Insurance: Mathematics and Economics 78 (2018) 246–254

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Valuation of variable long-term care Annuities with Guaranteed Lifetime Withdrawal Benefits: A variance reduction approach[☆]

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ARTICLE INFO

Article history:
Available online 28 September 2017

JEL classification:
G22

Keywords:
Variable annuity
Long-term care
Life Care Annuity
Guaranteed Lifetime Withdrawal Benefit
Variance reduction

ABSTRACT

This paper proposes a new product, the Variable Life Care Annuity with Guaranteed Lifetime Withdrawal Benefits (LCA-GLWB), and designs an efficient valuation algorithm. This innovative product provides a comprehensive retirement solution for both longevity risk and long-term care protection. It includes the benefits of guaranteed income streams with downside risk protection and long-term care expenses for retirees. However, the valuation of this type of product is both complex and time-consuming. In this paper, we propose a Monte Carlo valuation algorithm that uses the variance reduction technique. The numerical results indicate that the proposed valuation algorithm is very efficient under a broad range of asset return models. The proposed algorithm provides a general approach for the rapid valuation of similar products and can help provide life insurance companies offering innovative products with an appropriate valuation tool.

1. Introduction

The demand for annuities and Long-Term Care (LTC) insurance is expected to increase with improvements in medical technology and greater awareness of longevity risk since these products should become more attractive to risk-averse policyholders as longevity risk increases. However, the market shares of these retirement products are still limited because of the adverse selection and strict underwriting problems.[1] Previous studies indicate that a new innovative retirement product, the so-called Life Care Annuity (LCA), which is a combination of a lifetime annuity and long-term care insurance, may be able to resolve the problems associated with underwriting in the long-term care insurance market while offering consumers a lower price than traditional LTC products. A number of recent studies have indicated that the LCA can help inject new life into an otherwise stagnant long-term care insurance market (Murtaugh et al., 2001; Brown and Warshawsky, 2013; Webb, 2009).

Murtaugh et al. (2001) first demonstrated the advantages of combining LTC insurance and annuity, which can reduce the cost of both products and make LTC insurance available to more people. Brown and Warshawsky (2013) suggested that LCA products are better products for insurers, compared to traditional LTC products, because they provide lower adverse selection costs for annuities and fewer underwriting problems for LTC insurance. The creation of the LCA does an even better and more wide-ranging job of hedging the expenses associated with disability risk. Webb (2009) states that LCA products may also provide a cheaper product than buying long-term care insurance and an annuity separately in equilibrium. The literature has supported that LCA product provides many good features and advantages to the insurance market and consumers alike.

To meet the growing demand for long-term care insurance and annuity products, an innovation in the design of such LCA products can help to accelerate the growth of these insurance products. In this paper, we extend this line of research and

☆ Paper presented at Longevity 11: The Eleventh International Longevity Risk and Capital Markets Solutions Conference, 7–9 September 2015, Université Lyon 1, Lyon, France.

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[1] For annuity market in the US, the premiums totaled $235 billion in 2007 and it increased slowly to $252 billion in 2016 (NAIC, 2017). 2016 EY life-annuity insurance outlook suggests that individual annuity premium growth will be particularly sluggish, as consumers remain focused on retirement savings. For long-term care market, NAIC also reported that the long-term care industry has undergone significant contraction, both in terms of sales as well as insurers participating in the market (NAIC, 2016). The earned premiums of LTC insurance amounted to lightly less than $12 billion in 2014. However, the LTC expenditures exceeded $225 billion in 2014, which suggests that there is a great deal of room for growth in the LTC insurance market.

https://doi.org/10.1016/j.insmatheco.2017.09.017
0167-6687/

propose a new hybrid product LCA-GLWB[2] that combines long-term care insurance with a variable annuity incorporating Guaranteed Lifetime Withdrawal Benefits (VA-GLWB). Under the current low interest rate environment, general account insurance products are very expensive. As a consequence, variable annuities (VA) with embedded guarantees have been very popular with policy-holders since the 1990s. These investment-linked insurance products can eliminate downside risk while still providing upside retirement income potential. Faced with volatile financial markets and low interest rates, consumers look for higher returns but seek to avoid downside risk. VA-GLWB products provide a guaranteed withdrawal rate when the withdrawals have been initiated. For VA-GLWB, the payouts of annuity are made at certain rate, which is a percentage of the guaranteed minimum benefit base. The proposed LCA-GLWB products provide additional long-term care insurance benefits. The payouts of long-term insurance are made at specified rate, which is also a percentage of the guaranteed minimum benefit base.

The variable LCA products with GLWB have the advantage of providing upside income potential at a lower cost because they are separate account products.[3] Therefore, a variable LCA product with GLWB may provide a more comprehensive solution to retirement than traditional LTC products. The variable LCA products with GLWB provide upside income potential at a lower cost because they are separate account products. Besides, the LCA-GLWB product is only marginally more expensive than VA-GLWB based on our numerical results. Therefore, LCA products with GLWB may be more desirable to consumers because they have the advantage of two products and a GLWB option to reduce the downside investment risk.

However, the valuation of variable LCA products with GLWB or GMWB is very complex. The benefits of this proposed new product depend on the subaccount value and health status of the policy-holder and are thus path-dependent. Due to the complex model of health status transition, the underlying option pricing problems lack a straightforward closed-form solution. More precisely, the valuation of a path-dependent financial contract is a high-dimensional integration problem. Therefore, the previous literature has suggested that the Monte Carlo method is a feasible method that can be used for the valuation of these LCA products (Glasserman, 2004; Asmussen and Glynn, 2007).

However, when using the Monte Carlo method to evaluate the variable LCA product with GLWB, a large number of simulation paths are required, and the Monte Carlo method has the disadvantage of a slow convergence rate while also being time-consuming. Due to the drawback of slow computation costs, the insurer may be not willing to sell LCA-GLWB products. Therefore, we propose a fast valuation algorithm using the techniques of variance reduction. For well-structured problems, it is easy to construct efficient variance reduction algorithm. But this is not the case for problems with complex structures, such as the products considered here (Asmussen and Glynn, 2007). In this paper, we further compare the efficiency of the proposed approach with the crude Monte Carlo method. This algorithm can also apply to the valuation for the variable LCA product with Guaranteed Minimum Withdrawal Benefits (GMWB). As far as we know, this paper is the first study to investigate the valuation of the variable LCA with guaranteed insurance products.

Modeling the transition probability of the health state of the policy-holders is the key issue in LTC insurance, and has a huge impact on insurance premiums. Levikson and Mizrahi (1994) analyzed a general Markovian multi-state model for LTC insurance contracts and developed it into an actuarial model of disability. Other studies have focused on how Markov models can be used to develop an actuarial model of health status transition. Czado and Rudolph (2002) estimate transition intensities using the Cox proportional hazard model which allows for the inclusion of censored observations and time-dependent risk factors with a graduated approach. Albarran et al. (2005) calculate a disability-free survival probability and the disability survival probability in a multiple-state model of disability using the Spanish population. Pritchard (2006) presents a novel methodology for using interval-censored longitudinal data by parameterizing Markov models, and estimates the costs of the LTC insurance contract. Baione and Levantesi (2014) establish a parametric model to estimate transition intensities when data are limited and only aggregated information on mortality and morbidity is available.

Health insurance in the form of long-term care is generally structured by multiple-state models which allow us to represent the evolution of a given insurance policy. Multi-state models can be defined in both a time-continuous and a time-discrete context and offer a powerful tool for interpreting various practical calculation methods (Pitacco, 1995, 2014). In this paper, we adopt the health status transition model with a continuous-time Markov process, which is used by Manton et al. (1993), Pitacco (1995), Haberman and Pitacco (1998), Murtaugh et al. (2001), Czado and Rudolph (2002), Albarran et al. (2005), Pritchard (2006), Brown and Warshawsky (2013), and Baione and Levantesi (2014). We adopt the classification of health states and the transition intensity matrix developed by Pritchard (2006).

The problems associated with the fair valuation of GMWB have been discussed in many studies (Bacinello, 2003; Bacinello et al., 2011; Bauer et al., 2008; Chen and Forsyth, 2008; Chen et al., 2008; Dai et al., 2008; Holz et al., 2012; Milevsky and Salisbury, 2006; Peng et al., 2012; Yang and Dai, 2013). In addition, the problems associated with the fair valuation of GLWB have been discussed by Piscopo (2009), Bernard (2010), Holz et al. (2012), and Piscopo and Haberman (2011). In this paper, we assume a discrete withdrawal model which is closer to the actual way in which GMWBs and GLWBs are implemented in the market.

The remainder of this paper is organized as follows. In Section 2, we describe the product details and health state models. Section 3 develops our valuation models for variable LCA products with GLWB. Section 4 discusses the Monte Carlo method and variance reduction approach. Numerical results are provided in Section 5 followed by concluding remarks in Appendix.

2. Product specifications and other models

2.1. Product specifications

In this paper, we propose LCA-GLWB products, which provide payouts of annuity and long-term insurance. Payouts of LCA-GLWB are made at percentages of the guaranteed minimum benefit base when the withdrawals have been initiated. We now turn to describe LCA-GLWB as follows. In order to formulate more realistic assumptions for the dynamics of the account value W_t, we consider a discrete withdrawal model. In actual practice, many GLWB contracts are annually based on a discrete withdrawal scheme with fixed management fees K and guaranteed fees αW_t.[4] Let $S(t)$ be

[2] GLWB offer a lifelong withdrawal guarantee and minimum withdrawal guarantee.

[3] While a separate account product is more popular in a low interest rate environment, these VA-style guarantees also become more expensive if interest rates are low. We consider the effect of interest rates using sensitivity analysis in the numerical results.

[4] In this paper, we assume fixed management fees K and guaranteed fees αWt. In the real insurance market, there are some products that are so designed that the guaranteed fee α is a portion of the withdrawal benefit base. Other products may allow for an increase in management fees in the case of the good performance of the underlying fund.

the net asset value (NAV) of the invested mutual fund at time t. Then the annual return of the invested mutual fund over the tth year will be

$$R_t = \frac{S(t)}{S(t-1)}; t = 1, 2, \ldots..T \qquad (1) \tag{1}$$

The initial value of the sub-account W_0 equals w_0 and T denotes the time of death. At the beginning of year t ($t = 0, 1, 2, \ldots, T-1$), a guaranteed fee (α times the value of the sub-account) and a fixed management fee K are withdrawn from the sub-account by the insurer. We let $M_x(t)$ be the health status of the policy-holder after time t when he bought the policy at age x. At the end of year t ($t = 1, 2, \ldots, T-1$), the insurer pays $\theta_t(\mathrm{M_x(t)})$ on behalf of the insured for the LTC benefit according to the different health states and $g_t w_0$ for the withdrawal benefit at time t. In particular, θ_t and $g_t w_0$ are LTC benefits and withdrawal benefits at time t. Therefore, g_t is the withdrawal rate at time t. θ_t and g_t are product specifications and can be adjusted to reflect either realized inflation or indexation with a fixed inflation rate. When the insured dies at time T, the beneficiary can withdraw $g_T w_0 + \theta_T(M_x(T))$ and the remaining amount of the sub-account.

Let W_t^- denote the account value at year t before these withdrawals and W_t^+ the account value in year t after these withdrawals. The process of the account value can then be expressed as

$$W_0^- = w_0, \tag{2a}$$
$$W_0^+ = \max\left((1-\alpha) W_0^- - K, 0\right) \tag{2b}$$
$$W_t^- = R_t W_{t-1}^+, t = 1, 2, \ldots, T \tag{2c}$$
$$W_t^+ = \max\left(0, (1-\alpha) W_t^- - K - g_t w_0 - \theta_t(M_x(t))\right), \quad t = 1, 2, \ldots, T-1 \tag{2d}$$
$$W_T^+ = \max\left(g_T w_0 + \theta_T(M_x(T)), W_T^-\right) \tag{2e}$$

To be more precise, this contract provides the following cash-flows Y_t to the policy-holder,

$$Y_t = \left(g_t w_0 + \theta_t(M_x(t))\right), t = 1, 2, \ldots, T-1; \tag{3a}$$
$$Y_T = \max\left(\left(g_T w_0 + \theta_T(M_x(T))\right), W_T^-\right). \tag{3b}$$

We assume the insured deceases at time T. The cash-flow received by the beneficiary at time T can be decomposed into

$$Y_T = g_T w_0 + \theta_T(M_x(T)) + \max\left(0, W_T^- - \left(g_T w_0 + \theta_T(M_x(T))\right)\right) \tag{4}$$

The above final cash-flow Y_T is decomposed into the final payment of a LTC annuity $g_T w_0 + \theta_T(M_x(T))$ and an option-like payment $\max\left(0, W_T^- - \left(g_T w_0 + \theta_T(M_x(T))\right)\right)$. The fair value of the variable LCA with a GLWB contract is therefore the sum of the fair values of the LTC annuity and an option. We refer to the option-like payment as the LCA-GLWB option. The cash-flow Y_T can be decomposed into the final payment of a LTC annuity and LCA-GLWB option. This LCA-GLWB option is automatically exercised by the beneficiary at time T. Therefore, the above analysis reduces the problem of the valuation of the variable LCA with GLWB to that of the LCA-GLWB option and the LCA. Based on the risk-neutral valuation principle (Harrison and Kreps, 1979; Harrison and Pliska, 1981), the fair value of the LCA under a continuous-time Markov chain model is

$$E_Q\left(\sum_{t=0}^{\mathrm{T}} \left(g_t w_0 + \theta_t(M_x(t))\right)/B(t)\right) \tag{5a}$$

and the fair value of the LCA-GLWB option can be expressed as

$$E_Q\left[\frac{\max\left(0, W_T - \left(g_T w_0 + \theta_T(M_x(T))\right)\right)}{B(T)}\right] \tag{5b}$$

where E_Q denotes the expectations under a risk-neutral measure and $B(t)$ denotes the value of a money market account with an initial account value equal to 1 at time t. The stochastic variables used in pricing the LCA-GLWB option comprise the health state of the insured and the annual return that is calculated based on the invested mutual fund.

For real applications, each specific product needs tailor-made product specifications. The computed fair value of LCA-GLWB products should be designed to equal the initial account value w_0. The insurer need to adjust product specifications including annuity benefits withdrawal rate g and LTC benefits withdrawal rate c, fixed management fees K and guaranteed fees rate α.

2.2. Health state model

We adopt a continuous-time Markov model for the health state of the policy-holder based on the literature (Manton et al., 1993; Pitacco, 1995; Haberman and Pitacco, 1998; Murtaugh et al., 2001; Czado and Rudolph, 2002; Baione and Levantesi, 2014; Brown and Warshawsky, 2013). The continuous-time Markov model for the health state may be described as follows. Consider a policy-holder aged x and suppose that the individual moves independently among different health states, denoted by health state 1, health state 2..., health state h. Let $M_x(t)$ be the state occupied at time t by a randomly chosen individual starting at age x. For $0 \le s \le t$, let $P^x(s,t)$ be the $h \times h$ transition probability matrix with entries

$$p_{ij}^x(s,t) = \mathrm{P}\left\{M_x(t) = j | M_x(s) = i\right\}, \tag{6}$$

for health state $i, j = 1, \ldots\ldots, h$ with starting age x. The process can be specified in terms of the transition rates:

$$q_{ij}^x(t) = \lim_{\Delta t \to 0} p_{ij}^x(t, t+\Delta t)/\Delta t, i \neq j, \tag{7a}$$
$$q_{ii}^x(t) = \lim_{\Delta t \to 0} p_{ij}^x(t, t+\Delta t)/\Delta t, i = 1, \ldots\ldots, h. \tag{7b}$$

Under the model assumptions, we can describe the transition probabilities by Kolmogorov forward and backward equations. In what follows, we present the Kolmogorov forward equation

$$\frac{dp_{ij}^x(s,t)}{dt} = \sum_k p_{ik}^x(s,t)\, q_{kj}^x(t) \tag{8a}$$

and Kolmogorov backward equation

$$\frac{dp_{ij}^x(s,t)}{ds} = \sum_k q_{ik}^x(s)\, p_{kj}^x(s,t). \tag{8b}$$

Let $Q_x(t)$ be the $h \times h$ rate matrix with entries $q_{ij}^x(t)$. It is well known that $q_{ij}^x(t) \ge 0$ for $i \neq j$ and $\sum_{j=1}^h q_{ij}^x(t) = 0$. We assume that M_x is time-homogeneous during each year, i.e., for $s = 0, 1, 2, \ldots$.

$$Q_x(t) = Q_x(s), \quad \text{for } 0 \le t - s < 1 \tag{9a}$$

The transition probability matrix can then be computed via a rate matrix exponential

$$P^x(s,t) = e^{Q_x(s)t}, \tag{10}$$

where s is a non-negative integer and $0 \le t - s < 1$

2.3. Asset model

We assume that the invested mutual fund $S(t)$ follows a Levy process, which has stationary and independent increments (Asmussen and Glynn, 2007). There are a few variations of the Levy process, several of which have been used to describe the dynamics of asset prices. For example, Merton (1976) introduced a jump–diffusion model for derivative pricing. The model can be described through the stochastic differential equation

$$\frac{dS(t)}{S(t^-)} = \mu dt + \sigma dZ(t) + dJ(t), \tag{11}$$

where μ and σ are constants and Z is a standard Brownian motion and J is a jump process independent of Z that can be specified as

$$J(t) = \sum_{k=1}^{N(t)} (Y_k - 1), \tag{12}$$

where Y_k is a random variable and $N(t)$ a counting process. The jumps described by Eq. (11) can happen at any time t. If there is a jump at time t, $S(t^-)$ and $S(t)$ are used to distinguish the different asset prices before and after jump at time t. If there is no jump at time t, $S(t^-)$ and $S(t)$ are the same. Under the assumption that $S(t)$ follows jump–diffusion model, the annual returns on the invested mutual fund over each year are independent. Then we can simulate the asset price process by simulating the number of jumps, jump arrival time and jump size.

Pure jump processes have also been used in specifying the dynamics of asset prices (Samoradnitsky and Taqqu, 1994). Madan and Seneta (1990) proposed models based on gamma processes. They referred to the constructed process as a variance gamma (VG) process. Madan et al. (1998) used a VG process to estimate statistical and risk neutral densities using data based on the S&P500 index and the prices of options related to this index. They observed that the statistical density is symmetric with some kurtosis, while the risk neutral density is negatively skewed with a larger kurtosis. They also found that the additional parameters in the VG process correct the pricing biases of the Black–Scholes model. The distributions of logarithmic asset returns can often be well fitted by normal inverse Gaussian (NIG) distributions. Therefore, Barndorff-Nielsen (1997) proposed an NIG process to model the dynamics of asset prices. VG and NIG processes share some similarities. Their sample paths can be obtained through a Brownian motion characterized by a random time-change. Therefore, the generation of their sample paths is not much harder than that of a Brownian motion (see Asmussen and Glynn (2007) and Glasserman (2004).

3. Proposed Monte Carlo methods

Since we assume that the health state follows a CTMC process, we can simulate the health state standard using a stochastic simulation algorithm (Glasserman, 2004). The discrete skeleton of the health state process can be simulated as $M_x(1), M_x(2), \ldots, M_x(T)$.

It is clear that the process of the account value at time t depends on the entire path of R_t and $M_x(t)$. In particular,

$$W_T^- = f(R_1, R_2, \ldots, R_T, M_x(1), M_x(2), \ldots, M_x(T)) \tag{13}$$

where $f(.)$ is the function defined by recursions in Section 2. This makes the payoff of the LCA-GLWB option path-dependent. This also implies that the Monte Carlo method is the most cited approach for the valuation of this option (Boyle et al., 1997; Glasserman, 2004). We propose efficient Monte Carlo valuation methods by using variance reduction techniques. In particular, we use the control variates technique in accelerating the speed of the Monte Carlo methods. We provide a short description of the control variates below.

Suppose that we wish to estimate $\alpha = E(L)$, where L is the output of a complex stochastic process. A naïve Monte Carlo procedure would generate m independent copies of L, and produce the standard estimate

$$\alpha_{naïve} = \frac{1}{m}\sum_{i=1}^{m} L_i \tag{14}$$

where $L_1, \ldots, L_m$ are independent copies of L. Let X be a d by 1 random vector in which each component of X is correlated with L. Let (μ, Σ) denote the mean vector and covariance matrix of X. The mean vector μ is known. Suppose that the covariance between L_i and X is c_i and $c = (c_{1,\ldots,}c_d)^T$. We can define the control variates as

$$C = X - \mu. \tag{15}$$

It is clear that the mean vector of $C = 0$, the covariance matrix of $C = \Sigma$, and the covariance between L and C_i is c_i. Now define

$$L_C(\lambda) = L - \lambda^T C \tag{16}$$

It is obvious that

$$E[L_C(\lambda)] = 0 \tag{17}$$

and

$$Var[L_C(\lambda)] = \sigma_X^2 - 2\lambda^T c + \lambda^T \Sigma \lambda \tag{18}$$

The minimizer of above formula is

$$\lambda^* = \Sigma^{-1} c \tag{19}$$

and

$$Var\left[L_C(\lambda^*)\right] = \sigma_L^2 - 2\left(\Sigma^{-1}c\right)^T c + \left(\Sigma^{-1}c\right)^T \Sigma \left(\Sigma^{-1}c\right) \tag{20}$$

Hence

$$Var\left[L_C(\lambda^*)\right] = \sigma_L^2 - c^T \Sigma^{-1} c^T < \sigma_X^2 \tag{21}$$

Let $L_C^{(i)}(\lambda^*), i = 1, \ldots, m$ be independent copies of $L_C(\lambda^*)$. Then it is obvious that

$$\alpha_c = \frac{1}{m}\sum_{i=1}^{m} L_C^{(i)}\left(\lambda^*\right) \tag{22}$$

is a more efficient estimate for α. It is usually not possible to compute the exact value of λ^*, since Σ and c are usually unknown. However, accurate estimates of Σ and c are easy to compute from the simulation output, and therefore an accurate estimate of λ^* is also easy to obtain. The key step in applying control variates is to find suitable control variates.

In light of the payoff function of the LCA-GLWB option, we select efficient control variates as follows. First of all, we let

$$X_1 = ((1-\alpha) w_0 - K) R_1 \tag{23a}$$

$$X_t = ((1-\alpha) X_{t-1} - K - g_t w_0) R_t, t = 2, \ldots, T \tag{23b}$$

It is clear that $X_t = W_t$ if the account values are all positive times where $s < t$. Therefore, $X_T/B(T)$ is highly correlated with the discount payoff of the LCA-GLWB option. Since $R_1, R_2 \ldots\ldots R_T$ and $M_x(1), M_x(2) \ldots . . M_x(T)$ are independent, this implies that the expected value of X_T can be easily computed from the above recursions. Therefore, we use

$$C_1 = \frac{X_T}{B(T)} - E\left(\frac{X_T}{B(T)}\right) \tag{24}$$

as our key control variate. In addition, we consider the following control variates, which are easy to compute and are also correlated with the payoff of the LCA-GLWB option:

$$C_2 = \frac{S(T)}{S(0)} - E\left[\frac{S(T)}{S(0)}\right] = R_1 R_2 \cdots R_T - E[R_1 R_2 \cdots R_T] \tag{25}$$

Exhibit 1
Health states of disability model.

State	Health states	Benefit (amount)
1	Health	Annuity payment ($g_t w_0$)
2	I ADLs	Annuity payment ($g_t w_0$)
3	1–2 ADLs	Annuity payment ($g_t w_0$)
4	3–4 ADLs	Annuity payment($g_t w_0$)+ LTC insurance payment(θ_t)
5	5–6 ADLs	Annuity payment($g_t w_0$)+ LTC insurance payment(θ_t)
6	Institutionalized	Annuity payment($g_t w_0$)+ LTC insurance payment(θ_t)
7	Dead	0

$$C_3 = Y_t - \mathrm{E}\left[\sum_{t=1}^{T} Y_t\right] = \sum_{t=1}^{T} \left(g_t w_0 + \theta_t \left(M_x\left(t\right)\right)\right) - \mathrm{E}\left[\sum_{t=1}^{T} \left(g_t w_0 + \theta_t \left(M_x\left(t\right)\right)\right)\right] \tag{26}$$

$$C_4 = T - \mathrm{E}\left[T\right] \tag{27}$$

We can use these control variates simultaneously to increase the efficiency of the Monte Carlo procedure. For example, we can use $[C_1\ C_2]$, $[C_1\ C_2\ C_3]$, and $[C_1\ C_2\ C_3 C_4]$ as a set of control variates.

4. Numerical Results

Based on the valuation model and variance reduction technique described in the previous sections, we present the numerical results for the estimation of the fair value of the LTC annuity (i.e., traditional LCA) and LCA-GLWB options (i.e., the variable LCA with GLWB). In the numerical examples, we wish to show that our algorithm is much faster than the crude Monte Carlo method. We test C_1, C_2, C_3, C_4, $[C_1\ C_2]$, $[C_1\ C_2\ C_3]$, and $[C_1\ C_2\ C_3 C_4]$ as different sets of control variates, in which $C_1 = \frac{X_T}{B(T)} - E\left(\frac{X_T}{B(T)}\right)$, $C_2 = \frac{S(T)}{S(0)} - \mathrm{E}\left[\frac{S(T)}{S(0)}\right]$, $C_3 = Y_t - \mathrm{E}\left[\sum_{t=1}^{T} Y_t\right]$ and $C_4 = T - \mathrm{E}\left[T\right]$. To test the effectiveness of our algorithm, we apply it to the valuation problem of the LCA-GLWB option under the simple geometric Brownian motion (GBM) assumption asset models. We further test the effectiveness of the control variate sets $[C_1\ C_2\ C_3 C_4]$ under different types of asset model.

In the assumptions of the health state model, we adopt the transition rate matrix and classification of the health state from Pritchard (2006), and then generate the transition rate matrix using the parameters estimated by Pritchard (2006).[5] Exhibit 1 shows the classification of the health states. We classify the health states according to activities of daily living (ADL) and instrumental activities of daily living (IADL).[6] There are seven health state categories in our numerical example, which are denoted as follows: Healthy—state 1; 1 or more IADL—state 2; 1–2 ADLs—state 3; 3–4 ADLs—state 4; 5–6 ADLs only—state 5; institutionalized—state 6; and dead—state 7.

We design our numerical example as follows. For the annuity benefit, this contract offers a lifetime guarantee annuity $g_t w_0$ while the individual is alive. The LTC insurance benefit is a certain guaranteed amount per period as a person meets 3+ ADLs. Similar to actual LTC products, the LTC benefit includes indexation with a fixed inflation rate, which increases payments by π percent per year. LTC insurance benefits at time t amount to θ_t, which can

[5] Details of the health state model are provided in the appendix. We also report the transition probability matrix of different health states in Exhibit A.1.

[6] Activities of daily living (ADL) is used to measure people's daily self-care activities without needing assistance, which is important for determining the needs of long-term care. Instrumental activities of daily living (IADL) is used to measure people's ability to live independently in a community.

Exhibit 2
The actuarial values of LCA using standard numerical methods.

Entry age(x)	State at the start of contract (i): State 1 health		
	LTC	Annuity	LCA
60	7442	42,458	57,342
65	7021	32,868	46,909
70	6769	25,811	39,350
75	6613	20,472	33,699
80	6506	16,315	29,326

Note: Assuming Annuity benefit = 2000 (Indexation with a fixed inflation rate 0.05), LTC benefit = 6000 and $r = 0.04$.

be defined as a certain percentage c of the guaranteed minimum benefit base w_0, which includes indexation with a fixed inflation rate of π percent per year. This contract provides LTC insurance benefits θ_t under our numerical results as follows:

$$\theta_t\left[M_x\left(t\right), \pi, c, w_0\right] = 0, \quad M_x\left(t\right) = 1, 2, 3, 7 \tag{28a}$$

$$\theta_t\left[M_x\left(t\right), \pi, c, w_0\right] = c w_0 (1+\pi)^t, \quad M_x\left(t\right) = 4, 5, 6 \tag{28b}$$

We assume that an annuity benefits withdrawal rate g_t is 2% of w_0; an LTC benefits withdrawal rate c is 6% of w_0; an inflation protection rate of LTC benefits π is 5%; and a continuously compounded risk-free rate is 4%. In our numerical example, the initial account value w_0 is $100,000, which is used to compute the benefits for various products.

To understand how correlation affects the variance reduction efficiency of our variance reduction algorithm, we repeat the same calculation with different starting ages of 60, 65, 70, 75 and 80 with starting health state 1. The combination of a variable LCA with GLWB implies the difficulty associated with estimating the fair value of this product, and the distribution of this product is more complex than that of a traditional product. Based on the analysis in the previous section, the fair value of a variable LCA with GLWB can be decomposed into the fair value of an LCA-GLWB option and the fair value of an LCA. Our focus is on estimating the fair value of the LCA-GLWB options, which are vital inputs in calculating the fair value of a hybrid product combining variable LCA with GLWB.

In Exhibits 2 and 3, we present the numerical results for the estimation of the actuarial value of the LCA using standard numerical methods and Monte Carlo methods separately. The actuarial value of the annuity, long-term care insurance, and LCA products can be directly calculated using a transition probability matrix for different starting ages. The traditional LCA product includes the annuity benefits and LTC benefits. In this paper, we assume that the annuity benefit 2000 and LTC benefit 6000.[7]

The results of the standard numerical methods are presented in Exhibit 2. We also use the Monte Carlo valuation methods to calculate the actuarial value of an LCA product in Exhibit 3. We find the valuation results are similar for both the standard numerical methods and Monte Carlo valuation methods. For the insured at age 60, the actuarial value of the LCA is $57,342, and the estimated actuarial value of the same product calculated by the Monte Carlo

[7] For comparison purpose, we assume benefits of traditional LCA are equal to the assumption of the guaranteed withdrawal benefits of LCA-GLWB.

Exhibit 3
The estimated actuarial values of LCA using Monte Carlo methods.

Entry age(x)	State at the start of contract (i): State 1 health	
	p.e.	*s.e.*
60	56,956	394
65	47,024	370
70	39,367	337
75	33,874	306
80	29,312	279

Note: Assuming Annuity benefit = 2000 (Indexation with a fixed inflation rate 0.05), LTC benefit = 6000, $r = 0.04$, and the number of replicates = 10^6.

valuation method is $56,956. In addition, the value of the LCA product is decreasing with an increasing entry age (x). The actuarial values of the LCA products are $57,342, $46,909, $39,350, $33,699 and $29,326 where the entry age changes from 60 to 65, 70, 75 and 80, respectively.

In Exhibits 4 and 5, we present the results for different control variates and the variance reduction under the GBM process. We report the point estimates (p.e.) and standard errors (*s.e.*) for estimators with and without control variates, where the standard error is defined as $\frac{\sqrt{\frac{1}{m-1}\sum_{i=1}^{m}(Y_i-\alpha)^2}}{\sqrt{m}}$. In order to observe the efficiency achieved by the control variates, we calculate the variance reduction ratio (VRR). The VRR represents the efficiency gains of the control variates method relative to the standard method. Using the sample variance ratio, we can find evidence of a significant gain in efficiency between the crude estimator and the estimators based on our control variates technique. In each case, we generate 1,000,000 simulation runs for the naïve Monte Carlo approach and then compare the variance reduction ratio to quantify the statistical efficiency of our control variates.

In Exhibit 4, the control variates exhibit an efficiency gain in variance reduction. For example, according to the value of VRR, the tradition Monte Carlo method for the insured aged 60 takes 12.39 times as much work as using a single control variate C_1. By using the control variates simultaneously $[C_1\ C_2\ C_3\ C_4]$, the traditional Monte Carlo method takes 26.70 as much time as the variance reduction approach. The efficiency gain is more prominent when the age x is higher. Moreover, the efficiency of our variance reduction technique increases as the value of LCA-GLWB increases, which indicates that our algorithm reduces the variance of the estimator especially when there is only a remote possibility of an out-of-money event taking place.

In order to show the robustness of our variance reduction approach, we use different assumptions for the invested mutual fund process, different interest rates and different withdrawal rates. We further consider the Merton process, variance gamma (VG) process, and normal inverse Gaussian (NIG) process in our numerical results. As is evident in Exhibit 5, the estimator of the LCA-GLWB option for the insured aged 60, based on our control variates $[C_1\ C_2\ C_3\ C_4]$, is substantially more efficient than that estimated by crude Monte Carlo simulations for different kinds of invested mutual fund process. For example, the VRR is 26.84 under the assumption of the Merton process, is 27.15 under the assumption of the VG process and is 28.70 under the assumption of the NIG process.

We also use sensitivity analysis to discuss the impacts of different interest rate assumptions. In Exhibit 6, we can find that the estimated value of the LCA-GLWB option increases when the interest rate is lower. Our algorithm still results in a significant decrease in the variance. For all interest rate scenarios, our algorithm still has significant efficiency gains. For example, the VRRs of our control variates $[C_1\ C_2\ C_3\ C_4]$ are 19.69, 26.76, 38.24 and 56.01, when the interest rate moves, respectively, from 0.02 to 0.04, 0.06 and 0.08 in Exhibit 6.

Steinorth and Mitchell (2015) show that the insured who have a VA with GLWBs frequently withdraw more than the guaranteed withdrawal. Therefore, we further provide sensitive analysis of different withdrawal rates assumptions to demonstrate the robustness of our variance reduction approach. We report the results of different withdraw rate scenarios in Exhibit 7. The results show that significant gains in efficiency are obtained in every scenario in Exhibit 7. For example, the VRR of control variates $[C_1\ C_2\ C_3\ C_4]$ is 26.70 when the withdraw rate g_t is 0.02 for all t. The VRR is 8.06 when g_t is 0.01 for the first 10 years and 0.04 after that. The VRR is 17.80 when g_t is 0.04 for the first 10 years and 0.02 after that. Therefore, our numerical results indicate the proposed method consistently is much faster than crude Monte Carlo method in all cases.

In Exhibit 8, we compare the fair values between the LCA-GLWB and VA-GLWB. For easy comparison, the product specifications are set to be the same between these two products. In order to compute the fair value of LCA-GLWB, we decompose the benefits of LCA-GLWB into the benefits of LCA and LCA-GLWB option. By the same token, the fair value of the VA-GLWB product includes the annuity (guaranteed withdrawal benefits) and the VA-GLWB option. From Exhibit 8, we find the fair value of LCA is higher than that of the annuity. On the other hand, the fair value of the LCA-GLWB option is less than that of the VA-GLWB option. We can also find that LCA-GLWB is only marginally more expensive than VA-GLWB, making this a potentially attractive product for the insured.

For example, for the insured of age 60, the fair value of LCA-GLWB product is $107,022 and the fair value of VA-GLWB product is $101,053. The fair value of LCA-GLWB is only marginally higher

Exhibit 4
Variance reduction ratio of the LCA-GLWB option with different control variates under a GBM process.

Entry age(x)	Health state at the start of contract ((i) : State 1 health											
	Naïve			Control variates: C1			Control variates: C2			Control variates: C3		
	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR
60	49,680	57	–	49,702	16	12.39	49,710	44	1.71	49,687	54	1.13
65	55,660	54	–	55,752	17	10.66	55,690	43	1.56	55,685	50	1.19
70	60,458	51	–	60,512	17	9.33	60,498	42	1.49	60,450	46	1.24
75	64,346	48	–	64,337	17	8.32	64,308	40	1.47	64,381	42	1.28
80	67,701	45	–	67,653	16	7.61	67,688	37	1.47	67,670	39	1.33
Entry age(x)	Control Variates: C4			Control variates: [C1, C2]			Control variates: [C1, C2, C3]			Control variates: [C1, C2, C3, C4]		
	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR
60	49,682	54	1.11	49,696	15	15.05	49,703	12	22.62	49,701	11	26.70
65	55,690	50	1.15	55,757	15	13.57	55,763	11	22.30	55,758	10	27.32
70	60,456	47	1.19	60,505	14	12.46	60,504	11	22.78	60,501	10	28.74
75	64,385	44	1.22	64,354	14	11.83	64,360	10	23.94	64,356	9	31.28
80	67,683	40	1.25	67,648	13	11.83	67,637	9	26.25	67,636	8	36.01

Note: Assuming W0 = 100,000, $r = 0.04$, $g = 0.02$, $c = 0.06$, $\pi = 0.05$, σ of GBM = 0.16, $K = 300$, $\alpha = 0.008$ and the number of replicates = 10^6.

Exhibit 5
Variance reduction ratio of control variates under different types of invested mutual fund process.

		Naïve	Control variates						
			C1	C2	C3	C4	[C1, C2]	[C1, C2, C3]	[C1, C2, C3, C4]
GBM Process[a]	p.e.	49,642	49,677	49,653	49,642	49,645	49,679	49,677	49,678
	s.e.	57	16	44	54	54	15	12	11
	VRR	–	12.32	1.70	1.13	1.11	15.00	22.61	26.76
Merton Process[b]	p.e.	49,700	49,727	49,771	49,648	49,677	49,711	49,695	49,687
	s.e.	58	16	44	54	55	15	12	11
	VRR	–	12.56	1.71	1.13	1.11	15.17	22.85	26.84
VG Process[c]	p.e.	49,790	49,721	49,749	49,777	49,793	49,723	49,717	49,713
	s.e.	58	16	44	55	55	15	12	11
	VRR	–	12.81	1.73	1.12	1.10	15.48	23.18	27.15
NIG Process[d]	p.e.	49,708	49,729	49,727	49,707	49,730	49,726	49,727	49,715
	s.e.	60	16	45	57	58	15	12	11
	VRR	–	13.74	1.80	1.11	1.10	16.49	24.56	28.70

Note: Assuming $x = 60$, W0 $= 100{,}000$, $r = 0.04$, $g = 0.02$, $c = 0.06$, $\pi = 0.05$, $K = 300$, $\alpha = 0.008$ and the number of replicates$= 10^6$.
[a] Assuming the mean is 0.04 and the standard deviation is 0.16 under the GBM process.
[b] Assuming the mean is 0.04 and the standard deviation is 0.16, where sigma is 0.1, $a = 0$, $b = .10197$, and lambda $= 1.5$, under a Merton process.
[c] Assuming the mean is 0.04 and the standard deviation is 0.16, where C is 2, $G = 12.5$ and, $M = 12.5$, under a variance gamma (VG) process.
[d] Assuming the mean is 0.04 and the standard deviation is 0.16, where alpha $= 5$, beta $= 0$, and delta $= 0.128$, under a normal inverse Gaussian (NIG) process.

Exhibit 6
Variance reduction ratio of control variates under different assumptions of interest rates.

		Naïve	Control variates						
			C1	C2	C3	C4	[C1, C2]	[C1, C2, C3]	[C1, C2, C3, C4]
$r = 0.02$	p.e.	70,759	41,114	49,078	70,723	70,747	41,285	41,328	41,258
	s.e.	88	28	56	86	87	25	21	20
	VRR	–	9.99	2.46	1.06	1.03	12.58	17.36	19.69
$r = 0.04$	p.e.	49,642	49,677	49,653	49,642	49,645	49,679	49,677	49,678
	s.e.	57	16	44	54	54	15	12	11
	VRR	–	12.32	1.70	1.13	1.11	15.00	22.61	26.76
$r = 0.06$	p.e.	35,888	56,806	46,934	35,919	35,908	55,783	55,729	55,791
	s.e.	41	10	35	37	36	9	7	7
	VRR	–	16.83	1.33	1.21	1.25	19.38	31.64	38.24
$r = 0.08$	p.e.	26,528	62,243	43,172	26,530	26,530	60,253	60,260	60,152
	s.e.	31	6	29	27	26	6	5	4
	VRR	–	23.89	1.15	1.28	1.43	26.02	46.35	56.01

Note: Assuming W0 $= 100{,}000$, $g = 0.02$, $c = 0.06$, $\pi = 0.05$, σ of GBM $= 0.16$, $K = 300$, $\alpha = 0.008$, the mean is 0.04 and the standard deviation is 0.16 under the GBM process and the number of replicates $= 10^6$.

Exhibit 7
Variance reduction ratio of control variates under different assumptions of withdrawal rates.

Entry age(x)	Health state at the start of contract (i) : State 1 health Control Variates : [C1, C2, C3, C4]		
	Different withdrawal rates assumption		
	$g = 0.02$	$g = 0.01$ for first 10 years $g = 0.04$ after 10 years	$g = 0.04$ for first 10 years $g = 0.02$ after 10 years
	VRR	VRR	VRR
60	26.70	8.06	17.80
65	27.32	9.23	20.02
70	28.74	10.68	22.42
75	31.28	12.72	25.21
80	36.01	15.25	27.55

Note: Assuming W0 $= 100{,}000$, $r = 0.04$, $c = 0.06$, $\pi = 0.05$, σ of GBM $= 0.16$, $K = 300$, $\alpha = 0.008$ and the number of replicates $= 10^6$.

Exhibit 8
Comparing the fair values of LCA-GLWB and VA-GLWB.

Entry age(x)	LCA-GLWB			VA-GLWB		
	LCA	LCA-GLWB option	LCA-GLWB	Annuity	VA-GLWB option	VA-GLWB
60	57,342	49,680	107,022	42,458	58,595	101,053
65	46,909	55,660	102,569	32,868	65,318	98,186
70	39,350	60,458	99,808	25,811	70,816	96,627
75	33,699	64,346	98,045	20,472	75,053	95,525
80	29,326	67,701	97,027	16,315	78,613	94,928

Note: Assuming W0 $= 100{,}000$, $r = 0.04$, $g = 0.02$, $c = 0.06$, $\pi = 0.05$, σ of GBM $= 0.16$, $K = 300$, $\alpha = 0.008$ and the number of replicates $= 10^6$.

Exhibit A.1
Life expectancy for different health states and different ages.

Health states	Starting age					
	60	65	70	75	80	85
1 (Health)	19.05	14.99	11.94	9.58	7.72	6.21
2 (IADLs)	14.83	11.59	9.20	7.44	6.11	5.11
3 (1–2 ADLs)	13.01	10.34	8.36	6.88	5.74	4.84

than that of VA-GLWB. For the insured of age 70, the benefits of LCA-GLWB have a fair value of 99,808. The LCA-GLWB option is \$49,680 and the VA-GLWB option is \$58,595 for the insured at age 70. The fair value of LCA-GLWB option is lower than that of VA-GLWB option.

Furthermore, LCA-GLWB is a separate account product with minimum guaranteed withdrawal benefits of LCA. In addition to the LCA benefits, LCA-GLWB also provides a great upside investment potential to the beneficiary. Therefore, the fair value of LCA-GLWB is higher than traditional general account LCA product. For example, for the insured of age 60, the fair value of traditional general account LCA product is \$57,342. Thus, LCA-GLWB includes the guaranteed withdrawal benefits LCA \$57,342 and the VA-GLWB option \$49,680, which can use to provide more withdraw benefits according to the insured need. In real applications, each specific product needs tailor-made product specifications. At the time of withdraw, the computed fair value of LCA-GLWB products should be designed to equal the initial account value w_0. For example, when g_t and c remain the same and α increases from 0.008 to 0.0143, the value of LCA-GLWB for the insured at age 60 matches the initial account value \$100,000. Similarly, α decreases from 0.008 to 0.007, the value of LCA-GLWB for the insured at age 70 matches the initial account value. As we mentioned before, the insurer need to adjust suitable product specifications for consumers at the time of withdraw. Searching suitable product specifications requires an additional layer of computation. Therefore, a fast valuation algorithm is crucial for designing this type of products. The proposed approach in this paper provides an efficient way to achieve this function.

5. Conclusion

In this paper, we extend the literature on LCA products. We propose a new hybrid product, the LCA-GLWB, combining long-term care benefits and a variable annuity with GLWB. We believe that the LCA-GLWB provides a more comprehensive solution for retirement than the traditional LTC products, since it includes the benefits of guaranteed income streams and long-term care expenses for the retirees. Besides, this product combines the attraction of the LCA product with the advantages of the GLWB option. With the additional LTC benefit, the LCA-GLWB is only marginally more expensive than the VA-GLWB. Furthermore, the LCA-GLWB is much less expensive than purchasing traditional LTC product and VA-GLWB at the same time. Therefore, this innovative product should be more attractive to consumers because it has the advantages of providing upside income potential and can be purchased at lower cost. However, the valuation of this type of product is complex and time-consuming. As far as we know, this paper is the first study to investigate the valuation of the variable LCA with guaranteed insurance products.

For the valuation of the variable LCA with GLWB, we can use the Monte Carlo simulation for the path-dependent option. In order to improve the efficiency of simulation, we propose an efficient valuation algorithm for the variable LCA with GLWB by using the control variates technique. We select a set of efficient control variates. The numerical results show that the proposed valuation algorithm is very efficient and time-saving. The efficiency gain is more prominent when the age is higher. Moreover, the efficiency of the variance reduction technique increases as the value of LCA-GLWB increases. We also find that the fast Monte Carlo algorithm

Exhibit A.2
Transition probability matrix for different health states.

$P^{60}(0, 1)$							
Health states	Lagged health states						
	1	2	3	4	5	6	7
1	0.9840	0.0043	0.0084	0.0008	0.0015	0.0003	0.0006
2	0.2450	0.4288	0.2292	0.0299	0.0213	0.0008	0.0449
3	0.0951	0.1241	0.5764	0.0943	0.0396	0.0030	0.0675
4	0.0472	0.0380	0.2837	0.4483	0.0918	0.0023	0.0887
5	0.0504	0.0519	0.0547	0.0822	0.5720	0.0224	0.1664
6	0.0689	0.0115	0.0124	0.0083	0.0051	0.8568	0.0369
7	0	0	0	0	0	0	1
$P^{60}(10, 11)$							
Health states	Lagged health states						
	1	2	3	4	5	6	7
1	0.9306	0.0172	0.0143	0.0040	0.0036	0.0036	0.0266
2	0.1638	0.4217	0.2678	0.0350	0.0270	0.0209	0.0637
3	0.0694	0.1016	0.5813	0.1008	0.0378	0.0237	0.0852
4	0.0309	0.0247	0.2121	0.4774	0.1335	0.0261	0.0953
5	0.0345	0.0358	0.0518	0.0855	0.5580	0.0499	0.1845
6	0.0222	0.0095	0.0087	0.0077	0.0046	0.8203	0.1269
7	0	0	0	0	0	0	1
$P^{60}(20, 21)$							
Health states	Lagged health states						
	1	2	3	4	5	6	7
1	0.8480	0.0355	0.0307	0.0083	0.0082	0.0150	0.0542
2	0.0853	0.4089	0.3007	0.0412	0.0333	0.0451	0.0854
3	0.0465	0.0782	0.5690	0.1103	0.0401	0.0507	0.1052
4	0.0167	0.0172	0.1291	0.4719	0.1940	0.0633	0.1078
5	0.0211	0.0203	0.0463	0.0852	0.5382	0.0748	0.2141
6	0.0078	0.0075	0.0053	0.0067	0.0042	0.7591	0.2093
7	0	0	0	0	0	0	1

that we propose can be applied to very general asset models and contract designs. Therefore, our proposed algorithm provides a more efficient way of valuing and pricing LCA products and can help life insurance companies to offer this kind of innovative retirement products to the insurance market.

Appendix

In this paper, we generate the transition rate matrix using the parameters estimated by Pritchard (2006). The parameters are estimated in the graduation process using data from the National Long-Term Care Study in the United States. In order to provide more information on the health state model, we show life expectancy for individuals with different initial health states and different ages in Exhibit A.1. Life expectancy is decreasing with the increase in the initial ages. In Exhibit A.2, we report the transition probability matrix $P^x(s, t)$ for an individual starting at age 60.

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