國 立 政 治 大 學 應 用 數 學 系

碩 士 學 位 論 文

一個卡特蘭等式的組合證明

A Combinatorial Proof of a Catalan Identity

碩士班學生：劉映君　撰

指導教授：李陽明　博士

中華民國 106 年 6 月 16 日

國 立 政 治 大 學 應 用 數 學 系

劉映君君所撰之碩士學位論文

一個卡特蘭等式的組合證明
A Combinatorial Proof of a Catalan Identity

業經本委員會審議通過
論文考試委員會委員：

指導教授：

系主任：

中華民國 106 年 6 月 16 日

致謝

感謝這三年來中所有指導過我的老師，還有系上幫助過我的學長姐及同學們，幫助我順利完成學業。

首先感謝我的指導教授李陽明老師，謝謝老師對於我所投入的時間及心血，無論是平常上課、準備論文以及最後的論文修改，因為有老師的耐心指導，才能有這篇論文的誕生。

接著要感謝的是我最愛的家人，讀研究所的這三年來總是讓他們操煩，而且上研究所後與家人相處的時間減少許多，但他們仍給於我精神上的關懷，並給我很多支持與協助。

最後，感謝研究所的同學們，在我面對困難的時候伸出援手，也陪伴我度過這三年的研究所生活。

中文摘要

在這篇論文裡，我們探討卡塔蘭等式 $(n+2)C_{n+1} = (4n+2)C_2$ 的證明方法。以往都是用計算的方式來呈現卡塔蘭等式的由來，但我們選擇用組合的方法來證明卡塔蘭等式。

這篇論文主要是應用 C_{n+1} 壞路徑對應到打點 C_n 好路徑以及 C_{n+1} 好路徑對應到打點 C_n 壞路徑的方式來證明卡特蘭等式。

關鍵字：卡塔蘭等式

Abstract

In this thesis, we give another approach to prove Catalan identity, $(n+2)C_{n+1} = (4n+2)C_2$. In the past we use the method of computation to show Catalan Identity, in this thesis we choose a combinatorial proof of the Catalan identity.

This thesis is primary using the functions from C_{n+1} totally bad path to C_n dotted good path, and from C_{n+1} good path to C_n dotted totally bad path.

Keywords: Catalan Identity

Contents

List of Figures

Chapter 1

Introduction

Definition 1.1. A segment is either an east(e) or a north(n). A path consists of consecutive segments.

Definition 1.2. An (n, n) path is a path with n e's and n n's segments.

Definition 1.3. A good path means that all segments are below diagonal $y = x$. A bad path is a path that is not a good path.

Note : A bad path has at least one segment above diagonal $y = x$.

Definition 1.4. A totally bad path means that all segments are above diagonal $y = x$.

Catalan numbers are the number of good paths from the origin to the point (n, n), and we define Catalan number, C_n, by $C_n = \frac{1}{n+1}C_n^{2n}$, for $n \geq 0$. In this thesis, we focus on a combinatorial proof of a Catalan identity,

$(n+2)C_{n+1} = (4n+2)C_n$. [6]

In general, we obtain this formula by

$$\begin{aligned}(n+2)C_{n+1} &= \frac{(n+2)C^{2n+2}_{n+1}}{n+2} \\ &= \frac{(2n+2)!}{(n+1)!(n+1)!} \\ &= \frac{(2n+1)(2n)!(2n+2)}{(n+1)n!n!(n+1)} \\ &= \frac{2(2n+1)(2n)!(2n+2)}{(n+1)n!n!(2n+2)} \\ &= \frac{(4n+2)(2n)!}{(n+1)n!n!} \\ &= \frac{(4n+2)C^{2n}_{n}}{n+1} = (4n+2)C_n\end{aligned}$$

It is well-known that the number of paths with $n+1$ flaws, which has $n+1$ east and $n+1$ north segments above the diagonal $y = x$, is equal to the number of such paths with n flaws, which is equal to the number of such paths with $n-1$ flaws, and so on. In other words, we have split up the set of all paths into n + 2 equally sized classes. Since there are C^{2n+2}_{n+1} paths, we obtain the desired formula $C_{n+1} = \frac{1}{(n+2)}C^{2n+2}_{n+1}$. So the left side $(n+2)C_{n+1} = C^{2n+2}_{n+1}$.
By Pascal Identity, $C^{2n+2}_{n+1} = \underbrace{C^{2n+1}_{n}}_{\textit{Paths starting with north}} + \underbrace{C^{2n+1}_{n+1}}_{\textit{Paths starting with east}}$
On the right-hand side, $(4n+2)C_n = 2(2n+1)C_n = \underbrace{(2n+1)C_n}_{\textit{Dotted good paths}} + \underbrace{(2n+1)C_n}_{\textit{Dotted totally bad paths}}$.
In Chapter 2, we give a bijective proof between "Paths start with north" and "Dotted good paths". In Chapter 3, we give a bijective proof between "Paths start with east" and "Dotted totally bad paths".
Therefore, we complete the proof of $(n+2)C_{n+1} = (4n+2)C_n$ combinatorially.
For more details, we refer to [1–5, 7–10]

Chapter 2

Paths Start with North

Definition 2.1. The set X_1 consists of all $(n+1, n+1)$ paths which have at least one flaw. Each path in X_1 can be factorized into $GT\overrightarrow{e}Q$. The set Y_1 consists of all $(n+1, n+1)$ paths which has at most n flaws.

Define a function f_1 from X_1 into Y_1 by the following:

1. Starting from the bottom left, $(0,0)$, follow the path until it first travels above the diagonal $y = x$.
2. Continue to follow the path until it touches the diagonal $y = x$ again. Denote by $\overrightarrow{e}$, the first such segment that touches the diagonal $y = x$, in fact, $\overrightarrow{e}$ must be an east segment.
3. Swap the portion of the path before $\overrightarrow{e}$ with portion after $\overrightarrow{e}$.

i.e. $f_1(GT\overrightarrow{e}Q) = Q\overrightarrow{e}GT$,
where G is an (i,i) good path, $0 \leq i \leq n+1$, T is a $(j, j+1)$ totally bad path, $0 \leq j \leq n-i$, the east segment $\overrightarrow{e}$ is the first east which touches the diagonal $y = x$, and Q is an $(n-i-j, n-i-j)$ path.

NOTE: After using f_1, the flaws of path P decrease one.
i.e. If P has k flaws, $k \geq 1$, then $f_1(P)$ has $k-1$ flaws.

To show $f_1(GT\overrightarrow{e}Q) = Q\overrightarrow{e}GT$ by graph, we have:

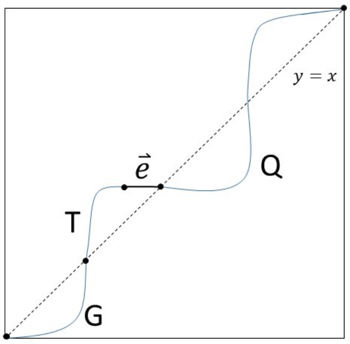


Figure 2.1: $GT\vec{e}Q$

Fix $\overrightarrow{e}$ and switch GT with Q, we have:

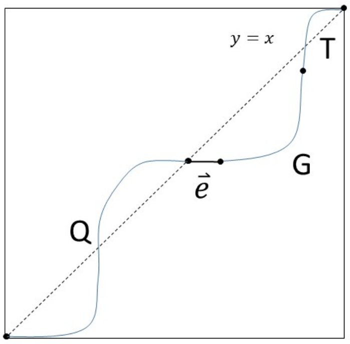


Figure 2.2: $Q\vec{e}GT$

Theorem 2.2. *Let $P = GT\vec{e}Q$ is an $(n+1, n+1)$ path. Define $f_1 : X_1 \longrightarrow Y_1$ by $f_1(GT\vec{e}Q) = Q\vec{e}GT$, where G is an (i,i) good path, $0 \leq i \leq n+1$, T is a $(j, j+1)$ totally bad path, $0 \leq j \leq n-i$, the east segment $\vec{e}$ is the first east which touches the diagonal $y = x$, and Q is an $(n-i-j, n-i-j)$ path. The function f_1 is one-to-one and onto.*

Proof. Claim: f_1 is one-to-one.

Let $P = GT\vec{e}Q$, $P' = G'T'\vec{e}Q'$, where T' is a an $(l, l+1)$ totally bad path, G' is a (k,k) good path, and Q' is an $(n-k-l, n-k-l)$ path.

$f_1(GT\vec{e}Q) = f_1(G'T'\vec{e}Q') \Rightarrow Q\vec{e}GT = Q'\vec{e}G'T'$.

Claim: $T = T'$

Case1: $l > j$

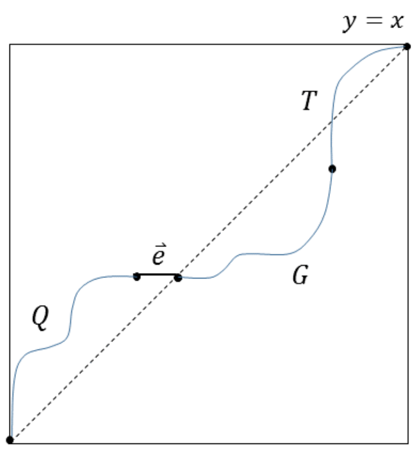


Figure 2.3: $Q\vec{e}GT$

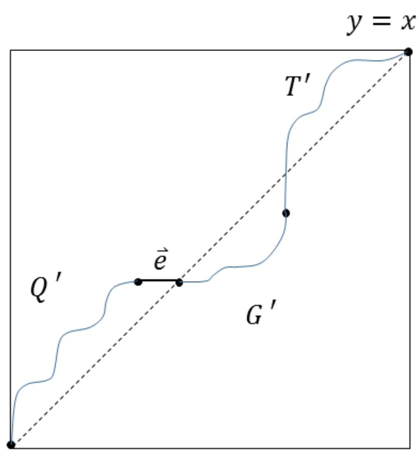


Figure 2.4: $Q'\vec{e}G'T'$

When we start on $(n+1, n+1)$, trace back the path.

We let two path both trace back to the point $(n-j+1, n-j+1)$, in the next step, the path in $Figure$ 2.3 is below the diagonal $y = x$, but the path in $Figure$ 2.4 is still above the diagonal $y = x$.

This is a contradiction, as two paths are the same.

Case2: $l < j$

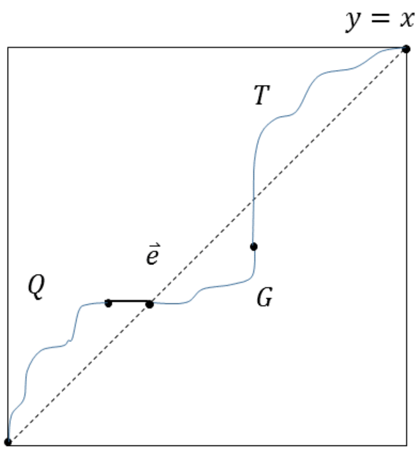


Figure 2.5: $Q\vec{e}GT$

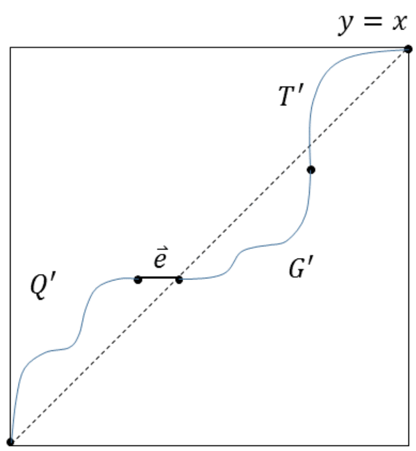


Figure 2.6: $Q'\vec{e}G'T'$

When we start on $(n+1, n+1)$, trace back the path.

We let two paths both trace back to the point $(n-l+1, n-l+1)$, in the next step, the path in $Figure$ 2.6 is below the diagonal $y = x$, but the path in $Figure$ 2.5 is still above diagonal $y = x$.

This is a contradiction, as two paths are the same.

Thus, we have proved that $l = j$.

$\therefore T = T'$.

Claim: $G = G'$.

Case1: $i > k$

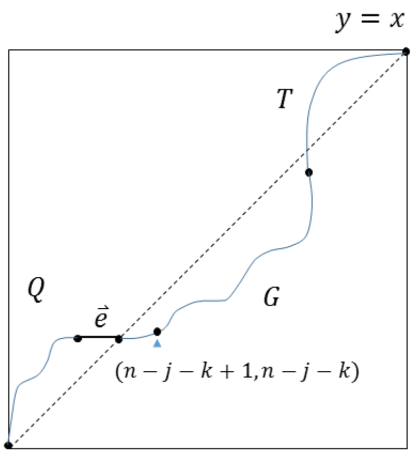


Figure 2.7: $Q\vec{e}GT$

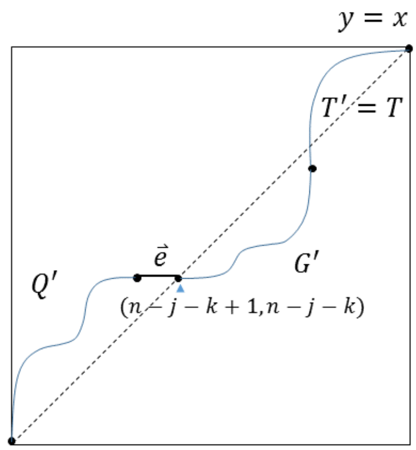


Figure 2.8: $Q'\vec{e}G'T'$

We both start on $(n-j+1, n-j)$.

Since $i > k$, when we let the path in $Figure$ 2.8 trace back to $(n-j-k+1, n-j-k)$, the next segment is $\overrightarrow{e}$, but the path in $Figure$ 2.7 is not $\overrightarrow{e}$.

This is a contradiction, as two paths are the same.

Case2: $i < k$

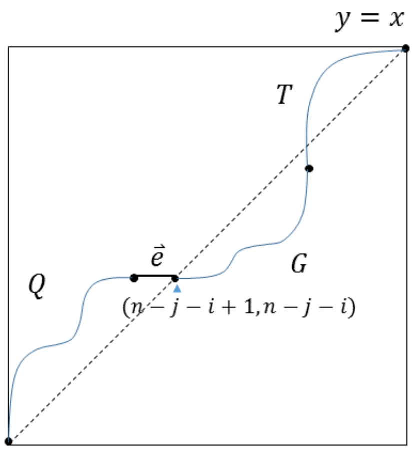


Figure 2.9: $Q'\vec{e}G'T'$

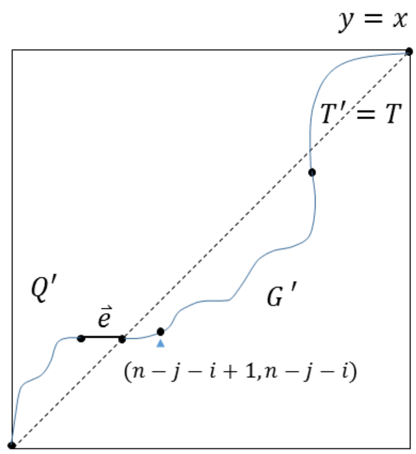


Figure 2.10: $Q\vec{e}GT$

We both start on $(n-j+1, n-j)$.

Since $i < k$, when we let the path in *Figure* 2.10 trace back to $(n-j-i+1, n-j-i)$, the next segment is not $\vec{e}$, but the path in *Figure* 2.9 is $\vec{e}$.

This is a contradiction, as two paths are the same.

Thus, we have proved that $i = k$.

$\therefore G = G'$.

Since $T = T'$ and $G = G'$.

$\because Q\vec{e}GT = Q'\vec{e}G'T' \Rightarrow Q\vec{e} = Q'\vec{e}$

$\therefore Q = Q' \Rightarrow GT\vec{e}Q = G'T'\vec{e}Q'$

$\therefore f_1$ is one-to one.

Claim: f_1 is onto.

i.e. For any path in Y_1, which has at most n flaws, we choose the last east leaving the diagonal $y = x$, denoted by $\hat{e}$, then we switch the portions before $\hat{e}$ and after $\hat{e}$. We can get a new path with at least one flaw, and the path is in X_1.

To show by graph:

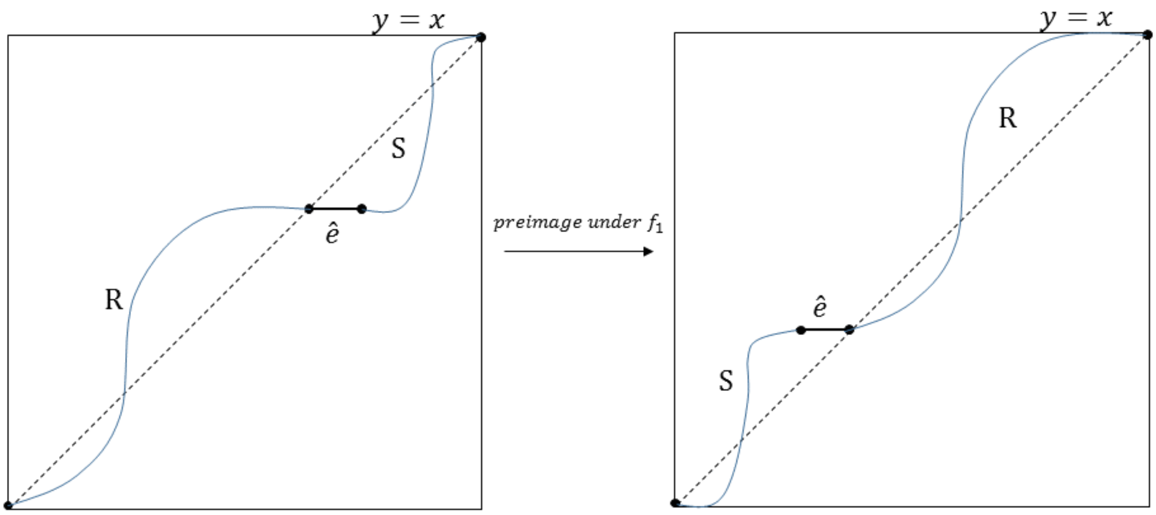


Figure 2.11: $R\hat{e}S \xrightarrow{\textit{preimage under } f_1} S\hat{e}R$

To show by formula:

$Q = R\hat{e}S \xrightarrow{\textit{preimage under } f_1} S\hat{e}R = P$,

where Q has at most n flaws, P has at least one flaw.

In fact, if Q has k flaws, P has $k+1$ flaws.

So, for every path Q in Y_1, we can find a path P in X_1 such that $f_1(P) = Q$.

Therefore, f_1 is one-to-one and onto.

□

NOTE: Let f_1^{-1} be the inverse function of f_1.

Lemma 2.3. *The first east $\vec{e}$ touching the diagonal $y = x$ in X_1 is below the diagonal $y = x$ in Y_1 after using f_1.*

Proof. Let $P = S\vec{e}R$, where S is a $(j, j+1)$ path, $\vec{e}$ is a $(1,0)$ east path, and R is an $(n-j, n-j)$ path.

After using f_1, we swap R and S, since R is an $(n-j, n-j)$ path, the next segment $\vec{e}$ is below the diagonal $y = x$.

Thus, we have proved that the first east $\vec{e}$ touching the diagonal $y = x$ in X_1 and it is below the diagonal $y = x$ in Y_1 after using f_1.

□

Lemma 2.4. *The last east $\hat{e}$ leaving from diagonal $y = x$ in Y_1 is the first east $\vec{e}$ touching the diagonal $y = x$ in X_1.*
i.e. If $\vec{e}$ is the first east touching the diagonal $y = x$ in X_1, then $\vec{e}$ is the last east leaving the diagonal $y = x$ in Y_1.

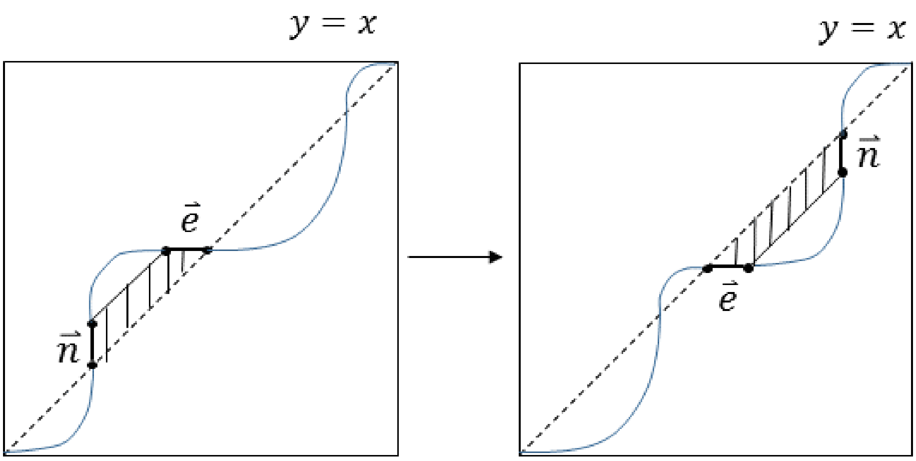


Figure 2.12: *Lemma*2.4

Proof. Since $\vec{e}$ is the first east touching the diagonal $y = x$, we can observe that there is a empty area enclosed by the first north that leaves the diagonal $y = x$, denoted by $\vec{n}$, the diagonal $y = x$, $\vec{e}$, and the diagonal $y = x + 1$.
After swapping two portions, another empty is enclosed by $\vec{e}$, the diagonal $y = x$, $\vec{n}$, and the diagonal $y = x - 1$, so that there is no east segment can touch the diagonal $y = x$ between $\vec{e}$ and $\vec{n}$.
And the remain segments which are behind $\vec{n}$ are at most touching the diagonal $y = x$ but not be below the diagonal $y = x$. Therefore, the last east $\hat{e}$ leaving from diagonal $y = x$ in Y_1 is the first east $\vec{e}$ touching the diagonal $y = x$ in X_1. □

Definition 2.5. The set A_1 consists of all paths which first segment is north, and the first touching the diagonal $y = x$ east is marked.
The set B_1 consists of all paths which are good path.

Define g_1 from A_1 into B_1 by $g_1(P) = f_1^{(k)}(P)$, where P has k flaws, and $f_1^{(k)} = \underbrace{f_1 \circ f_1 \circ ... \circ f_1}_{k}$.

Example 2.6. The following example is one of $g_1(P)$:

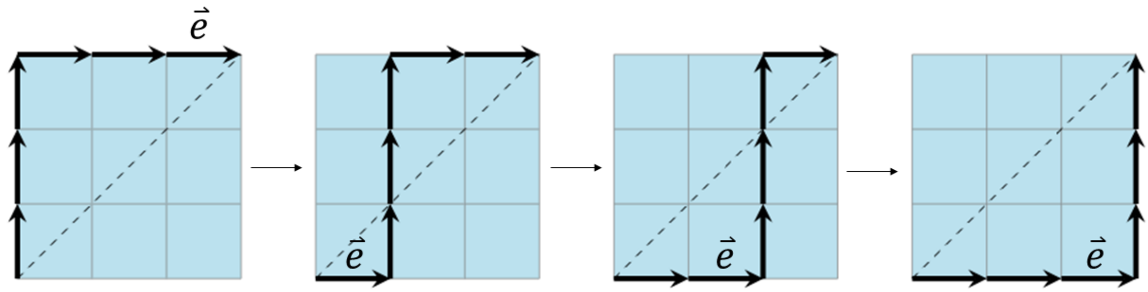


Figure 2.13: $n = 3$

Lemma 2.7. *In $g_1(P)$, after using the first f_1, the first east which is denoted by $\overrightarrow{e}$, connects with the first segment of P in A_1. And this part will not be separated afterward.*

Proof. First, we prove the first part.

Since using f_1 will swap the portion berfore and after $\overrightarrow{e}$, and the first segment of path is north, after using f_1, $\overrightarrow{e}$ connects with the north segment.

Thus, we have proved.

Next, we prove the second part.

Notice that after using the first f_1, the part of $\overrightarrow{e}$ and the north segment is below the diagonal $y = x$.

There is another first touching the diagonal $y = x$ east, and the part is in a (j, j) path after that east, in the next step, we use f_1 again, so this part will be swapped to before that east, since it is (j, j) path, the part is still below the diagonal $y = x$.

Therefore, no matter how many times we use f_1, $\overrightarrow{e}$ connects with the first segment of P in A_1 and they are not be separated afterward. $\square$

Theorem 2.8. *g_1 is one-to-one and onto.*

Proof. Claim: g_1 is one-to-one.

$g_1(P) = g_1(Q) \Rightarrow f_1^{(k)}(P) = f_1^{(h)}(Q)$, where P has k flaws and Q has h flaws.

Case1: $k < h$

$f_1^{(k)}(P) = f_1^{(h)}(Q) \Rightarrow f_1(f_1^{(k-1)}(P)) = f_1(f_1^{(h-1)}(Q))$

$\because f_1$ is one-to-one.

$\therefore f_1^{(k-1)}(P) = f_1^{(h-1)}(Q)$.

$f_1(f_1^{(k-2)}(P)) = f_1(f_1^{(h-2)}(Q)) \Rightarrow f_1^{(k-2)}(P) = f_1^{(h-2)}(Q)$, since f_1 is one-to-one.

Use this way for $k-1$ times, we have $f_1(P) = f_1^{(h-(k-1))}(Q) = f_1(f_1^{(h-k)}(Q))$

$\Rightarrow P = f_1^{(h-k)}(Q)$

The first $\vec{e}$ of P is above the diagonal $y = x$, but the first $\vec{e}$ of $f_1^{(h-k)}(Q)$ is below the diagonal $y = x$ by Lemma 2.7.

This is a contradiction.

Case2: $k > h_1$

$f_1^{(k)}(P) = f_1^{(h)}(Q) \Rightarrow f_1(f_1^{(k-1)}(P)) = f_1(f_1^{(h-1)}(Q))$

$\because f_1$ is one-to-one.

$\therefore f_1^{(k-1)}(P) = f_1^{(h-1)}(Q)$.

$f_1(f_1^{(k-2)}(P)) = f_1(f_1^{(h-2)}(Q)) \Rightarrow f_1^{(k-2)}(P) = f_1^{(h-2)}(Q)$, since f_1 is one-to-one.

Use this way for $h-1$ times, we have $f_1^{(k-(h-1))}(P) = f_1(f_1^{(k-h)}(P)) = f_1(Q)$

$\Rightarrow f_1^{(k-h)}(P) = Q$

The first $\vec{e}$ of Q is above the diagonal $y = x$, but the first $\vec{e}$ of $f_1^{(k-h)}(P)$ is below the diagonal $y = x$ by Lemma 2.7.

This is a contradiction.

Case3: $k = h$

$f_1^{(k)}(P) = f_1^{(h)}(Q) \Rightarrow f_1(f_1^{(k-1)}(P)) = f_1(f_1^{(h-1)}(Q))$

$\Rightarrow f_1^{(k-1)}(P) = f_1^{(h-1)}(Q)$ ($\because f_1$ is one-to-one.)

Use this way for $k-1$ times, we have $f_1(P) = f_1(Q) \Rightarrow P = Q$

Therefore, g_1 is one-to-one.

Claim: g_1 is onto.

Given $Q \in B_1$.

Define $f_1^{(-k)} = \underbrace{f_1^{-1} \circ f_1^{-1} \circ ... \circ f_1^{-1}}_{k}$.

$f_1^{-1}(Q)$ is a preimage of Q under f_1 and $f_1^{-1}(Q)$ has 1 flaw.

We can use this way for $n+1$ times, until the segment $\vec{e}$ is above the diagonal $y = x$ by Lemma 2.7.

So we have $f_1^{-(n+1)}(Q) = P$, where P has $n+1$ flaws, $P \in A_1$.

Thus, g_1 is onto. Therefore, g_1 is one-to-one and onto. $\square$

Definition 2.9. The set C_1 consists of all (n,n) paths which are replaced the marked east and the next north segment in B_1 with a dot, and all paths in B_1 are $(n+1, n+1)$ path.

Let h_1 be the function from B_1 into C_1.

i.e. $P = R\vec{e}\,\vec{n}S$ is an $(n+1, n+1)$ path, where R and S are all good paths.

$h_1(P) = h_1(R\vec{e}\,\vec{n}S) = R \bullet S$

Theorem 2.10. *h_1 is one-to-one and onto.*

Proof. It is clearly obvious that h_1 is one-to-one and onto.

Given $Q = R \bullet S \in C_1$.

We can change $\bullet$ into $\vec{e}\,\vec{n}$.

Thus, $R \bullet S \Rightarrow R\vec{e}\,\vec{n}S \in B_1$.

Therefore, h_1 is one-to-one and onto. $\square$

Chapter 3

Paths Start with East

Definition 3.1. The set X_2 consists of all $(n+1, n+1)$ paths which have at most n flaw. Each path in X_2 can be factorized into $TG\vec{n}Q$. The set Y_2 consists of all $(n+1, n+1)$ paths which has at least one flaws.

Define a function f_2 from X_2 into Y_2 by the following:

1. Starting from the bottom left, $(0,0)$, follow the path until it first travels below the diagonal $y = x$.
2. Continue to follow the path until it touches the diagonal $y = x$ again. Denote by $\vec{n}$, the first such segment that touches the diagonal $y = x$, in fact, $\vec{n}$ must be an north segment.
3. Swap the portion of the path before $\vec{n}$ with portion after $\vec{n}$.

i.e. $f_2(TG\vec{n}Q) = Q\vec{n}TG$,
where T is an (i,i) totally bad path, $0 \leq i \leq n+1$, G is a $(j+1, j)$ good path, $0 \leq j \leq n-i$, the north segment $\vec{n}$ is the first north touching diagonal $y = x$, and Q is an $(n-i-j, n-i-j)$ path.

NOTE: After using f_2, the flaws of path P increase one.
i.e. if P has k flaws, $k \leq n$, then $f_2(P)$ has $k+1$ flaws.

To show $f_2(TG\overrightarrow{n}Q) = Q\overrightarrow{n}TG$ by graph, we have:

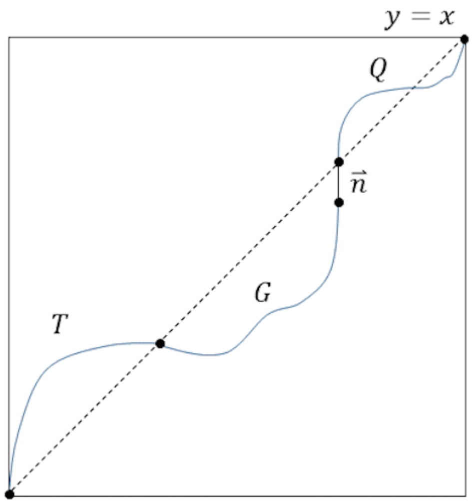


Figure 3.1: $TG\vec{n}Q$

Fix $\overrightarrow{n}$ and switch TG with Q, we have:

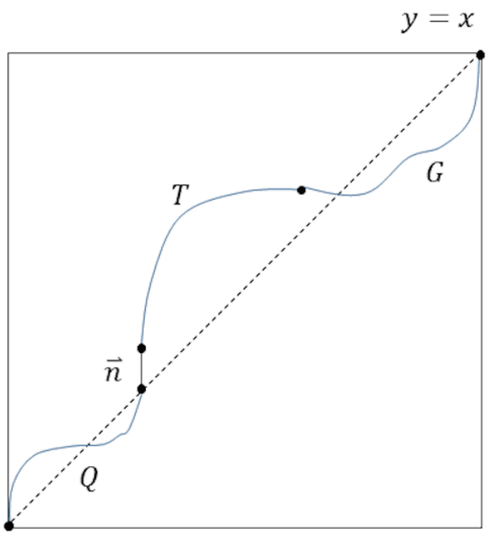


Figure 3.2: $Q\vec{n}TG$

Theorem 3.2. *Let $P = TG\overrightarrow{n}Q$ is an $(n+1, n+1)$ path. Define $f_2 : X_2 \longrightarrow Y_2$ by $f_2(TG\overrightarrow{n}Q) = Q\overrightarrow{n}TG$, where T is an (i,i) totally bad path, $0 \leq i \leq n+1$, G is a $(j+1, j)$ good path, $0 \leq j \leq n-i$, the north segment $\overrightarrow{n}$ is the first north which touches the diagonal $y = x$, and Q is an $(n-i-j, n-i-j)$ path. The function f_2 is one-to-one and onto.*

Proof. Claim: f_2 is one-to-one.

Let $P = TG\overrightarrow{n}Q$, $P' = T'G'\overrightarrow{n}Q'$, where G' is a an $(l+1, l)$ good path, T' is a (k,k) totally bad path, and Q' is an $(n-k-l, n-k-l)$ path.

If $f_2(TG\overrightarrow{n}Q) = f_2(T'G'\overrightarrow{n}Q') \Rightarrow Q\overrightarrow{n}TG = Q'\overrightarrow{n}T'G'$.

Claim: $G = G'$.

Case1: $l > j$

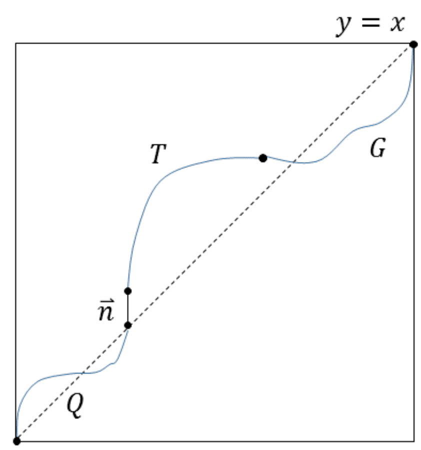


Figure 3.3: $Q\vec{n}TG$

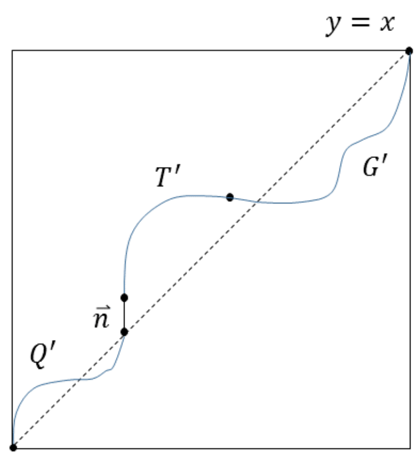


Figure 3.4: $Q'\vec{n}T'G'$

When we start on $(n+1, n+1)$, trace back the path.

We let two paths both trace back to the point $(n-j, n-j+1)$, in the next step, the path in *Figure* 3.3 is above the diagonal $y = x$, but the path in *Figure* 3.4 is still below diagonal $y = x$.

This is a contradiction, as two paths are the same.

Case2: $l < j$

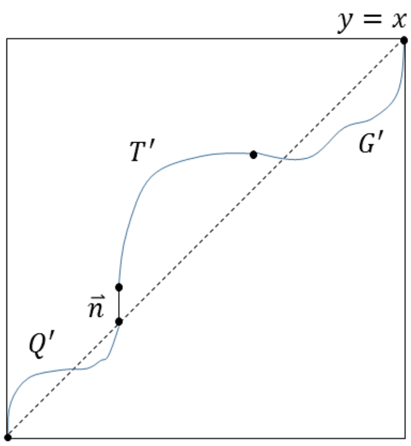


Figure 3.5: $Q^{'}\vec{n}T^{'}G^{'}$

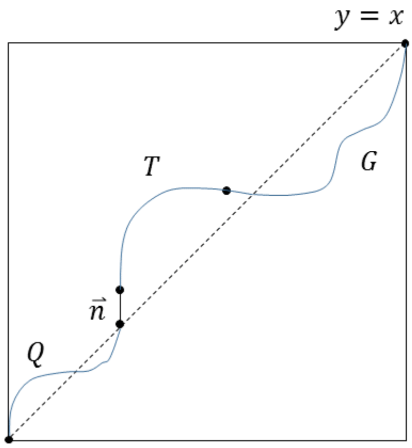


Figure 3.6: $Q\vec{n}TG$

When we start on $(n+1, n+1)$, trace back the path.

Ww let two paths both trace back to the point $(n-l, n-l+1)$, in the next step, the path in $Figure$ 3.5 is above the diagonal $y = x$, but the path in $Figure$ 3.6 is still below diagonal $y = x$.

This is a contradiction,as two paths are the same.

Thus, we have proved that $l = j$.

$\therefore G = G^{'}$.

Claim: $T = T^{'}$.

Case1: $i > k$

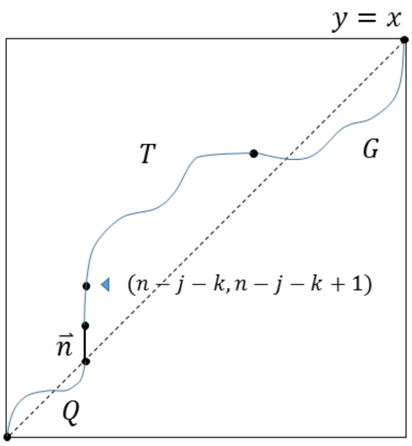


Figure 3.7: $Q\vec{n}TG$

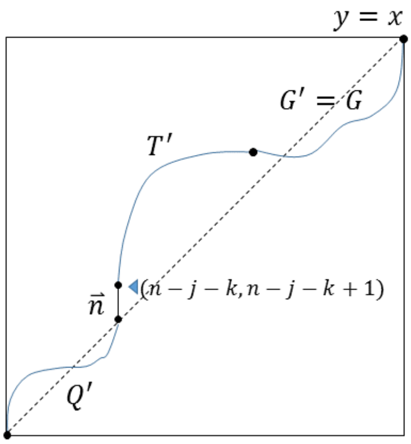


Figure 3.8: $Q'\vec{n}T'G'$

We both start on $(n-j, n-j+1)$.

Since $i > k$, when we let the path in $Figure$ 3.8 trace to $(n-j-k, n-j-k+1)$, the next segment is $\vec{n}$, but the path in $Figure$ 3.7 is not $\vec{n}$.

This is a contradiction,as two paths are the same.

Case2: $i < k$

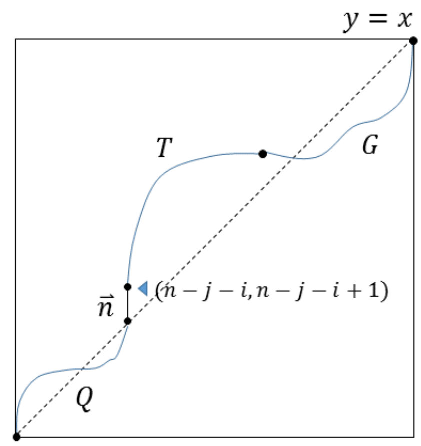


Figure 3.9: $Q\vec{n}TG$

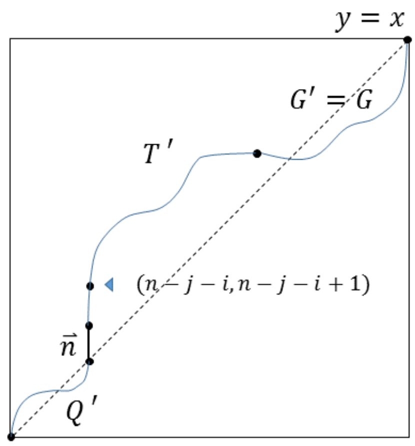


Figure 3.10: $Q'\vec{n}T'G'$

We both start on $(n-j+1, n-j)$.

Since $i < k$, when we let the path in $Figure$ 3.10 trace back to $(n-j-i, n-j-i+1)$, the next segment is not $\overrightarrow{n}$, but the path in $Figure$ 3.9 is $\overrightarrow{n}$,.

This is a contradiction, as two paths are the same.

Thus, we have proved that $i = k$.

$\therefore T = T^{'}$.

Since $G = G^{'}$ and $T = T^{'}$.

$\because Q\overrightarrow{n}TG = Q^{'}\overrightarrow{n}T^{'}G^{'} \Rightarrow Q\overrightarrow{n} = Q^{'}\overrightarrow{n}$

$\therefore Q = Q^{'} \Rightarrow TG\overrightarrow{n}Q = T^{'}G^{'}\overrightarrow{n}Q^{'}$

$\therefore f_2$ is one-to one.

Claim: f_2 is onto.

$i.e.$ For any path in Y_2, which has at least one flaw, we choose the last north leaving the diagonal $y = x$, denoted by $\hat{n}$, then we switch the portions before $\hat{n}$ and after $\hat{n}$. We can get a new path with at most n flaws which is in X_2.

To show by graph:

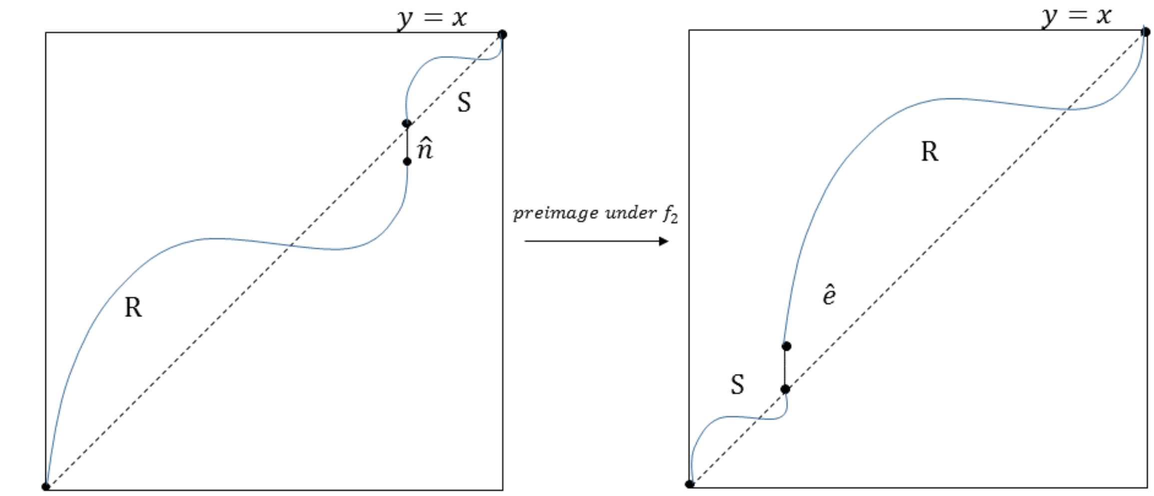


Figure 3.11: $R\hat{n}S \xrightarrow{\text{preimage under } f_2} S\hat{n}R$

To show by formula:

$Q = R\hat{n}S \xrightarrow{\text{preimage under } f_2} S\hat{n}R = P,$

where Q has at least one flaw, P has at most n flaws.

In fact, if Q has k flaws, P has $k-1$ flaws.

So, for every path Q in Y_2, we can find a path P in X_2 such that $f_2(P) = Q$.

Therefore, f_2 is one-to-one and onto.

□

Lemma 3.3. *The first north $\vec{n}$ touching the diagonal $y = x$ in X_2 is above the diagonal $y = x$ in Y_2 after using f_2.*

Proof. Let $P = S\vec{n}R$, where S is a $(j+1, j)$ path, $\vec{n}$ is a $(0,1)$ north path, and R is an $(n-j, n-j)$ path.

After using f_2, we swap R and S, since R is an $(n-j, n-j)$ path, the next segment $\vec{n}$ is above the diagonal $y = x$.

Thus, we have proved that the first north $\vec{n}$ touching the diagonal $y = x$ in X_2 and it is above the diagonal $y = x$ in Y_2 after using f_2.

□

Lemma 3.4. *The last north $\hat{n}$ leaving from diagonal $y = x$ in Y_2 is the first north $\overrightarrow{n}$ touching the diagonal $y = x$ in X_2.*

i.e. If $\overrightarrow{n}$ is the first north touching the diagonal $y = x$ in X_2, then $\overrightarrow{n}$ is the last north leaving the diagonal $y = x$ in Y_2.

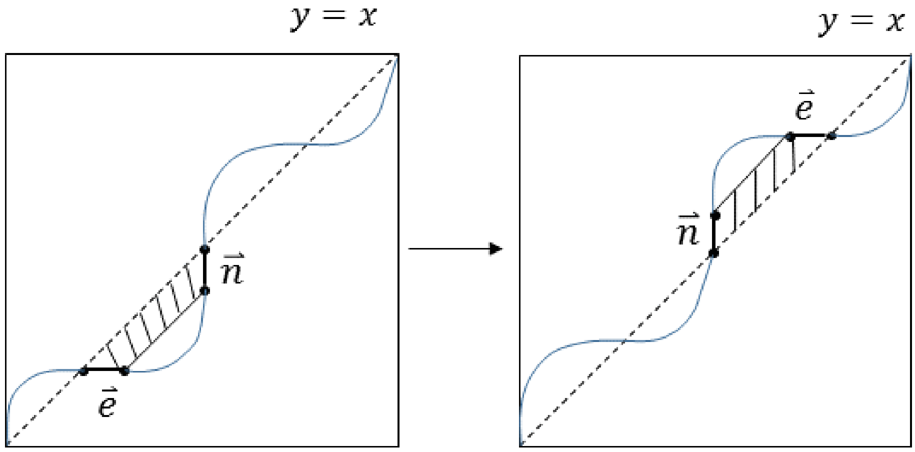


Figure 3.12: *Lemma*3.7 *2nd part*

Proof. Since $\overrightarrow{n}$ is the first north touching the diagonal $y = x$, we can observe that there is a empty area enclosed by the first east that leaves the diagonal $y = x$, denoted by $\overrightarrow{e}$, the diagonal $y = x$, $\overrightarrow{n}$, and the diagonal $y = x - 1$.

After swapping two portions, another empty area is enclosed by $\overrightarrow{n}$, the diagonal $y = x$, $\overrightarrow{e}$, and the diagonal $y = x + 1$, so that there is no north segment can touch the diagonal $y = x$ between $\overrightarrow{n}$ and $\overrightarrow{e}$.

And the remain segments which are behind $\overrightarrow{e}$ are at most touching the diagonal $y = x$ but not be above the diagonal $y = x$. Therefore, the last north $\hat{n}$ leaving from diagonal $y = x$ in Y' is the first north $\overrightarrow{n}$ touching the diagonal $y = x$ in Y_2. $\square$

Definition 3.5. The set A_2 consists of all paths which first segment is east, and the first touching the diagonal $y = x$ north is marked.

The set B_2 consists of all paths which are totally bad path.

Define g_2 from A_2 into B_2 by $g_2(P) = f^{(k)}(P)$, where P has k flaws, and $f^{(k)} = \underbrace{f_2 \circ f_2 \circ ... \circ f_2}_{k}$.

Example 3.6. The following example is one of $g_2(P)$:

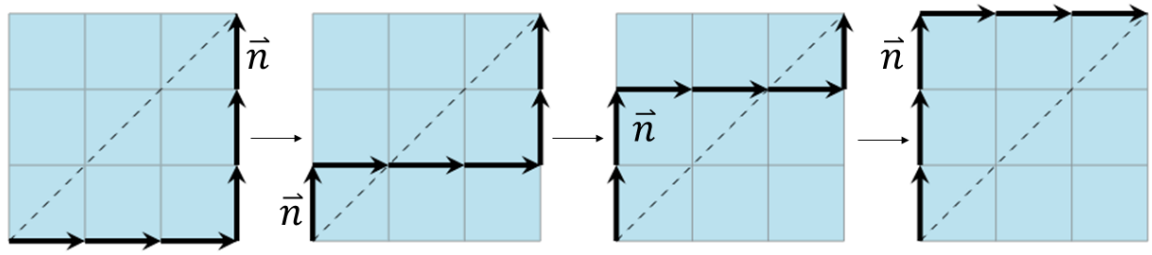


Figure 3.13: $n = 3$

Lemma 3.7. *In $g_2(P)$, after using the first f_2, the first north which is denoted by $\overrightarrow{n}$, connects with the first segment of P in A_2. And this part will not be separated afterward.*

Proof. First, we prove the first part.

Since using f_2 will swap the portion berfore and after $\overrightarrow{n}$, and the first segment of path is east, after using f_2, $\overrightarrow{n}$ connects with the east segment.

Thus, we have proved.

Then we prove the second part.

Notice that after using the first f_2, the part of $\overrightarrow{n}$ and the east segment is above the diagonal $y = x$.

There is another first touching the diagonal $y = x$ north, and the part is in a (j, j) path after that north, in the next step, we use f_2 again, so this part will be swapped to before that north and since it is (j, j) path, the part is still above the diagonal $y = x$.

Therefore, no matter how many times you use f_2, $\overrightarrow{n}$ connects with the first segment of P in A_2 and they are not be separated afterward. □

Theorem 3.8. *g_2 is one-to-one and onto.*

Proof. Claim: g_2 is one-to-one.

$g_2(P) = g_2(Q) \Rightarrow f_2^{(k)}(P) = f_2^{(h)}(Q)$, where P has k flaws and Q has h flaws.

Case1: $k < h$

$f_2^{(k)}(P) = f_2^{(h)}(Q) \Rightarrow f_2(f_2^{(k-1)}(P)) = f_2(f_2^{(h-1)}(Q))$

$\because f_2$ is one-to-one.

$\therefore f_2^{(k-1)}(P) = f_2^{(h-1)}(Q)$.

$f_2(f_2^{(k-2)}(P)) = f_2(f_2^{(h-2)}(Q)) \Rightarrow f_2^{2(k-2)}(P) = f_2^{(h-2)}(Q)$. Since f_2 is one-to-one.

Use this way for $k-1$ times, we have $f_2(P) = f_2^{(h-(k-1))}(Q) = f_2(f_2^{(h-k)}(Q))$

$\Rightarrow P = f_2^{(h-k)}(Q)$

The first $\vec{n}$ of P is below the diagonal $y = x$, but the first $\vec{n}$ of $f_2^{(h-k)}(Q)$ is above the diagonal $y = x$ by Lemma 3.7.

This is a contradiction.

Case2: $k > h$

$f_2^{(k)}(P) = f_2^{(h)}(Q) \Rightarrow f_2(f_2^{(k-1)}(P)) = f_2(f_2^{(h-1)}(Q))$

$\because f_2$ is one-to-one.

$\therefore f_2^{(k-1)}(P) = f_2^{(h-1)}(Q)$.

$f_2(f_2^{(k-2)}(P)) = f_2(f_2^{(h-2)}(Q)) \Rightarrow f_2^{(k-2)}(P) = f_2^{(h-2)}(Q)$. Since f_2 is one-to-one.

Use this way for $h-1$ times, we have $f_2^{(k-(h-1))}(P) = f_2(f_2^{(k-h)}(P)) = f_2(Q)$

$\Rightarrow f_2^{(k-h)}(P) = Q$

The first $\vec{n}$ of Q is below the diagonal $y = x$, but the first $\vec{n}$ of $f_2^{(k-h)}(P)$ is above the diagonal $y = x$ by Lemma 3.7.

This is a contradiction.

Case3: $k = h$

$f_2^{(k)}(P) = f_2^{(h)}(Q) \Rightarrow f_2(f_2^{(k-1)}(P)) = f_2(f_2^{(h-1)}(Q))$

$\Rightarrow f_2^{(k-1)}(P) = f_2^{(h-1)}(Q)$ ($\because f_2$ is one-to-one.)

Use this way for $k-1$ times, we have $f_2(P) = f_2(Q) \Rightarrow P = Q$

Therefore, g_2 is one-to-one.

Claim: g_2 is onto.

Given $Q \in B_2$.

Define $f_2^{(-k)} = \underbrace{f_2^{-1} \circ f_2^{-1} \circ ... \circ f_2^{-1}}_{k}$.

Since $f_2^{-1}(Q)$ is a preimage of Q under f_2 and $f_2^{-1}(Q)$ has $n-1$ flaw.

We can use this way for $n+1$ times, until the segment $\vec{n}$ is below the diagonal $y = x$ by Lemma 3.7.

So we have $f_2^{-(n+1)}(Q) = P$, where P has no flaw, $P \in A_2$.

Thus, g_2 is onto. Therefore, g_2 is one-to-one and onto. $\square$

Definition 3.9. The set C_2 consists of all (n, n) paths which are replaced the marked north and the next east segment in B_2 with a dot, and all paths in B_2 are $(n+1, n+1)$ path.

Let h_2 be the function from B_2 into C_2.

i.e. $P = R\vec{n}\,\vec{e}\,S$ is an $(n+1, n+1)$ path, where R and S are all totally bad paths.

$h_2(P) = h_2(R\vec{n}\,\vec{e}\,S) = R \bullet S$

Theorem 3.10. *h_2 is one-to-one and onto.*

Proof. It is clearly obvious that h_2 is one-to-one and onto.

Given $Q = R \bullet S \in C_2$.

We can change $\bullet$ into $\vec{n}\,\vec{e}$.

Thus, $R \bullet S \Rightarrow R\vec{n}\,\vec{e}\,S \in B_2$.

Therefore, h_2 is one-to-one and onto. $\square$

Chapter 4

Summary

In this thesis, we prove the Catalan identity in a combinatorial way. We split the paths into two portions according to the first segment. Then we construct the functions in $A_1 \xrightarrow{g_1} B_1 \xrightarrow{h_1} C_1$ which the first segment is north.

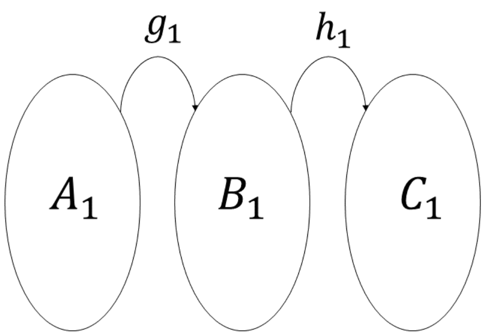


Figure 4.1: $A_1 \xrightarrow{g_1} B_1 \xrightarrow{h_1} C_1$

And the other functions in $A_2 \xrightarrow{g_2} B_2 \xrightarrow{h_2} C_2$ which the first segment is east.

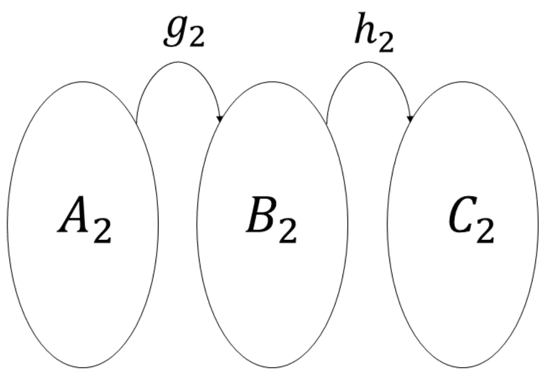


Figure 4.2: $A_2 \xrightarrow{g_2} B_2 \xrightarrow{h_2} C_2$

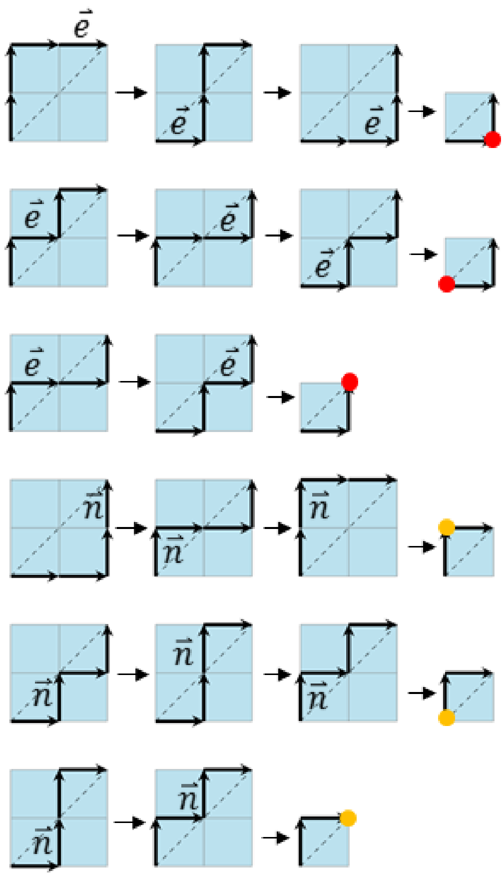
In chapter 3, we can also use reflection along the diagonal $y = x$ to prove the paths with east segment, since the paths in chapter 3 is reflection along the diagonal $y = x$ to the paths in chapter 2. But it will be less clear. In this thesis, we can obverse more details and easier to understand.

Appendix A

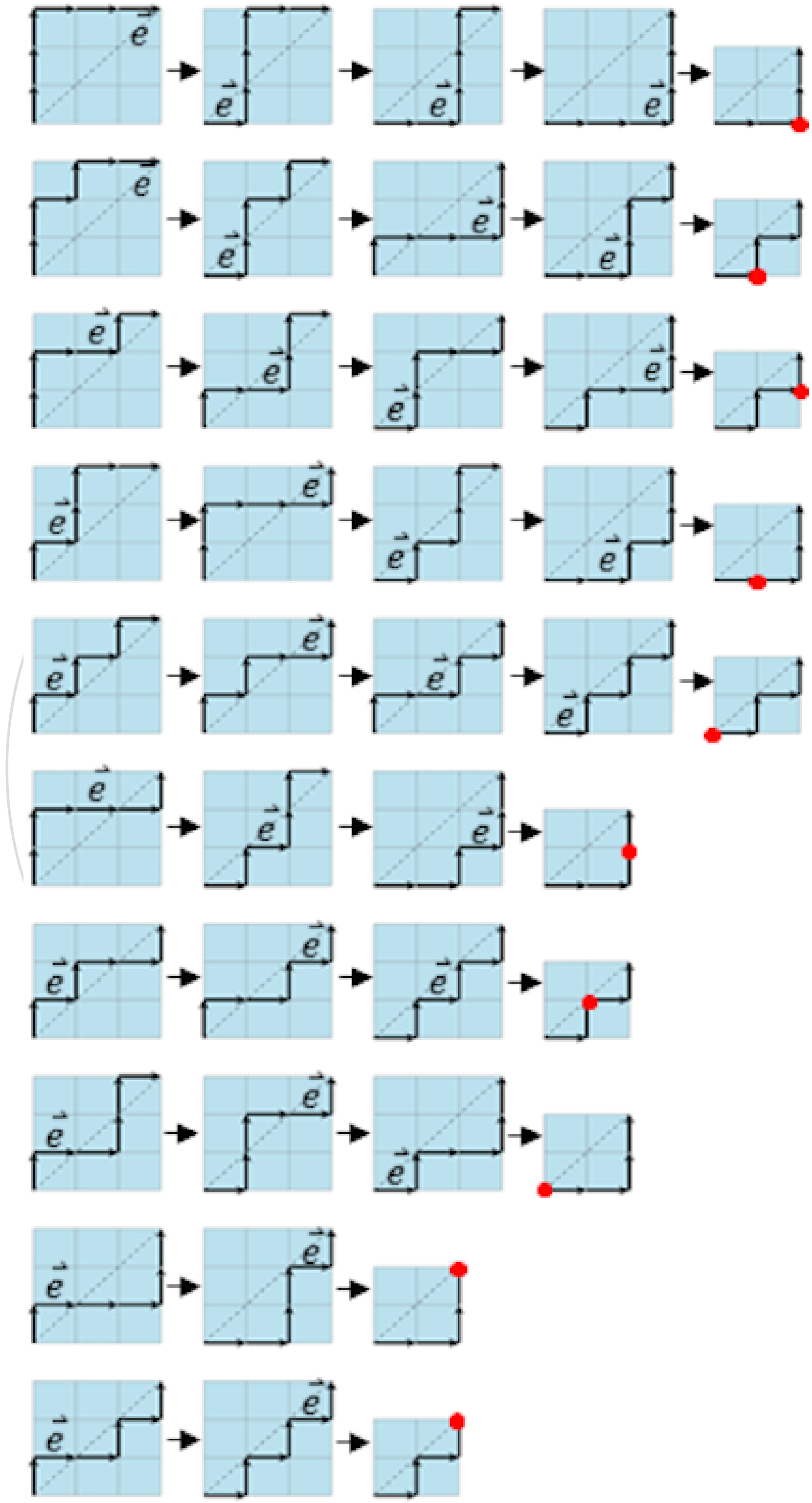
Some examples of Catalan identity

$$(n+2)C_{n+1} = (4n+2)C_n$$

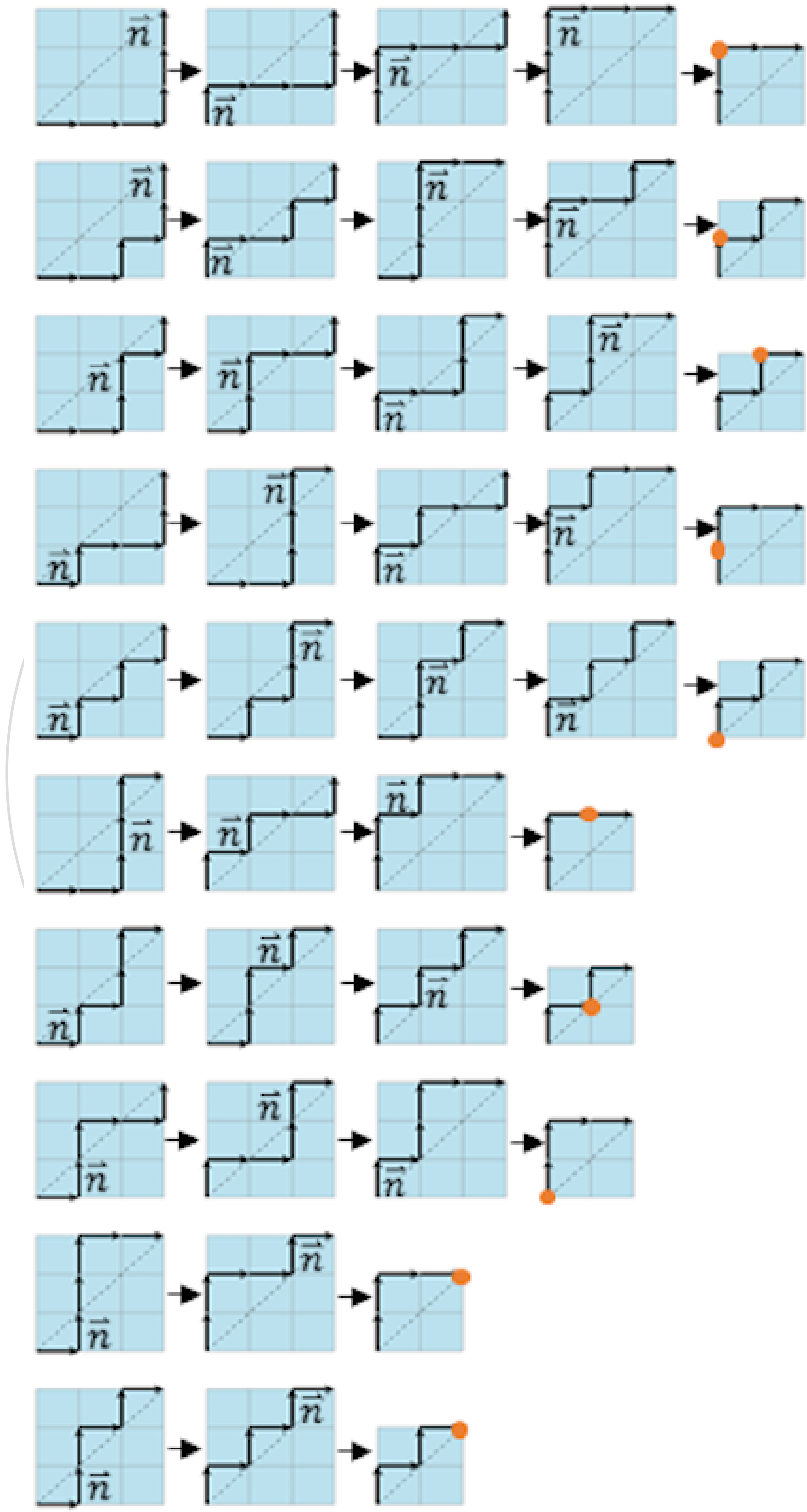
$n = 1$



$n = 2$, the first segment is north.



$n = 2$, the first segment is east.



Bibliography

[1] Ronald Alter. Some remarks and results on Catalan numbers. pages 109–132, 1971.

[2] Ronald Alter and K. K. Kubota. Prime and prime power divisibility of Catalan numbers. *J. Combinatorial Theory Ser. A*, 15:243–256, 1973.

[3] Federico Ardila. Catalan numbers. *Math. Intelligencer*, 38(2):4–5, 2016.

[4] Young-Ming Chen. The Chung-Feller theorem revisited. *Discrete Math.*, 308(7):1328–1329, 2008.

[5] Ömer Eğecioğlu. A Catalan-Hankel determinant evaluation. In *Proceedings of the Fortieth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 195, pages 49–63, 2009.

[6] R. Johnsonbaugh. *Discrete Mathematics*. Pearson/Prentice Hall, 2009.

[7] Thomas Koshy. *Catalan numbers with applications*. Oxford University Press, Oxford, 2009.

[8] Tamás Lengyel. On divisibility properties of some differences of the central binomial coefficients and Catalan numbers. *Integers*, 13:Paper No. A10, 20, 2013.

[9] Youngja Park and Sangwook Kim. Chung-Feller property of Schröder objects. *Electron. J. Combin.*, 23(2):Paper 2.34, 14, 2016.

[10] Matej Črepinšek and Luka Mernik. An efficient representation for solving Catalan number related problems. *Int. J. Pure Appl. Math.*, 56(4):589–604, 2009.