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Asymptotic behavior for a long-range Domany–Kinzel model

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HIGHLIGHTS

- We consider a long-range Domany–Kinzel model for directed bond percolation on the rectangular lattice.
- The critical aspect ratio for this percolation model in the thermodynamic limit is corrected.
- The asymptotic behavior of this percolation model near the critical aspect ratio is obtained.
- We investigate the cases with infinite bonds from one vertex to the next row for this model.
- A special case with infinite bonds is studied in details, and its critical power is derived.

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ABSTRACT

We consider a long-range Domany–Kinzel model proposed by Li and Zhang (1983), such that for every site (i, j) in a two-dimensional rectangular lattice there is a directed bond present from site (i, j) to $(i+1, j)$ with probability one. There are also $m+1$ directed bounds present from (i, j) to $(i-k+1, j+1)$, $k = 0, 1, \ldots, m$ with probability $p_k \in [0, 1)$, where m is a non-negative integer. Let $\tau_m(M, N)$ be the probability that there is at least one connected-directed path of occupied edges from $(0, 0)$ to (M, N). Defining the aspect ratio $\alpha = M/N$, we derive the correct critical value $\alpha_{m,c} \in \mathbb{R}$ such that as $N \to \infty$, $\tau_m(M, N)$ converges to 1, 0 and 1/2 for $\alpha > \alpha_{m,c}$, $\alpha < \alpha_{m,c}$ and $\alpha = \alpha_{m,c}$, respectively, and we study the rate of convergence. Furthermore, we investigate the cases in the infinite m limit. Specifically, we discuss in details the case such that $p_n \in [0, 1)$ with $n \in \mathbb{Z}_+$ and $p_n \approx_{n\to\infty} pn^{-s}$ for $p \in (0, 1)$ and $s > 0$. We find that the behavior of $\lim_{m\to\infty} \tau_m(M, N)$ for this case highly depends on the value of s and how fast one approaches to the critical aspect ratio. The present study corrects and extends the results given in Li and Zhang (1983).

1. Introduction

Directed percolation, or oriented percolation, can be thought of simply as a percolation process on a directed lattice in which connections are allowed only in a preferred direction. It was first studied by Broadbent and Hammersley in 1957 [1], and it has remained to this day as one of the most outstanding interesting problems in probability and statistical mechanics. Furthermore, directed percolation is closely related to the Reggeon field theory in high-energy physics and the Markov processes with branching, recombination and absorption that occur in chemistry and biology [2,3], etc. Various properties, results and conjectures of directed percolation can be found in [4,5] and the references therein. However, very little is known in the way of exact solutions for the directed percolation problem.

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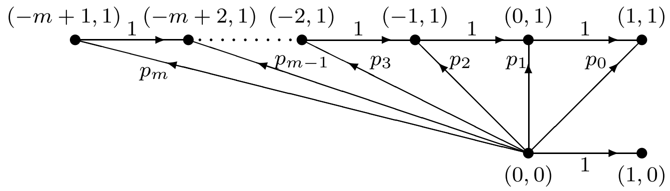

Fig. 1. The long-range directed bond percolation model.

Domany and Kinzel [6] defined a solvable version of compact directed percolation on the square lattice in 1981 as follows. Given a fixed $p \in (0, 1)$, each vertical bond is directed upward with occupation probability p (independently of the other bonds), while each horizontal bond is directed rightward with occupation probability 1. Furthermore, it is known that the boundary of the Domany–Kinzel model has the same distribution as the one-dimensional last passage percolation model [7]. A three-dimensional version of Domany–Kinzel model with occupation probability 1 along two spatial directions was considered in Ref. [8]. Recently, the model has been considered in more general cases. For example, the directed percolation model on the square lattice whose vertical edges occupied with a probability p_v and horizontal edges in the nth row occupied with a probability 1 if n is even and p_h if n is odd [9]; the directed percolation model on the triangular lattice in terms of a square lattice with vertical probability y, horizontal probabilities 1 and x alternatively, and diagonal edges from lower-left to upper-right or from lower-right to upper-left with probability d [10]; the directed percolation model on the honeycomb lattice as bricks such that vertical edges are directed upward with probability y, and horizontal edges are directed rightward with probabilities 1 and x in alternate rows [11]; the directed percolation model on square lattice such that horizontal edges are directed rightward with probabilities one, and vertical edges are directed upward with probabilities p_1, p_2 alternatively in even rows and probabilities p_2, p_1 alternatively in odd rows [12].

In this article, we consider the long-range directed bond percolation models on the square lattice as follows. The horizontal edges are directed rightward with probabilities one, and the directed bond from (i, j) to $(i - k + 1, j + 1)$, $k = 0, 1, \ldots, m$ with respective probabilities $p_k \in (0, 1)$, where m is a non-negative integer (see Fig. 1). Notice that this model has been considered in [13]. However, the critical value derived there is questionable, and the rate of convergence has not yet been discussed.

The vertices (sites) of the square lattice are located at a two-dimensional rectangular net $\{(i, j) \in \mathbb{Z} \times \mathbb{Z}_+ : i \le M$ and $0 \le j \le N\}$. Throughout this article, we denote $q_k = 1 - p_k$, $k = 0, 1, \ldots, m$ for convenience. We say that the vertex (i, j) is percolating if there is at least one connected-directed path of occupied edges from $(0, 0)$ to (i, j). Given any $\alpha \in \mathbb{R}$, denote $N_\alpha = \lfloor \alpha N \rfloor = \sup\{i \in \mathbb{Z} : i \le \alpha N\}$ with $N \in \mathbb{Z}_+$. Let $\mathbb{P}$ be the probability distribution of the bond variables, and define the two point correlation function

$$\tau_m(N_\alpha, N) = \mathbb{P}((N_\alpha, N) \text{ is percolating}) . \tag{1.1}$$

2. Main results

For convenience, define

$$\bar{q}_m = q_0 \cdots q_{m-1} q_m ,$$

and the notation $a_1 \approx a_2$ means that $a_1/a_2 \in (0, \infty)$. $f(N) \approx_{N\to\infty} g(N)$ and $f(\alpha) \approx_{\alpha\to\alpha_c} g(\alpha)$ mean that the limits $\lim_{N\to\infty} f(N)/g(N)$ and $\lim_{\alpha\to\alpha_c} f(\alpha)/g(\alpha)$ are bounded from zero and infinite, respectively. We have the following main theorem.

Theorem 2.1. *Given a finite $m \in \mathbb{Z}_+$ and $p_k \in (0, 1)$, $k = 0, 1, 2, \ldots, m$, there is a critical aspect ratio*

$$\alpha_{m,c} = \frac{\prod_{j=1}^m q_j^j}{1 - \bar{q}_m} - \sum_{k=2}^m \left(1 - q_k q_{k+1}^2 \cdots q_m^{m-k+1}\right) , \tag{2.1}$$

such that

$$\tau_m(N_{\alpha_{m,c}}, N) = \frac{1}{2} + O(\frac{1}{\sqrt{N}}) \tag{2.2}$$

in the large N limit, and when α is close to $\alpha_{m,c}$ but not equal to $\alpha_{m,c}$

$$\tau_m(N_\alpha, N) \le e^{-NI(\alpha)} \quad \textit{for} \quad \alpha < \alpha_{m,c} ,$$
$$1 - \tau_m(N_\alpha, N) \le e^{-NI(\alpha)} \quad \textit{for} \quad \alpha > \alpha_{m,c} ,$$

where

$$I(\alpha) \approx (\alpha - \alpha_{m,c})^2 \,. \tag{2.3}$$

Notice that the rate function $I(\alpha)$ in (2.3) is optimal [14], which gives the upper bound of $\tau_m(N_\alpha, N)$ or $1 - \tau_m(N_\alpha, N)$ as $e^{-NI(\alpha)}$ when α is smaller or larger than $\alpha_{m,c}$, respectively.

Remark 2.2. (2.1) can be written as

$$\alpha_{m,c} = \frac{\sum_{k=1}^{m} q_k q_{k+1}^2 \cdots q_m^{m-k+1} + \bar{q}_m \sum_{k=2}^{m}(1 - q_k q_{k+1}^2 \cdots q_m^{m-k+1}) - (m-1)}{1 - \bar{q}_m} \,, \tag{2.4}$$

which corrects the expression

$$\alpha_{m,c} = \frac{\sum_{k=1}^{m} q_k q_{k+1} \cdots q_m - (m-1)}{1 - \bar{q}_m} \tag{2.5}$$

given in [13]. These two expressions of $\alpha_{m,c}$ are equal only when $m = 0$ or $m = 1$. In particular, $\alpha_{2,c}$ should be $\frac{q_1 q_2^2}{1-q_0q_1q_2} - p_2$ using (2.4), rather than $\frac{q_1q_2+q_2-1}{1-q_0q_1q_2} = \frac{q_1q_2(p_0+q_0q_2)}{1-q_0q_1q_2} - p_2$ using (2.5). From Theorem 2.1, we obtain $\alpha_{0,c} = \frac{1}{p_0}$, and $\alpha_{1,c} = \frac{1-p_0}{p_0+p_1-p_0p_1}$ corresponds to that for the triangle lattice in [15]. While Ref. [13] only gave the limiting behavior of τ_m, here in Theorem 2.1 we obtain the rate of convergence of τ_m as well, namely, the main result of [13] is extended.

Next result is the investigation of the asymptotic phenomena of $\tau_m(N_{\alpha^-_{m,N}}, N)$ and $\tau_m(N_{\alpha^+_{m,N}}, N)$, where $\alpha^+_{m,N} \downarrow \alpha_{m,c}$ and $\alpha^-_{m,N} \uparrow \alpha_{m,c}$ as $N \uparrow \infty$. A sequence $\{\ell_n\}_{n=1}^\infty$ is called a regularly varying sequence if $\lim_{n\to\infty} \ell_{\lfloor \lambda n \rfloor}/\ell_n = 1$ for any $\lambda \in (0, \infty)$. We obtain the following corollary, which can be shown by the argument used in the proof of Theorem 2.3 in [12]. In the following expressions of upper bounds, c stands for a certain positive constant whose value varies from one equation to another.

Corollary 2.3. *Given a finite $m \in \mathbb{Z}_+$ and $p_k \in (0, 1)$, $k = 0, 1, 2, \ldots, m$, let $\alpha^-_{m,N} = \alpha_{m,c} - N^{-\frac{\rho}{2}}\ell_N$ and $\alpha^+_{m,N} = \alpha_{m,c} + N^{-\frac{\rho}{2}}\ell_N$, where $\rho \in (0, \infty)$ and $\{\ell_N\}_{N=1}^\infty$ is a positive regularly varying sequence. We obtain*

$$\tau_m(N_{\alpha^-_{m,N}}, N),\ 1 - \tau_m(N_{\alpha^+_{m,N}}, N)$$
$$\begin{cases} \le \exp\big(-cN^{1-\rho}\ell_N^2\big), & \text{if } \rho \in (0,1), \\ \le \exp\big(-c\ell_N^2\big), & \text{if } \rho = 1, \text{ and } \lim_{N\to\infty} \ell_N = \infty, \\ = \Psi(L) + O(1)\max\{\ell_N - L, \dfrac{1}{\sqrt{N}}\}, & \text{if } \rho = 1, \text{ and } \lim_{N\to\infty} \ell_N = L \in [0,\infty), \\ = \dfrac{1}{2} + O\big(\dfrac{\ell_N}{N^{\frac{\rho}{2}-\frac{1}{2}}}\big), & \text{if } \rho \in (1,2], \\ = \dfrac{1}{2} + O\big(\dfrac{1}{\sqrt{N}}\big), & \text{if } \rho > 2. \end{cases} \tag{2.6}$$

It is interesting to investigate the behavior in the limit $m \to \infty$, and the rest part of this section is devoted to such limit. Again we define

$$\tau(N_\alpha, N) = \mathbb{P}((N_\alpha, N) \text{ is percolating}) \,. \tag{2.7}$$

It is necessary that $p_k \to 0$ as $k \to \infty$, we can rewrite

$$\bar{q}_m = e^{\sum_{k=0}^{m} \log(1-p_k)}, \quad q_k q_{k+1}^2 \cdots q_m^{m-k+1} = e^{\sum_{j=1}^{m-k+1} j \log(1-p_{k+j-1})} \,, \tag{2.8}$$

and

$$-p_k(1+p_k) \le \log(1-p_k) \le -p_k \quad \text{for} \quad p_k \in (0, 0.6838026238) \,,$$

so that

$$\bar{q}_m \underset{m\to\infty}{\approx} e^{-\sum_{k=0}^{m} p_k} \tag{2.9}$$

and for $k \ge 1$

$$q_k q_{k+1}^2 \cdots q_m^{m-k+1} \underset{m\to\infty}{\approx} e^{-\sum_{j=1}^{m-k+1} j p_{k+j-1}} \,. \tag{2.10}$$

Therefore by (2.1), under the condition $p_k \to 0$ as $k \to \infty$ we obtain

$$\alpha_c := \lim_{m\to\infty} \alpha_{m,c} \approx \lim_{m\to\infty} \Big(\frac{\exp(-\sum_{j=1}^{m} jp_j)}{1 - e^{-\sum_{k=0}^{m} p_k}} - \sum_{k=2}^{m} \big(1 - \exp(-\sum_{j=1}^{m-k+1} jp_{k+j-1})\big) \Big) . \tag{2.11}$$

Hereafter we shall omit the subscript m for the limit $m \to \infty$.

Let us consider the following three cases for $m \to \infty$, i.e., $p_n \in [0, 1)$ with $n \in \mathbb{Z}_+$. (i) $p_n = \frac{a}{a+1+n}$ with $a > 0$; (ii). $p_n = \frac{\beta}{m}$ with $\beta \in (0, m)$. (iii) $p_n \approx_{n\to\infty} \frac{p}{n^s}$ with $p \in (0, 1)$, $s > 0$.

Consider the case (i), it is easy to see that $\lim_{m\to\infty} \sum_{j=1}^{m} jp_j = \infty$ and $\lim_{m\to\infty} \sum_{j=1}^{m} p_j = \infty$. Moreover,

$$\lim_{m\to\infty} \sum_{k=2}^{m} \exp\big(-\sum_{j=1}^{m-k+1} jp_{k+j-1}\big) = \lim_{m\to\infty} \sum_{k=2}^{m} \exp\big(-\sum_{j=1}^{m-k+1} \frac{ja}{a+k+j}\big)$$
$$= \lim_{m\to\infty} \sum_{k=2}^{m} \exp\big(-a(m-k+1) + a(a+k)[\Psi(a+m+2) - \Psi(a+k+1)]\big) ,$$

where

$$\Psi(k) = \frac{\Gamma'(k)}{\Gamma(k)}$$

is the digamma function. It follows that $\lim_{m\to\infty} \sum_{k=2}^{m} \exp\big(-\sum_{j=1}^{m-k+1} jp_{k+j-1}\big) < \infty$, and by (2.11) we have

$$\alpha_c := \lim_{m\to\infty} \alpha_{m,c} = -\infty . \tag{2.12}$$

For the case (ii), by (2.11) again it is easy to see that $\alpha_c = -\infty$.

Case (iii) is most interesting, because its behavior is quite different from the case with finite m. We obtain the following result.

Theorem 2.4. *Let $p_n \in [0, 1)$ with $n \in \mathbb{Z}_+$ and $p_n \approx_{n\to\infty} \frac{p}{n^s}$ with $p \in (0, 1)$ and $s > 0$. For $s \in (0, 3]$ we have*

$$\tau(N_\alpha, N) \to 1, \quad \text{as } N \to \infty \tag{2.13}$$

for any $\alpha \in \mathbb{R}$. For $s > 3$, we have

$$\alpha_c \in (-\infty, 1) \tag{2.14}$$

and

$$\tau(N_{\alpha_c}, N) \begin{cases} \in (0, 1), & s \in (3, 4), \\ = \frac{1}{2} + O(\frac{1}{\log N}), & s = 4, \\ = \frac{1}{2} + O(\frac{1}{N^{\frac{s-4}{2}}}), & s \in (4, 5), \\ = \frac{1}{2} + O(\frac{\log N}{\sqrt{N}}), & s = 5, \\ = \frac{1}{2} + O(\frac{1}{\sqrt{N}}), & s > 5, \end{cases} \tag{2.15}$$

in the large N limit. Moreover for $s > 3$, when α is close to α_c but not equal to α_c, we obtain

$$\tau(N_\alpha, N) \le \frac{1}{(\alpha_c - \alpha)^{s-2} N^{s-3}} \quad \text{for} \quad \alpha < \alpha_c ,$$
$$1 - \tau(N_\alpha, N) \le e^{-NI(\alpha)} \quad \text{for} \quad \alpha > \alpha_c , \tag{2.16}$$

where

$$I(\alpha) \approx \begin{cases} (\alpha - \alpha_c)^{1+\frac{1}{s-3}} & \text{for } s \in (3, 4] , \\ (\alpha - \alpha_c)^2 & \text{for } s > 4 . \end{cases} \tag{2.17}$$

The critical power s_c can be defined such that $\alpha_c = -\infty$ if $s \le s_c$, and $\alpha_c \in (-\infty, 1)$ if $s > s_c$. We find here that the critical power s_c is equal to 3, which corrects the value 2 given in [13]. For $s \in (3, 4)$, the convergence of $\tau(N_{\alpha_c}, N)$ is too weak to decide its value as $N \to \infty$, such that $\tau(N_{\alpha_c}, N)$ in (2.15) may not always tend to $\frac{1}{2}$ as $N \to \infty$. We shall discuss this further in Remark 5.1. It is appropriate to define some standard critical exponents and to sketch the phenomenological scaling theory of $\tau(N_\alpha, N)$. For $\alpha > \alpha_c$ and α close to α_c, let us write the upper bound of $\tau(N_\alpha, N)$ as (c.f. [16])

$$\tau(N_\alpha, N) \le \exp(\frac{-BN}{(\alpha_c-\alpha)^{-\nu}}) . \tag{2.18}$$

By Theorem 2.4, if $s \in (3, 4)$ we have $\nu = 1 + \frac{1}{s-3} > 2$, and if $s \ge 4$ we have $\nu = 2$. For $\alpha < \alpha_c$ and α close to α_c, the decay of the upper bound for $\tau(N_\alpha, N)$ is not exponential.

The last result of this article is the investigation of the asymptotic phenomena of $\tau(N_{\alpha_N^-}, N)$ and $\tau(N_{\alpha_N^+}, N)$ where $\alpha_N^+ \downarrow \alpha_c$ and $\alpha_N^- \uparrow \alpha_c$ as $N \uparrow \infty$. For convenience, we denote $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du$ as the standard cumulative distribution function of *Gaussian* distribution with mean 0, variance 1 and let $\Psi(x) = 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{u^2}{2}}\, du$. It is not difficult to see that

$$\Psi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi} x}\big(1 + O(x^{-2})\big) \quad \text{for large } x\, , \tag{2.19}$$

$$\frac{1}{2} - \Psi(x) = \frac{x}{\sqrt{2\pi}}\big(1 + O(x^{-2})\big) \quad \text{for small } x\, . \tag{2.20}$$

In the following expressions of upper bounds, c stands for certain positive constant whose value varies from one equation to another, and is independent of p_n and n. We have the following theorem.

Theorem 2.5. *Given $p_n \in [0, 1)$ with $n \in \mathbb{Z}_+$ and $p_n \approx_{n\to\infty} \frac{p}{n^s}$ with $p \in (0, 1)$ and $s > 3$, let $\alpha_N^- = \alpha_c - N^{-\frac{\rho(s\wedge 4-3)}{(s\wedge 4-2)}} \ell_N$ and $\alpha_N^+ = \alpha_c + N^{-\frac{\rho(s\wedge 4-3)}{(s\wedge 4-2)}} \ell_N$, where $\rho \in (0, \infty)$ and $\{\ell_N\}_{N=1}^{\infty}$ is a positive regularly varying sequence. We obtain the following results in the large N limit. For $\rho < 1$, we have*

$$\tau(N_{\alpha_N^-}, N) \le \frac{1}{N^{(s-3)(1-\rho)} \ell_N^{s-2}}, \quad 1 - \tau(N_{\alpha_N^+}, N) \le \exp\big(-cN^{1-\rho} \ell_N^{\frac{s\wedge 4-2}{s\wedge 4-3}}\big). \tag{2.21}$$

For $s \in (3, 4)$ and $\rho \ge 1$, we have

$$1 - \tau(N_{\alpha_N^+}, N) \underset{m\to\infty}{\approx} \begin{cases} \exp(-c\ell_N^{\frac{s-2}{s-3}}), & \rho = 1,\ \lim_{N\to\infty} \ell_N = \infty, \\ \in (0, 1), & \rho = 1,\ \lim_{N\to\infty} \ell_N \in (0, \infty), \quad \text{or} \quad \rho > 1, \end{cases}$$

and

$$\tau(N_{\alpha_N^-}, N) \le \begin{cases} \frac{1}{\ell_N^{s-2}}, & \rho = 1,\ \lim_{N\to\infty} \ell_N = \infty, \\ \in (0, 1), & \rho = 1,\ \lim_{N\to\infty} \ell_N \in (0, \infty), \quad \text{or} \quad \rho > 1. \end{cases}$$

For $s \ge 4$ and $\rho = 1$, we have both

$$\tau(N_{\alpha_N^-}, N),\ 1 - \tau(N_{\alpha_N^+}, N)$$
$$= \begin{cases} O(1)\max\{\frac{\sqrt{\log N}}{\ell_N} e^{-\frac{c\ell_N^2}{\log N}}, \frac{1}{\log N}\}, & \text{if } s = 4 \text{ and } \lim_{N\to\infty} \frac{\ell_N}{\sqrt{\log N}} = \infty, \\ \lambda_1 + O(1)\max\{|\frac{\ell_N}{\sqrt{\log N}} - \mathcal{L}|, \frac{1}{\log N}\}, & \text{if } s = 4 \text{ and } \lim_{N\to\infty} \frac{\ell_N}{\sqrt{\log N}} = \mathcal{L} \in [0, \infty), \\ O(1)\max\{\frac{1}{\ell_N} e^{-\frac{\ell_N^2}{2\sigma^2}}, \frac{1}{N^{\frac{s-4}{2}}}\}, & \text{if } s \in (4, 5),\ \lim_{N\to\infty} \ell_N = \infty, \\ \lambda_2 + O(1)\max\{|\ell_N - L|, \frac{1}{N^{\frac{s-4}{2}}}\}, & \text{if } s \in (4, 5),\ \lim_{N\to\infty} \ell_N = L \in [0, \infty), \\ O(1)\max\{\frac{1}{\ell_N} e^{-\frac{\ell_N^2}{2\sigma^2}}, \frac{\log N}{\sqrt{N}}\}, & \text{if } s = 5,\ \lim_{N\to\infty} \ell_N = \infty, \\ \lambda_3 + O(1)\max\{|\ell_N - L|, \frac{\log N}{\sqrt{N}}\}, & \text{if } s = 5,\ \lim_{N\to\infty} \ell_N = L \in [0, \infty), \\ O(1)\max\{\frac{1}{\ell_N} e^{-\frac{\ell_N^2}{2\sigma^2}}, \frac{1}{\sqrt{N}}\}, & \text{if } s > 5,\ \lim_{N\to\infty} \ell_N = \infty, \\ \lambda_4 + O(1)\max\{|\ell_N - L|, \frac{1}{\sqrt{N}}\}, & \text{if } s > 5,\ \lim_{N\to\infty} \ell_N = L \in [0, \infty), \end{cases} \tag{2.22}$$

where λ_i with $i = 1, 2, 3, 4$ are different constants between $(0, 1)$, σ^2 is a finite constant and stands for the variance of a certain random variable.

For $s \geq 4$ and $\rho > 1$, we have both

$$\tau(N_{\alpha_N^-}, N),\, 1 - \tau(N_{\alpha_N^+}, N) = \begin{cases} \frac{1}{2} + O(\frac{1}{\log N}), & \text{if } s = 4, \\ \frac{1}{2} + O(\frac{\ell_N}{N^{\frac{\rho}{2}-\frac{1}{2}}}), & \text{if } s \in (4,5),\ \rho \in (1, s-3], \\ \frac{1}{2} + O(\frac{1}{N^{\frac{s-4}{2}}}), & \text{if } s \in (4,5),\ \rho > s-3, \\ \frac{1}{2} + O(\frac{\ell_N}{N^{\frac{\rho}{2}-\frac{1}{2}}}), & \text{if } s = 5,\ \rho \in (1,2), \\ \frac{1}{2} + O(1)\max\{\frac{\ell_N}{\sqrt{N}}, \frac{\log N}{\sqrt{N}}\}, & \text{if } s = 5,\ \rho = 2, \\ \frac{1}{2} + O(\frac{\log N}{\sqrt{N}}), & \text{if } s = 5,\ \rho > 2, \\ \frac{1}{2} + O(\frac{\ell_N}{N^{\frac{\rho}{2}-\frac{1}{2}}}), & \text{if } s > 5,\ \rho \in (1,2], \\ \frac{1}{2} + O(\frac{1}{\sqrt{N}}), & \text{if } s > 5,\ \rho > 2. \end{cases} \tag{2.23}$$

3. Derivation of $\alpha_{m,c}$ and σ_m^2

For any $N \in \mathbb{N}$, we say that an occupied vertical edge in a bond configuration is wet if it lies on a percolating path where (N_α, N) is percolating. For a certain occupied vertical edge ending at (k, n), we say that it is *primary* wet if it is the wet edge with smallest k value for that n. In a percolating configuration where (N_α, N) is percolating, there is one *primary wet* edge for each $n \in \{1, 2, \ldots, N\}$. Define $C_{m,N}(k)$ as the probability that the *primary wet* edge for $n = N$ ending at (k, N), and let us formally define $C_{m,0}(k) = \delta_{0,k}$ where δ is the Kronecker delta. Since the primary wet edge can occur at any value of $k \leq N_\alpha$, we have

$$\tau_m(N_\alpha, N) = \sum_{k \leq N_\alpha} C_{m,N}(k) \tag{3.1}$$

for $N \in \mathbb{N}$.

By the definition of our model, for any $k \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$C_{m,n+1}(k) = \sum_{j \in \mathbb{Z}} C_{m,n}(k-j) D_m(j)\,, \tag{3.2}$$

where

$$D_m(j) = \begin{cases} 0, & \text{if } \ j \leq -m, \\ 1 - q_m & \text{if } \ j = -m+1, \\ U_m(j) & \text{if } \ j \in \{-m+2, \ldots, 0\}, \\ (\bar{q}_m)^{j-1}(\prod_{l=1}^{m} q_l^l)(1-\bar{q}_m) & \text{if } \ j \geq 1, \end{cases} \tag{3.3}$$

with

$$U_m(j) = q_{-j+2} q_{-j+3}^2 \cdots q_m^{m+j-1}(1 - q_{-j+1} \cdots q_m), \quad \text{if } \ j \in \{-m+2, \ldots, 0\}\,.$$

As an example, we illustrate $D_2(0)$ in Fig. 2 and $D_2(1)$ in Fig. 3 for $m = 2$.

For any probability distribution $f : \mathbb{Z}_+ \to \mathbb{R}_+$, its generating function can be defined as

$$\hat{f}(t) = \sum_{j \in \mathbb{Z}} f(j) t^j, \qquad \text{where } |t| \text{ is less than the radius of convergence.} \tag{3.4}$$

It follows that

$$\hat{C}_{m,n}(t) = \hat{D}_m(t)^n. \tag{3.5}$$

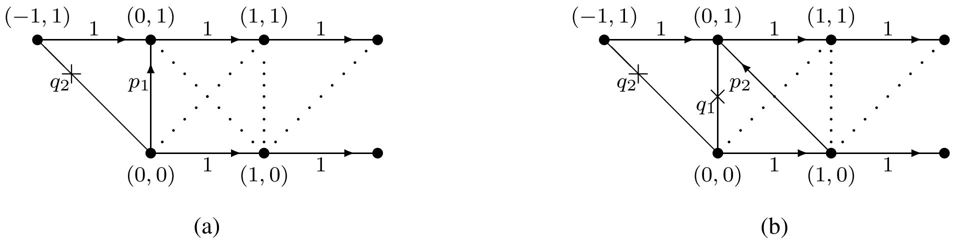

Fig. 2. Illustrations for $D_2(0)$, including (a) $P(\{(0,0) \to (0,1)\}) = p_1 q_2$ and (b) $P(\{(1,0) \to (0,1)\}) = p_2(q_1 q_2)$. The connection with an arrow is occupied, while the others with a cross is unoccupied. The dotted lines can be either occupied or unoccupied.

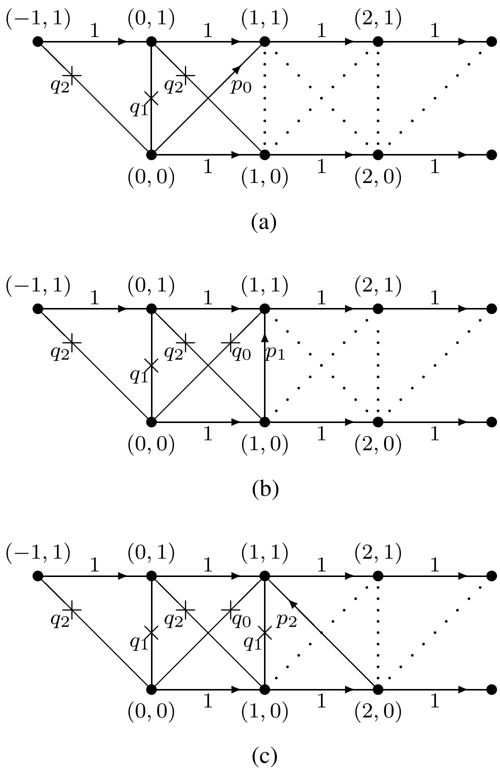

Fig. 3. Illustrations for $D_2(1)$, including (a) $P(\{(0,0) \to (1,1)\}) = (p_0 q_1 q_2) q_2$, (b) $P(\{(1,0) \to (1,1)\}) = (q_0 q_1 q_2)(p_1 q_2)$, (c) $P(\{(2,0) \to (1,1)\}) = (q_0 q_1 q_2)(q_1 q_2) p_2$. The connection with an arrow is occupied, while the others with a cross is unoccupied. The dotted lines can be either occupied or unoccupied.

By (3.3), we have

$$\hat{D}_m(t) = \hat{U}_m(t) + (1 - q_m) t^{-m+1} + \frac{(\prod_{l=1}^{m} q_l^l)(1 - \bar{q}_m) t}{1 - \bar{q}_m t} , \tag{3.6}$$

with

$$\hat{U}_m(t) = \sum_{j=0}^{m-2} q_{j+2} q_{j+3}^2 \cdots q_m^{m-j-1} (1 - q_{j+1} \cdots q_m) t^{-j} .$$

Note that

$$(1 - q_m) + \hat{U}_m(1) = 1 - q_1 q_2^2 \cdots q_m^m = 1 - \prod_{j=1}^{m} q_j^j , \tag{3.7}$$

such that $\hat{D}_m(1) = 1$.

Define the average mean of 1-step walk as

$$\mu_m = \sum_{j\in\mathbb{Z}} jD_m(j) = \hat{D}'_m(1). \tag{3.8}$$

We shall show in next section that

$$\alpha_{m,c} = \mu_m. \tag{3.9}$$

By (3.6), we have

$$\frac{d}{dt}\hat{D}_m(t) = \frac{d}{dt}\hat{U}_m(t) - (m-1)(1-q_m)t^{-m} + \frac{(\prod_{j=1}^m q_j^j)(1-\bar{q}_m)}{(1-\bar{q}_m t)^2}\,, \tag{3.10}$$

where

$$\frac{d}{dt}\hat{U}_m(t) = -\sum_{j=1}^{m-2} jq_{j+2}q_{j+3}^2\cdots q_m^{m-j-1}(1-q_{j+1}\cdots q_m)t^{-j-1}\,.$$

Taking $t=1$, we have

$$\begin{aligned}
&\Big(\frac{d}{dt}\hat{U}_m(t) - (m-1)(1-q_m)t^{-m}\Big)\Big|_{t=1}\\
&= -(m-1)(1-q_m) - (m-2)q_m(1-q_{m-1}q_m) - \cdots - q_3q_4^2\cdots q_m^{m-2}(1-q_2q_3\cdots q_m)\\
&= -(m-1) + \sum_{k=2}^{m}(q_kq_{k+1}^2\cdots q_m^{m-k+1}).
\end{aligned}$$

Therefore from (3.9) and (3.10), we obtain

$$\begin{aligned}
\alpha_{m,c} &= \frac{\prod_{j=1}^m q_j^j}{1-\bar{q}_m} - (m-1) + \sum_{k=2}^{m} q_kq_{k+1}^2\cdots q_m^{m-k+1}\\
&= \frac{\prod_{j=1}^m q_j^j}{1-\bar{q}_m} - \sum_{k=2}^{m}\big(1-q_kq_{k+1}^2\cdots q_m^{m-k+1}\big).
\end{aligned} \tag{3.11}$$

Furthermore, the variance of 1-step walk is defined as

$$\sigma_m^2 = \sum_{j\in\mathbb{Z}} j^2D_m(j) - \mu_m^2\,, \tag{3.12}$$

so the variance of the two-point function is given by

$$\sigma_m^2 = \frac{d^2}{dt^2}\hat{D}_m(1) + \alpha_{m,c} - \alpha_{m,c}^2\,. \tag{3.13}$$

By (3.10), we have

$$\frac{d^2}{dt^2}\hat{D}_m(t)|_{t=1} = \frac{d^2}{dt^2}\hat{U}_m(t)|_{t=1} + m(m-1)(1-q_m) + \frac{2\bar{q}_m(\prod_{j=1}^m q_j^j)}{(1-\bar{q}_m)^2}\,, \tag{3.14}$$

where

$$\frac{d^2}{dt^2}\hat{U}_m(t) = \sum_{j=1}^{m-2} j(j+1)q_{j+2}q_{j+3}^2\cdots q_m^{m-j-1}(1-q_{j+1}\cdots q_m)t^{-j-2}\,.$$

Therefore, we find

$$\frac{d^2}{dt^2}\hat{U}_m(t)|_{t=1} + m(m-1)(1-q_m) = m(m-1) - 2\sum_{k=1}^{m-1} kq_{k+1}^1\cdots q_m^{m-k}\,,$$

such that

$$\sigma_m^2 = m(m-1) - 2\sum_{k=1}^{m-1} kq_{k+1}^1\cdots q_m^{m-k} + \frac{2\bar{q}_m(\prod_{j=1}^m q_j^j)}{(1-\bar{q}_m)^2} + \alpha_{m,c} - (\alpha_{m,c})^2. \tag{3.15}$$

4. Proof of Theorem 2.1

As we have obtained the expression of $\alpha_{m,c}$ in the previous section, here we shall study the behavior of $\tau_m(N_\alpha, N)$ when α is close to this value. Let $m \in \mathbb{N}$ be fixed in this section. Define a N-step random walk $S_{m,N}$ where the distribution of each step is given by D_m and the probability Prob_m such that $\mathrm{Prob}_m(S_{m,N} = j) = C_{m,N}(j)$ with $j \in \mathbb{Z}$ and $\mathrm{Prob}_m(S_{m,0} = j) = \delta_{0,j}$. The expectation for Prob_m is denoted by Exp_m. It is easy to see that $S_{m,n}$ is the n-sum of these independent random variables and each random variable has finite third moment. By the law of large number, we have

$$\frac{S_{m,N}}{N} \to \alpha_{m,c} \quad \text{a.s.} \quad \text{when } N \to \infty\,, \tag{4.1}$$

where a.s. stands for "almost surely". Notice that the variance of $S_{m,N}$ is given by $N\sigma_m^2$. Berry–Esseen theorem (c.f. [17]) asserts that

$$\Big|\mathrm{Prob}_m\Big(\frac{S_{m,N}-\alpha_{m,c}N}{\sqrt{N\sigma_m^2}} \le \frac{N(\alpha-\alpha_{m,c})}{\sqrt{N\sigma_m^2}}\Big) - \int_{-\infty}^{\frac{N(\alpha-\alpha_{m,c})}{\sqrt{N\sigma_m^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du\Big| \le O\Big(\frac{\sum_{j\in\mathbb{Z}}|j|^3 D_m(j)}{\sqrt{N}\sigma_m^3}\Big). \tag{4.2}$$

With the definition of N_α given in the introduction, we have

$$\mathrm{Prob}_m\big(S_{m,N} \le \alpha N - 1\big) \le \tau_m(N_\alpha, N) = \mathrm{Prob}_m(S_{m,N} \le N_\alpha) \le \mathrm{Prob}_m\big(S_{m,N} \le \alpha N\big). \tag{4.3}$$

Setting $\alpha = \alpha_{m,c}$ and using $\sum_{j\in\mathbb{Z}}|j|^3 D_m(j) < \infty$, we obtain

$$\tau_m(N_{\alpha_{m,c}}, N) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du + O\Big(\frac{1}{\sqrt{N}}\Big) = \frac{1}{2} + O\Big(\frac{1}{\sqrt{N}}\Big)\,, \tag{4.4}$$

which gives (2.2).

In the following part of this section, we shall consider a general $\alpha \ne \alpha_{m,c}$. When $\alpha < \alpha_{m,c}$, we set $\eta = -\log t > 0$ and use Chernov inequality to have

$$\begin{aligned}\mathrm{Prob}_m(S_{m,N} \le N_\alpha) &\le \inf_{\eta>0} \frac{\mathrm{Exp}_m(e^{-\eta S_{m,N}})}{e^{-\eta\alpha N}} = \inf_{t\in(0,1)} \frac{\hat{S}_{m,N}(t)}{t^{\alpha N}} \\ &\le e^{-N I_m(\alpha)},\end{aligned} \tag{4.5}$$

where

$$I_m(\alpha) = \sup_{t>0}\Big\{\alpha\log t - \log\hat{D}_m(t)\Big\} := \alpha\log t_\alpha - \log\hat{D}_m(t_\alpha)\,. \tag{4.6}$$

Similarly, when $\alpha > \alpha_{m,c}$, we set $\eta = \log t > 0$ to have

$$\begin{aligned}\mathrm{Prob}_m(S_{m,N} > N_\alpha) &= \inf_{\eta>0} \mathrm{Prob}_m(e^{\eta S_{m,N}} > e^{\eta\alpha N - \eta c_\alpha}) \\ &\underset{N\to\infty}{\approx} \inf_{\eta>0} \mathrm{Prob}_m(e^{\eta S_{m,N}} > e^{\eta\alpha N}) \\ &\le \inf_{\eta>0} \frac{\mathrm{Exp}_m(e^{\eta S_{m,N}})}{e^{\eta\alpha N}} \\ &\le e^{-N I_m(\alpha)},\end{aligned} \tag{4.7}$$

where $c_\alpha \in [0, 1)$.

By (4.6), we have

$$\frac{\alpha}{t_\alpha} = \frac{\hat{D}'_m(t_\alpha)}{\hat{D}_m(t_\alpha)}\,, \tag{4.8}$$

such that

$$I'_m(\alpha) = \log t_\alpha - \Big(\frac{\hat{D}'_m(t_\alpha)}{\hat{D}_m(t_\alpha)} - \frac{\alpha}{t_\alpha}\Big)\frac{d\,t_\alpha}{d\,\alpha} = \log t_\alpha\,. \tag{4.9}$$

As $\hat{D}_m(t = 1) = 1$ and $d\hat{D}_m(t)/dt|_{t=1} = \alpha_{m,c}$, setting $t_\alpha = 1$ in (4.8) leads to $t_{\alpha_{m,c}} = \hat{D}_m(t_{\alpha_{m,c}}) = 1$ and hence $I_m(\alpha_{m,c}) = I'_m(\alpha_{m,c}) = 0$. Furthermore, by the definition of the generating function and (4.8), we have

$$\alpha = \frac{\sum_{n\in\mathbb{Z}} nD_m(n)t_\alpha^n}{\sum_{n\in\mathbb{Z}} D_m(n)t_\alpha^n} = \frac{t_\alpha \hat{D}'_m(t_\alpha)}{\hat{D}_m(t_\alpha)} . \tag{4.10}$$

Taking derivative with respect to α on both sides of (4.8) leads to

$$\frac{1}{t_\alpha} - \frac{\alpha \frac{d\,t_\alpha}{d\,\alpha}}{t_\alpha^2} = \Big(\frac{\hat{D}''_m(t_\alpha)}{\hat{D}_m(t_\alpha)} - \big(\frac{\hat{D}'_m(t_\alpha)}{\hat{D}_m(t_\alpha)}\big)^2\Big)\frac{d\,t_\alpha}{d\,\alpha} .$$

It follows that

$$I''_m(\alpha) = \frac{\frac{d\,t_\alpha}{d\,\alpha}}{t_\alpha} = \frac{1}{V_m(\alpha)}, \quad \text{where } V_m(\alpha) = \frac{\sum_{n\in\mathbb{Z}} n^2 D_m(n)t_\alpha^n}{\hat{D}_m(t_\alpha)} - \alpha^2 \in (0,\infty) \quad \text{for all } \alpha \in \mathbb{R}. \tag{4.11}$$

Therefore, $I_m(\alpha)$ is a strictly convex function with local minimum at $\alpha_{m,c}$ and $I_m(\alpha_{m,c}) = 0$.

As t_α is continuous with respect to α, we have $|t_\alpha - t_{\alpha_{m,c}}| \in (0, 1)$ when α is close to α_c. By (4.9) we get

$$I'_m(\alpha) = \log t_\alpha = \log\big(t_{\alpha_{m,c}} + (t_\alpha - t_{\alpha_{m,c}})\big) = (t_\alpha - t_{\alpha_{m,c}}) + O(1)(t_\alpha - t_{\alpha_{m,c}})^2,$$

while using mean value theorem and (4.11), we have $I'_m(\alpha) \approx I''_m(\alpha_{m,c})(\alpha - \alpha_{m,c})$. It follows that

$$t_\alpha - t_{\alpha_{m,c}} \underset{\alpha\to\alpha_{m,c}}{\approx} \alpha - \alpha_{m,c} ,$$

so that

$$I_m(\alpha) = \int_{\alpha_{m,c}}^{\alpha} I'_m(u)\,d\,u \underset{\alpha\to\alpha_{m,c}}{\approx} \int_{\alpha_{m,c}}^{\alpha} (t_u - t_{\alpha_{m,c}})\,d\,u \underset{\alpha\to\alpha_{m,c}}{\approx} \int_{\alpha_{m,c}}^{\alpha} (u - \alpha_{m,c})\,d\,u = \frac{(\alpha - \alpha_{m,c})^2}{2} .$$

This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.4

In this section, we shall consider $p_n \in [0, 1)$ with $n \in \mathbb{Z}_+$ and $p_n \approx_{n\to\infty} \frac{p}{n^s}$ with $p \in (0, 1)$ and $s > 0$. Since

$$\sum_{j=1}^{m} jp_j \underset{m\to\infty}{\approx} \sum_{j=1}^{m} \frac{pj}{j^s} \begin{cases} = \infty & \text{if } s \le 2, \\ \in (0,\infty) & \text{if } s > 2, \end{cases}$$

and

$$\sum_{j=1}^{m-k+1} jp_{k+j-1} \underset{m\to\infty}{\approx} \sum_{j=1}^{m-k+1} \frac{pj}{(k+j)^s} \begin{cases} = \infty & \text{if } s \le 2, \\ \approx pk^{2-s} & \text{if } s > 2, \end{cases}$$

by (2.11) we obtain

$$\alpha_c \begin{cases} = -\infty & \text{if } s \in (0, 2], \\ \approx \lim_{m\to\infty} \sum_{k=2}^{m} -\big(1 - e^{-pk^{2-s}}\big) + p & \text{if } s > 2, \end{cases}$$
$$\begin{cases} = -\infty & \text{if } s \in (0, 2], \\ \approx \lim_{m\to\infty} \sum_{k=2}^{m} -pk^{2-s} + p & \text{if } s > 2, \end{cases}$$
$$\begin{cases} = -\infty & \text{if } s \in (0, 3], \\ \in (-\infty, 1) & \text{if } s > 3. \end{cases} \tag{5.1}$$

That is, α_c exists for any $p \in (0, 1)$ when $s > 3$. We find $\alpha_c = -\infty$ when $s \in (0, 3]$, and $\tau(N_\alpha, N) \to 1$ for any $\alpha \in \mathbb{R}$. Therefore, we shall only consider $s > 3$ in the following discussion of this section.

Let us consider the variance of $D(j)$. We need to analyze the first two terms of (3.15) in the limit $m \to \infty$. Similar to (2.10), for $s > 3$ as $m \to \infty$ we have

$$m(m-1) - 2\sum_{k=1}^{m-1} k q_{k+1}^{1} \cdots q_m^{m-k} = 2\sum_{k=1}^{m-1} k(1 - q_{k+1}^{1}\cdots q_m^{m-k}) \underset{m\to\infty}{\approx} \sum_{k=1}^{m-1} k\left(1 - e^{-pk^{2-s}}\right) \underset{m\to\infty}{\approx} p\sum_{k=1}^{m-1} k^{3-s}. \tag{5.2}$$

By (2.9) and (2.10), it is easy to see that $\lim_{m\to\infty} \frac{\bar{q}_m(\prod_{j=1}^{m} q_j^j)}{(1-\bar{q}_m)^2}$ is finite for $s > 3$, and so is α_c. Using (5.2) in (3.15), we obtain

$$\sigma^2 := \lim_{m\to\infty} \sigma_m^2 \begin{cases} = \infty & s \in (3,4], \\ \in (0,\infty), & s > 4. \end{cases} \tag{5.3}$$

More precisely, the asymptotic behavior of the variance in the infinite m limit is given by

$$\sigma_m^2 \underset{m\to\infty}{\approx} \begin{cases} m^{4-s}, & \text{if } s \in (3,4), \\ \log m, & \text{if } s = 4, \end{cases} \tag{5.4}$$

and

$$\sigma_m^2 - \sigma^2 \underset{m\to\infty}{\approx} \frac{1}{m^{s-4}}, \quad \text{if } s > 4. \tag{5.5}$$

By (3.3) with infinite m, we have

$$D(-j) \underset{j\to\infty}{\approx} e^{-\frac{p}{j^{s-2}}}\left(1 - e^{-\frac{p}{j^{s-1}}}\right) \underset{j\to\infty}{\approx} \frac{p}{j^{s-1}}. \tag{5.6}$$

Note that combining (3.4) and (5.6), the generating function

$$\hat{D}(t) = \sum_{j=-1}^{\infty} D(-j)t^{-j} \approx \sum_{j=1}^{\infty} \frac{p}{j^{s-1}} t^{-j} \tag{5.7}$$

is well defined for $t \geq 1$ and $s > 2$.

Denote the probability Prob. such that $\text{Prob.}(S_N = j) = C_N(j)$ with $j \in \mathbb{Z}$ and $\text{Prob.}(S_0 = j) = C_0(j) = \delta_{0,j}$ where $C_N(k) = \sum_{j\in\mathbb{Z}} C_{N-1}(k-j)D(j)$ for $N \geq 1$. The expectation for Prob. is denoted by Exp. It is easy to see that S_n is the n-sum of independent random variables. Since $\sigma^2 = \infty$ for $s \in (3,4]$, we cannot use Berry–Esseen theorem directly and the probability should be separated into two parts,

$$\text{Prob.}(S_N \leq \alpha_c N) = \text{Prob.}(\tilde{S}_N \leq \alpha_c N) + \text{Prob.}(S_N \leq \alpha_c N, \exists j \in \{1, 2\ldots, N\} \text{ such that } Y_j < -N_T(s)).$$

The first term is the truncation with the definition $\tilde{S}_N = \sum_{k=1}^{N} \tilde{Y}_k$, where $\tilde{Y}_k$ are i.i.d. random variables with distribution

$$\text{Prob.}(\tilde{Y}_k = j) = \begin{cases} bD(j) & \text{if } j \geq -N_T(s), \\ 0 & \text{if } j < -N_T(s), \end{cases}$$

where b is a normalization constant and

$$N_T(s) = \begin{cases} (-\alpha_c \vee 1)N^{\frac{1}{s\wedge 4-2}}, & s > 3 \text{ and } s \neq 4, \\ (-\alpha_c \vee 1)\sqrt{N\log N}, & s = 4, \end{cases} \tag{5.8}$$

for any $N \geq 2$. By (2.11), (5.6) and using $\alpha_c \in (-\infty, 1)$, we find

$$\alpha_{N_T} - \alpha_c \underset{N\to\infty}{\approx} \begin{cases} \dfrac{1}{N^{\frac{s-3}{s\wedge 4-2}}}, & s > 3 \text{ and } s \neq 4, \\ \dfrac{1}{\sqrt{N\log N}}, & s = 4, \end{cases} \tag{5.9}$$

where $\alpha_{N_T} = \text{Exp.}(\tilde{Y}_1)$. From (5.4) and (5.5), we have

$$\sigma_{N_T}^2 = \text{var}(\tilde{Y}_1) \underset{N\to\infty}{\approx} \begin{cases} N^{\frac{4-s}{s-2}}, & s \in (3,4), \\ \log\sqrt{N\log N}, & s = 4, \\ 1 & s > 4. \end{cases} \tag{5.10}$$

More precisely,

$$\sigma_{N_T}^2 - \sigma^2 \underset{N\to\infty}{\approx} \frac{1}{N^{\frac{s-4}{2}}}, \quad \text{if } s > 4 \tag{5.11}$$

that will be used in next section. Similarly we also obtain

$$\text{Exp.}(\tilde{Y}_1)^3 \underset{N\to\infty}{\approx} \begin{cases} N^{\frac{5-s}{s-2}}, & s \in (3,4), \\ \sqrt{N\log N}, & s = 4, \\ N^{\frac{5-s}{2}}, & s \in (4,5), \\ \log\sqrt{N}, & s = 5, \\ 1, & s > 5. \end{cases} \tag{5.12}$$

Note that the variance of $\tilde{S}_N$ is $N \times \text{var}(\tilde{Y}_1)$. Berry–Esseen theorem (c.f. [17]) asserts that for any $\alpha \in \mathbb{R}$

$$\text{Prob.}(\tilde{S}_N \leq \alpha N) = \int_{-\infty}^{\frac{N(\alpha-\alpha_{N_T})}{\sqrt{N\sigma_{N_T}^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du + O\Big(\frac{\text{Exp.}(\tilde{Y}_1)^3}{\sqrt{N}\sigma_{N_T}^3}\Big). \tag{5.13}$$

Combining (5.10) and (5.12), the error term in (5.13) is given by

$$\frac{\text{Exp.}(\tilde{Y}_1)^3}{\sqrt{N}\sigma_{N_T}^3} \underset{m\to\infty}{\approx} \begin{cases} 1, & s \in (3,4), \\ (\log N)^{-1}, & s = 4, \\ N^{-\frac{s-4}{2}}, & s \in (4,5), \\ \frac{\log N}{\sqrt{N}}, & s = 5, \\ \frac{1}{\sqrt{N}}, & s > 5. \end{cases} \tag{5.14}$$

Combining (5.9) and (5.10) and setting $\alpha = \alpha_c$, the upper limit of the integral in (5.13) is given by

$$\frac{N(\alpha_c - \alpha_{N_T})}{\sqrt{N\sigma_{N_T}^2}} \underset{N\to\infty}{\approx} \begin{cases} 1, & s \in (3,4), \\ -\frac{1}{\log N}, & s = 4, \\ -\frac{1}{N^{\frac{s-4}{2}}}, & s > 4, \end{cases}$$

such that (5.13) becomes

$$\begin{aligned} \text{Prob.}(\tilde{S}_N \leq \alpha_c N) &= \int_{-\infty}^{\frac{N(\alpha_c-\alpha_{N_T})}{\sqrt{N\sigma_{N_T}^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du + O\Big(\frac{\text{Exp.}(\tilde{Y}_1)^3}{\sqrt{N}\sigma_{N_T}^3}\Big) \\ &= \begin{cases} \int_{-\infty}^{O(1)} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du + O(1), & s \in (3,4), \\ \frac{1}{2} + O\big(\frac{1}{\log N}\big), & s = 4, \\ \frac{1}{2} + O\big(\frac{1}{N^{\frac{s-4}{2}}}\big), & s \in (4,5), \\ \frac{1}{2} + O\big(\frac{\log N}{\sqrt{N}}\big), & s = 5, \\ \frac{1}{2} + O\big(\frac{1}{\sqrt{N}}\big), & s > 5. \end{cases} \end{aligned} \tag{5.15}$$

The second part of the probability has the following upper bound by (5.6)

$$\begin{aligned}
&\text{Prob.}(S_N \le \alpha_c N : \exists Y_j < -N_T(s)) \\
\le\ & N\text{Prob.}(S_N \le \alpha_c N \text{ and } Y_1 < -N_T(s)) \\
\le\ & N\text{Prob.}(Y_1 < -N_T(s)) \\
\underset{N\to\infty}{\approx}\ & \frac{N}{N_T(s)^{s-2}} \\
\underset{N\to\infty}{\approx}\ & \begin{cases} 1, & s \in (3,4), \\ (\log N)^{-1}, & s = 4, \\ N^{-\frac{s-4}{2}}, & s > 4. \end{cases}
\end{aligned} \tag{5.16}$$

Combining (5.15) and (5.16), we find

$$\text{Prob.}(S_N \le \alpha_c N) \begin{cases} \in (0,1), & s \in (3,4), \\ = \dfrac{1}{2} + O(\dfrac{1}{\log N}), & s = 4, \\ = \dfrac{1}{2} + O(\dfrac{1}{N^{\frac{s-4}{2}}}), & s \in (4,5), \\ = \dfrac{1}{2} + O(\dfrac{\log N}{\sqrt{N}}), & s = 5, \\ = \dfrac{1}{2} + O(\dfrac{1}{\sqrt{N}}), & s > 5. \end{cases} \tag{5.17}$$

With the definition of N_α given in the introduction, (2.15) is proved.

Remark 5.1. The probability is undecided when $s \in (3,4)$ in (5.17). In order to see if this probability can be determined, let us redefine $N_T(s) = (-\alpha_c \vee 1)N^{\frac{1}{s-2}} w(N)$ for a certain positive sequence $\{w(N)\}$. Note that (5.8) corresponds to the choice such that all $\{w(N)\}$ are equal to one. Using the same argument of (5.15), we have

$$\text{Prob.}(\tilde{S}_N \le \alpha_c N) = \int_{-\infty}^{-w(N)^{1-\frac{s}{2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du + O(w(N)^{\frac{s-2}{2}})\,,$$

where we must choose $w(N) \to 0$ as $N \to \infty$ to diminish the error term. On the other hand, using the same argument of (5.16), we get

$$\text{Prob.}(S_N \le \alpha_c N : \exists Y_j < -N_T(s))\ N \overset{\approx}{\to} \infty\ w(N)^{2-s}\,,$$

here we should choose $w(N) \to \infty$ as $N \to \infty$ to avoid divergent. The probability is the sum of these two contributions, and there is no appropriate sequence $\{w(N)\}$ we can use to settle the value.

In the following part of this section, we shall consider a general $\alpha \neq \alpha_c$. Let us deal with $\alpha > \alpha_c$ first. By the same argument of (4.7), we obtain

$$\text{Prob.}(S_N > N_\alpha) \le e^{-NI(\alpha)}, \tag{5.18}$$

where

$$I(\alpha) = \sup_{t>1}\Big\{\alpha \log t - \log \hat{D}(t)\Big\} := \alpha \log t_\alpha - \log \hat{D}(t_\alpha)\,. \tag{5.19}$$

According to (5.19), we have

$$\frac{\alpha}{t_\alpha} = \frac{\frac{d\hat{D}(t_\alpha)}{dt}}{\hat{D}(t_\alpha)} := \frac{\hat{D}'(t_\alpha)}{\hat{D}(t_\alpha)} \quad \text{and} \quad I'(\alpha) = \log t_\alpha. \tag{5.20}$$

As $\hat{D}(t=1) = 1$ and $\hat{D}'(1) = \alpha_c$, setting $t_\alpha = 1$ in (5.20) leads to $t_{\alpha_c} = \hat{D}(t_{\alpha_c}) = 1$ and hence $I(\alpha_c) = \frac{d}{d\alpha} I'(\alpha)|_{\alpha=\alpha_c} = 0$. It is easy to see that $I(\alpha)$ and t_α are both strictly increasing function for $\alpha > \alpha_c$. Since $\frac{d^2\hat{D}(t)}{dt^2}|_{t=t_{\alpha_c}=1} = \sum_{k\in\mathbb{Z}} k(k-1)D(k) \approx p\sum_{k=1}^{\infty} k^2/k^{s-1} = \infty$ for $s \in (3,4]$ using (5.6), $\frac{dt_\alpha}{d\alpha}|_{\alpha=\alpha_c}$ does not exist and so does $I''(\alpha)$ in (4.11). We must turn to the *Riemann–Liouville functional derivative* (c.f. [18]) as follows:

$$\frac{d^\gamma f(\alpha)}{d\alpha^\gamma} = \frac{1}{\Gamma(1-\gamma)} \frac{d}{d\alpha} \int_0^\alpha (\alpha-u)^{-\gamma} f(u)\, du, \quad \text{for } \gamma \in (0,1)\,.$$

It is easy to check that $\frac{d^{1+\epsilon}\hat{D}(t)}{dt^{1+\epsilon}}|_{t=t_{\alpha_c}} = \frac{d^\epsilon \hat{D}'(t)}{dt^\epsilon}|_{t=t_{\alpha_c}} = O(1)\sum_{n\in\mathbb{Z}} n^{1+\epsilon} D(n)$ exists for $\epsilon \in (0, s-3)$ with $s \in (3,4]$.

Under the condition $\alpha - \alpha_c \in (0, 1)$, by Taylor formula we have

$$\begin{aligned} & \alpha \log t - \log \hat{D}(t) \\ = & \alpha \log\big(t_{\alpha_c} + (t - t_{\alpha_c})\big) - \log\big(\hat{D}(t_{\alpha_c}) + \hat{D}'(t_{\alpha_c})(t - t_{\alpha_c}) + O(t - t_{\alpha_c})^{1+\epsilon}\big) \\ = & \alpha \log\big(1 + (t - t_{\alpha_c})\big) - \log\big(1 + \alpha_c(t - t_{\alpha_c}) + O(t - t_{\alpha_c})^{1+\epsilon}\big) \\ = & (\alpha - \alpha_c)(t - t_{\alpha_c}) + O(t - t_{\alpha_c})^{1+\epsilon} \end{aligned}$$

for $\epsilon \in (0, s-3)$ with $s \in (3, 4]$ and $t > 1$. Taking $t = t_\alpha$, we have

$$I(\alpha) = (\alpha - \alpha_c)(t_\alpha - t_{\alpha_c}) + O(t_\alpha - t_{\alpha_c})^{1+\epsilon}. \tag{5.21}$$

Therefore although $\frac{d t_\alpha}{d\alpha}|_{\alpha=\alpha_c}$ does not exist, we find that $\frac{d^\epsilon t_\alpha}{d\alpha^\epsilon}$ exists for $\epsilon \in (0, s-3)$ with $s \in (3, 4]$ by (5.20). Moreover, (5.20) can be rewritten as $\alpha \hat{D}(t_\alpha) = t_\alpha \hat{D}'(t_\alpha)$ for all t_α. It follows that

$$(\alpha - \alpha_c)\hat{D}(t_{\alpha_c}) + \alpha\big(\hat{D}(t_\alpha) - \hat{D}(t_{\alpha_c})\big) = t_\alpha\big(\hat{D}'(t_\alpha) - \hat{D}'(t_{\alpha_c})\big) + \alpha_c(t_\alpha - t_{\alpha_c}). \tag{5.22}$$

Using Taylor formula as α is close to α_c, we have $\hat{D}(t_\alpha) - \hat{D}(t_{\alpha_c}) = \alpha_c(t_\alpha - t_{\alpha_c}) + O(t_\alpha - t_{\alpha_c})^{1+\epsilon}$ and $\hat{D}'(t_\alpha) - \hat{D}'(t_{\alpha_c}) \approx_{\alpha\to\alpha_c} (t_\alpha - t_{\alpha_c})^\epsilon$ for $\epsilon \in (0, s-3)$. When $\alpha_c - \alpha \in (0, 1)$, the order of the left hand side of (5.22) is $(\alpha - \alpha_c) + \alpha\alpha_c(t_\alpha - t_{\alpha_c})$, while the order of the right hand side of (5.22) is $t_\alpha(t_\alpha - t_{\alpha_c})^\epsilon$. It follows that $t_\alpha - t_{\alpha_c} \approx_{\alpha\to\alpha_c} (\alpha - \alpha_c)^{\frac{1}{\epsilon}}$. From (5.21), we have

$$0 < I(\alpha) \underset{\alpha\to\alpha_c}{\approx} (\alpha - \alpha_c)^{1+\frac{1}{\epsilon}}.$$

For $0 < \alpha - \alpha_c < 1$, the upper bound of $1 - \tau(N_\alpha, N)$ is tightest as ϵ approaches to $s - 3$.

$$1 - \tau(N_\alpha, N) \le \lim_{\epsilon\uparrow s-3} e^{-cN(\alpha-\alpha_c)^{1+\frac{1}{\epsilon}}} = e^{-cN(\alpha-\alpha_c)^{1+\frac{1}{s-3}}}$$

for some $c \in (0, \infty)$.

Next consider the condition $\alpha < \alpha_c$. We cannot use the same argument of (4.5) since $\hat{D}(t)$ is divergent for any $t < 1$ by (5.7). Let us separate the probability into two cases as follows. The first case allows at least one of the random variables assumes a large value, while the second case does not.

$$\begin{aligned} \text{Prob.}(S_N \le \alpha N) = \; & \text{Prob.}(S_N \le \alpha N, \exists j \in \{1, 2 \ldots, N\} \text{ such that } Y_j < (\alpha - \alpha_c)N) \\ & + \text{Prob.}(S_N \le \alpha N, \forall j \in \{1, 2 \ldots, N\} \text{ such that } Y_j \ge (\alpha - \alpha_c)N)\,. \end{aligned} \tag{5.23}$$

For the first case, we use the same argument of (5.16) to have

$$\begin{aligned} & \text{Prob.}(S_N \le \alpha N : \exists Y_j < (\alpha - \alpha_c)N) \\ \le\; & N\text{Prob.}(S_N \le \alpha N \text{ and } Y_1 < (\alpha - \alpha_c)N) \\ \le\; & N\text{Prob.}(Y_1 < (\alpha - \alpha_c)N) \\ \underset{N\to\infty}{\approx}\; & \int_{(\alpha_c-\alpha)N}^{\infty} \frac{N}{x^{s-1}}\,dx \\ \underset{N\to\infty}{\approx}\; & \frac{1}{(\alpha_c - \alpha)^{s-2}N^{s-3}}. \end{aligned} \tag{5.24}$$

For the second case $\text{Prob.}(S_N \le \alpha N, \forall j \in \{1, 2 \ldots, N\}$ such that $Y_j \ge (\alpha - \alpha_c)N)$, let us define $\bar{S}_N = \sum_{k=1}^N \bar{Y}_k$ where $\bar{Y}_k$ are i.i.d. random variables with distribution

$$\text{Prob.}(\bar{Y}_k = j) = \begin{cases} b'D(j) & \text{if } j \ge (\alpha - \alpha_c)N, \\ 0 & \text{if } j < (\alpha - \alpha_c)N, \end{cases}$$

where b' is a normalization constant. Then the argument of (4.7) can be used since $\hat{D}(t)$ is well-defined for $t < 1$. Using the same argument from (5.18) to (5.21), we have

$$\text{Prob.}(S_N \le \alpha N, \forall j \in \{1, 2 \ldots, N\} \text{ such that } Y_j \ge (\alpha - \alpha_c)N) \le e^{-N(\alpha_c-\alpha)^{1+\frac{1}{\epsilon}}}. \tag{5.25}$$

Comparing (5.24) and (5.25) in (5.23), the first case dominates so that

$$\text{Prob.}(S_N \le \alpha N) \le \frac{O(1)}{(\alpha_c - \alpha)^{s-2}N^{s-3}}\,.$$

Notice that this upper bound should be optimal since by Proposition A.2 of [19] we find

$$\text{Prob.}(S_N \le N_\alpha) \ge \int_{N(\alpha_c-\alpha)}^{\infty} \frac{N}{x^{s-1}}\,dx \approx_{N\to\infty} \frac{N}{\big(N(\alpha_c - \alpha)\big)^{s-2}} = \frac{1}{(\alpha_c - \alpha)^{s-2}N^{s-3}}.$$

6. Proof of Theorem 2.5

According to the definition stated in Theorem 2.5,

$$\alpha_N^+ - \alpha_c = \alpha_c - \alpha_N^- = \begin{cases} N^{-\rho\frac{s-3}{s-2}}\ell_N & s < 4\,, \\ N^{-\rho/2}\ell_N & s \geq 4\,. \end{cases}$$

Let us consider $s \in (3, 4)$ first. From (2.16) and (2.17), we have

$$\begin{aligned} 1 - \tau(N_{\alpha_N^+}, N) &\leq \exp\left(-cN^{1-\rho}\ell_N^{\frac{s-2}{s-3}}\right) \\ &\underset{N\to\infty}{\approx} \begin{cases} \exp\left(-cN^{1-\rho}\ell_N^{\frac{s-2}{s-3}}\right) & \rho \in (0, 1), \\ \exp(-c\ell_N^{\frac{s-2}{s-3}}), & \rho = 1,\ \lim_{N\to\infty} \ell_N = \infty, \end{cases} \end{aligned} \tag{6.1}$$

and

$$\tau(N_{\alpha_N^-}, N) \leq \begin{cases} \dfrac{1}{N^{(s-3)(1-\rho)}\ell_N^{s-2}}, & \rho \in (0, 1), \\ \dfrac{1}{\ell_N^{s-2}}, & \rho = 1,\ \lim_{N\to\infty} \ell_N = \infty\,. \end{cases}$$

Moreover, for $\rho = 1$ with $\lim_{N\to\infty}\ell_N \in (0, \infty)$ or $\rho > 1$, we have

$$\lim_{N\to\infty} \tau(N_{\alpha_N^-}, N) = 1 - \lim_{N\to\infty} \tau(N_{\alpha_N^+}, N) = \lim_{N\to\infty} \tau(N_{\alpha_c}, N) \in (0, 1).$$

when $s \in (3, 4)$ again.

Similarly, when $s \geq 4$ and $\rho \in (0, 1)$, it is easy to see that

$$\tau(N_{\alpha_N^-}, N) \leq \frac{1}{N^{(s-3)(1-\rho/2)-\rho/2}\ell_N^{s-2}}, \quad 1 - \tau(N_{\alpha_N^+}, N) \leq \exp\left(-cN^{1-\rho}\ell_N^2\right).$$

Consider $s = 4$ and $\rho \geq 1$, we have $\sigma_{N_T}^2 \approx \log N$ by (5.10). Using (5.13) and (5.14) with $\alpha = \alpha_N^-$ so that $\alpha_N^- - \alpha_{N_T} = \alpha_N^- - \alpha_c + \alpha_c - \alpha_{N_T}$ and using (5.16), we have

$$\begin{aligned} \tau(N_{\alpha_N^-}, N) &= \int_{-\infty}^{-\frac{cN^{\frac{1-\rho}{2}}\ell_N}{\sqrt{\log N}} - \frac{O(1)}{\log N}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du + O(\frac{1}{\log N}) \\ &= \begin{cases} O(1)\max\{\dfrac{\sqrt{\log N}}{\ell_N} e^{-\frac{c\ell_N^2}{\log N}}, \dfrac{1}{\log N}\}, & \rho = 1,\ \text{if } \lim_{N\to\infty} \dfrac{\ell_N}{\sqrt{\log N}} = \infty, \\ \lambda_1 + O(1)\max\{|\dfrac{\ell_N}{\sqrt{\log N}} - \mathcal{L}|, \dfrac{1}{\log N}\}, & \rho = 1,\ \text{if } \lim_{N\to\infty} \dfrac{\ell_N}{\sqrt{\log N}} = \mathcal{L} \in [0, \infty), \\ \dfrac{1}{2} + O(\dfrac{1}{\log N}), & \rho > 1, \end{cases} \end{aligned} \tag{6.2}$$

where $\lambda_1 \in (0, 1)$. Therefore we obtain (2.22) and (2.23) for $s = 4$.

When $s > 4$, we have $\sigma_{N_T}^2 = \sigma^2 + O(N^{\frac{4-s}{2}})$ by (5.11). Consider $s \in (4, 5)$ and $\rho \geq 1$, by (4.2) and the same argument of (6.2), we have

$$\begin{aligned} \tau(N_{\alpha_N^-}, N) &= \int_{-\infty}^{\frac{-N\left(N^{-\frac{\rho}{2}}\ell_N + O(N^{\frac{3-s}{2}})\right)}{\sigma(1+O(1)N^{\frac{4-s}{4}})\sqrt{N}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du + O(\frac{1}{N^{\frac{s-4}{2}}}) \\ &= \begin{cases} O(1)\max\{\dfrac{\sigma}{\ell_N} e^{-\frac{\ell_N^2}{2\sigma^2}}, \dfrac{1}{N^{\frac{s-4}{2}}}\}, & \rho = 1,\ \lim_{N\to\infty} \ell_N = \infty, \\ \lambda_2 + O(1)\max\{|\ell_N - L|, \dfrac{1}{N^{\frac{s-4}{2}}}\}, & \rho = 1,\ \lim_{N\to\infty} \ell_N = L \in [0, \infty), \\ \dfrac{1}{2} + O(\dfrac{\ell_N}{N^{\frac{\rho}{2}-\frac{1}{2}}}), & \rho \in (1, s-3], \\ \dfrac{1}{2} + O(\dfrac{1}{N^{\frac{s-4}{2}}}), & \rho > s-3, \end{cases} \end{aligned} \tag{6.3}$$

where $\lambda_2 \equiv \Psi(\frac{L}{\sigma})$ is a constant. Therefore we obtain (2.22) and (2.23) for $s \in (4, 5)$.

Consider $s=5$ and $\rho \geq 1$, by (4.2) and the same argument of (6.2), we have

$$\tau(N_{\alpha_N^-},N) = \int_{-\infty}^{\frac{-N\left(N^{-\frac{\rho}{2}}\ell_N+O(N^{-\frac{1}{2}})\right)}{\sigma(1+O(1)N^{\frac{-1}{4}})\sqrt{N}}} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\,du + O(\frac{\log N}{\sqrt{N}})$$

$$= \begin{cases} O(1)\max\{\frac{\sigma}{\ell_N}e^{-\frac{\ell_N^2}{2\sigma^2}}, \frac{\log N}{\sqrt{N}}\}, & \rho=1,\ \text{if } \lim_{N\to\infty}\ell_N=\infty, \\ \lambda_3+O(1)\max\{|\ell_N-L|, \frac{\log N}{\sqrt{N}}\}, & \rho=1,\ \text{if } \lim_{N\to\infty}\ell_N=L\in[0,\infty), \\ \frac{1}{2}+\frac{\ell_N}{N^{\frac{\rho}{2}-\frac{1}{2}}}, & \rho\in(1,2), \\ \frac{1}{2}+O(1)\max\{\frac{\ell_N}{\sqrt{N}}, \frac{\log N}{\sqrt{N}}\}, & \rho=2, \\ \frac{1}{2}+O(\frac{\log N}{\sqrt{N}}), & \rho>2, \end{cases} \tag{6.4}$$

where $\lambda_3 \equiv \Psi(\frac{L}{\sigma})$ is a constant. Therefore we obtain (2.22) and (2.23) for $s=5$.

Consider $s>5$ and $\rho \geq 1$, by (4.2) and the same argument of (6.2), we have

$$\tau(N_{\alpha_N^-},N) = \int_{-\infty}^{\frac{-N\left(N^{-\frac{\rho}{2}}\ell_N+O(N^{-\frac{s-3}{2}})\right)}{\sigma(1+O(1)N^{\frac{4-s}{4}})\sqrt{N}}} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\,du + O(\frac{1}{\sqrt{N}})$$

$$= \begin{cases} O(1)\max\{\frac{\sigma}{\ell_N}e^{-\frac{\ell_N^2}{2\sigma^2}}, \frac{1}{\sqrt{N}}\}, & \rho=1, \text{if } \lim_{N\to\infty}\ell_N=\infty, \\ \lambda_4+O(1)\max\{|\ell_N-L|, \frac{1}{\sqrt{N}}\}, & \rho=1,\ \text{if } \lim_{N\to\infty}\ell_N=L\in[0,\infty), \\ \frac{1}{2}+O(\frac{\ell_N}{N^{\frac{\rho}{2}-\frac{1}{2}}}), & \rho\in(1,2], \\ \frac{1}{2}+O(\frac{1}{\sqrt{N}}), & \rho>2, \end{cases} \tag{6.5}$$

where $\lambda_4 \equiv \Psi(\frac{L}{\sigma})$ is a constant. Therefore we obtain (2.22) and (2.23) for $s>5$.

The corresponding result for $\alpha=\alpha_N^+$ can be obtained by the same method, and the proof of Theorem 2.5 is completed.

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