) satisfies condition (S1) under condition (62). Hence, the assertion of this theorem follows directly from Theorem 1. □

The following corollary originates from Theorem 3, by requiring strict diagonal-dominance of $\mathbf{M}^*$ in (63), which is a sufficient condition for the convergence of the Gauss–Seidel iteration for (63). The criterion in this corollary can be verified directly, without computing the eigenvalue λ^*, defined in (64).

Corollary 1. *System (51) attains global cluster synchronization in spite of time delays τ_1^I, τ_2^I, τ_{12} and τ_{21} if*

$$1 > \max\{a_1, a_2 + \beta_2 + \gamma_2\}, \qquad 1 + \beta_1 g'(\rho_{1,1}^*) > a_1 + \gamma_1, \tag{78}$$

and τ_{11} and τ_{22} are sufficiently small such that

$$\tau_{11} < \min\{\tau_1^*, \tilde{\tau}_1^*\}, \qquad \tau_{22} < \min\{\tau_2^*, \tilde{\tau}_2^*\}, \tag{79}$$

where

$$\tilde{\tau}_1^* := \frac{1 + \beta_1 g'(\rho_{1,1}^*) - a_1 - \gamma_1}{\beta_1[2 + (1 + g'(\rho_{1,1}^*))\beta_1]}, \tag{80}$$

$$\tilde{\tau}_2^* := \frac{1 + \beta_2 g'(\rho_{2,1}^*) - a_2 - \beta_2 - \gamma_2}{\beta_2[2 + (1 + g'(\rho_{2,1}^*))\beta_2]}, \tag{81}$$

τ_1^* and τ_2^* are defined in (58) and (59), respectively, and $\rho_{1,1}^*$ and $\rho_{2,1}^*$ are defined in (60).

Proof. Obviously, condition (79) guarantees that condition (62) holds and $\tau_{rr} < \tilde{\tau}_r^*$, for $r = 1, 2$. It is not difficult to verify that matrix $\mathbf{M}^*$ is strictly diagonally dominant if condition (78) holds, and $\tau_{rr} < \tilde{\tau}_r^*$, for $r = 1, 2$. Accordingly, conditions (78) and (79) imply that condition (62) holds, and the Gauss–Seidel iteration for system (63) converges to zero, the unique solution of (63). Hence, the assertion of this corollary hence follows directly from Theorem 3. □

Remark 3. As summarized in the Introduction, Cao and Li (2009), Lu et al. (2010a), Ma et al. (2006), Wu and Chen (2009), Zhang, Ma, and Chen (2014), and Zhang, Ma, and Zhang (2013) investigated the cluster synchronization of linearly coupled systems. Song and Zhao (2014) established the cluster synchronization of nonlinearly coupled identical systems with homogeneous coupling delays. On the other hand, as seen from Remark 2, the connection matrix for system (51) does not satisfy the requirements in Cao and Li (2009), Ma et al. (2006), Song and Zhao (2014), Wu and Chen (2009), Zhang, Ma, and Chen (2014), and Zhang, Ma, and Zhang (2013). As far as we could establish, the cluster synchronization of system (51) cannot be concluded by the methods employed in those papers, even if system (51) has homogeneous delays.

The following two examples illustrate synchronization scenarios of system (51), by using Corollary 1 and Theorem 3, respectively.

Example 1. Consider system (51) with $a_1 = 0.99$, $a_2 = 0.2$, $\beta_1 = 0.4$, $\beta_2 = 0.2$, $\gamma_1 = 0.05$, $\gamma_2 = 0.04$, $\tau_1^I = 20$, $\tau_2^I = 10$, $\tau_{11} = 0.04$, $\tau_{12} = 5$, $\tau_{21} = 10$, $\tau_{22} = 0.5$.

It is obvious that

$$1 > \max\{a_1, a_2 + \beta_2 + \gamma_2\} = 0.99. \tag{82}$$

We obtain directly from (60) that

$$\rho_{1,1}^* = 1.44, \qquad \rho_{2,1}^* = 0.44. \tag{83}$$

By (83), a direct computation yields

$$1 + \beta_1 g'(\rho_{1,1}^*) - (a_1 + \gamma_1) \approx 0.0405, \tag{84}$$

recalling $g(\xi) = \tanh(\xi)$. Moreover, by (58), (59), (80), (81), and (83), we obtain

$$\tau_1^* \approx 0.5606, \qquad \tau_2^* \approx 1.0791, \qquad \tilde{\tau}_1^* \approx 0.0408, \qquad \tilde{\tau}_2^* \approx 1.5339. \tag{85}$$

Notably, (82), (84), and (85) imply that conditions (78) and (79) hold. By Corollary 1, the system attains global cluster synchronization. Fig. 1(a)–(c) demonstrate that the three solutions either converge to various nontrivial equilibria, or remain oscillating, depending on their initial conditions; moreover, their corresponding synchronization errors $Err(t)$ all approach zero.

If we consider $\tau_{11} = 20$ and $\tau_{22} = 10$ instead of $\tau_{11} = 0.04$ and $\tau_{22} = 0.5$, then condition (79) does not hold. In this case, Fig. 1(d) demonstrates that for some solution, the corresponding synchronization error $Err(t)$ does not tend to zero.

We note that, from our simulation, the dynamics for each isolated neuron in this system (i.e., system (50), for $r = 1$ or 2) admits a stable equilibrium at the origin as the only attractor. This example appears to indicate that the coupling can promote the oscillation and multistability, and large coupling delays between neurons within the same community can induce asynchrony.

Example 2. Consider system (51), with $a_1 = 1.02$, $a_2 = 0.2$, $\beta_1 = 0.5$, $\beta_2 = 0.2$, $\gamma_1 = 0.1$, $\gamma_2 = 0.05$, $\tau_1^I = 20$, $\tau_2^I = 15$, $\tau_{11} = \tau_{22} = 0.01$, $\tau_{12} = 5$, $\tau_{21} = 10$.

We obtain from (60) directly that

$$\rho_{1,1}^* = 1.62, \qquad \rho_{2,1}^* = 0.45, \qquad \rho_{12}^* = 1.02, \qquad \rho_{22}^* = 0.2. \tag{86}$$

Based on (86), we compute (58), (59), and (61) to obtain

$$\eta_1^* \approx 1.0597, \qquad \eta_2^* \approx 1.1597, \qquad \tau_1^* \approx 0.4722, \qquad \tau_2^* \approx 1.0678. \tag{87}$$

By (87), we can verify that condition (62) holds. With $a_1 = 1.02$, $a_2 = 0.2$, $\beta_1 = 0.5$, $\beta_2 = 0.2$, $\gamma_1 = 0.1$, $\gamma_2 = 0.05$, and $\eta_1^* \approx 1.0597$ and $\eta_2^* \approx 1.1597$ in (87), we then estimate $\mathbf{M}^*$ defined in (57), as follows:

$$\mathbf{M}^* = D_{\mathbf{M}^*} - L_{\mathbf{M}^*} - U_{\mathbf{M}^*}$$

where

$$D_{\mathbf{M}^*} \approx \begin{pmatrix} 1.0597 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.1597 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{88}$$

$$L_{\mathbf{M}^*} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1.02 & 0 & 0 & 0 & 0 & 0 \\ 0.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0.05 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \end{pmatrix}, \tag{89}$$

$$U_{\mathbf{M}^*} = \begin{pmatrix} 0 & 1.02 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{90}$$

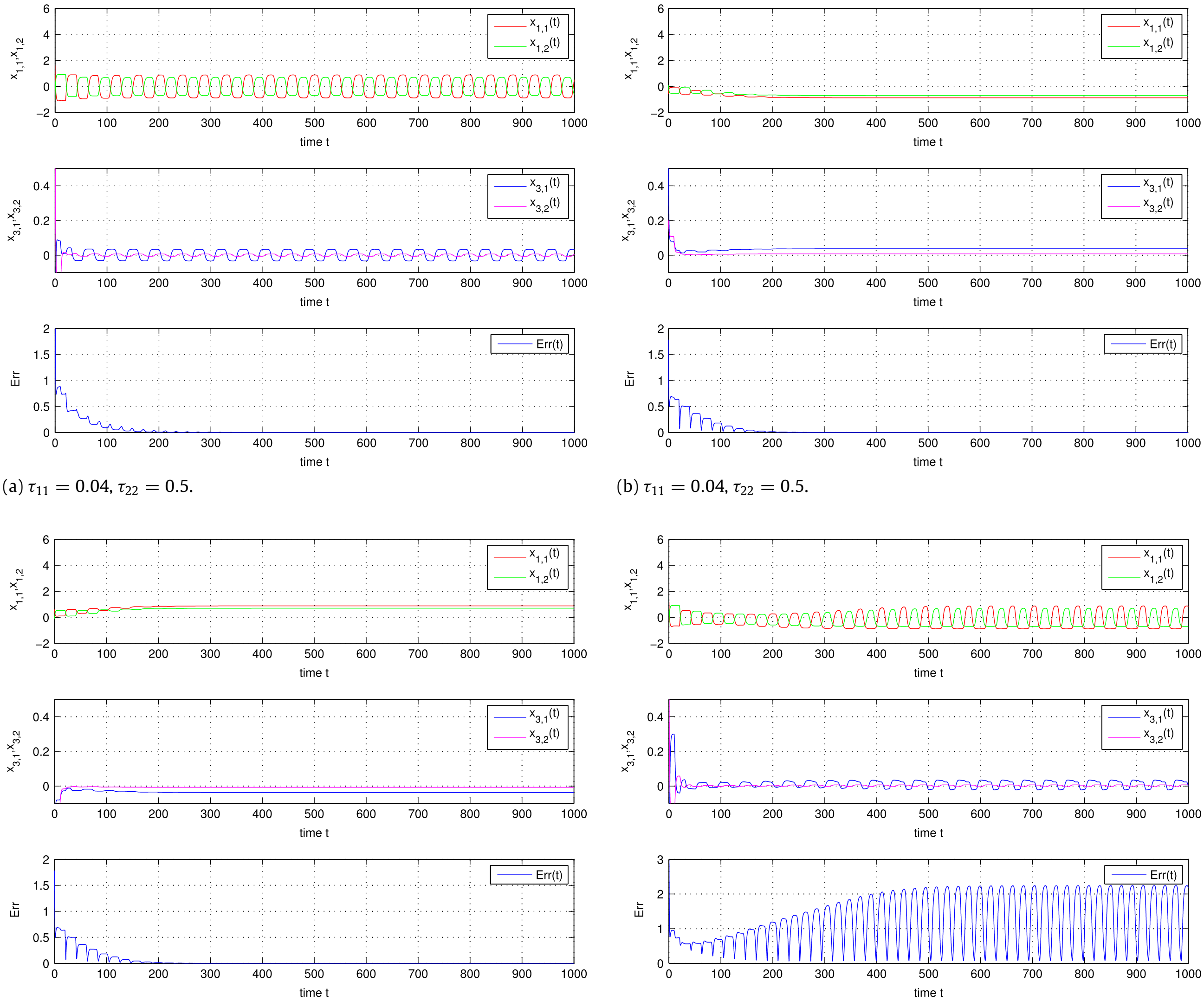
By (88)–(90), a numerical computation yields

$$\{\lambda : \lambda \text{ is an eigenvalue of } (D_{\mathbf{M}^*} - L_{\mathbf{M}^*})^{-1} U_{\mathbf{M}^*}\} = \{\lambda_i,\ i = 1, \ldots, 6\},$$

where $\lambda_i \approx 0.000$, $i = 1, 2, 3$, $\lambda_4 \approx 0.988$, $\lambda_5 \approx 0.093$, and $\lambda_6 \approx 0.015$. Consequently, we can compute $\lambda^* \approx 0.988$, defined in (64). Accordingly, the system satisfies conditions (62) and (64); hence, it achieves global cluster synchronization by Theorem 3. Fig. 2(a)–(c) show that the three solutions either converge to various nontrivial equilibria, or remain oscillating, depending on their initial conditions, and their corresponding synchronization errors $Err(t)$ all tend to zero.

If we consider $\tau_{11} = \tau_{22} = 20$ instead of $\tau_{11} = \tau_{22} = 0.01$, then condition (79) does not hold. In this case, Fig. 1(d) demonstrates that the synchronization error $Err(t)$ corresponding to some solution does not approach zero.

Our simulation shows that the dynamics of each isolated neuron in the first community of the system (i.e. system (50)



(a) $\tau_{11} = 0.04$, $\tau_{22} = 0.5$.

(b) $\tau_{11} = 0.04$, $\tau_{22} = 0.5$.

(c) $\tau_{11} = 0.04$, $\tau_{22} = 0.5$.

(d) $\tau_{11} = 20$, $\tau_{22} = 10$.

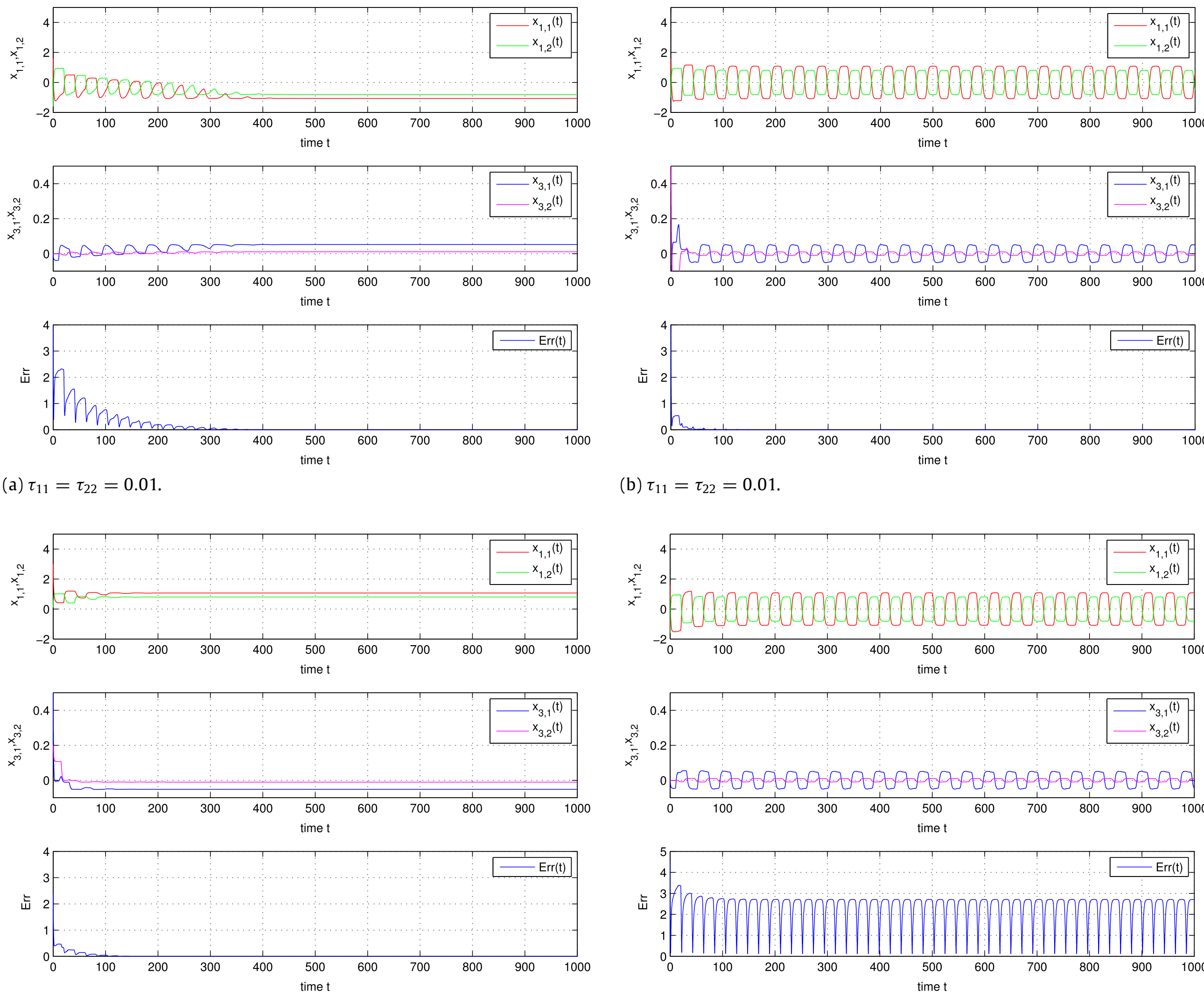
Fig. 1. Evolutions of $(x_{1,1}(t), x_{1,2}(t))^T$, $(x_{3,1}(t), x_{3,2}(t))^T$, and $Err(t) = \sqrt{\sum_{k\in\{1,2\}}\sum_{i\in\{1,3,4\}}[x_{i,k}(t) - x_{i+1,k}(t)]^2}$ for solutions $(x_{i,j}(t))$ of system (51), with various τ_{11} and τ_{22}, in Example 1. The solutions in (a) and (d) both start from $(1.6, -1, 0.2, -1.6, -1.6, 1, 1.6, -1, 0.1, -0.1)$, at $t_0 = 0$, whereas those in (b) and (c) start from $(-0.6, 0.1, -0.1, -0.6, 0.6, 0.2, -0.3, 0.6, -0.1, -0.6)$ and $(0.6, -0.1, 0.1, 0.6, -0.6, -0.2, 0.3, -0.6, 0.1, 0.6))$, at $t_0 = 0$, respectively.

with $r = 1$) admits a globally asymptotically stable equilibrium. In contrast, the dynamics of each isolated neuron in the second community of the system (i.e. system (50) with $r = 2$) exhibits convergence to multiple stable equilibria. Thus, this example demonstrates that the coupling can promote oscillation and multistability, and that large coupling delays between neurons within the same community may induce asynchrony.

4. Conclusion

This investigation establishes the global cluster synchronization of complex networks based on an iterative approach. With the dissipative property in a network system, our approach first derives component-wise convergent properties, i.e., a preliminary attracting set of cluster synchronous manifold, for the system. Through an iteration scheme, this approach subsequently formulates delay-dependent and delay-independent criteria for the global convergence of dynamics to the cluster synchronous manifold, and hence global cluster synchronization, for the system. The proposed framework accommodates a large class of network systems in the form of (1). The units comprising the network can either be identical or non-identical. The coupling configuration of the network can be rather general with coupling terms that could be nonlinear and with heterogeneous coupling time delays. The connection matrix could be time-dependent and could contain mixed signs of off-diagonal entries. The unique assumption imposed on the connection matrix is assumption (I). To the best of our knowledge, this is a prerequisite requirement, hence the weakest condition, imposed on the connection matrix for the cluster synchronization problem in the existing literature, cf. Remark 2. Accordingly, our approach can be applied to diffusively or non-diffusively coupled systems. In addition, the coupled systems considered under our framework can accommodate excitatory and inhibitory connections simultaneously.

We implemented this approach to study the cluster synchronization of a nonlinearly coupled neural network with heterogeneous coupling delays, which cannot be treated by existing approaches, cf. Remark 3. As illustrated in Examples 1 and 2, the present approach allows coupled neural networks to exhibit new and rich collective behavior, which is distinct from the individual behavior of isolated neurons, under the synchronization criteria. In addition, our examples also illustrated that coupled neural networks may lose synchrony if the coupling delays between neurons



(a) $\tau_{11} = \tau_{22} = 0.01$.

(b) $\tau_{11} = \tau_{22} = 0.01$.

(c) $\tau_{11} = \tau_{22} = 0.01$.

(d) $\tau_{11} = \tau_{22} = 20$.

Fig. 2. Evolutions of $(x_{1,1}(t), x_{1,2}(t))^T$, $(x_{3,1}(t), x_{3,2}(t))^T$, and $Err(t) = \sqrt{\sum_{k\in\{1,2\}} \sum_{i\in\{1,3,4\}} [x_{i,k}(t) - x_{i+1,k}(t)]^2}$ for solutions $(x_{i,j}(t))$ of system (51), with various τ_{11} and τ_{22}, in Example 2. The solutions in (a) and (d) start from $(1.6 + 0.1\sin(t), -1.6 + 0.1t, -1.6 + 0.1t, 1.6 + 0.1t, 0, 0, 0, 0, 0, 0)$, at $t_0 = 0$; the solution in (b) (resp., (c)) starts from $(1.6, -1, 1.6, -1.6, -1.6, 1, 1.6, -1, 0.1, -0.1)$ (resp., $(3, 0.1, 4, 0.6, 0.6, 0.2, -0.9, 0.6, -0.1, -0.6)$), at $t_0 = 0$.

within the same community are too large, such that the synchronization criteria are violated. In the literature, the coupling delay may be considered to be larger if the distance between two interacting neurons is larger, cf. Crook et al. (1997). In this situation, our examples indicate that coupled neural networks may lose synchrony if neurons within the same community are too far away from each other.

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Appendix. Scalar delay-differential equation

We denote by t_0 the initial time and by $\tau_M > 0$ the upper bound of the delay magnitude. Let $w(t)$ be a bounded continuous function for $t \geq t_0$, and let $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{h} : \mathcal{C}([-\tau_M, 0]; \mathbb{R}) \times \mathcal{C}([-\tau_M, 0]; \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Let $x_t, y_t \in \mathcal{C}([-\tau_M, 0]; \mathbb{R})$ for $t \geq t_0$, and set $x(t + \theta) = x_t(\theta)$, $y(t + \theta) = y_t(\theta)$ for $\theta \in [-\tau_M, 0]$. We assume that $x(t)$ and $y(t)$ eventually enter and then remain in some closed and bounded interval $[\check{\varrho}, \hat{\varrho}]$; namely, $x(t)$ and $y(t)$ lie in $[\check{\varrho}, \hat{\varrho}]$ for all $t \geq \tilde{t}_0$, for some $\tilde{t}_0 \geq t_0$. We suppose that $z(t) = x(t) - y(t)$ satisfies the following scalar equation:

$$\dot{z}(t) = h(x(t), y(t), t) + \tilde{h}(x_t, y_t, t) + w(t), \quad t \geq t_0. \tag{91}$$

Set

$$|w|^{\max}(T) := \sup\{|w(t)| : t \geq T\},$$
$$|w|^{\max}(\infty) := \lim_{T\to\infty} |w|^{\max}(T).$$

We introduce the following conditions:

Condition (H_0): There exist $\hat{\mu}, \check{\mu}, \check{\beta}, \hat{\beta} \in \mathbb{R}$, $\rho^h > 0$, and $0 \leq \bar{\tau} \leq \tau_M$, such that for each $\phi, \psi \in \{\varphi \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}) : \varphi(\theta) \in [\check{\varrho}, \hat{\varrho}], \theta \in [-\tau_M, 0]\}$, the following properties hold for all $t \geq t_0$:

(H_0-i):

$$\begin{cases} \check{\mu} \leq h(\phi(0), \psi(0), t)/[\phi(0) - \psi(0)] \leq \hat{\mu} & \text{if } \phi(0) - \psi(0) \neq 0, \\ h(\phi(0), \psi(0), t) = 0 \quad \text{if } \phi(0) - \psi(0) = 0, \end{cases}$$

(H_0-ii): $|\tilde{h}(\phi, \psi, t)| \le \rho^h$, and there exists a $\tau = \tau(\phi, \psi, t) \in [0, \bar{\tau}]$, such that

$$\begin{cases} \check{\beta} \le \tilde{h}(\phi, \psi, t)/[\phi(-\tau) - \psi(-\tau)] \le \hat{\beta} \\ \quad \text{if } \phi(-\tau) - \psi(-\tau) \neq 0, \\ \tilde{h}(\phi, \psi, t) = 0 \quad \text{if } \phi(-\tau) - \psi(-\tau) = 0. \end{cases}$$

Condition (A1): $\hat{\mu} + \hat{\beta} < 0$ and $\bar{\beta}\bar{\tau} < 3\rho^h(\hat{\mu} + \hat{\beta})/[(\hat{\mu} + \check{\mu} + \hat{\beta} + \check{\beta})(3\rho^h + |w|^{\max}(\tilde{t}_0))]$, where $\bar{\beta} := \max\{|\check{\beta}|, |\hat{\beta}|\}$.

Condition (A2): $0 \le \bar{\beta} < -\hat{\mu}/[1 + |w|^{\max}(\tilde{t}_0)/\rho^h]$.

The following two propositions are based directly on Propositions 2.3 and 2.4, respectively, that were reported by Shih and Tseng (2013).

Proposition A.1. *If $z(t)$ satisfies (23), then $z(t)$ converges to interval $[-\bar{\nu}, \bar{\nu}]$ as $t \to \infty$, under conditions (H_0) and (A1). Moreover,*

$$0 \le \bar{\nu} \le |w|^{\max}(\infty)/[-\hat{\mu} - \hat{\beta} + \bar{\beta}\bar{\tau}(\check{\mu} + \hat{\mu} + \check{\beta} + \hat{\beta})].$$

Proposition A.2. *If $z(t)$ satisfies (23), then $z(t)$ converges to interval $[-\tilde{\nu}, \tilde{\nu}]$ as $t \to \infty$, under conditions (H_0) and (A2). Moreover,*

$$0 \le \tilde{\nu} \le |w|^{\max}(\infty)/(-\hat{\mu} - \bar{\beta}).$$

The assertion in Proposition 2.3 *(resp.,* 2.4*) in Shih and Tseng (2013) uses t_0 instead of $\tilde{t}_0$ in condition* (A1) *(resp.,* (A2)*). From the arguments for Proposition* 2.3 *(resp.,* 2.4*) in Shih and Tseng (2013), it can seen that t_0 in condition* (A1) *(resp.,* (A2)*) for Proposition* 2.3 *(resp.,* 2.4*) therein can be replaced by $\tilde{t}_0$ to weaken the condition, which then implies Proposition* A.1 *(resp.,* A.2*).*

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