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MAXIMIZING THE VARIANCE OF THE TIME TO RUIN IN A MULTIPLAYER GAME WITH SELECTION

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Abstract

We consider a game with $K \geq 2$ players, each having an integer-valued fortune. On each round, a pair (i, j) among the players with nonzero fortunes is chosen to play and the winner is decided by flipping a fair coin (independently of the process up to that time). The winner then receives a unit from the loser. All other players' fortunes remain the same. (Once a player's fortune reaches 0, this player is out of the game.) The game continues until only one player wins all. The choices of pairs represent the control present in the problem. While it is known that the expected time to ruin (i.e. expected duration of the game) is independent of the choices of pairs (i, j) (the strategies), our objective is to find a strategy which maximizes the variance of the time to ruin. We show that the maximum variance is uniquely attained by the (optimal) strategy, which always selects a pair of players who have currently the largest fortunes. An explicit formula for the maximum value function is derived. By constructing a simple martingale, we also provide a short proof of a result of Ross (2009) that the expected time to ruin is independent of the strategies. A brief discussion of the (open) problem of minimizing the variance of the time to ruin is given.

Keywords: Gambler's ruin; martingale; dynamic programming; stochastic control

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1. Introduction and results

We consider a game with $K \geq 2$ players, each having an integer-valued fortune. On each round, a pair (i, j) among the players with *nonzero* fortunes is chosen to play and the winner is decided by flipping a fair coin (independently of the process up to that time). The winner then receives a unit from the loser. All other players' fortunes remain the same. (Once a player's fortune reaches 0, this player is out of the game.) The game continues until only one player wins all. The choices of pairs represent the control present in the problem. It is known [10] that the expected time to ruin $\mathbb{E}(T)$ (i.e. expected duration of the game) is independent of the choices of pairs (i, j). It is then meaningful to investigate the relation among possible strategies of picking the pairs in terms of the variance of the time to ruin.

The gambler's ruin model (GRM) has been used in genetic algorithms (GAs) [8], where T is the *time of convergence* of the GA. Our model is an idealization, which applies as well to the estimation of the time to reach fixation in an evolutionary model with multiple genotypes. The players represent the competing genotypes and the time to ruin models the extinction of one

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type or, equivalently, completing an evolutionary step in favor of the type with better fitness. This is irreversible, as in *Muller's ratchet* model [5], [7], motivating the GRM.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space on which a sequence of independent and identically distributed (i.i.d.) Bernoulli random variables with probability of success equal to $\frac{1}{2}$ is defined such that for each $t = 1, 2, \ldots$, the first t Bernoulli random variables are measurable with respect to $\mathcal{F}_t$ and all the later Bernoulli random variables are independent of $\mathcal{F}_t$. (In particular, all the Bernoulli random variables are independent of $\mathcal{F}_0$.)

Thereafter, we shall denote by $\eta = (\eta_1, \ldots, \eta_K)$ a configuration with η_i the fortune of player i and e_i the K-dimensional vector with entries (components) equal to 0 except at i, where the entry is 1. In this way

$$\eta^{ij} = \eta + e_i - e_j \tag{1}$$

is the transformation occurring when we pick the pair (i, j) and player i wins. Thus, when the pair (i, j) is chosen, the configuration η will move to either η^{ij} (in the case when i wins) or η^{ji} (in the case when j wins) with equal probability.

Any sequence of pairs $\{(i(t), j(t))\}_{t\geq 0}$ designating the pair picked at (the end of) time $t \geq 0$ (to play at time $t+1$) will generate a random process denoted by $\eta(t) = (\eta_1(t), \eta_2(t), \ldots, \eta_K(t))$, the vector of fortunes $\eta_r(t)$ of players r, $1 \leq r \leq K$, at time t by updating the configuration $\eta(t)$ to $\eta(t+1) = (\eta(t))^{ij}$ if the pair (i, j) is selected at (the end of) time t, plays at time $t+1$, and i wins by sampling the $(t+1)$th term of the Bernoulli random sequence.

We shall assume that $(i(t), j(t)) \in \mathcal{F}_t$, i.e. the pair to play at time $t+1$ is selected according to the information available up to and including time t of the game. Additionally, we assume that *a zero entry in the vector η cannot be selected*. Such a random sequence is said to be a *strategy* (or *policy*) and will be generally denoted by S. The set of all strategies is denoted $\mathcal{S}$. For any such strategy, the process $\{\eta(t)\}_{t\geq 0}$ is adapted to the filtration. In the case when the strategy $(i(t), j(t))$ depends only on $\eta(t)$ for all $t \geq 0$, then $\{\eta(t)\}_{t\geq 0}$ is a Markov chain.

For any strategy $S \in \mathcal{S}$ and an initial configuration η, we denote by $\mathbb{E}_\eta^S[\cdot]$ the expected value of the process starting with fortune $\eta_0 = \eta$ which follows the strategy S.

Let $N = |\eta| := \sum_{i=1}^K \eta_i$ be the sum of all fortunes in configuration η, and T be the time to ruin of all but one player (i.e. T is the duration of the game). Note that for any strategy, the time T to ruin is a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. We also note that trivially $N = |\eta(t)|$ remains constant.

Proposition 1. *For any strategy S with T denoting the time to ruin, the process*

$$q_1(t) = \sum_{i=1}^K (\eta_i(t))^2 - 2t, \qquad K \geq 2, \tag{2}$$

is an $\mathcal{F}_t$-martingale up to T. For $K = 2$, there is only one possible strategy, to pick both players at all times, denoted by S_0, and the process

$$q_2(t) = (q_1(t))^2 - \tfrac{1}{6}(\eta_1(t) - \eta_2(t))^4 + \tfrac{8}{3}t, \qquad K = 2,$$

is an $\mathcal{F}_t$-martingale up to T.

Proof. It is easily shown that $q_1(t)$ is a martingale up to T. Recalling that $(i(t), j(t)) \in \mathcal{F}_t$, we have almost surely (a.s.) on $\{t < T\}$,

$$\mathbb{E}_\eta^S[(q_1(t+1))^2 \mid \mathcal{F}_t] = (q_1(t))^2 + 4[\eta_{i(t)}(t) - \eta_{j(t)}(t)]^2, \qquad K \geq 2, \tag{3}$$

and

$$\mathbb{E}_\eta^{S_0}[(\eta_1(t+1)-\eta_2(t+1))^4 \mid \mathcal{F}_t] = (\eta_1(t)-\eta_2(t))^4 + 24(\eta_1(t)-\eta_2(t))^2 + 16, \qquad K=2,$$

from which it follows that $q_2(t)$ is a martingale up to T. □

Note that (3) for $K > 2$, while not needed for the proof of Proposition 1, will be called for later. Now for $K \geq 2$, noting that

$$q_1(0) = \sum_{i=1}^{K} (\eta_i)^2 \quad \text{and} \quad q_1(T) = N^2 - 2T = \left(\sum_{i=1}^{K} \eta_i\right)^2 - 2T,$$

the first martingale gives

$$\mathbb{E}_\eta^S[T] = \frac{1}{2}\left[\left(\sum_{i=1}^{K} \eta_i\right)^2 - \sum_{i=1}^{K} (\eta_i)^2\right] = \sum_{1\leq i<j\leq K} \eta_i \eta_j, \tag{4}$$

which is *independent of the strategy* (cf. [10]). For $K = 2$, the sole strategy S_0 is deterministic and the second martingale yields the following formula for the variance of T:

$$\operatorname{var}_\eta^{S_0}(T) = \frac{\eta_1\eta_2}{3}((\eta_1)^2 + (\eta_2)^2 - 2), \qquad K = 2. \tag{5}$$

Hereafter, we write $V^S(\eta) = \operatorname{var}_\eta^S(T)$ for notational simplicity.

Since the expected value of T is finite, the stopping time T is finite a.s. While the expected time to ruin is independent of the strategy, the variance depends on S for $K \geq 3$. In this case, we would like to solve the problems

(i) find S_+ such that

$$V^{S_+}(\eta) = \sup_{S\in\mathcal{S}} V^S(\eta); \tag{6}$$

(ii) find S_- such that

$$V^{S_-}(\eta) = \inf_{S\in\mathcal{S}} V^S(\eta). \tag{7}$$

Remark 1. Since $\mathbb{E}_\eta^S(T)$ does not depend on S, optimizing the variance $V^S(\eta)$ is equivalent to optimizing the second moment $\mathbb{E}_\eta^S(T^2)$, but it is more convenient to work directly with the variance. Also, it is not difficult to show (cf. the proof of [10, Lemma 1]) that there exist $0 < \rho < 1$ and $0 < C < \infty$ such that $\mathbb{P}_\eta^S(T > t) \leq C\rho^t$ for all $t \geq 0$ and all $S \in \mathcal{S}$, implying that $\sup_{S\in\mathcal{S}} V^S(\eta) < \infty$. Moreover, $V^{S_+}(\eta)$ and $V^{S_-}(\eta)$ are invariant with respect to permutations of η.

Adopting the terminology from the literature of dynamic programming and Markov decision processes (see, e.g. [3]), a *stationary* strategy S is determined by a (deterministic) mapping s from the configuration space to the set of pairs $\{(i,j)\colon 1 \leq i < j \leq K\}$ such that the pair $(i(t), j(t))$ is given by $s(\eta(t))$. Then the following recurrence holds:

$$V^S(\eta) = \tfrac{1}{2}(V^S(\eta^{ij}) + V^S(\eta^{ji})) + (\eta_i - \eta_j)^2, \qquad (i,j) = s(\eta), \tag{8}$$

which follows from the strong Markov property and the well-known conditional variance formula

$$\operatorname{var}(X) = \mathbb{E}[\operatorname{var}(X \mid \mathcal{G})] + \operatorname{var}(\mathbb{E}[X \mid \mathcal{G}])$$

for any σ-field $\mathcal{G}$ and random variable X with $\mathbb{E}(X^2) < \infty$.

In particular, when $K = 2$ the variance (5) satisfies trivially (8).

While any stationary strategy satisfies (8), only the optimal strategy satisfies (9) below.

Proposition 2. *The dynamic programming equation for the maximization problem (6) is*

$$V(\eta) = \max_{(i,j)} \{ \tfrac{1}{2}(V(\eta^{ij}) + V(\eta^{ji})) + (\eta_i - \eta_j)^2 \}, \qquad V(\eta^f) = 0, \tag{9}$$

where the maximum is taken over all pairs (i, j) with $\eta_i \eta_j > 0$, and η^f is any final (terminal) configuration, i.e. with all but one entry equal to 0. The dynamic programming equation for the minimization problem (7) is (9) with $\max_{(i,j)}$ replaced by $\min_{(i,j)}$.

Proposition 3. *Assume that a real-valued function $V(\eta)$ defined on the (finite) configuration space satisfies $V(\eta^f) = 0$ for any final configuration η^f and*

$$V(\eta) \geq \tfrac{1}{2}(V(\eta^{ij}) + V(\eta^{ji})) + (\eta_i - \eta_j)^2 \quad \textit{for any } (i, j) \textit{ with } \eta_i \eta_j > 0. \tag{10}$$

Then $V(\eta) \geq V^S(\eta)$ for any $S \in \mathcal{S}$. If there exists $S' \in \mathcal{S}$ such that $V(\eta) = V^{S'}(\eta)$ then $S_+ = S'$ and V is the solution to the maximization problem (6). The same holds for the minimization problem (7) by replacing $\geq$ with $\leq$ in all inequalities.

Proof. For any real-valued function $f(\eta)$, by conditional probability, we have

$$\mathbb{E}_\eta^S[f(\eta(t+1)) \mid \mathcal{F}_t] = \tfrac{1}{2}(f((\eta(t))^{i(t)j(t)}) + f((\eta(t))^{j(t)i(t)})), \qquad 0 \leq t \leq T - 1.$$

Applying this relation to $f = V$ and using (3), the process

$$M(t) = V(\eta(t)) + \tfrac{1}{4}(q_1(t))^2$$

is a super-martingale up to T and comparison between the expected values at $t = 0$ and $t = T$ shows the claim of the proposition. □

Theorem 1. *Let the stationary strategy S_+ be defined by $s_+(\eta) = (i, j)$, where η_i and η_j are the largest two values in η. (In case of ties, any pair corresponding to the largest two values may be selected.) Then $(S_+, V^{S_+}(\eta))$ solves the maximization problem (6). Furthermore, the maximum variance of the time to ruin cannot be attained by any strategy that ever selects a pair which does not correspond to the largest two values in the current configuration.*

Proof. By Proposition 3, the proposed strategy S_+ solves (6) if $V^{S_+}(\eta)$ satisfies (10). Note that by (8) and the definition of S_+, both sides of (10) with $V = V^{S_+}$ are equal if (i, j) with $\eta_i \eta_j > 0$ is such that η_i and η_j are the largest two values in η. Thus, it suffices to consider those pairs (i, j) for which $\eta_i \eta_j > 0$ and $\{\eta_i, \eta_j\}$ is not the set of the largest two values in η.

Indeed, we will show that $V^{S_+}(\eta)$ satisfies the following (stronger) strict inequality:

$$V^{S_+}(\eta) > \tfrac{1}{2}(V^{S_+}(\eta^{ij}) + V^{S_+}(\eta^{ji})) + (\eta_i - \eta_j)^2 \tag{11}$$

for any pair (i, j) with $\eta_i \eta_j > 0$ and $\{\eta_i, \eta_j\} \neq \{\eta_{M1}, \eta_{M2}\}$, where η_{M1} and η_{M2} denote, respectively, the largest and second largest values in η. Here, $\{\eta_i, \eta_j\}$ and $\{\eta_{M1}, \eta_{M2}\}$ are interpreted as multisets counting multiplicities of elements (cf. [9, p. 483]). (Note that η_{M1} and η_{M2} are equal if two or more entries tie for the maximum value in η.) The proof is achieved by induction on K, and for fixed K, by induction on N as stated in Proposition 7 of Section 5 where the induction step is proven. The most difficult case of the induction step is proven separately in Section 6. This case requires a lemma, proved in Section 7. The verification step corresponding to $K = 2$ is performed in Section 4. Thus the proof is complete. □

Theorem 1 shows that S_+ uniquely attains the maximum variance of the duration of the game, i.e. S_+ along with the corresponding value function $V^{S_+}(\eta)$ solves the maximization problem (6). However, for the minimization problem (7), we have yet to find a strategy that attains the minimum variance of the duration of the game. A natural candidate strategy is the 'minimal' strategy $\tilde{S}$ that is stationary, defined by $\tilde{s}(\eta) = (i, j)$, where η_i and η_j are the smallest two values in η. Unfortunately, this strategy does not attain the minimum variance. As an example, consider the case $\eta = (\eta_1, \eta_2, \eta_3) = (1, 2, 2)$. We take the convention that in case of ties, $\tilde{S}$ picks the lower-indexed players. Then under $\tilde{S}$, players 1 and 2 continue to play until one of them has fortune 0 (and is out of the game). The survivor (with fortune 3) and player 3 (with fortune 2) then play for the rest of the game. It is readily seen that under $\tilde{S}$, the duration T can be decomposed as $T = T_1 + T_2$, where T_1 and T_2 are independent and T_1 (T_2, respectively) is the duration of a game with $K = 2$ and configuration $(1, 2)$ ($(3, 2)$, respectively). From (5), it follows that

$$V^{\tilde{S}}(\eta) = \mathrm{var}(T_1) + \mathrm{var}(T_2) = \tfrac{1\cdot 2}{3}(1^2 + 2^2 - 2) + \tfrac{3\cdot 2}{3}(3^2 + 2^2 - 2) = 24.$$

On the other hand, consider the strategy that selects players 2 and 3 to play at $t = 1$ and then selects the loser and player 1 to play at $t = 2$. At the end of $t = 2$, only two players survive with fortunes 2 and 3. Thus, by (5) again, the variance of the duration of the game under this strategy equals $\frac{2\cdot 3}{3}(2^2 + 3^2 - 2) = 22 < 24 = V^{\tilde{S}}(\eta)$. (The referee of this paper suggests another candidate strategy $\tilde{S}'$, which always picks the pair (i, j) with η_i the minimal and η_j the maximal value in η. Since $\tilde{S}$ and $\tilde{S}'$ are identical for $\eta = (1, 2, 2)$, $\tilde{S}'$ does not attain the minimum variance in general.) It is also of interest to consider the constrained case where, once a pair is chosen, it must play until one of the two players is defeated and eliminated from the game (the player whose fortune reaches 0). After that, another pair is chosen and the game continues with the new pair until one is defeated, and so on, until all but one are defeated, at time T. We are interested in finding a 'constrained' strategy which maximizes or minimizes the variance of T. The constrained maximization problem can be readily solved. Indeed, the optimal 'constrained' strategy is to pick a pair of players who have currently the largest fortunes whenever selection of a pair is called for (which may be viewed as the constrained version of S_+). However, as for the minimization problem (7), the constrained minimization problem does not seem to admit a simple solution.

We conclude this section by reviewing some relevant literature. The so-called K-tower problem is concerned with the strategy S_R that, at each time t, a pair is chosen at random among all players remaining in the game and the game stops as soon as one player's fortune drops to 0. For $K = 3$, Engel [6] obtained a simple formula for the expected duration with the help of extensive computer calculations, while Stirzaker [12] used martingale theory to derive the formula. Bruss *et al.* [4] later derived the variance and the probability distribution of the duration for $K = 3$, and also argued convincingly that no simple formula for the expected duration can be expected for $K \geq 4$. Engel [6] and Stirzaker [12] also considered the ruin problem where the game stops when one player wins all, and found the expected duration under S_R for general K (cf. (4)). Later Ross [10] showed among other things that the expected duration is the same for all strategies. We gave a short proof of this result by constructing a simple martingale (cf. (2)).

There are other versions of the multiplayer gamblers' ruin problem. In particular, the so-called multiplayer ante one game consists of K players each with initial (integer-valued) fortune η_i, $i = 1, \ldots, K$. At each time $t = 1, 2, \ldots$, each player with positive fortune puts one unit in a pot, which is then won (with equal probability) by one of them. Players whose fortunes drop

to 0 are eliminated. Let $T^{(i)}$ be the total time player i stays in the game. (Equivalently, $T^{(i)}$ is the first time when player i's fortune either drops to 0 or reaches the maximum $|\eta|$.) Let $T = \max_i T^{(i)}$ be the duration of the game. Let T_j be the total time when exactly j players are in the game. Note that $T = T_K + \cdots + T_2$ and that T_K is equivalent to the first time when at least one player's fortune drops to 0. Martingale theory has been used to derive $\mathbb{E}(T)$, $\mathbb{E}(T^{(i)})$, and $\mathbb{E}(T_j)$ for $K = 3$; see [2], [6], and [11]. No simple formulas are available for $K \geq 4$. See also [1] and [13] for related results.

The rest of this paper is organized as follows. In Section 2 we provide several useful reduction formulas, which provide an effective way to perform explicit calculations in Section 3 and to permit the induction argument in Sections 4–7. In particular, *an explicit formula for the maximal value function* is presented in Section 3; see Theorem 2.

2. Reduction formulas

Let $\eta = (\eta_1, \ldots, \eta_K)$ be a configuration with K components and total fortune $N = |\eta|$. A configuration is said to be *extremal* if all except possibly one component are equal and the unequal one (if it exists) has a greater value. In other words, a configuration is extremal if either all the components are equal or all except the (unique) greatest component are equal. Since the total number N is known, we shall specify only the common value of the smaller components, so ζ_c designates the configuration with $K - 1$ components equal to c and one component equal to $N - (K - 1)c$ (which is greater than or equal to c). Thus, c must satisfy the condition $Kc \leq N$. In particular, when $c = 0$, the extremal configuration ζ_0 has only one nonzero component and is referred to as a *final* configuration.

For notational simplicity, we shall write $\mathbb{P}^+$ for $\mathbb{P}^{S_+}$ and $V^+(\eta)$ for $V^{S_+}(\eta)$ (the variance of T under the strategy S_+ with initial configuration η). Two configurations are said to be *indistinguishable* if they are identical up to an ordering (i.e. one is a permutation of the other). Note that indistinguishability is an equivalence relation, so the equivalence class of η is the set of all configurations that are indistinguishable from η. Clearly, $V^+(\eta) = V^+(\eta')$ if η and η' are indistinguishable, i.e. V^+ is invariant with respect to permutations. A configuration ζ (or more precisely, the equivalence class of ζ) is said to be *accessible from* η if under the strategy S_+ it is reached before or at the time T to ruin with probability 1, i.e. the hitting time τ_ζ of (the equivalence class of) ζ has the property $\mathbb{P}^+_\eta(\tau_\zeta \leq T) = 1$. In what follows, the words 'the equivalence class of' will be omitted unless necessary for clarity purposes.

Among all extremal configurations we single out the one with $c = 1$. We shall deal separately with $V^+(\zeta_1) = V^{S_+}(\zeta_1)$. First, we look at $m(\eta) := \min_i \eta_i$.

Proposition 4. *For any η with $m(\eta) \geq 1$, ζ_1 is accessible from η.*

Proof. The $K = 2$ case is trivial. Below we assume that $K \geq 3$. Under S_+, the components with values equal to $m = m(\eta)$ will not be touched as long as there exist two larger components.

If η is such that $M = M(\eta) := \max_i \eta_i = m$ (all flat), we have two possibilities. In case $M = m = 1$, $\eta = \zeta_1$ in the special case when $N = K$. In case $M = m \geq 2$ we play one turn under the strategy S_+ and the two resulting configurations will be indistinguishable, denoted η', for which we have $M(\eta') > m(\eta') \geq 1$.

Thus we can assume without loss of generality that η has $M > m \geq 1$. If there exists exactly one component greater than m then $\eta = \zeta_m$. If there are two or more components greater than m, we may view m as a baseline and the set of those components greater than the baseline continues to evolve under S_+ until all components (except one) are equal to m. The strategy S_+ will simply not look at components equal to m until the process reaches the extremal ζ_m, which

shows that ζ_m is accessible. Note that this process up to the hitting time τ_{ζ_m} of ζ_m is exactly the same as the ruin problem with the initial configuration $\eta - \overline{m} = (\eta_1 - m, \dots, \eta_K - m)$, where $\overline{m}$ is the K-dimensional configuration with all entries equal to m (i.e. τ_{ζ_m} has the same distribution as the time to ruin when the initial configuration is $\eta - \overline{m}$). As such, $\mathbb{P}_\eta^+(\tau_{\zeta_m} < T) = 1$.

On the other hand, under S_+ with $K \geq 3$, we have $m(\eta(t)) - 1 \leq m(\eta(t+1)) \leq m(\eta(t))$ for $0 \leq t < T$, i.e. $m(\eta(t))$ is nonincreasing and can move down by one unit only. Since ζ_0 (the final configuration when the game stops) has $m(\zeta_0) = 0$, it follows that a configuration with $m = 1$ will be reached a.s., and based on the preceding reasoning on τ_{ζ_m}, the configuration ζ_1 will be reached before T with probability 1 as well. □

Proposition 5. *For any η with $m(\eta) \geq 1$,*

$$V^+(\eta) = V^+(\eta - \bar{1}) + V^+(\zeta_1). \tag{12}$$

Proof. By Proposition 4, for a given initial configuration η with $m(\eta) \geq 1$, we have $0 \leq \tau_{\zeta_1} < T < \infty$ a.s. under S_+. Write $T = \tau_{\zeta_1} + (T - \tau_{\zeta_1})$. It follows from the strong Markov property that τ_{ζ_1} and $T - \tau_{\zeta_1}$ are independent. Moreover, τ_{ζ_1} has the same distribution as the time to ruin when the initial configuration is $\eta - \bar{1}$, while $T - \tau_{\zeta_1}$ has the same distribution as the time to ruin when the initial configuration is ζ_1. So,

$$V^+(\eta) = \mathrm{var}_\eta^{S_+}(T) = \mathrm{var}_\eta^{S_+}(\tau_{\zeta_1}) + \mathrm{var}_\eta^{S_+}(T - \tau_{\zeta_1}) = V^+(\eta - \bar{1}) + V^+(\zeta_1),$$

proving (12). □

Let η be a configuration given in ordered form and let $c \geq 0$ with the property

$$\eta_1 \leq \cdots \leq \eta_i \leq c < \eta_{i+1} \leq \cdots \leq \eta_K, \tag{13}$$

where the strict inequality is to be interpreted that *there exists at least one entry strictly larger than c*. We define the configuration *flattened up to level c*, denoted $\eta^{|c}$, by

$$\eta_r^{|c} = \eta_r, \quad 1 \leq r \leq i, \qquad \eta_r^{|c} = c, \quad i < r \leq K-1, \qquad \eta_K^{|c} = \sum_{r=i+1}^{K} \eta_r - (K-i-1)c,$$

which results from following S_+ until all the last $K - i$ entries (except one) are reduced to the level c. (We remark that the exceptional entry in the resulting configuration has a value equal to $\sum_{r=i+1}^{K} \eta_r - (K-i-1)c > c$, which is not necessarily the Kth entry. Thus, we should interpret $\eta^{|c}$ as a configuration up to an ordering.) Let $(\eta - \bar{c})_+ := ((\eta_1 - c)_+, \dots, (\eta_K - c)_+)$, where $(x)_+ := \max\{x, 0\}$. If the configuration η was restricted to entries η_r $(r > i)$ and shifted by c to $\eta_r - c$, i.e. $(\eta - \bar{c})_+$, then the configuration $\eta^{|c}$ (restricted to the last $K - r$ entries and shifted by c) coincides, up to an ordering, with the final configuration of the restricted process, while all other entries η_r, $1 \leq r \leq i$, are left unchanged. This shows that $\eta^{|c}$ is accessible from η.

Proposition 6. *Let η be a configuration and c as in (13) such that there is at least one entry greater than c. Then $\eta^{|c}$ is accessible from η and*

$$V^+(\eta) = V^+((\eta - \bar{c})_+) + V^+(\eta^{|c}). \tag{14}$$

In particular, for $c = m = m(\eta)$ and η not constant, we have $\eta^{|c} = \zeta_m$ and

$$V^+(\eta) = V^+((\eta - \overline{m})_+) + V^+(\zeta_m). \tag{15}$$

Proof. If there is exactly one entry strictly larger than c then $\eta = \eta^{|c}$ and (14) holds since $V^+((\eta - \bar{c})_+) = 0$. (Note that $(\eta - \bar{c})_+$ is a final configuration.)

Suppose there are at least two entries greater than c. The reasoning is almost identical to the proofs of Propositions 4 and 5. Since $(\eta - \bar{c})_+$ has at least two nonzero components, the process evolving under the strategy S_+ will not touch any entry $\eta_r \leq c$ until the entries above c are flattened out, i.e. until the configuration $\eta^{|c}$ is reached, which we know happens with probability 1. So $\eta^{|c}$ is accessible from η. Now write $T = \tau + (T - \tau)$, where $\tau = \tau_{\eta^{|c}}$ is the hitting time of $\eta^{|c}$. By the strong Markov property, τ and $T - \tau$ are independent. Moreover, τ has the same distribution as the time to ruin with initial configuration $(\eta - \bar{c})_+$, while $T - \tau$ has the same distribution as the time to ruin with initial configuration $\eta^{|c}$, from which (14) follows. This completes the proof. □

2.1. A reduction formula that gives insight but we do not use in the proof

In the next lemma we show that we can prove (11) for any configuration and pair having at least two entries dominating the members of the pair by proving it for a simplified configuration $\eta^{|c}$. The reader should think of the case $c > \max\{\eta_i, \eta_j\}$ and should understand the condition that there must exist two entries exceeding strictly $\max\{\eta_i, \eta_j\} + 1$, to prevent interference when we commute the operation of 'moving' between two entries and flattening at level c described formally in (16) and (17) below.

Recall (1) that the transformation of η consisting of a move from entry j to entry i is denoted $\eta^{ij} = \eta + e_i - e_j$.

Lemma 1. *Let η, $c \geq 0$, and (i, j) be such that $\max\{\eta_i, \eta_j\} < c$ and there exist at least two entries of η greater than c. Then*

$$(\eta - \bar{c})_+ = (\eta^{ij} - \bar{c})_+ = (\eta^{ji} - \bar{c})_+,$$

$$V^+(\eta^{ij}) = V^+((\eta^{ij} - \bar{c})_+) + V^+((\eta^{ij})^{|c}) = V^+((\eta - \bar{c})_+) + V^+((\eta^{|c})^{ij}), \tag{16}$$

$$V^+(\eta^{ji}) = V^+((\eta^{ji} - \bar{c})_+) + V^+((\eta^{ji})^{|c}) = V^+((\eta - \bar{c})_+) + V^+((\eta^{|c})^{ji}). \tag{17}$$

Proof. The operation $\eta \to \eta^{|c}$ involves only entries exceeding c. The lemma follows from Proposition 6 and the fact that η, η^{ij}, and η^{ji} differ only in the ith and jth entries, which are all less than or equal to c since $\max\{\eta_i, \eta_j\} < c$. □

3. Explicit formula for the maximal value function

For given K (the number of entries) and N (the total sum of entries) for $0 \leq c \leq N/K$, recall that ζ_c is the extremal configuration being all flat at c except possibly one maximal value. We may write $\zeta_c = (N - Kc + c, c, c, \ldots, c)$ up to an ordering. The values of V^+ at these extremal configurations will allow us to calculate $V^+(\eta)$ for general η. In some sense, we need to develop a rudimentary calculus for these structures as presented below.

To make the dependence on K and N explicit, we write

$$\zeta_c = \zeta_{c,K,N} = (N - Kc + c, c, c, \ldots, c)$$

(with K entries summing up to N), and introduce the convenient notation

$$W_K(N, c) := V^+(\zeta_{c,K,N}) = V^+(N - Kc + c, c, c, \ldots, c), \qquad Kc \leq N. \tag{18}$$

We start writing a formula for $W_K(N, c)$. Based on (12), we have

$$\begin{aligned} W_K(N, c) &= V^+(N - Kc + c, c, \dots, c) \\ &= V^+(N - Kc + c - 1, c - 1, \dots, c - 1) + V^+(N - K + 1, 1, \dots, 1), \\ &= W_K(N - K, c - 1) + W_K(N, 1). \end{aligned}$$

Repeating the same argument,

$$\begin{aligned} W_K(N - K, c - 1) &= W_K(N - 2K, c - 2) + W_K(N - K, 1), \\ &\vdots \\ W_K(N - (c - 2)K, 2) &= W_K(N - (c - 1)K, 1) + W_K(N - (c - 2)K, 1). \end{aligned}$$

Summing up yields

$$W_K(N, c) = \sum_{r=0}^{c-1} W_K(N - rK, 1). \tag{19}$$

In these formulas the parameter K does not change. The simpler function $W_K(d, 1)$ with $d \geq K$ can be obtained by applying the recurrence formula (8) for V^S to $V^+ = V^{S_+}$ as follows. For $d \geq K$, we have, by (8),

$$\begin{aligned} W_K(d, 1) &= V^+(d - K + 1, 1, \dots, 1) \\ &= \tfrac{1}{2} V^+(d - K + 2, 0, 1, \dots, 1) + \tfrac{1}{2} V^+(d - K, 2, 1, \dots, 1) + (d - K)^2 \\ &= \tfrac{1}{2} W_{K-1}(d, 1) + \tfrac{1}{2} V^+(d - K, 2, 1, \dots, 1) + (d - K)^2. \end{aligned} \tag{20}$$

By (12), we have for $d > K$,

$$\begin{aligned} V^+(d - K, 2, 1, \dots, 1) &= V^+(d - K - 1, 1, 0, \dots, 0) + V^+(d - K + 1, 1, \dots, 1) \\ &= V^+(d - K - 1, 1, 0, \dots, 0) + W_K(d, 1), \end{aligned}$$

which together with (20) implies that

$$W_K(d, 1) = W_{K-1}(d, 1) + V^+(d - K - 1, 1, 0, \dots, 0) + 2(d - K)^2. \tag{21}$$

By (5) for the two-player case for which there is only one strategy denoted S_0, we have

$$\begin{aligned} V^+(d - K - 1, 1, 0, \dots, 0) &= V^{S_0}(d - K - 1, 1) \\ &= \tfrac{1}{3}(d - K - 1)((d - K - 1)^2 + 1 - 2) \\ &= \tfrac{1}{3}(d - K)(d - K - 1)(d - K - 2). \end{aligned}$$

From (21), it follows that for $d > K$,

$$W_K(d, 1) = W_{K-1}(d, 1) + Q(d - K), \tag{22}$$

where $Q(x) = \frac{1}{3}x(x - 1)(x - 2) + 2x^2 = \frac{1}{3}x(x + 1)(x + 2)$.

Note that for $d = K$, we have, by (20),

$$\begin{aligned} W_K(K, 1) &= \tfrac{1}{2} W_{K-1}(K, 1) + \tfrac{1}{2} W_{K-1}(K, 1) + (K - K)^2 \\ &= W_{K-1}(K, 1) \\ &= W_{K-1}(K, 1) + Q(K - K), \end{aligned}$$

so (22) also holds for $d = K$. Applying (22) repeatedly yields

$$W_K(d, 1) = W_2(d, 1) + \sum_{r=3}^{K} Q(d - r) = \sum_{r=2}^{K} Q(d - r), \tag{23}$$

where we have used the fact that $W_2(d, 1) = V^{S_0}(d - 1, 1) = Q(d - 2)$.

Remark 2. For convenience, we define

$$W_1(d, 1) := 0 \quad \text{for all } d, \qquad V^+(\eta) := 0 \quad \text{for all } \eta \text{ of dimension } 1, \tag{24}$$

which is consistent with (18).

We are now ready to derive a formula for $V^+(\eta)$ for general η. Let $0 < \eta'_1 < \eta'_2 < \cdots < \eta'_p$ be the distinct values present in η, in increasing order. Let $1 \le p = p(\eta) \le K$ be the total number of such values and let ℓ_k, $1 \le k \le p$ be the multiplicities of the values η'_k. Note that

$$\sum_{k=1}^{p} \ell_k = K, \qquad |\eta| := \sum_{k=1}^{p} \ell_k \eta'_k = N. \tag{25}$$

Here we have assumed that η has no zero entries, i.e. $m(\eta) \ge 1$. In case $m(\eta) = 0$, we simply reduce η to a lower-dimensional configuration by deleting all zero entries. By (15) with $m = \eta'_1$ and (18), we have

$$V^+(\eta) = V^+(\eta - \overline{\eta'_1}) + V^+(\zeta_{\eta'_1, K, N}) = V^+(\eta - \overline{\eta'_1}) + W_K(N, \eta'_1).$$

Since with all zero entries removed, $\eta - \overline{\eta'_1}$ reduces to a lower-dimensional configuration with $K - \ell_1$ entries summing up to $N - K\eta'_1$ and the minimal entry value being $\eta'_2 - \eta'_1$, we have, by (15) with $m = \eta'_2 - \eta'_1$,

$$\begin{aligned} V^+(\eta - \overline{\eta'_1}) &= V^+((\eta - \overline{\eta'_2})_+) + V^+(\zeta_{\eta'_2 - \eta'_1, K - \ell_1, N - K\eta'_1}) \\ &= V^+((\eta - \overline{\eta'_2})_+) + W_{K - \ell_1}(N - K\eta'_1, \eta'_2 - \eta'_1). \end{aligned}$$

Repeating this argument, we have for $r = 0, \ldots, p - 1$,

$$\begin{aligned} V^+((\eta - \overline{\eta'_r})_+) &= V^+((\eta - \overline{\eta'_{r+1}})_+) + V^+(\zeta_{\eta'_{r+1} - \eta'_r, K_r, N_r}) \\ &= V^+((\eta - \overline{\eta'_{r+1}})_+) + W_{K_r}(N_r, \eta'_{r+1} - \eta'_r), \end{aligned} \tag{26}$$

where $\eta'_0 := 0$, $V^+((\eta - \overline{\eta'_p})_+) := 0$, and

$$\begin{aligned} K_r &:= K - \sum_{i=1}^{r} \ell_i, \\ N_r &:= N - \sum_{i=1}^{p} \ell_i \min\{\eta'_i, \eta'_r\} = \sum_{i=r+1}^{p} \ell_i(\eta'_i - \eta'_r), \qquad r = 0, \ldots, p - 1. \end{aligned} \tag{27}$$

Note that $K_0 = K$ and $N_0 = N$. Summing up (26) over $r = 0, \ldots, p - 1$ yields the following formula for $V^+(\eta)$.

Theorem 2. *The maximal value function is given by (5) if $K = 2$. For $K \geq 3$ and for η with p distinct nonzero values $0 < \eta'_1 < \cdots < \eta'_p$ and multiplicities ℓ_r, $r = 1, \ldots, p$ satisfying (25), we have*

$$V^+(\eta) = \sum_{r=0}^{p-1} W_{K_r}(N_r, \eta'_{r+1} - \eta'_r),$$

where K_r and N_r are given in (27) and $W_K(N, c)$ is given in (19), which can be reduced to the special case $c = 1$, as shown in (23) for $W_K(d, 1)$.

4. Cases $K = 2$ and $K = 3$

Case $K = 2$. In this case, (11) is trivially satisfied, since there is only one pair $(1, 2)$ and, hence, there is no (i, j) such that $\{\eta_i, \eta_j\} \neq \{\eta_{M1}, \eta_{M2}\} = \{\eta_1, \eta_2\}$. It is useful to look at the next simplest case $K = 3$, which can be calculated explicitly.

Case $K = 3$. Let $\eta = (a, b, c)$ with $\min\{a, b\} \geq c \geq 1$ and $N = a + b + c$. By (15) and (19), we have

$$\begin{aligned} V^+(a, b, c) &= V^+(a - c, b - c, 0) + W_3(N, c) \\ &= V^+(a - c, b - c, 0) + \sum_{r=0}^{c-1} W_3(N - 3r, 1). \end{aligned} \tag{28}$$

By (5) and (23), we have

$$\begin{aligned} V^+(a - c, b - c, 0) &= \tfrac{1}{3}(a - c)(b - c)[(a - c)^2 + (b - c)^2 - 2], \\ W_3(N - 3r, 1) &= Q(N - 3r - 2) + Q(N - 3r - 3) \\ &= \tfrac{1}{3}(N - 3r - 2)(N - 3r - 1)(2N - 6r - 3), \end{aligned}$$

implying, by (28), that for $\min\{a, b\} \geq c \geq 1$ and $N = a + b + c$,

$$V^+(a, b, c) = U(a - c, b - c) + \frac{1}{3}\sum_{r=0}^{c-1}(N - 3r - 2)(N - 3r - 1)(2N - 6r - 3), \tag{29}$$

where

$$U(x, y) := \tfrac{1}{3}xy(x^2 + y^2 - 2). \tag{30}$$

For $a \geq b \geq c > 0$, let

$$\Delta_{23} := V^+(a, b, c) - \tfrac{1}{2}[V^+(a, b + 1, c - 1) + V^+(a, b - 1, c + 1)] - (b - c)^2,$$

which is the difference between the two sides of (11) with $(i, j) = (2, 3)$. Now for $a \geq b \geq c + 2$, letting $\alpha := a - c$ and $\beta := b - c$, we have, by (29),

$$\begin{aligned} \Delta_{23} = U(\alpha, \beta) &- \tfrac{1}{2}(U(\alpha + 1, \beta + 2) + U(\alpha - 1, \beta - 2)) - \beta^2 \\ &+ \tfrac{1}{6}((N - 3c + 1)(N - 3c + 2)(2N - 6c + 3) \\ &\quad - (N - 3c - 2)(N - 3c - 1)(2N - 6c - 3)) \end{aligned}$$

$$
\begin{aligned}
&= U(\alpha,\beta) - \tfrac{1}{2}(U(\alpha+1,\beta+2) + U(\alpha-1,\beta-2)) - \beta^2 \\
&\quad + \tfrac{1}{6}((\alpha+\beta+1)(\alpha+\beta+2)(2\alpha+2\beta+3) \\
&\qquad - (\alpha+\beta-2)(\alpha+\beta-1)(2\alpha+2\beta-3)) \\
&= \alpha^2 + \alpha\beta \\
&> 0.
\end{aligned}
$$

For $a \geq b = c+1$ and $a > b = c$, it can be shown that $\Delta_{23} = (a-c)(a-c+1) > 0$. Let

$$\Delta_{13} := V^+(a,b,c) - \tfrac{1}{2}[V^+(a+1,b,c-1) + V^+(a-1,b,c+1)] - (a-c)^2,$$

which is the difference between the two sides of (11) with $(i,j) = (1,3)$. Similarly, by (29) for $a \geq b > c$, it can be shown that $\Delta_{13} = (a-c)(b-c) + (b-c)^2 > 0$. This proves that (11) holds for $K = 3$. (It should be noted that the induction step in the next section covers $K = 3$.)

5. The induction step

Let $\mathbb{S}(K)$ be the following induction statement: for any $K' \leq K$ and any $N > 0$, the function $V^+(\cdot)$ satisfies

$$V^+(\eta) > \tfrac{1}{2}(V^+(\eta^{ij}) + V^+(\eta^{ji})) + (\eta_i - \eta_j)^2 \tag{31}$$

for any pair (i,j) with $\eta_i\eta_j > 0$ and $\{\eta_i,\eta_j\} \neq \{\eta_{M1},\eta_{M2}\}$, where η_{M1} and η_{M2} denote the largest two values in η.

Note that (31) is (11) where $V^+ = V^{S_+}$. As remarked before, by the definition of the strategy S_+, both sides of (31) are equal if (i,j) is such that $\eta_i\eta_j > 0$ and $\{\eta_i,\eta_j\} = \{\eta_{M1},\eta_{M2}\}$.

For $K = 2$, $\mathbb{S}(K)$ holds trivially. The next result concludes the proof of Theorem 1.

Proposition 7. *For each $K \geq 3$, if $\mathbb{S}(K')$ holds for all $K' < K$ then it holds for K.*

Proof. For $K \geq 3$ fixed, we start an induction on N. Note that for $m = m(\eta) = 0$, η has at least one zero entry, which reduces to a K'-dimensional configuration for some $K' < K$, so that (31) holds by the induction hypothesis. Furthermore, if $m = m(\eta) = M = M(\eta) := \max_r \eta_r$ then all entries in η are equal, so there is no (i,j) such that $\{\eta_i,\eta_j\} \neq \{\eta_{M1},\eta_{M2}\}$, implying that (31) holds trivially. Thus, it suffices to consider the case $M > m \geq 1$ (implying that $N > K$). In light of (12), we shall prepare by observing that for pair (i,j), $\eta^{ij} + \bar{1} = (\eta+\bar{1})^{ij}$ is well defined if $m \geq 1$, and $\eta^{ij} - \bar{1} = (\eta-\bar{1})^{ij}$ is well defined if $m \geq 2$. With this in mind we now proceed by induction on N (with $K \geq 3$ fixed).

Assume that (31) holds for all $N' < N$ $(N > K)$ and we want to prove it for N.

Case $m \geq 2$. We have $m(\eta^{ij}) \geq 1$ and $m(\eta^{ji}) \geq 1$ for any pair (i,j), so, by (12),

$$V^+(\eta) = V^+(\eta - \bar{1}) + V^+(\zeta_1), \tag{32}$$

$$V^+(\eta^{ij}) = V^+(\eta^{ij} - \bar{1}) + V^+(\zeta_1) = V^+((\eta-\bar{1})^{ij}) + V^+(\zeta_1), \tag{33}$$

$$V^+(\eta^{ji}) = V^+(\eta^{ji} - \bar{1}) + V^+(\zeta_1) = V^+((\eta-\bar{1})^{ji}) + V^+(\zeta_1). \tag{34}$$

This gives, for any pair (i,j) with $\{\eta_i,\eta_j\} \neq \{\eta_{M1},\eta_{M2}\}$,

$$
\begin{aligned}
V^+(\eta) - \tfrac{1}{2}(V^+(\eta^{ij}) + V^+(\eta^{ji})) &= V^+(\eta-\bar{1}) - \tfrac{1}{2}(V^+((\eta-\bar{1})^{ij}) + V^+((\eta-\bar{1})^{ji})) \\
&> ((\eta_i - 1) - (\eta_j - 1))^2 \\
&= (\eta_i - \eta_j)^2,
\end{aligned} \tag{35}
$$

where the inequality is due to the induction hypothesis applied to the configuration $\eta - \bar{1}$ for which the total fortune is $N - K < N$ and

$$\{(\eta - \bar{1})_i, (\eta - \bar{1})_j\} = \{\eta_i - 1, \eta_j - 1\} \neq \{\eta_{M1} - 1, \eta_{M2} - 1\} = \{(\eta - \bar{1})_{M1}, (\eta - \bar{1})_{M2}\}.$$

Case $m = 1$. If the pair (i, j) does not contain any minimum (i.e. $\min\{\eta_i, \eta_j\} \geq 2$), the above argument for $m \geq 2$ (i.e. (32)–(35)) works identically. Also recall that it suffices to consider $M > m \geq 1$. Thus, it is the case $M > m = 1$ and $\min\{\eta_i, \eta_j\} = 1$ that will be done separately in Section 6. □

Remark 3. The case $m = 1$ stands apart from all others because either (33) or (34) cannot be applied when the pair (i, j) contains a minimal value, say $\eta_i = m = 1$. It is perfectly correct to drop one unit on all entries based on (12) which is done in (32). Applying the transformation η^{ij} which increases the ith entry to 2 and shifting by $\bar{1}$ is possible as they clearly commute. However, the transformation η^{ji} which lowers the ith entry to 0 would not commute with the one unit shift as $(\eta - \bar{1})^{ji}$ is not properly defined. Moreover, (12) does not apply to η^{ji} since $m(\eta^{ji}) = 0$. Thus, (34) does not hold. As a result, the induction step (35) breaks down.

6. The case $m = 1 < M$ and $\min\{\eta_i, \eta_j\} = 1$

In this section we consider the case $m = 1 < M$ and $\min\{\eta_i, \eta_j\} = 1$. We shall assume without loss of generality that $i = 1$ and $\eta_1 = 1$. Thus $j \geq 2$ and $\eta_j \geq 1$. With ℓ denoting the multiplicity of 1 in η, we write the configuration $\eta = (\bar{1}_\ell, \xi)$, where the subscript to the vector of 1s marks its dimension. (This notation is sometimes suppressed when no danger of confusion can arise.) Then ξ is a $(K - \ell)$-dimensional vector with the total sum of entries $|\xi| = |\eta| - |\bar{1}_\ell| = N - \ell$ and the minimum value $m(\xi) \geq 2$. (Note that $\ell < K$ since $m(\eta) = 1 < M(\eta)$.)

We write $V_K^+(\eta) = V^+(\eta)$ with the subscript K denoting the dimension of the argument η. This is necessary in order to keep track of the reduction formulas of the type (39) below.

Case $\ell \geq 2$ *and* $\eta_j = 1$. This treats the case when we pick two minima. Without loss of generality, assume that $\eta_1 = \eta_2 = 1$, $j = 2$. (Note that $\{\eta_1, \eta_2\} = \{1, 1\} \neq \{\eta_{M1}, \eta_{M2}\}$ since $M = \eta_{M1} > 1$.)

We need to prove (31), i.e.

$$V_K^+(\eta) - \tfrac{1}{2}(V_K^+(\eta - e_1 + e_2) + V_K^+(\eta + e_1 - e_2)) > 0. \tag{36}$$

The inequality contains no $(\eta_i - \eta_j)^2$ term since the two entries are equal. As $\eta - e_1 + e_2$ and $\eta + e_1 - e_2$ are indistinguishable, (36) reduces to

$$V_K^+(\bar{1}_\ell, \xi) > V_K^+(2, 0, \bar{1}_{\ell-2}, \xi) = V_{K-1}^+(2, \bar{1}_{\ell-2}, \xi), \tag{37}$$

where we have used the projection identity $V_K^+(\xi, \bar{0}_\ell) = V_{K-\ell}^+(\xi)$ which removes the zero entries by lowering the dimension K correspondingly.

We have, by (12) (recalling $W_K(N, c) := V^+(\zeta_{c,K,N})$ in (18)),

$$V_K^+(\eta) = V_K^+(\bar{1}_\ell, \xi) = V_K^+(\bar{0}_\ell, \xi - \bar{1}) + W_K(N, 1) = V_{K-\ell}^+(\xi - \bar{1}) + W_K(N, 1), \tag{38}$$

where for notational simplicity we have suppressed the subscript $K - \ell$ to the vector $\bar{1}$ in $\xi - \bar{1}$.

Similarly, by (12),

$$\begin{aligned}
V_K^+(2, 0, \bar{1}_{\ell-2}, \xi) &= V_{K-1}^+(2, \bar{1}_{\ell-2}, \xi) \\
&= V_{K-1}^+(1, \bar{0}_{\ell-2}, \xi - \bar{1}) + W_{K-1}(N, 1) \\
&= V_{K-\ell+1}^+(1, \xi - \bar{1}) + W_{K-1}(N, 1) \\
&= V_{K-\ell+1}^+(0, \xi - \bar{2}) + W_{K-\ell+1}(N - K + 1, 1) + W_{K-1}(N, 1) \\
&= V_{K-\ell}^+(\xi - \bar{2}) + W_{K-\ell+1}(N - K + 1, 1) + W_{K-1}(N, 1) \\
&= V_{K-\ell}^+(\xi - \bar{1}) - W_{K-\ell}(N - K, 1) + W_{K-\ell+1}(N - K + 1, 1) \\
&\quad + W_{K-1}(N, 1). \qquad (39)
\end{aligned}$$

The last expression has replaced $V_{K-\ell}^+(\xi - \bar{2})$ by $V_{K-\ell}^+(\xi - \bar{1}) - W_{K-\ell}(N-K, 1)$. This follows from (12) applied to $V_{K-\ell}^+(\xi - \bar{1})$, where $|\xi - \bar{1}| = |\xi| - (K-\ell) = (N-\ell) - (K-\ell) = N-K$. Note that if $K - \ell = 1$, we have $V_{K-\ell}^+(\xi - \bar{2}) = V_{K-\ell}^+(\xi - \bar{1}) = W_{K-\ell}(N - K, 1) := 0$ (cf. (24)).

The advantage is that both the last expressions in (38) and (39) contain $V_{K-\ell}^+(\xi - \bar{1})$, and the rest are known computable quantities. Now (37) is equivalent to

$$W_K(N, 1) - W_{K-1}(N, 1) > W_{K-\ell+1}(N - K + 1, 1) - W_{K-\ell}(N - K, 1),$$

which by (23) is equivalent to

$$Q(N - K) > Q(N - K - 1).$$

This holds since $Q(x)$ is an increasing function for $x \geq 0$.

Case $\ell \geq 2$ and $\eta_j > 1$. Note that $(i, j) = (1, j)$, $\eta = (\bar{1}_\ell, \xi)$, $\eta^{ij} = (\bar{1}_\ell, \xi) - e_j + e_1$, and $\eta^{ji} = (\bar{1}_\ell, \xi) + e_j - e_1$. Note also that the entry $\eta_j > 1$ is an entry of ξ. We show (31) using the following terms. First,

$$V_K^+(\bar{1}_\ell, \xi) = V_K^+(\bar{0}_\ell, \xi - \bar{1}) + W_K(N, 1) = V_{K-\ell+1}^+(0, \xi - \bar{1}) + W_K(N, 1),$$

where we have intentionally kept one zero entry resulting in the dimension $K - \ell + 1$. This can be expressed as

$$V_K^+(\bar{1}_\ell, \xi) = V_{K-\ell+1}^+(1, \xi) - W_{K-\ell+1}(N - \ell + 1, 1) + W_K(N, 1), \qquad (40)$$

where we have used the identity $V_{K-\ell+1}^+(1, \xi) = V_{K-\ell+1}^+(0, \xi - \bar{1}) + W_{K-\ell+1}(N - \ell + 1, 1)$ which follows from (12).

Second, with e_j' denoting the $(K - \ell)$-dimensional vector of 0s except for a 1 at the location where η_j appears in ξ,

$$\begin{aligned}
V_K^+((\bar{1}_\ell, \xi) - e_j + e_1) &= V_K^+(2, \bar{1}_{\ell-1}, \xi - e_j') \\
&= V_K^+(1, \bar{0}_{\ell-1}, \xi - e_j' - \bar{1}) + W_K(N, 1) \\
&= V_{K-\ell+1}^+(1, \xi - e_j' - \bar{1}) + W_K(N, 1) \\
&= V_{K-\ell+1}^+(2, \xi - e_j') - W_{K-\ell+1}(N - \ell + 1, 1) + W_K(N, 1) \\
&= V_{K-\ell+1}^+((1, \xi) - e_j'' + e_1) - W_{K-\ell+1}(N - \ell + 1, 1) \\
&\quad + W_K(N, 1), \qquad (41)
\end{aligned}$$

where $e''_j = (0, e'_j)$ (the $(K-\ell+1)$-dimensional vector of 0s except for a 1 at the location where η_j appears in $(1,\xi)$), and the fourth equality follows from

$$V^+_{K-\ell+1}(2, \xi - e'_j) = V^+_{K-\ell+1}(1, \xi - e'_j - \bar{1}) + W_{K-\ell+1}(N-\ell+1, 1).$$

Third,

$$\begin{aligned}
V^+_K((\bar{1}_\ell, \xi) + e_j - e_1) &= V^+_K(0, \bar{1}_{\ell-1}, \xi + e'_j) \\
&= V^+_{K-1}(\bar{1}_{\ell-1}, \xi + e'_j) \\
&= V^+_{K-1}(\bar{0}_{\ell-1}, \xi + e'_j - \bar{1}) + W_{K-1}(N, 1) \\
&= V^+_{K-\ell}(\xi + e'_j - \bar{1}) + W_{K-1}(N, 1) \\
&= V^+_{K-\ell}(\xi + e'_j) - W_{K-\ell}(N-\ell+1, 1) + W_{K-1}(N, 1) \\
&= V^+_{K-\ell+1}(0, \xi + e'_j) - W_{K-\ell}(N-\ell+1, 1) + W_{K-1}(N, 1) \\
&= V^+_{K-\ell+1}((1, \xi) + e''_j - e_1) - W_{K-\ell}(N-\ell+1, 1) \\
&\quad + W_{K-1}(N, 1),
\end{aligned} \tag{42}$$

where $e''_j = (0, e'_j)$. Note that we have brought the same terms in terms of $K \to K - \ell + 1$. Since the pair $(i, j) = (1, j)$ satisfies $\{\eta_1, \eta_j\} \neq \{\eta_{M1}, \eta_{M2}\}$, it follows that the two entries 1 and η_j in $(1, \xi)$ are not the pair consisting of the largest two values in $(1, \xi)$. By the induction hypothesis applied to the configuration $(1, \xi)$ of dimension $K - \ell + 1 < K$, we have

$$V^+_{K-\ell+1}(1, \xi) - \tfrac{1}{2}(V^+_{K-\ell+1}((1, \xi) - e''_j + e_1) + V^+_{K-\ell+1}((1, \xi) + e''_j - e_1)) - (1 - \eta_j)^2 > 0.$$

It remains to show that the extra terms that appear in (40)–(42) add up to a nonnegative term, which means that

$$\begin{aligned}
&[-W_{K-\ell+1}(N-\ell+1, 1) + W_K(N, 1)] - \tfrac{1}{2}([-W_{K-\ell+1}(N-\ell+1, 1) + W_K(N, 1)] \\
&\qquad\qquad + [-W_{K-\ell}(N-\ell+1, 1) + W_{K-1}(N, 1)]) \\
&\quad \geq 0
\end{aligned}$$

or, equivalently,

$$W_K(N, 1) - W_{K-1}(N, 1) \geq W_{K-\ell+1}(N-\ell+1, 1) - W_{K-\ell}(N-\ell+1, 1). \tag{43}$$

By (22), the left- and right-hand sides of (43) both equal $Q(N-K)$. Thus, (43) holds as an equality.

Case $\ell = 1$ *and* $\eta_j \geq 3$. This treats a subcase of $\eta_r \geq 2$ for all $r > 1$. The subcase $\eta_j = 2$ is treated in the next subsection. The reason we adopt $\eta_j \geq 3$ is (45) where we reduce the configuration by two units. Writing $\eta = (1, \xi)$, note that all entries of ξ are greater than 1. Since $\eta_j \geq 3$, all entries in $\xi - e'_j - \bar{2}$ are nonnegative, where e'_j denotes the vector of all entries equal to 0 except for a 1 at the location where η_j appears in ξ.

First,

$$V^+_K(1, \xi) = V^+_K(0, \xi - \bar{1}) + W_K(N, 1) = V^+_{K-1}(\xi - \bar{1}) + W_K(N, 1). \tag{44}$$

Second, using $\eta_j \geq 3$,

$$\begin{aligned} V_K^+((1,\xi) - e_j + e_1) &= V_K^+(2, \xi - e_j') \\ &= V_K^+(0, \xi - e_j' - \bar{2}) + W_K(N, 2) \quad \text{(by (15) with } m = 2) \\ &= V_{K-1}^+(\xi - e_j' - \bar{2}) + W_K(N, 2) \\ &= V_{K-1}^+(\xi - \bar{1} - e_j') - W_{K-1}(N - K - 1, 1) + W_K(N, 2), \end{aligned} \tag{45}$$

where we have used the identity

$$V_{K-1}^+(\xi - \bar{1} - e_j') = V_{K-1}^+(\xi - e_j' - \bar{2}) + W_{K-1}(N - K - 1, 1).$$

Third,

$$\begin{aligned} V_K^+((1,\xi) + e_j - e_1) &= V_K^+(0, \xi + e_j') \\ &= V_{K-1}^+(\xi + e_j') \\ &= V_{K-1}^+(\xi - \bar{1} + e_j') + W_{K-1}(N, 1). \end{aligned} \tag{46}$$

We need to establish that

$$V_K^+(1,\xi) - \tfrac{1}{2}(V_K^+((1,\xi) - e_j + e_1) + V_K^+((1,\xi) + e_j - e_1)) - (\eta_j - 1)^2 > 0. \tag{47}$$

By (44)–(46), the left-hand side of (47) equals $A + B$, where

$$A = V_{K-1}^+(\xi - \bar{1}) - \tfrac{1}{2}(V_{K-1}^+(\xi - \bar{1} + e_j') + V_{K-1}^+(\xi - \bar{1} - e_j')) - (\eta_j - 1)^2 \tag{48}$$

and

$$B = W_K(N, 1) - \tfrac{1}{2}(-W_{K-1}(N - K - 1, 1) + W_K(N, 2) + W_{K-1}(N, 1)). \tag{49}$$

By (19), (23), and (49),

$$\begin{aligned} 2B &= 2W_K(N, 1) - (-W_{K-1}(N - K - 1, 1) + W_K(N, 1) + W_K(N - K, 1) \\ &\qquad + W_{K-1}(N, 1)) \\ &= (W_K(N, 1) - W_{K-1}(N, 1)) - (W_K(N - K, 1) - W_{K-1}(N - K - 1, 1)) \\ &= Q(N - K) - Q(N - K - 2) \\ &= 2(N - K)^2. \end{aligned}$$

So we have

$$B = (N - K)^2. \tag{50}$$

By (48) and (50), (47) is equivalent to

$$V_{K-1}^+(\xi - \bar{1}) - \tfrac{1}{2}(V_{K-1}^+(\xi - \bar{1} + e_j') + V_{K-1}^+(\xi - \bar{1} - e_j')) - (\eta_j - 1)^2 + (N - K)^2 > 0,$$

which follows from Lemma 2 in Section 7 (and concludes the proof of the most difficult case). Note that $N - K$, $K - 1$, $\xi - \bar{1}$, $\eta_j - 1$, and e_j' here should be identified, respectively, with N, K, η, η_j, and e_j in (56) of Lemma 2. Note also that $m(\xi) \geq 2$ implies that $m(\xi - \bar{1}) \geq 1$ and $N - K \geq K - 1$ as required by Lemma 2.

We are left with the case $\ell = 1$ and $\eta_j = 2$. Without loss of generality, assume that $j = 2$. Let ℓ' be the multiplicity of $\eta_2 = 2$.

Case $\ell = 1$, $\eta_j = 2$, *and* $\ell' = 1$. Write $\eta = (1, 2, \xi)$ where all entries in ξ are greater than 2. (Note that ξ cannot be vacuous since $K \geq 3$.) We have, by (12),

$$\begin{aligned} V_K^+(1,2,\xi) &= V_K^+(0,1,\xi-\bar{1}) + W_K(N,1) \\ &= V_{K-1}^+(1,\xi-\bar{1}) + W_K(N,1) \\ &= V_{K-1}^+(0,\xi-\bar{2}) + W_{K-1}(N-K,1) + W_K(N,1) \\ &= V_{K-2}^+(\xi-\bar{2}) + W_{K-1}(N-K,1) + W_K(N,1), \end{aligned} \tag{51}$$

and, by (15) (with $m = 3$),

$$\begin{aligned} V_K^+(0,3,\xi) &= V_{K-1}^+(3,\xi) \\ &= V_{K-1}^+(0,\xi-\bar{3}) + W_{K-1}(N,3) \\ &= V_{K-2}^+(\xi-\bar{3}) + W_{K-1}(N,3) \\ &= V_{K-2}^+(\xi-\bar{2}) - W_{K-2}(N-2K+1,1) + W_{K-1}(N,3), \end{aligned} \tag{52}$$

where the last line follows from

$$V_{K-2}^+(\xi-\bar{2}) = V_{K-2}^+(\xi-\bar{3}) + W_{K-2}(N-2K+1,1).$$

Since $V_K^+((1,2,\xi) - e_1 + e_2) = V_K^+(0,3,\xi)$ and $V_K^+((1,2,\xi) + e_1 - e_2) = V_K^+(1,2,\xi)$, (31) is equivalent to $V_K^+(1,2,\xi) > V_K^+(0,3,\xi) + 2$. By (19), we have

$$W_{K-1}(N,3) = W_{K-1}(N,1) + W_{K-1}(N-K+1,1) + W_{K-1}(N-2K+2,1).$$

By (23), (51), and (52),

$$\begin{aligned} &V_K^+(1,2,\xi) - V_K^+(0,3,\xi) \\ &\quad = [W_K(N,1) - W_{K-1}(N,1)] \\ &\qquad - [W_{K-1}(N-K+1,1) - W_{K-1}(N-K,1)] \\ &\qquad - [W_{K-1}(N-2K+2,1) - W_{K-2}(N-2K+1,1)] \\ &\quad = Q(N-K) - [Q(N-K-1) - Q(N-2K+1)] - Q(N-2K) \\ &\quad = [Q(N-K) - Q(N-K-1)] + [Q(N-2K+1) - Q(N-2K)] \\ &\quad \geq 12 \\ &\quad > 2, \end{aligned}$$

since $Q(x) - Q(x-1) = x(x+1) \geq 12$ for $x \geq 3$ and $N - K \geq 3$.

Case $\ell = 1$, $\eta_j = 2$, *and* $\ell' \geq 2$. Let $\eta = (1, 2, \bar{2}_{\ell'-1}, \xi)$, where ξ is possibly vacuous. We single out one 2 to explicitly carry out the transform corresponding to the pair (1, 2). Note that ξ is of dimension $K - \ell' - 1$ with $|\xi| = N - 2\ell' - 1$. We have, by (12),

$$\begin{aligned} V_K^+(1,2,\bar{2}_{\ell'-1},\xi) &= V_K^+(0,1,\bar{1}_{\ell'-1},\xi-\bar{1}) + W_K(N,1) \\ &= V_{K-1}^+(1,\bar{1}_{\ell'-1},\xi-\bar{1}) + W_K(N,1) \\ &= V_{K-1}^+(0,\bar{0}_{\ell'-1},\xi-\bar{2}) + W_{K-1}(N-K,1) + W_K(N,1) \\ &= V_{K-\ell'-1}^+(\xi-\bar{2}) + W_{K-1}(N-K,1) + W_K(N,1) \end{aligned} \tag{53}$$

and

$$\begin{aligned}
V_K^+(0,3,\bar{2}_{\ell'-1},\xi) &= V_{K-1}^+(3,\bar{2}_{\ell'-1},\xi)\\
&= V_{K-1}^+(1,\bar{0}_{\ell'-1},\xi-\bar{2}) + W_{K-1}(N,2)\\
&= V_{K-\ell'}^+(1,\xi-\bar{2}) + W_{K-1}(N,2)\\
&= V_{K-\ell'}^+(0,\xi-\bar{3}) + W_{K-\ell'}(N-2K+2,1) + W_{K-1}(N,2)\\
&= V_{K-\ell'-1}^+(\xi-\bar{3}) + W_{K-\ell'}(N-2K+2,1) + W_{K-1}(N,2)\\
&= V_{K-\ell'-1}^+(\xi-\bar{2}) - W_{K-\ell'-1}(N-2K+1,1)\\
&\quad + W_{K-\ell'}(N-2K+2,1) + W_{K-1}(N,2),
\end{aligned} \tag{54}$$

where the last equality follows from

$$V_{K-\ell'-1}^+(\xi-\bar{2}) = V_{K-\ell'-1}^+(\xi-\bar{3}) + W_{K-\ell'-1}(N-2K+1,1).$$

(Note that if ξ is vacuous, $V_K^+(1,2,\bar{2}_{\ell'-1},\xi)$ and $V_K^+(0,3,\bar{2}_{\ell'-1},\xi)$ reduce to $W_{K-1}(N-K,1)+W_K(N,1)$ and $W_{K-1}(N,2)$, respectively.) As in the preceding case, (31) is equivalent to

$$V_K^+(1,2,\bar{2}_{\ell'-1},\xi) > V_K(0,3,\bar{2}_{\ell'-1},\xi) + 2. \tag{55}$$

Noting by (19) that $W_{K-1}(N,2) = W_{K-1}(N,1) + W_{K-1}(N-K+1,1)$, we have, by (53) and (54),

$$\begin{aligned}
&V_K^+(1,2,\bar{2}_{\ell'-1},\xi) - V_K^+(0,3,\bar{2}_{\ell'-1},\xi)\\
&\quad = [W_K(N,1) - W_{K-1}(N,1)] + [W_{K-1}(N-K,1) - W_{K-1}(N-K+1,1)]\\
&\qquad - [W_{K-\ell'}(N-2K+2,1) - W_{K-\ell'-1}(N-2K+1,1)]\\
&\quad = Q(N-K) + [Q(N-2K+1) - Q(N-K-1)] - Q(N-2K) \quad \text{(by (23))}\\
&\quad \geq Q(N-K) - Q(N-K-1)\\
&\quad = (N-K)(N-K+1)\\
&\quad \geq 6,
\end{aligned}$$

since $N-K \geq 2$, establishing (55). This completes the proof. □

7. Lemma 2

In this section we have the same notation $V_K^+(\eta)$ for the value function under S_+ in dimension K as in Section 6.

Lemma 2. *For $N \geq K \geq 2$, for configuration η with $m(\eta) \geq 1$ and $N = |\eta|$,*

$$V_K^+(\eta) - \tfrac{1}{2}(V_K^+(\eta+e_j) + V_K^+(\eta-e_j)) - \eta_j^2 + N^2 > 0. \tag{56}$$

Proof. We first consider the special case $N = K \geq 2$ and $\eta = \bar{1}_K$. By (23),

$$\begin{aligned}
&2V_K^+(\eta) - V_K^+(\eta+e_j) - V_K^+(\eta-e_j)\\
&\quad = 2W_K(K,1) - W_K(K+1,1) - W_{K-1}(K-1,1)\\
&\quad = [W_K(K,1) - W_{K-1}(K-1,1)] - [W_K(K+1,1) - W_K(K,1)]
\end{aligned}$$

$$
\begin{aligned}
&= Q(K-2) - Q(K-1) \\
&= -K(K-1) \\
&> 2(1-K^2) \\
&= 2(\eta_j^2 - N^2),
\end{aligned}
$$

establishing (56) for this case. Next for the case $K = 2$, note by (5) that $V_K^+(\eta_1, \eta_2) = U(\eta_1, \eta_2)$, where $U(x, y) = (\frac{1}{3})xy(x^2 + y^2 - 2)$ as defined in (30). Then for $N \geq K = 2$, the left-hand side of (56) (with $j = 1$) can be expressed as

$$
\begin{aligned}
&V_K^+(\eta) - \tfrac{1}{2}(V_K^+(\eta + e_1) + V_K^+(\eta - e_1)) - \eta_1^2 + N^2 \\
&\quad = U(\eta_1, \eta_2) - \tfrac{1}{2}(U(\eta_1 + 1, \eta_2) + U(\eta_1 - 1, \eta_2)) - \eta_1^2 + (\eta_1 + \eta_2)^2 \\
&\quad = -\eta_1\eta_2 - \eta_1^2 + (\eta_1 + \eta_2)^2 \\
&\quad = \eta_1\eta_2 + \eta_2^2 > 0,
\end{aligned}
$$

establishing (56). (The case with $j = 2$ is done by symmetry.) Thus, we have shown that (56) holds for all $2 \leq K \leq N$ with either $K = 2$ or $N = K$.

To prove the general case with $N > K \geq 3$ in a similar fashion as in the proof of Proposition 7, we perform induction on K, and for fixed K, induction on N. Specifically, with $3 \leq K < N$ fixed, suppose that (56) holds for all $2 \leq K' \leq N'$ with either $K' < K$ or $K' = K$ and $N' < N$. Then we need to prove that (56) holds for (K, N).

Case $m = m(\eta) \geq 2$ or $m = 1 < \eta_j$. As in (32)–(34), if $m \geq 2$ or $m = 1 < \eta_j$, we may apply the induction step immediately for $\eta - \bar{1}$ for which K' (the number of nonzero entries in $\eta - \bar{1}$) is at most K and $N' := |\eta - \bar{1}| = N - K < N$. Clearly, $N' \geq K' \geq 1$. In the special case $K' = 1$ (arising when $m = 1 < \eta_j$ and $\eta_i = 1$ for all $i \neq j$) for which the induction hypothesis does not apply, we note that η, $\eta + e_j$, and $\eta - e_j$ are, respectively, the extremal configurations $\zeta_{1,K,N}$, $\zeta_{1,K,N+1}$, and $\zeta_{1,K,N-1}$, so

$$
V_K^+(\eta) = W_K(N, 1), \qquad V_K^+(\eta + e_j) = W_K(N + 1, 1), \qquad V_K^+(\eta - e_j) = W_K(N - 1, 1).
$$

The left-hand side of (56) equals $C + D$, where $C := -\eta_j^2 + N^2 = -(N - K + 1)^2 + N^2$ (since $\eta_j = N - K + 1$), and

$$
\begin{aligned}
D :&= W_K(N, 1) - \tfrac{1}{2}(W_K(N + 1, 1) + W_K(N - 1, 1)) \qquad (57) \\
&= \tfrac{1}{2}([W_K(N, 1) - W_K(N - 1, 1)] - [W_K(N + 1, 1) - W_K(N, 1)]) \\
&= \tfrac{1}{2}([Q(N - 2) - Q(N - K - 1)] - [Q(N - 1) - Q(N - K)]) \quad \text{(by (23))} \\
&= \tfrac{1}{2}([Q(N - K) - Q(N - K - 1)] - [Q(N - 1) - Q(N - 2)]) \\
&= \tfrac{1}{2}(N - K)(N - K + 1) - \tfrac{1}{2}N(N - 1). \qquad (58)
\end{aligned}
$$

It follows that $C + D = g(N) - g(N - K + 1) > 0$, where

$$
g(x) := x^2 - \tfrac{1}{2}x(x - 1) \tag{59}
$$

is an increasing function in $x > 0$. This establishes (56) for $K' = 1$.

We now consider $N' \geq K' \geq 2$. Let η' denote the K'-dimensional vector derived from $\eta - \bar{1}$ by deleting all zero entries, and let e'_j be the K'-dimensional vector of 0s except for a 1 at the location where $\eta_j - 1$ appears in η'. Then $m(\eta') \geq 1$ and $|\eta'| = N - K = N' \geq K' \geq 2$. By the induction hypothesis applied to η' (for which either $2 \leq K' < K$ or $K' = K$ and $N' < N$), we have

$$\begin{aligned}\alpha :&= V_K^+(\eta - \bar{1}) - \tfrac{1}{2}(V_K^+(\eta - \bar{1} + e_j) + V_K^+(\eta - \bar{1} - e_j)) \\ &= V_{K'}^+(\eta') - \tfrac{1}{2}(V_{K'}^+(\eta' + e'_j) + V_{K'}^+(\eta' - e'_j)) \\ &> (\eta_j - 1)^2 - (N - K)^2. \end{aligned} \tag{60}$$

Since, by (12),

$$V_K^+(\eta) = V_K^+(\eta - \bar{1}) + W_K(N, 1), \qquad V_K^+(\eta + e_j) = V_K^+(\eta - \bar{1} + e_j) + W_K(N + 1, 1),$$
$$V^+(\eta - e_j) = V_K^+(\eta - \bar{1} - e_j) + W_K(N - 1, 1),$$

the left-hand side of (56) can be expressed as

$$\begin{aligned}&\alpha + W_K(N, 1) - \tfrac{1}{2}(W_K(N + 1, 1) + W_K(N - 1, 1)) - \eta_j^2 + N^2 \\ &\quad > ((\eta_j - 1)^2 - (N - K)^2) + D - \eta_j^2 + N^2 \quad \text{(by (57) and (60))} \\ &\quad = D + N^2 - (N - K)^2 - 2\eta_j + 1 \\ &\quad \geq D + N^2 - (N - K)^2 - 2(N - K + 1) + 1 \quad (\text{since } \eta_j \leq N - K + 1) \\ &\quad = \tfrac{1}{2}(N - K)(N - K + 1) - \tfrac{1}{2}N(N - 1) + N^2 - (N - K + 1)^2 \quad \text{(by (58))} \\ &\quad = g(N) - g(N - K + 1) \\ &\quad > 0 \quad \text{(by (59))}.\end{aligned}$$

This completes the proof for the case $m \geq 2$ or $m = 1 < \eta_j$.

Case $m = 1 = \eta_j$. Let ℓ be the multiplicity of the minimum value 1. Without loss of generality, assume that $j = 1$. Write $\eta = (\bar{1}_\ell, \xi) = (1, \bar{1}_{\ell-1}, \xi)$, where ξ is a vector of dimension $K - \ell$ with $m(\xi) \geq 2$. Note that ξ cannot be vacuous (since $N > K$), so $K - \ell > 0$.

By (12),

$$V_K^+(1, \bar{1}_{\ell-1}, \xi) = V_K^+(0, \bar{0}_{\ell-1}, \xi - \bar{1}) + W_K(N, 1) = V_{K-\ell}^+(\xi - \bar{1}) + W_K(N, 1), \tag{61}$$

$$\begin{aligned}V_K^+(0, \bar{1}_{\ell-1}, \xi) &= V_{K-1}^+(\bar{1}_{\ell-1}, \xi) = V_{K-1}^+(\bar{0}_{\ell-1}, \xi - \bar{1}) + W_{K-1}(N - 1, 1) \\ &= V_{K-\ell}^+(\xi - \bar{1}) + W_{K-1}(N - 1, 1),\end{aligned} \tag{62}$$

$$\begin{aligned}V_K^+(2, \bar{1}_{\ell-1}, \xi) &= V_K^+(1, \bar{0}_{\ell-1}, \xi - \bar{1}) + W_K(N + 1, 1) \\ &= V_{K-\ell+1}^+(1, \xi - \bar{1}) + W_K(N + 1, 1) \\ &= V_{K-\ell+1}^+(0, \xi - \bar{2}) + W_{K-\ell+1}(N - K + 1, 1) + W_K(N + 1, 1) \\ &= V_{K-\ell}^+(\xi - \bar{2}) + W_{K-\ell+1}(N - K + 1, 1) + W_K(N + 1, 1) \\ &= V_{K-\ell}^+(\xi - \bar{1}) - W_{K-\ell}(N - K, 1) + W_{K-\ell+1}(N - K + 1, 1) \\ &\quad + W_K(N + 1, 1),\end{aligned} \tag{63}$$

where the last equality follows from the identity

$$V_{K-\ell}^+(\xi - \bar{1}) = V_{K-\ell}^+(\xi - \bar{2}) + W_{K-\ell}(N - K, 1).$$

(Note that, by (24), if $K-\ell = 1$, we have $V^+_{K-\ell}(\xi-\bar{2}) = V^+_{K-\ell}(\xi-\bar{1}) = W_{K-\ell}(N-K, 1) = 0$.) By (61)–(63),

$$\begin{aligned}
\beta :&= V^+_K(1, \bar{1}_{\ell-1}, \xi) - \tfrac{1}{2}(V^+_K(0, \bar{1}_{\ell-1}, \xi) + V^+_K(2, \bar{1}_{\ell-1}, \xi)) \\
&= W_K(N, 1) - \tfrac{1}{2}(W_{K-1}(N-1, 1) - W_{K-\ell}(N-K, 1) \\
&\qquad\qquad + W_{K-\ell+1}(N-K+1, 1) + W_K(N+1, 1)) \\
&= D + \tfrac{1}{2}(W_K(N-1, 1) - W_{K-1}(N-1, 1)) \\
&\quad - \tfrac{1}{2}(W_{K-\ell+1}(N-K+1, 1) - W_{K-\ell}(N-K, 1)) \quad \text{(by (57))} \\
&= D + \tfrac{1}{2}Q(N-K-1) - \tfrac{1}{2}Q(N-K-1) \\
&= D.
\end{aligned}$$

Thus, the left-hand side of (56) is $\beta - \eta_1^2 + N^2$, which can be expressed as

$$\begin{aligned}
D - 1 + N^2 &\geq D - (N-K+1)^2 + N^2 \\
&= \tfrac{1}{2}(N-K)(N-K+1) - \tfrac{1}{2}N(N-1) - (N-K+1)^2 + N^2 \quad \text{(by (58))} \\
&= g(N) - g(N-K+1) \\
&> 0 \quad \text{(by (59))}.
\end{aligned}$$

This concludes the proof of the lemma. □

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References

[1] Amano, K., Tromp, J., Vitányi, P. M. B. and Watanabe, O. (2001). On a generalized ruin problem. In *Approximation, Randomization, and Combinatorial Optimization* (Lecture Notes Comput. Sci. **2129**), Springer, Berlin, pp. 181–191.
[2] Bach, E. (2007). Bounds for the expected duration of the monopolist game. *Inform. Process. Lett.* **101,** 86–92.
[3] Blackwell, D. (1970). On stationary policies. *J. R. Statist. Soc. Ser. A* **133,** 33–37.
[4] Bruss, F. T., Louchard, G. and Turner, J. W. (2003). On the N-tower problem and related problems. *Adv. Appl. Prob.* **35,** 278–294.
[5] Bürger, R. and Ewens, W. J. (1995). Fixation probabilities of additive alleles in diploid populations. *J. Math. Biol.* **33,** 557–575.
[6] Engel, A. (1993). The computer solves the three tower problem. *Amer. Math. Monthly* **100,** 62–64.
[7] Felsenstein, J. (1974). The evolutionary advantage of recombination. *Genetics* **78,** 737–756.
[8] Harik, G., Cantú-Paz, E., Goldberg, D. E. and Miller, B. L. (1999). The gambler's ruin problem, genetic algorithms, and the sizing of populations. *Evolutionary Computation* **7,** 231–253.
[9] Knuth, D. E. (1998). *The Art of Computer Programming*, Vol. 2, 3rd edn. Addison-Wesley, Reading, MA.
[10] Ross, S. M. (2009). A simple solution to a multiple player gambler's ruin problem. *Amer. Math. Monthly* **116,** 77–81.
[11] Ross, S. M. (2011). The multiple-player ante one game. *Prob. Eng. Inf. Sci.* **25,** 343–353.
[12] Stirzaker, D. (1994). Tower problems and martingales. *Math. Scientist* **19,** 52–59.
[13] Swan, Y. C. and Bruss, F. T. (2006). A matrix-analytic approach to the N-player ruin problem. *J. Appl. Prob.* **43,** 755–766.