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# Testing for central dominance: Method and application

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ABSTRACT

Central dominance (CD) introduced in Gollier (1995, *Journal of Economic Theory*) is a risk concept that differs from stochastic dominance (SD) in an important way. In particular, CD implies a deterministic comparative static of a change in decision when risk changes, but SD does not have such an implication. In this paper, we propose the first test of central dominance, which amounts to checking a functional inequality. We derive the asymptotic distribution of a lower bound of the proposed test and suggest a bootstrap procedure to compute the critical values. We also conduct simulations to evaluate the performance of this test. Our empirical study finds clear evidence of CD relations between the S&P 500 index return distributions during 2001–2013 and results in unambiguous implications for investment decisions.



## 1. Introduction

A major building block of modern risk theory is the notion of stochastic dominance (SD) introduced in Hadar and Russell (1969), Hanoch and Levy (1969), and Rothschild and Stiglitz (1970); see Levy (1992) for a survey. It is well known that SD can determine a preference ordering of different risks. In particular, Rothschild and Stiglitz (1970) show that all risk-averse agents, i.e., those with increasing and concave utility functions, prefer risk $A$ to risk $B$ if, and only if, the distribution associated with $A$ second-order stochastically dominates that of $B$. Yet, SD does not imply a change in demand when risk changes. For example, Rothschild and Stiglitz (1971) find that risk-averse agents need not reduce their demand for a risky asset when its risk increases in the sense of second-order SD; also see Eeckhoudt and Gollier (2000) for a numerical example. Thus, SD offers limited practical direction for adjusting investment decisions after the distribution of risk changes.

Among many researchers that try to link risk and demand directly,[1] Gollier (1995) makes an important contribution by introducing the new concept, "central dominance" (CD), and shows that risk-averse investors demand less of a risky asset if, and only if, its risk increases (the associated distribution being dominated) in the sense of CD. Hollifield and Kraus (2009) further elaborate on this idea and analyze the condition under which a demand-reducing change in risk makes all risk-averse investors worse off. It must be emphasized that, while CD implies a deterministic comparative static of a change in decision when the risk (distribution) changes, SD does not have a similar implication. It has also been shown that second-order SD is neither sufficient nor necessary for CD (Gollier, 1995).

Despite the practical relevance of CD, testing CD has not been considered in the literature, to the best of our knowledge. According to Gollier (1995), CD is defined as the existence of some parameter such that a functional inequality holds; yet, it is not easy to construct a test for an inequality constraint. The study of CD has been limited partly because there has been no test of CD available.[2] This paper intends to fill this gap and proposes a test of CD. We first transform the functional inequality in the definition of CD into an equality and then construct a test on this equality condition based on the maximum of an integral process.[3] We derive the asymptotic

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1 See, e.g., Sandmo (1971), Eeckhoudt and Hansen (1980), Meyer and Ormiston (1985), Black and Bulkley (1989), Landsberger and Meilijson (1990), Dionne and Gollier (1992), Eeckhoudt and Gollier (1995), Gollier (1995, 1997), and Tzeng (2001).

2 The study of SD suffers from a similar difficulty. Note that testing SD, which also requires checking an inequality constraint, has received more attention only recently; see, e.g., Anderson (1996), Davidson and Duclos (1997, 2000), Barrett and Donald (2003), Linton et al. (2005), Horváth et al. (2006), Bennett (2007), Linton et al. (2010), and Donald and Hsu (2016).

3 Chen and Szroeter (2009) and Linton et al. (2010) also construct tests by transforming moment inequalities into equalities.

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distribution of the proposed test and suggest a bootstrap procedure to compute the critical values. Simulations are then conducted to evaluate the performance of this test.

In the empirical study, we apply the proposed test to the daily return distributions of the S&P 500 index from 2001 to 2013. Our empirical study finds clear evidence of CD relations during that period of time and results in unambiguous implications for investment decisions. We find, for example, that the return distributions in 2003, 2004, 2006 and 2010 centrally dominate, respectively, those in 2004, 2005, 2007 and 2011. These findings suggest that the optimal investment amounts in 2004, 2005, 2007 and 2011 should be lower than what they were in the previous year. We also find that the return distributions in 2006, 2010, 2012 and 2013 centrally dominate, respectively, those in 2005, 2009, 2011 and 2012, so that the optimal investment amounts in 2006, 2010, 2012 and 2013 should be higher than what they were in the previous year.

This paper is organized as follows. In Section 2, we review the conditions and properties of CD; examples are also provided to illustrate the difference between SD and CD. In Section 3, we introduce the proposed test and establish its asymptotic properties. Monte Carlo simulation results are reported in Section 4. An empirical study on S&P 500 index return distributions based on the proposed test is presented in Section 5. Section 6 concludes the paper. All proofs are collected in the Appendix.

## 2. Central dominance

### 2.1. Theory

Consider a representative agent who faces the optimal decision problem with respect to a change in risk. We follow the setup in Gollier (1995), and make the following assumptions.

**Assumption 1.** Assume that:

1. The individual has an increasing, concave, and twice differentiable von Neumann–Morgenstern utility function $u(z(\alpha, x))$, where $z(\alpha, x)$ is a payoff function.

2. The payoff of the individual has the form $z(\alpha, x) = \alpha x + z_0$, which is determined by a decision variable $\alpha$ and a risk variable $x$, where $z_0$ is an exogenous parameter.

3. The range of $\alpha$ is normalized to [0, 1]. The random variable $x$ is defined on $[a, b]$ with $a < 0 < b$ and has a continuous distribution function $F$ with $E[x] > 0$.[4]

The first condition of Assumption 1 ensures that the individual is risk averse. In the second condition, we choose a particular form of the payoff function which entails the standard portfolio problem, the problem of a competitive firm with a constant marginal cost, and the insurance problem. For more details and examples, see Gollier (1995). Note that the third condition is required to avoid a boundary solution for $\alpha$.

When the distribution function $F$ is known to the individual, he/she chooses the optimal $\alpha^*(u; F)$ to maximize his/her expected utility. The following proposition due to Gollier (1995) gives a deterministic change in the optimal decision after a certain change in risk.

**Proposition 2.1.** *All individuals have their $\alpha^*(u; F) \geq \alpha^*(u; G)$ after the change in the risk distribution from F to G if, and only if, there exists $\gamma \in \mathbb{R}$ such that*

$$\gamma T(x; F) \geq T(x; G), \text{ for all } x \in [a, b]. \tag{1}$$

*Here $T(x; F) = \int_a^x tdF(t)$, and $T(x; G) = \int_a^x tdG(t)$.*

Proposition 2.1 provides a necessary and sufficient condition, hereafter Condition (1), for *all* individuals to decrease their decision variable after a change in risk from distribution $F$ to $G$. When Condition (1) holds, we say that $F$ centrally dominates $G$, denoted as $F \overset{CD}{\succ} G$.[5]

Since

$$T(x; F) = \int_a^x tdF(t) = E_F[t|t \leq x]F(x),$$

$T(x; F)$ could be viewed as the conditional expectation of $t$ given $t \leq x$ multiplied by the probability of $t \leq x$. It follows that Condition (1) can be rewritten as:

There exists a real number $\gamma$ satisfying

$$\gamma E_F[t|t \leq x]F(x) \geq E_G[t|t \leq x]G(x), \text{ for all } x \in [a, b].$$

This is a continuum of the conditional moment inequality plus an existence condition.

### 2.2. Example: Investment decision

CD and SD are two distinct concepts. CD implies a deterministic change in the optimal decision variable, but SD does not have similar implications. The following example illustrates that SD and CD do not imply each other.

Consider a traditional portfolio problem: there are two assets in the market, one is risk free with the rate of return $r_f$ and the other is risky with the rate of return $y$, where $y \in [\underline{y}, \bar{y}]$. An investor with initial wealth $W$ chooses to invest $\alpha$ in the risky asset. The final wealth of this individual is then

$$\begin{aligned}\alpha(1 + y) + (W - \alpha)(1 + r_f) &= \alpha(y - r_f) + W(1 + r_f)\\ &= \alpha x + z_0,\end{aligned}$$

where $x = y - r_f$ is the excess return and $z_0 = W(1 + r_f)$.

Let $F$ and $G$ represent two distributions of the excess return with $F \overset{CD}{\succ} G$, which means that there exists a real number $\gamma$ satisfying

$$\gamma \int_a^x tdF(t) \geq \int_a^x tdG(t), \text{ for all } x \in [a, b],$$

where $a = \underline{y} - r_f$ and $b = \bar{y} - r_f$. In addition, let $u(\cdot)$ be an increasing and concave utility function of the investor. The objective of the investor under distribution $F$ is to choose an $\alpha$ to maximize the expected utility:

$$E_F \left[u\left(\alpha x + z_0\right)\right]. \tag{2}$$

Thus, the first-order condition of the problem (2) can be written as

$$E_F \left[xu'\left(\alpha x + z_0\right)\right] = 0.$$

By integration by parts, the first-order condition can be further rewritten as

$$u'\left(\alpha b + z_0\right) T(b; F) - \int_a^b \alpha u''\left(\alpha x + z_0\right) T(x; F)dx = 0. \tag{3}$$

[4] The assumption of bounded support for $x$, while ruling out unbounded distributions, is made for simplicity. Although this is a limitation of our result, we note that similar conditions are also frequently adopted in testing functional inequalities, such as tests of SD, see, e.g. Barrett and Donald (2003) and Donald and Hsu (2016).

[5] In Gollier (1995), it is stated that "$G$ is centrally riskier than $F$" and is denoted as $F$ CR $G$ when Condition (1) holds. For more discussions, examples, and illustrations about this proposition, see Gollier (1995).

Then, by Proposition 2.1, we have $\alpha^*(u; F) \geq \alpha^*(u; G)$, which suggests that the optimal amount invested in the risky asset should be higher for distribution $F$ than for distribution $G$. By contrast, Eeckhoudt and Gollier (2000) provide a numerical example for portfolio selection and show that an increase in risk in the sense of first-order SD and second-order SD does not imply a decrease in the demand for a risk-averse individual.

## 3. Test for central dominance

In this section, we propose a test for the null hypothesis $F \overset{CD}{\succ} G$ against the alternative hypothesis $F \overset{CD}{\nsucc} G$. A bootstrap procedure is also proposed to compute the critical values.

### 3.1. The proposed test

We first make the following assumption on $\gamma$ in Condition (1).

**Assumption 2.** $\gamma$ in Condition (1) is in $\mathcal{C}$, a compact subset of $\mathbb{R}$ specified by researchers.

This assumption is convenient for checking the existence condition of $\gamma$ and deriving the asymptotic results. It can be seen that $\gamma$ has a natural lower and upper bounds determined by $F$ and $G$. To see this, note that $a < 0 < b$ implies $T(x; F) < 0$ when $x$ is close enough to $a$, and that $E[x] > 0$ implies $T(b; F) > 0$. That is, $T(x; F)$ must alternate in sign by Assumption 1. Another way to verify that $T(x; F)$ alternates in sign is through the first-order condition of the problem. Take the investment decision in Section 2.2 as an example. By Assumption 1, we have $u' > 0$, $u'' < 0$ and $\alpha > 0$. From the first-order condition (3), $T(x; F)$ must alternate in sign to support the existence of an interior solution. Thus, $\{x : T(x; F) < 0\}$ and $\{x : T(x; F) > 0\}$ are both nonempty sets, and we have the following bounds for $\gamma$ under Condition (1):

$$\inf_{\{x:T(x;F)<0\}} \frac{T(x; G)}{T(x; F)} \geq \gamma \geq \sup_{\{x:T(x;F)>0\}} \frac{T(x; G)}{T(x; F)}.$$

As $\mathcal{C}$ can be chosen to be as large as possible, this assumption is not really restrictive.

Given the distribution functions $F$ and $G$, let $\mu(\gamma, x) = \gamma T(x; F) - T(x; G)$. Then, for some $\gamma_0 \in \mathcal{C}$, $\gamma_0 T(x; F) \geq T(x; G)$ for all $x \in [a, b]$ is equivalent to $\mu(\gamma_0, x) \geq 0$ for all $x \in [a, b]$. In this case, $Q(\gamma_0) = 0$, where

$$Q(\gamma) = \int_a^b \min\{\mu(\gamma, x), 0\}dx. \tag{4}$$

The results below are straightforward:

**Proposition 3.1.** *The function $Q(\gamma)$ defined in Eq. (4) satisfies:*

(a) $Q(\gamma) \leq 0$ *for all* $\gamma \in \mathbb{R}$;

(b) $Q(\gamma)$ *is continuous in* $\gamma$;

(c) $Q(\gamma_0) = 0$ *if, and only if,* $\gamma_0 T(x; F) \geq T(x; G)$, *for all* $x \in [a, b]$;

(d) $F \overset{CD}{\succ} G$ *if, and only if,* $\max_{\gamma \in \mathcal{C}} Q(\gamma) = 0$.

By Proposition 3.1(d), the existence condition in Proposition 2.1 can be transformed into the equality condition: $\max_{\gamma \in \mathcal{C}} Q(\gamma) = 0$. Thus, $F \overset{CD}{\succ} G$ if, and only if,

$$\Gamma = \{\gamma \in \mathcal{C} : Q(\gamma) = 0\}$$

is a non-empty set. More can be said about $\Gamma$, as shown in the result below.

**Lemma 3.2.** *When $F \overset{CD}{\succ} G$, $\Gamma$ is either a singleton or a closed interval.*

Consider now the following assumption regarding data samples.

**Assumption 3.** $\{x_{1i}\}_{i=1}^{N_1}$ and $\{x_{2j}\}_{j=1}^{N_2}$ are random samples taken from, respectively, the distributions $F$ and $G$ such that $\lim_{N_1,N_2 \to \infty} N_2/(N_1 + N_2) = \lambda$, $\lambda \in (0, 1)$.

The assumption of a random sample is quite strong and excludes dependent and heterogeneous data, such as time series data. It is possible to relax this assumption and allow for data with weak dependence (e.g., $\alpha$-mixing); see, e.g., Linton et al. (2005). To reduce technicality, we do not pursue this generalization in this paper.

Let the empirical distribution functions of $F(x)$ and $G(x)$ be, respectively,

$$\hat{F}_{N_1}(x) = \frac{1}{N_1}\sum_{i=1}^{N_1} I\{x_{1i} \leq x\}, \quad \text{and} \quad \hat{G}_{N_2}(x) = \frac{1}{N_2}\sum_{j=1}^{N_2} I\{x_{2j} \leq x\},$$

where $I\{A\}$ denotes the indicator function of the event $A$. Also define the empirical processes of $T(x; F)$ and $T(x; G)$ as:

$$T(x; \hat{F}_{N_1}) = \frac{1}{N_1}\sum_{i=1}^{N_1} x_{1i} I\{x_{1i} \leq x\},$$

$$T(x; \hat{G}_{N_2}) = \frac{1}{N_2}\sum_{j=1}^{N_2} x_{2j} I\{x_{2j} \leq x\}.$$

Then, $\hat{\mu}_{N_1,N_2}(\gamma, x) := \gamma T(x; \hat{F}_{N_1}) - T(x; \hat{G}_{N_2})$. The proposed test statistic is based on the sample counterpart of $\max_{\gamma \in \mathcal{C}} Q(\gamma)$:

$$\begin{aligned} \hat{S}_{N_1,N_2} &= \sqrt{\frac{N_1 N_2}{N_1 + N_2}} \max_{\gamma \in \mathcal{C}} \hat{Q}_{N_1,N_2}(\gamma) \\ &= \sqrt{\frac{N_1 N_2}{N_1 + N_2}} \max_{\gamma \in \mathcal{C}} \int_a^b \min\{\hat{\mu}_{N_1,N_2}(\gamma, x), 0\}dx. \end{aligned} \tag{5}$$

By the functional central limit theorem, as $N_1, N_2 \to \infty$, $\sqrt{N_1}[\hat{F}_{N_1} - F] \rightsquigarrow B_1(F(x)) := B_F$ and $\sqrt{N_2}[\hat{G}_{N_2} - G] \rightsquigarrow B_2(G(x)) := B_G$, where $\rightsquigarrow$ denotes weak convergence, and $B_1$ and $B_2$ are two independent Brownian bridges. Note that $B_F$ and $B_G$ are mean zero Gaussian processes defined on $[a, b]$ with the covariance functions: $\mathrm{E}[B_F(x_1)B_F(x_2)] = F(x_1 \wedge x_2) - F(x_1)F(x_2)$, and $\mathrm{E}[B_G(x_1)B_G(x_2)] = G(x_1 \wedge x_2) - G(x_1)G(x_2)$.[6] Moreover, the following limiting results hold:

**Lemma 3.3.** *As $N_1, N_2 \to \infty$,*

$$\sqrt{N_1}[T(x; \hat{F}_{N_1}) - T(x; F)] \rightsquigarrow T(x; B_F),$$
$$\sqrt{N_2}[T(x; \hat{G}_{N_2}) - T(x; G)] \rightsquigarrow T(x; B_G),$$

*where $T(x; B_F)$ and $T(x; B_G)$ are two Gaussian processes with mean zero and respective covariance functions:*

$$E[T(x_1; B_F)T(x_2; B_F)] = \int_a^{x_1 \wedge x_2} t^2 dF(t) - \int_a^{x_1} t dF(t) \int_a^{x_2} t dF(t),$$

$$E[T(x_1; B_G)T(x_2; B_G)] = \int_a^{x_1 \wedge x_2} t^2 dG(t) - \int_a^{x_1} t dG(t) \int_a^{x_2} t dG(t).$$

[6] For a more detailed discussion about the asymptotic properties of empirical processes, see Kosorok (2008).

When $\Gamma = \{\gamma^*\}$ is a singleton (Lemma 3.2), from Lemma 3.3 we have

$$\begin{aligned}\sqrt{\frac{N_1N_2}{N_1+N_2}}[\hat{\mu}_{N_1,N_2}(\gamma^*,x)-\mu(\gamma^*,x)] &= \sqrt{\lambda}\gamma^*\sqrt{N_1}[T(x;\hat{F}_{N_1})-T(x;F)] \\ &\quad -\sqrt{1-\lambda}\sqrt{N_2}[T(x;\hat{G}_{N_2})-T(x;G)]+o_P(1) \\ &\rightsquigarrow \sqrt{\lambda}\gamma^*T(x;B_F)-\sqrt{1-\lambda}T(x;B_G), \end{aligned} \tag{6}$$

where $o_P(1)$ holds uniformly on $[a,b]$. The limit of $\hat{S}_{N_1,N_2}$ follows from (6) and the continuous mapping theorem. It is worth mentioning that, in the limit, the integral over $[a,b]$ in (5) is essentially determined by the integral over the set:

$$\mathcal{B}^0_{\gamma^*} = \{x\in[a,b]:\mu(\gamma^*,x)=0\},$$

which is also known as the "contact set" of $\gamma^*T(x;F)$ and $T(x;G)$.[7]

The result below shows that, when $\Gamma$ is a singleton, we can find the asymptotic distribution of a lower bound for $\hat{S}_{N_1,N_2}$; otherwise, $\hat{S}_{N_1,N_2}$ is degenerate at zero in the limit.

**Theorem 3.4.** *When $F \overset{CD}{\succ} G$, we have one of the following results:*

(a) *If $\Gamma$ is a singleton $\{\gamma^*\}$, then $0 \ge \hat{S}_{N_1,N_2} \ge \sqrt{\frac{N_1N_2}{N_1+N_2}}\hat{Q}_{N_1,N_2}(\gamma^*)$ and*

$$\sqrt{\frac{N_1N_2}{N_1+N_2}}\hat{Q}_{N_1,N_2}(\gamma^*) \xrightarrow{d} \int_{\mathcal{B}^0_{\gamma^*}} \min\left\{\sqrt{\lambda}\gamma^*T(x;B_F)-\sqrt{1-\lambda}T(x;B_G),\ 0\right\}dx.$$

(b) *If $\Gamma$ is a closed interval: $[\underline{\gamma},\bar{\gamma}]$, then $\hat{S}_{N_1,N_2} \xrightarrow{a.s.} 0$.*

Since $\sqrt{(N_1N_2)/(N_1+N_2)}\hat{Q}_{N_1,N_2}(\gamma^*)$ is a lower bound of $\hat{S}_{N_1,N_2}$, we have, as $N_1,N_2\to\infty$,

$$P(\hat{S}_{N_1,N_2}\le c_\delta) \le P\left(\sqrt{\frac{N_1N_2}{N_1+N_2}}\hat{Q}_{N_1,N_2}(\gamma^*)\le c_\delta\right)=\delta,$$

where $c_\delta$ is the $\delta$-percentile of the asymptotic distribution in Theorem 3.4(a). Thus, by letting $\hat{S}_{N_1,N_2}$ reject the null hypothesis when it is less than the critical value, we obtain a conservative test for the null hypothesis. To see Theorem 3.4(b), we first note that, if the contact set $\mathcal{B}^0_{\gamma^*}$ turns out to be of measure zero, $\hat{S}_{N_1,N_2}$ would converge to zero with a probability of one by Theorem 3.4(a). When $\Gamma$ is a closed interval, it can be shown that there are at most countably many $\gamma$ in $\Gamma$ such that their respective contact sets have a positive measure. In other words, there are still uncountably many $\gamma$ with contact sets of measure zero. As a result, the sample counterpart of $\max_{\gamma\in\mathcal{C}} Q(\gamma)$ converges to zero.

On the other hand, when $F$ does not centrally dominate $G$, $\Gamma$ is an empty set. The result below shows that the proposed test diverges to minus infinity and hence can reject the null hypothesis with a probability approaching one.

**Theorem 3.5.** *When $F \overset{CD}{\nsucc} G$, $\hat{S}_{N_1,N_2}$ diverges to $-\infty$ in probability.*

### 3.2. Bootstrapping the critical values

In this section, we show how the $\delta$-percentile of the asymptotic distribution in Theorem 3.4(a) can be obtained by bootstrapping. The bootstrap procedure is:

1. When $\max_{\gamma\in\mathcal{C}}\hat{Q}_{N_1,N_2}(\gamma)\neq 0$, let
$$\hat{\gamma}^* = \arg\max_{\gamma\in\mathcal{C}}\hat{Q}_{N_1,N_2}(\gamma);$$
otherwise, accept the null hypothesis (a trivial case).
2. Take a sequence $\{c_{N_1,N_2}\}$ with $c_{N_1,N_2}\to 0$ and $\sqrt{N_1N_2/(N_1+N_2)}\,c_{N_1,N_2}\to\infty$ as $N_1,N_2\to\infty$,[8] and define
$$\hat{\mathcal{B}}^0_{\hat{\gamma}^*} = \{x\in[a,b]:|\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*,x)|<c_{N_1,N_2}\}.$$
3. For $b=1,\ldots,B$, draw $\{x^*_{1i,b}\}_{i=1}^{N_1}$ with replacement from $\{x_{1i}\}_{i=1}^{N_1}$ and draw $\{x^*_{2j,b}\}_{j=1}^{N_2}$ with replacement from $\{x_{2j}\}_{j=1}^{N_2}$. Then compute $\hat{S}^*_{N_1,N_2,b}$ based on the bootstrapped samples $\{x^*_{1i,b}\}_{i=1}^{N_1}$ and $\{x^*_{2j,b}\}_{j=1}^{N_2}$, where
$$\hat{S}^*_{N_1,N_2,b} = \sqrt{\frac{N_1N_2}{N_1+N_2}}\int_{\hat{\mathcal{B}}^0_{\hat{\gamma}^*}}\min\left\{\hat{\mu}^*_{N_1,N_2,b}(\hat{\gamma}^*,x) - \hat{\mu}_{N_1,N_2}(\hat{\gamma}^*,x),\ 0\right\}dx.$$
4. The bootstrapped critical values for a test of size $\delta$ are computed as
$$c^*_{\delta,N_1,N_2,B} \equiv \sup\left\{t:\frac{1}{B}\sum_{b=1}^{B}I\{\hat{S}^*_{N_1,N_2,b}>t\}\ge 1-\delta\right\}.$$

The null hypothesis is then rejected if $\hat{S}_{N_1,N_2} < c^*_{\delta,N_1,N_2,B}$. Note that in bootstrapping the critical values, the integral in $\hat{S}^*_{N_1,N_2,b}$ is taken over the estimated contact set $\hat{\mathcal{B}}^0_{\hat{\gamma}^*}$, which usually helps improve the test power.

Let $c^*_{\delta,N_1,N_2,\infty}$ denote the limit of the bootstrapped critical value, $c^*_{\delta,N_1,N_2,B}$, when $B$ tends to infinity. This would be the proper critical value asymptotically if $\Gamma$ is a singleton. Yet, if $\Gamma$ is a closed interval, both $\hat{S}_{N_1,N_2}$ and $c^*_{\delta,N_1,N_2,\infty}$ would converge to 0. To prevent the critical value from converging to 0, we introduce a negative number $c_0$, its magnitude however small, and set $c^\dagger_{\delta,N_1,N_2} = c^*_{\delta,N_1,N_2,\infty}+c_0$.[9] We show below that the proposed test based on $c^\dagger_{\delta,N_1,N_2}$ has proper size asymptotically, and the test can reject the null hypothesis with a probability approaching one when there is no CD.

**Theorem 3.6.** *Let $\delta$ be in $(0,1/2]$ and $c_0$ be a negative number of small magnitude.*

(a) *If $F \overset{CD}{\succ} G$, then $\lim_{N_1,N_2\to\infty} P(\hat{S}_{N_1,N_2} < c^\dagger_{\delta,N_1,N_2}) \le \delta$.*

(b) *If $F \overset{CD}{\nsucc} G$, then $\lim_{N_1,N_2\to\infty} P(\hat{S}_{N_1,N_2} < c^\dagger_{\delta,N_1,N_2}) \to 1$.*

### 3.3. Extensions

The bootstrapping procedure may be extended to allow for correlated samples and time series data with serial dependence.

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[7] Similar results can be found in the literature, e.g., Hansen (2005), Chernozhukov et al. (2007), Chen and Szroeter (2009), and Andrews and Soares (2010) for finite dimensional moment inequality tests, and Linton et al. (2010) and Donald and Hsu (2016) for functional inequality tests.

[8] For example, $c_{N_1,N_2}$ may be $\kappa_0 k^{-1/2}\log k$ with $\kappa_0$ an arbitrary positive constant and $k = N_1N_2/(N_1+N_2)$. The choice of $\kappa_0$ does not affect asymptotic properties. In our paper, we choose $\kappa_0$ according to the sample variance and length of the corresponding estimated contact set.

[9] In our subsequent analysis, we set $c_0 = -0.001$.

**Assumption 4.** $\{(x_{1i}, x_{2i})\}_{i=1}^{N}$ is a sample with i.i.d. observations from the joint distribution $F(x_1, x_2)$ defined on $[a, b] \times [a, b]$, with the marginal distributions $F$ and $G$, respectively.

This assumption allows the samples of $x_1$ and $x_2$ from a joint distribution and hence may be correlated. In this case, step 3 in the bootstrap procedure in Section 3.2 is modified as follows.

3′. For $b = 1, \ldots, B$, draw $\{(x^*_{1i,b}, x^*_{2i,b})\}_{i=1}^{N}$ with replacement from $\{(x_{1i}, x_{2i})\}_{i=1}^{N}$ and compute $\hat{F}_{N,b}$ and $\hat{G}_{N,b}$ based on the bootstrapped samples. $\hat{S}^*_{N,b}$ is then computed as:

$$\hat{S}^*_{N,b} = \sqrt{N} \int_{\hat{\mathscr{B}}^0_{\hat{\gamma}^*}} \min\left\{\hat{\mu}^*_{N,b}(\hat{\gamma}^*, x) - \hat{\mu}_N(\hat{\gamma}^*, x),\ 0\right\} dx.$$

**Assumption 5.** $\{(x_{1i}, x_{2i})\}_{i=1}^{N}$ is a sample with strictly stationary time series observations from the joint distribution $F(x_1, x_2)$ defined on $[a, b] \times [a, b]$, with marginal distributions $F$ and $G$, respectively. In addition, Assumption 1 of Linton et al. (2005) holds.[10]

Given this assumption, the bootstrap procedure can be modified by invoking the overlapping blockwise bootstrap method.

3″. Let $L$ be the length of blocks which is positively proportional to $N^k$, $0 < k < 1$. There are $N - L + 1$ different overlapping blocks in the sample: the $j$th block is $\{(x_{1j}, x_{2j}), \ldots, (x_{1j-L+1}, x_{2j-L+1}))\}$. We draw with replacement from the $N - L + 1$ blocks and by laying them end-to-end until the resulting sample size is larger than or equal to $N$, then drop the unnecessary observations until the sample size is equal to $N$.
For $b = 1, \ldots, B$, we draw $\{(x^*_{1i,b}, x^*_{2i,b})\}_{i=1}^{N}$ as in step 3′ and compute $\hat{F}_{N,b}$ and $\hat{G}_{N,b}$ from the bootstrapped samples. $\hat{S}^*_{N,b}$ is then computed as:

$$\hat{S}^*_{N,b} = \sqrt{N} \int_{\hat{\mathscr{B}}^0_{\hat{\gamma}^*}} \min\left\{\hat{\mu}^*_{N,b}(\hat{\gamma}^*, x) - \hat{\mu}_N(\hat{\gamma}^*, x),\ 0\right\} dx.$$

It is straightforward to show that Theorem 3.6 remains valid when the critical values are bootstrapped in these two ways.

## 4. Simulations

In this section, we evaluate the performance of the proposed test using simulations. We consider three density functions: $f(x) = 1, x \in [-0.3, 0.7]$,

$$g_1(x) = \begin{cases} 0.6 & \text{if } x \in [-0.3, 0.3), \\ 1.6 & \text{if } x \in [0.3, 0.7], \end{cases}$$

$$g_2(x) = \begin{cases} 0.9 & \text{if } x \in [-0.3, 0.3), \\ 2 & \text{if } x \in [0.3, 0.4), \\ 0.6 & \text{if } x \in [0.4, 0.5), \\ 1 & \text{if } x \in [0.5, 0.7], \end{cases}$$

with respective distribution functions: $F$, $G_1$, and $G_2$. The sample sizes are: $N_1 = N_2 = 50, 250, 1000$, and the number of replications is 1000.

Our simulation design consists of the following 5 setups, with CD in the first 3 setups and no CD in the remaining two.

1. $G_1 \overset{\text{CD}}{\succ} F$, and $Q(\gamma) = 0$ for $\gamma \in [0.68, 1.66]$.
2. $G_2 \overset{\text{CD}}{\succ} F$, and $Q(\gamma) = 0$ for $\gamma \in [0.93, 1.11]$.
3. $F \overset{\text{CD}}{\succ} F$, and $Q(\gamma) = 0$ for $\gamma = 1$.
4. $F \overset{\text{CD}}{\not\succ} G_1$, and $Q(\gamma) < 0$ for all $\gamma$.
5. $F \overset{\text{CD}}{\not\succ} G_2$, and $Q(\gamma) < 0$ for all $\gamma$.

The means and standard deviations of $\hat{S}_{N_1,N_2}$ under these setups are summarized in Table 1. It can be seen that, under the first two setups, both the mean and standard deviation of $\hat{S}_{N_1,N_2}$ decrease with the sample size. These results are consistent with Theorem 3.4(b) that $\hat{S}_{N_1,N_2}$ converges to zero. Under setup 3, the mean and standard deviation of $\hat{S}_{N_1,N_2}$ remain stable across different samples; this is consistent with Theorem 3.4(a). Under setups 4 and 5, there is no CD, and hence the sample mean of $\hat{S}_{N_1,N_2}$ diverges, as suggested by Theorem 3.5.

To further evaluate the finite-sample performance of the proposed test, we simulate the empirical sizes under setups 1 and 3 and the empirical power under setup 4. The number of replications is still 1000. To compute the critical values, we employ i.i.d. bootstrap with the number of bootstraps $B = 1000$ and estimate the contact set by setting $c_{N_1,N_2} = \kappa_0 k^{-1/2} \log k$, with $\kappa_0 = 0.05, 0.075, 0.1$ and $k = N_1 N_2/(N_1 + N_2)$. The actual lengths of the contact set in setup 1, setup 3, and setup 4 are 0, 1, and 0, respectively. The resulting rejection proportions are reported in Table 2.

Since the length of the estimated contact set increases in $\kappa_0$, the corresponding critical value decreases in $\kappa_0$, and so does the rejection proportion. When $\Gamma$ is an interval (setup 1), the proposed test is undersized for all values of $\kappa_0$ in all samples. This is consistent with Theorem 3.4(b) since our test statistic converges to zero. When $\Gamma$ is a singleton (setup 3), we find that, for each $\kappa_0$, the test tends to be undersized in larger samples, yet it may be oversized when the samples are really small ($N_1 = N_2 = 50$). This shows that the proposed test is indeed conservative. The finite-sample power under setup 4 also improves when the sample increases. This ought to be the case because, by Theorem 3.5, this test diverges when there is no CD.

**Table 1**
Mean and standard deviation of $\hat{S}_{N_1,N_2}$.

| $N_1 = N_2$ | Setup 1 | Setup 2 | Setup 3 | Setup 4 | Setup 5 |
|---|---|---|---|---|---|
| 50 | −0.0164<br>(0.0343) | −0.0386<br>(0.0611) | −0.0664<br>(0.0788) | −0.1731<br>(0.1207) | −0.1018<br>(0.0982) |
| 250 | −0.0078<br>(0.0158) | −0.0265<br>(0.0444) | −0.0676<br>(0.0816) | −0.3332<br>(0.1502) | −0.1532<br>(0.1190) |
| 1000 | −0.0030<br>(0.0071) | −0.0130<br>(0.0279) | −0.0692<br>(0.0818) | −0.6169<br>(0.1476) | −0.2600<br>(0.1291) |

*Note*: Each entry is the average of $\hat{S}_{N_1,N_2}$ over 1000 replications, with the standard deviation in the parentheses.

**Table 2**
Empirical size and power of the proposed test.

| | | Size | | Power |
|---|---|---|---|---|
| $N_1 = N_2$ | $\kappa_0$ | Setup 1 | Setup 3 | Setup 4 |
| 50 | 0.05 | 2.0 (0.634) | 16.8 (0.728) | 51.1 (0.585) |
| | 0.075 | 0.3 (0.799) | 12.2 (0.865) | 37.0 (0.758) |
| | 0.1 | 0.0 (0.884) | 9.3 (0.936) | 28.9 (0.875) |
| 250 | 0.05 | 0.0 (0.554) | 7.6 (0.847) | 81.6 (0.416) |
| | 0.075 | 0.1 (0.758) | 5.1 (0.954) | 71.6 (0.585) |
| | 0.1 | 0.0 (0.861) | 2.2 (0.982) | 61.1 (0.750) |
| 1000 | 0.05 | 0.0 (0.397) | 5.7 (0.919) | 100 (0.325) |
| | 0.075 | 0.0 (0.568) | 2.4 (0.983) | 99.9 (0.433) |
| | 0.1 | 0.0 (0.699) | 1.9 (0.996) | 99.9 (0.537) |

*Note*: Each entry is the rejection proportion (as a percentage), with the average length of the estimated contact set in the parentheses.

[10] The assumption of Linton et al. (2005) is a regularity condition of weak dependence on time series data.

## 5. Empirical study

In our empirical study, we consider testing the S&P 500 index return distributions during 2001–2013, based on the model discussed in Section 2. The daily returns are taken from the CRSP database with 3269 observations; see Table 3 for the summary statistics of these returns for each year. As these distributions can only be known *ex post*, the testing results cannot be used to form decisions for future investment. Yet, one may evaluate the performance of investment experts or fund managers by checking if their decisions are in line with the implications of CD.

For example, suppose that there is a mutual fund whose objective is characterized as Eq. (2). Now, the test of CD could be used to evaluate whether the fund manager makes a correct decision. Specifically, if the return distribution of year $t+1$ dominates that of year $t$ in terms of CD, then all non-satiable and risk-averse investors should increase their investment in stocks. If the fund manager invests less in stocks, then the fund manager should not be rewarded as long as the mutual fund is established for non-satiable and risk-averse investors. On the other hand, if the return distributions of year $t+1$ and year $t$ do not result in a CD, then we cannot conclude whether the fund manager makes a mistake no matter he/she increases or decreases the investment in stocks. Some non-satiable and risk-averse investors prefer to increasing the investment in stocks, whereas others prefer to decreasing. Thus, whether the fund manager should be punished or rewarded need to be further checked on whether the fund manager follows the investment guidelines which satisfy the specific preferences of the mutual fund.

Recall that the excess return distribution of year $i$ ($F_i$) centrally dominates that of year $j$ ($F_j$), i.e., $F_i \overset{\text{CD}}{\succ} F_j$, if and only if there exists $\gamma \in \mathbb{R}$ such that

$$\gamma \int_a^x t dF_i(t) \geq \int_a^x t dF_j(t), \text{ for all } x \in [a, b].$$

In our study, we consider 3 different risk free rates: $r_f = 3\%$, 2%, and 1%; there is no further assumption as to the value of $W$. We estimate the excess return distribution in each year using daily return data and test both the null hypotheses $F_i \overset{\text{CD}}{\succ} F_{i+1}$ and $F_{i+1} \overset{\text{CD}}{\succ} F_i$ for all $i$. Testing these paired hypotheses may lead to four possible outcomes: (accept, accept), (accept, reject), (reject, accept), and (reject, reject). The first outcome suggests that the two distributions are close to each other. The second and third outcomes occur when there is only one CD relation between $F_i$ and $F_j$, so that there is no ambiguity regarding the CD relation. The fourth outcome indicates that there is no CD relation between these distributions.[11]

It is worth mentioning that when a return distribution has a negative expected value (which must be less than the risk-free rate), it would be optimal for any risk-averse investor to put all of his/her money in the risk-free asset. In this case, the optimal decision variable $\alpha^*$ would be zero, and it would be unnecessary to test for central dominance. For example, given that the sample average of the risk premium in 2002 is negative, the optimal demand for the risky asset in 2002 would be 0, the lower bound of $\alpha^*$. We then have $F_{2003} \overset{\text{CD}}{\succ} F_{2002}$ because $\alpha^*_{2002} \leq \alpha^*_{2003}$ for all risk-averse investors. On the other hand, $F_{2002}$ cannot centrally dominate $F_{2003}$. For if it does, $\alpha^*_{2003}$ cannot exceed $\alpha^*_{2002}$ and hence must also be zero for all risk-averse investors. However, this cannot be true because there should be some positive optimal investment in 2003 when the sample average of the risk premium in 2003 is positive. From Table 3 we can see that the sample means in 2001, 2002 and 2008 are negative. We therefore do not consider these years in our CD testing.

**Table 3**
Descriptive statistics of S&P 500 index daily returns.

| Year | Mean | s.d. | Min | Max | Obs. |
|---|---|---|---|---|---|
| 2001 | −0.041% | 1.361% | −4.882% | 5.025% | 248 |
| 2002 | −0.086% | 1.643% | −4.173% | 5.754% | 252 |
| 2003 | 0.106% | 1.073% | −3.507% | 3.557% | 252 |
| 2004 | 0.044% | 0.696% | −1.629% | 1.625% | 252 |
| 2005 | 0.022% | 0.647% | −1.667% | 1.940% | 252 |
| 2006 | 0.060% | 0.632% | −1.824% | 2.134% | 251 |
| 2007 | 0.027% | 1.005% | −3.448% | 2.955% | 251 |
| 2008 | −0.148% | 2.564% | −8.996% | 11.513% | 253 |
| 2009 | 0.107% | 1.699% | −5.225% | 7.033% | 252 |
| 2010 | 0.062% | 1.133% | −3.857% | 4.362% | 252 |
| 2011 | 0.018% | 1.465% | −6.653% | 4.742% | 252 |
| 2012 | 0.063% | 0.801% | −2.471% | 2.517% | 250 |
| 2013 | 0.114% | 0.695% | −2.489% | 2.558% | 252 |

Table 4 summarizes the $p$-values of the testing results under different risk free rates; the numbers in the parentheses in this table are the values of $\hat{\gamma}^*$, the estimated point at which $\hat{Q}(\gamma)$ attains its maximum.[12] To compute our test, we adopt a block bootstrap with block length 20.[13] We can see that, for example, the $p$-values for $F_{03} \overset{\text{CD}}{\succ} F_{04}$ and $F_{04} \overset{\text{CD}}{\succ} F_{03}$ are 0.364 and 0.004, respectively. These indicate that we do not reject the former hypothesis but reject the latter at 1% significance level. We therefore conclude that the distribution in 2003 centrally dominates that in 2004 but not the other way around. Table 4 shows the following CD relations: $F_{03} \overset{\text{CD}}{\succ} F_{04}$, $F_{04} \overset{\text{CD}}{\succ} F_{05}$, $F_{06} \overset{\text{CD}}{\succ} F_{05}$, $F_{06} \overset{\text{CD}}{\succ} F_{07}$, $F_{10} \overset{\text{CD}}{\succ} F_{09}$, $F_{10} \overset{\text{CD}}{\succ} F_{11}$, $F_{12} \overset{\text{CD}}{\succ} F_{11}$, and $F_{13} \overset{\text{CD}}{\succ} F_{12}$, while their corresponding opposite null hypotheses are all strongly rejected at 1% level. Note that the testing results are consistent across different risk-free rates. These results suggest that the optimal investment in the S&P 500 index in 2004, 2005, 2007 and 2011 could have been lower than what it was in the previous year, and that the optimal investment in 2006, 2010, 2012 and 2013 could have been higher than what it was in the previous year.

To further illustrate the testing results that support the CD hypothesis in Table 4, we plot their paired $T(x; F_i)$ with $r_f = 2\%$ in Figs. 1 and 2. For example, Fig. 1(a) contains the plots for "0.38 $T(x, F_{03})$ vs. $T(x, F_{04})$", and Fig. 1(b) contains the plots for "0.44 $T(x, F_{04})$ vs. $T(x, F_{05})$", and so on. The multiplier, $\hat{\gamma}^*$, for each test, is taken directly from Table 4 (the number in parentheses). It is interesting to note that, in some cases, the CD relation may be determined by visual inspection, because one $T$ function is completely above the other $T$ function. This happens when $F_{06} \overset{\text{CD}}{\succ} F_{07}$, $F_{10} \overset{\text{CD}}{\succ} F_{11}$, $F_{12} \overset{\text{CD}}{\succ} F_{11}$, and $F_{13} \overset{\text{CD}}{\succ} F_{12}$. For the remaining cases, the CD relation is not so obvious from these figures and must be determined by the proposed test.

## 6. Concluding remarks

CD is an important concept in risk theory but has not been examined empirically due to a lack of proper econometric tools. In this paper, we propose the first test of CD and provide the first empirical evidence of CD relations in financial data. It is expected that,

11 Note that the daily index returns typically do not satisfy the assumption of a random sample imposed in Section 3. Hence, our results should be interpreted as suggestions, rather than definite conclusions, of possible CD relations.

12 Note that when the maximal value of $Q(\gamma)$ is zero, the set $\Gamma$ could be a closed interval. In such cases, we list the lower bound of the set.

13 We also tried block length 10 and an i.i.d. bootstrap, the inference remained unchanged.

**Table 4**
Test results of the null hypotheses $F_i \overset{CD}{\succ} F_{i-1}$ and $F_i \overset{CD}{\succ} F_{i+1}$.

| Year | $r_f = 3\%$ | | $r_f = 2\%$ | | $r_f = 1\%$ | |
|---|---|---|---|---|---|---|
| | $F_i \overset{CD}{\succ} F_{i-1}$ | $F_i \overset{CD}{\succ} F_{i+1}$ | $F_i \overset{CD}{\succ} F_{i-1}$ | $F_i \overset{CD}{\succ} F_{i+1}$ | $F_i \overset{CD}{\succ} F_{i-1}$ | $F_i \overset{CD}{\succ} F_{i+1}$ |
| 2003 | | 0.364 (0.37) | | 0.342 (0.38) | | 0.339 (0.4) |
| 2004 | 0.004* (1.56) | 0.886 (0.39) | 0.004* (1.56) | 0.805 (0.44) | ≤0.001* (1.58) | 0.748 (0.47) |
| 2005 | ≤0.001* (1) | ≤0.001* (0.69) | ≤0.001* (1) | ≤0.001* (0.73) | ≤0.001* (1) | ≤0.001* (0.75) |
| 2006 | 0.702 (0.39) | 0.556 (0.37) | 0.698 (0.44) | 0.594 (0.41) | 0.632 (0.47) | 0.542 (0.43) |
| 2007 | ≤0.001* (0.01) | 0.028* (1.62) | 0.016* (0.01) | 0.002* (1.63) | ≤0.001* (0.01) | 0.004* (1.63) |
| 2009 | 0.002* (0.1) | ≤0.001* (0.18) | ≤0.001* (0.1) | ≤0.001* (0.19) | ≤0.001* (0.1) | ≤0.001* (0.21) |
| 2010 | 0.471 (1.52) | 0.516 (0.19) | 0.48 (1.52) | 0.491 (0.22) | 0.464 (1.52) | 0.452 (0.26) |
| 2011 | ≤0.001* (0.01) | ≤0.001* (0.01) | ≤0.001* (0.01) | ≤0.001* (0.01) | ≤0.001* (0.01) | ≤0.001* (0.01) |
| 2012 | 0.485 (0.18) | ≤0.001* (0.79) | 0.485 (0.22) | ≤0.001* (0.8) | 0.487 (0.26) | ≤0.001* (0.8) |
| 2013 | 0.477 (0.58) | | 0.465 (0.59) | | 0.464 (0.6) | |

*Note:* The numbers in the table are $p$-values of the tests; an asterisk signifies statistical significance. The numbers in the parentheses are $\hat{\gamma}^*$, the estimated points at which $\hat{Q}(\gamma)$ attains its maximum. As the sample for 2008 is not included in the study, the distribution for 2007 is tested against 2006 and 2009, whereas the distribution for 2009 is tested against 2007 and 2010.

based on this work, more econometric research on the tests of CD and more empirical studies on the CD relations will be developed.

There are some future research directions. First, we may extend the asymptotic results to allow for time series dependence in the data. Second, we may extend our test to allow for nonlinear payoff functions. Third, the power of the proposed test may be improved. What we have now is a conservative test when $\Gamma$ is a singleton, because we obtain only the asymptotic distribution of a lower bound of $\hat{S}_{N_1,N_2}$. The test power can be improved by finding the asymptotic distribution of $\hat{S}_{N_1,N_2}$. To this end, we need to analyze the estimation effect of the estimator $\hat{\gamma}^*$ in $\hat{S}_{N_1,N_2}$. This is challenging because the true parameter is defined by a functional inequality; see Andrews and Shi (2013, 2014) for related research. These topics are being investigated.

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## Appendix

**Proof of Lemma 3.2.** When $F \overset{CD}{\succ} G$, $\Gamma$ is a non-empty set. Based on the definition of CD, $\Gamma$ is closed. The assertion holds provided that $\Gamma$ is convex set. That is, given any $\gamma_1, \gamma_2 \in \Gamma$, we must show that $\gamma^* = \alpha\gamma_1 + (1-\alpha)\gamma_2$ is also in $\Gamma$ for any $\alpha \in (0, 1)$. Clearly, when $\gamma_1$ and $\gamma_2$ are in $\Gamma$, $Q(\gamma_1) = 0$ and $Q(\gamma_2) = 0$, so that $\gamma_1 T(x; F) \geq T(x; G)$ and $\gamma_2 T(x; F) \geq T(x; G)$ for all $x \in [a, b]$. It follows that $\gamma^* T(x; F) \geq T(x; G)$ for all $x \in [a, b]$, proving that $\gamma^*$ is in $\Gamma$. □

**Proof of Lemma 3.3.** Since $f(t) = t$ is a continuous function defined on the closed interval $[a, b]$, it is measurable and uniformly bounded. In addition, the collection of the indicator functions $\{I(t \leq x), x \in [a, b]\}$ is a Donsker class and bounded by 1 for all probability measures. By Corollary 9.32 of Kosorok (2008), the collection of functions $tI(t \leq x), x \in [a, b]\}$ is also a Donsker class. The result follows directly from the Donsker Theorem. □

**Lemma A.1.** *For some fixed $\gamma_0 \in \Gamma$, we have*

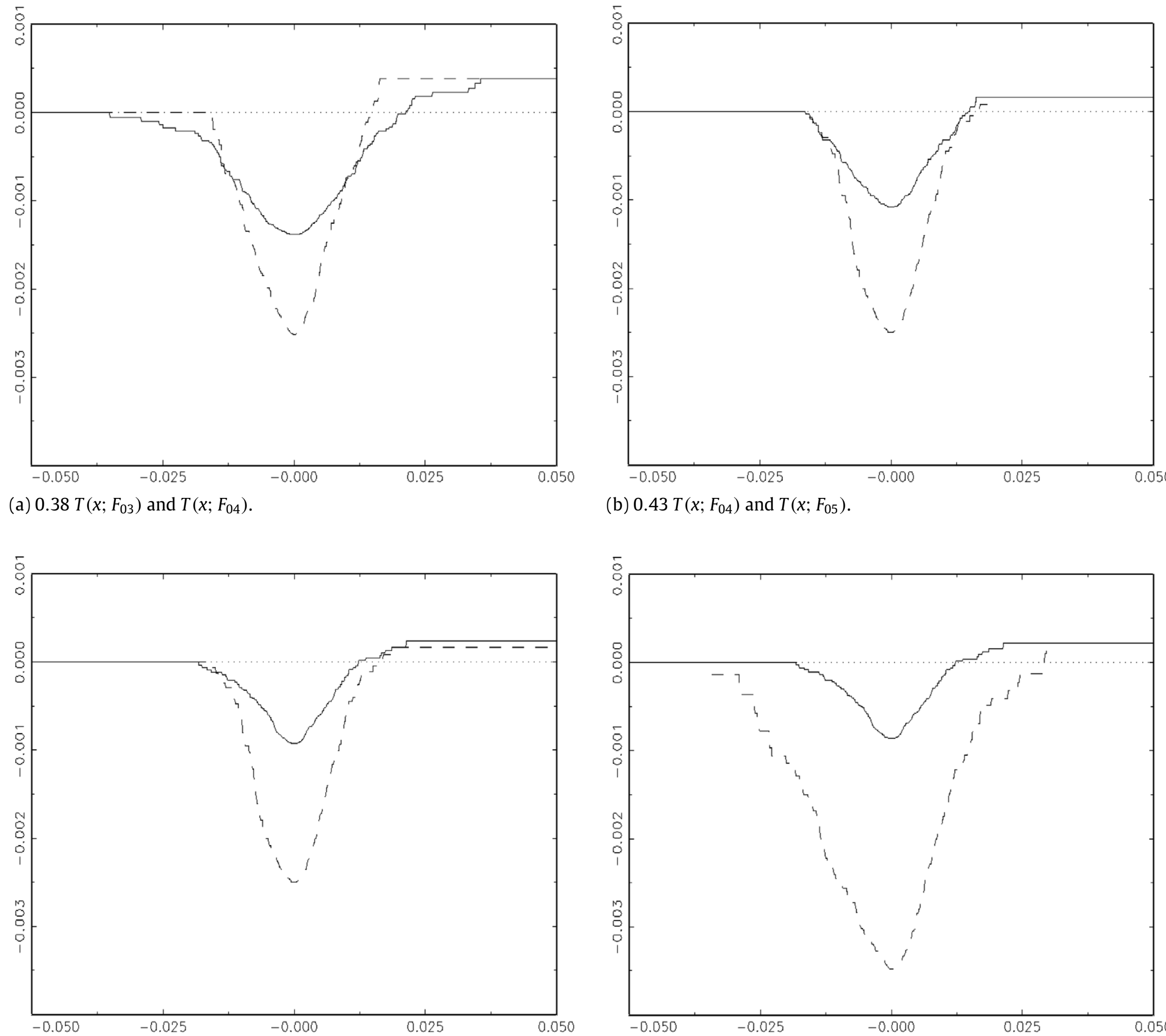
$$\sqrt{\frac{N_1 N_2}{N_1+N_2}}\hat{Q}_{N_1,N_2}(\gamma_0) \xrightarrow{d} \int_{\mathcal{B}^0_{\gamma_0}} \min\{\sqrt{\lambda}\gamma_0 T(x; B_F) - \sqrt{1-\lambda}T(x; B_G), 0\}dx.$$

**Proof of Lemma A.1.** Since $\gamma_0 \in \Gamma$, $\gamma_0 T(x; F) \geq T(x; G)$ for all $x \in [a, b]$. By the definition of $\mathcal{B}^0_{\gamma_0}$, for all $x \in [a, b] \setminus \mathcal{B}^0_{\gamma_0}$, we have $\gamma_0 T(x; F) > T(x; G)$, and hence $\mu(\gamma_0, x) > 0$ and

$$P\left\{\lim_{N_1,N_2\to\infty} \hat{\mu}_{N_1,N_2}(\gamma_0, x) > 0\right\} = 1.$$

Then by Lemma 3.3 and the continuous mapping theorem,

$$\begin{aligned}
&\sqrt{\frac{N_1 N_2}{N_1+N_2}}\hat{Q}_{N_1,N_2}(\gamma_0) \\
&= \sqrt{\frac{N_1 N_2}{N_1+N_2}} \int_a^b \min\{\hat{\mu}_{N_1,N_2}(\gamma_0, x), 0\}dx \\
&= \sqrt{\frac{N_1 N_2}{N_1+N_2}} \left[\int_{[a,b]\setminus\mathcal{B}^0_{\gamma_0}} \min\{\hat{\mu}_{N_1,N_2}(\gamma_0, x), 0\}dx \right. \\
&\quad \left. + \int_{\mathcal{B}^0_{\gamma_0}} \min\{\hat{\mu}_{N_1,N_2}(\gamma_0, x), 0\}dx\right] \\
&= \sqrt{\frac{N_1 N_2}{N_1+N_2}} \int_{[a,b]\setminus\mathcal{B}^0_{\gamma_0}} \min\{\hat{\mu}_{N_1,N_2}(\gamma_0, x), 0\}dx
\end{aligned}$$




(a) 0.38 $T(x; F_{03})$ and $T(x; F_{04})$.

(b) 0.43 $T(x; F_{04})$ and $T(x; F_{05})$.

(c) 0.44 $T(x; F_{06})$ and $T(x; F_{05})$.

(d) 0.41 $T(x; F_{06})$ and $T(x; F_{07})$.

**Fig. 1.** Paired $T(x; F_i)$ with $r_f = 2\%$; the solid lines are the dominating functions and the dashed lines are the dominated functions.

$$\xrightarrow{d} \int_{\mathcal{B}^0_{\gamma_0}} \min\{\sqrt{\lambda}\gamma_0 T(x; B_F) - \sqrt{1-\lambda}T(x; B_G), 0\}dx,$$

where the last equality follows from the fact that $\mu(\gamma_0, x) = 0$ for all $x \in \mathcal{B}^0_{\gamma_0}$. □

**Lemma A.2.** *If* $\Gamma = \{[\underline{\gamma}, \bar{\gamma}]\}$, *then*

$$\max_{\gamma\in\mathcal{C}} \hat{Q}_{N_1,N_2}(\gamma) \to 0 \textit{ in probability},$$

*and*

$$\sqrt{\frac{N_1N_2}{N_1+N_2}} \max_{\gamma\in\mathcal{C}} \hat{Q}_{N_1,N_2}(\gamma) \to 0 \textit{ in probability}.$$

**Proof of Lemma A.2.** Let the Lebesgue measure of set $\mathcal{B}$ be $|\mathcal{B}|$, and let $\Gamma_0 \equiv \{\gamma \in \Gamma : |\mathcal{B}^0_\gamma| = 0\}$ and $\Gamma_1 \equiv \{\gamma \in \Gamma : |\mathcal{B}^0_\gamma| > 0\}$, then $\Gamma_0 \cap \Gamma_1 = \phi$ and $\Gamma_0 \cup \Gamma_1 = \Gamma$. We want to prove that $\Gamma_1$ is countable and hence $\Gamma_0$ is non-empty.

(1) Note that for all $\gamma_i \in \Gamma_1$, we can construct disjoint open intervals $(a_{\gamma_i}, b_{\gamma_i}) \subset B^0_{\gamma_i}$. This is because $|B^0_{\gamma_i}| > 0$ and $|B^0_{\gamma_i} \cap B^0_{\gamma_j}| = 0$ for all distinct $\gamma_i, \gamma_j \in \Gamma_1$.

(2) Since disjoint open intervals in $[a, b]$ are countable, the points in $\Gamma_1$ are countable. Hence, $\Gamma_0$ is non-empty, $|\Gamma_0| = |\Gamma|$, and $|\Gamma_1| = 0$.

(3) By Lemma A.1, for all $\gamma_0 \in \Gamma_0$, $\gamma_1 \in \Gamma_1$, we know that $\sqrt{\frac{N_1N_2}{N_1+N_2}}\hat{Q}_{N_1,N_2}(\gamma_0)$ converges in probability to 0 and $\sqrt{\frac{N_1N_2}{N_1+N_2}}\hat{Q}_{N_1,N_2}(\gamma_1)$ converges in distribution to some negative distribution. Hence

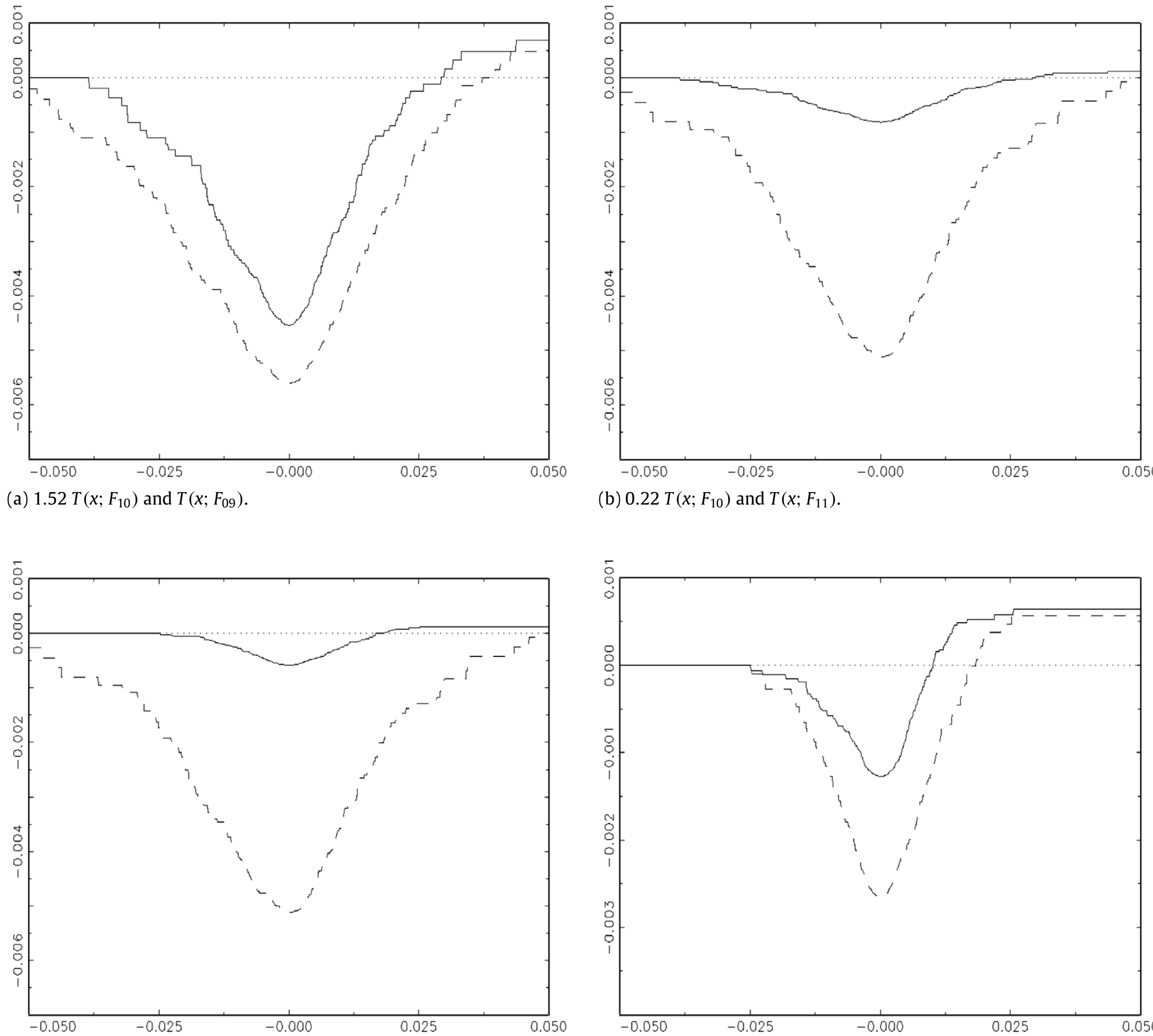
$$\lim_{N_1,N_2\to\infty} P(\hat{Q}_{N_1,N_2}(\gamma_0) > \hat{Q}_{N_1,N_2}(\gamma_1)) = 1.$$

(4) Based on our previous argument, we know that $P(\arg\max_{\gamma\in\Gamma} \hat{Q}_{N_1,N_2}(\gamma) \in \Gamma_0)$ is no less than $P$ (choose a point randomly in $\Gamma$, and it lies in $\Gamma_0$). In addition, if we randomly choose a point in $\Gamma$, then it lies in $\Gamma_0$ with probability 1, since $|\Gamma_0| = |\Gamma|$. Hence

$$P(\arg\max_{\gamma\in\Gamma} \hat{Q}_{N_1,N_2}(\gamma) \in \Gamma_0) = 1,$$

and the Lemma follows. □

**Proof of Theorem 3.4.** If $\Gamma$ is not an empty set, then $\Gamma$ is a singleton or a closed interval by Lemma 3.2. The result follows from Lemmas A.1 and A.2. □




(a) 1.52 $T(x; F_{10})$ and $T(x; F_{09})$.

(b) 0.22 $T(x; F_{10})$ and $T(x; F_{11})$.

(c) 0.22 $T(x; F_{12})$ and $T(x; F_{11})$.

(d) 0.59 $T(x; F_{13})$ and $T(x; F_{12})$.

**Fig. 2.** Paired $T(x; F_i)$ with $r_f = 2\%$; the solid lines are the dominating functions and the dashed lines are the dominated functions.

**Proof of Theorem 3.5.** By the law of large numbers and the continuous mapping theorem, $\max_\gamma \hat{Q}_{N_1,N_2}(\gamma) \to \max_\gamma Q(\gamma)$, hence $\sqrt{N_1} \max_\gamma \hat{Q}_{N_1,N_2}(\gamma) \to -\infty$ if $\max_\gamma Q(\gamma) < 0$. □

Let the bootstrapped empirical distribution function of $\hat{F}_{N_1}$ and $\hat{G}_{N_2}$ be $\hat{F}_{N_1,b}$ and $\hat{G}_{N_2,b}$ respectively, $b = 1, \ldots, B$. Let the bootstrapped empirical processes with respect to $T(x; \hat{F}_{N_1})$, $T(x; \hat{G}_{N_2})$, $\hat{\mu}_{N_1,N_2}(\gamma, x)$, and $\hat{Q}_{N_1,N_2}(\gamma)$ be $T(x; \hat{F}_{N_1,b})$, $T(x; \hat{G}_{N_2,b})$, $\hat{\mu}_{N_1,N_2,b}(\gamma, x)$, and $\hat{Q}_{N_1,N_2,b}(\gamma)$, $b = 1, \ldots, B$. Remember that $B_F$ and $B_G$ are mean zero Gaussian processes defined in Section 3.

**Lemma A.3.** $\sqrt{\frac{N_1N_2}{N_1+N_2}}[\hat{\mu}_{N_1,N_2,b}(\gamma, x) - \hat{\mu}_{N_1,N_2}(\gamma, x)]$ *weakly converges to a mean zero Gaussian process that has the same distribution as the limiting process of* $\sqrt{\frac{N_1N_2}{N_1+N_2}}[\hat{\mu}_{N_1,N_2}(\gamma, x) - \mu(\gamma, x)]$.

**Proof of Lemma A.3.** Based on the central limit theorem for a bootstrap, for independent samples drawn (with replacement) from distribution $F$, we have

$$\sqrt{N_1}[\hat{F}_{N_1,b} - \hat{F}_{N_1}] \rightsquigarrow B'_F \stackrel{d}{=} B_F.$$

Similarly,

$$\sqrt{N_2}[\hat{G}_{N_2,b} - \hat{G}_{N_2}] \rightsquigarrow B'_G \stackrel{d}{=} B_G.$$

Here $B'_F$ (respectively $B'_G$) is a Brownian bridge with respect to distribution $F$ (respectively $G$) and hence equals $B_F$ (respectively $B_G$) in the distribution. The assertion follows from the definitions of $\hat{\mu}_{N_1,N_2}(\gamma, x)$ and $\mu(\gamma, x)$. □

For some $\epsilon > 0$ and given $\gamma_0$, let $B^{\epsilon}_{\gamma_0} = \{x \in [a, b] : |\mu(\gamma_0, x)| < \epsilon\}$. The following lemma discusses the relationship between $B^{\epsilon}_{\gamma^*}$ and $\hat{B}^0_{\hat{\gamma}^*}$ when $\epsilon \downarrow 0$.

**Lemma A.4.** *If* $\Gamma = \{\gamma^*\}$ *is a single point, then for any* $\epsilon > 0$,

$$\lim_{N_1,N_2\to\infty} P(B^{(1-\epsilon)c_{N_1,N_2}}_{\gamma^*} \subset \hat{B}^0_{\hat{\gamma}^*} \subset B^{(1+\epsilon)c_{N_1,N_2}}_{\gamma^*}) = 1.$$

**Proof of Lemma A.4.** (1) By triangular inequality, we have:

$$|\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*, x) - \mu(\gamma^*, x)| \le |\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*, x) - \mu(\hat{\gamma}^*, x)| + |\mu(\hat{\gamma}^*, x) - \mu(\gamma^*, x)|,$$

for all $x \in [a, b]$. By Lemma A.1, we have

$$\sqrt{\frac{N_1N_2}{N_1+N_2}}[\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*, x) - \mu(\hat{\gamma}^*, x)] \rightsquigarrow \sqrt{\lambda}\hat{\gamma}^* T(x; B_F) - \sqrt{1-\lambda}T(x; B_G),$$

and hence

$$\sqrt{\frac{N_1N_2}{N_1+N_2}}|\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*, x) - \mu(\hat{\gamma}^*, x)|$$

is stochastically bounded. If $\Gamma = \{\gamma^*\}$, then

$$\sqrt{\frac{N_1N_2}{N_1+N_2}}|\mu(\hat{\gamma}^*, x) - \mu(\gamma^*, x)| = \sqrt{\frac{N_1N_2}{N_1+N_2}}|(\hat{\gamma}^* - \gamma^*)T(x; F)|$$

is also stochastically bounded since $\hat{\gamma}^*$ is an M-estimator of $\gamma^*$. Hence

$$\sqrt{\frac{N_1N_2}{N_1+N_2}}|\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*, x) - \mu(\gamma^*, x)|$$

is stochastically bounded.

(2) By assumption, $\sqrt{\frac{N_1N_2}{N_1+N_2}}c_{N_1,N_2} \to \infty$, so

$$|\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*, x) - \mu(\gamma^*, x)| < \epsilon c_{N_1,N_2},$$

for all $\epsilon > 0$, if $N_1, N_2$ large enough.

Note that $|\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*, x) - \mu(\gamma^*, x)| < \epsilon c_{N_1,N_2}$ implies both $|\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*, x)| - |\mu(\gamma^*, x)| < \epsilon c_{N_1,N_2}$ and $|\mu(\gamma^*, x)| - |\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*, x)| < \epsilon c_{N_1,N_2}$, which imply $B_{\gamma^*}^{(1-\epsilon)c_{N_1,N_2}} \subset \hat{B}^0_{\hat{\gamma}^*}$ and $\hat{B}^0_{\hat{\gamma}^*} \subset B_{\gamma^*}^{(1+\epsilon)c_{N_1,N_2}}$ respectively. So we have

$$|\hat{\mu}_{N_1,N_2}(\hat{\gamma}^*, x) - \mu(\gamma^*, x)| < \epsilon c_{N_1,N_2} \Rightarrow B_{\gamma^*}^{(1-\epsilon)c_{N_1,N_2}} \subset \hat{B}^0_{\hat{\gamma}^*} \subset B_{\gamma^*}^{(1+\epsilon)c_{N_1,N_2}},$$

and the lemma follows. □

For notational simplicity, define $k_{\gamma*}(x) = \min\{\sqrt{\lambda}\gamma^* T(x; B_F) - \sqrt{1-\lambda}T(x; B_G), 0\}$, and note that $k_{\gamma*}(x) \le 0$ by definition.

**Lemma A.5.** *If $\Gamma = \{\gamma^*\}$ is a singleton, then*

$$\int_{\hat{B}^0_{\hat{\gamma}^*}} k_{\gamma*}(x)dx \xrightarrow{a.s.} \int_{B^0_{\gamma^*}} k_{\gamma*}(x)dx.$$

**Proof of Lemma A.5.** Since $k_{\gamma*}(x) \le 0$, we have

$$\int_{B_{\gamma^*}^{(1-\epsilon)c_{N_1,N_2}}} k_{\gamma*}(x)dx \ge \int_{B_{\gamma^*}^{(1+\epsilon)c_{N_1,N_2}}} k_{\gamma*}(x)dx.$$

In addition, since $c_{N_1,N_2} \downarrow 0$, both the functions $I\{|\mu(\gamma^*, x)| < (1-\epsilon)c_{N_1,N_2}\}$ and $I\{|\mu(\gamma^*, x)| < (1+\epsilon)c_{N_1,N_2}\}$ are decreasing and converge to $I\{\mu(\gamma^*, x) = 0\}$. According to the monotone convergence theorem, we have

$$\int_{B_{\gamma^*}^{(1-\epsilon)c_{N_1,N_2}}} k_{\gamma*}(x)dx \xrightarrow{\text{a.s.}} \int_{B^0_{\gamma^*}} k_{\gamma*}(x)dx, \quad \text{and}$$

$$\int_{B_{\gamma^*}^{(1+\epsilon)c_{N_1,N_2}}} k_{\gamma*}(x)dx \xrightarrow{\text{a.s.}} \int_{B^0_{\gamma^*}} k_{\gamma*}(x)dx.$$

Together with Lemma A.4, we have $\int_{\hat{B}^0_{\hat{\gamma}^*}} k_{\gamma*}(x)dx \xrightarrow{\text{a.s.}} \int_{B^0_{\gamma^*}} k_{\gamma*}(x)dx$. □

**Proof of Theorem 3.6.** If $\Gamma = \{\gamma^*\}$, by Lemmas A.3 and A.5

$$\sqrt{\frac{N_1N_2}{N_1+N_2}}\int_{\hat{B}^0_{\hat{\gamma}^*}} \min\{[\hat{\mu}_{N_1,N_2,b}(\gamma, x) - \hat{\mu}_{N_1,N_2}(\gamma, x)]\}dx$$

has the same asymptotic distribution as

$$\sqrt{\frac{N_1N_2}{N_1+N_2}}\int_{B^0_{\gamma^*}} \min\{[\hat{\mu}_{N_1,N_2}(\gamma, x) - \mu(\gamma, x)]\}dx.$$

If $\Gamma = \{[\underline{\gamma}, \bar{\gamma}]\}$, then our test statistic converges to 0 almost surely, and the critical value converges to $c_0$, which is negative. Hence, the probability of accepting the null hypothesis converges to 1. If $F \overset{\text{CD}}{\not\succ} G$, our test statistic converges to negative infinity, but the critical value does not. It follows that the probability of rejecting the null hypothesis also converges to 1. □

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