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Oscillation in Nonlinear Difference Equations

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Abstract—In this paper, we shall discuss oscillatory behavior of the solutions of difference equations, including the self-adjoint second-order linear equation and the discrete version of the nonlinear wave equation. Our work is to give sufficient conditions such that every nontrivial solution of the equations oscillates. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The principal objective of this paper is to establish oscillation theorems for ordinary and partial difference equations.

We are concerned with the self-adjoint second-order linear difference equation

$$\Delta(P_{n-1}\Delta x_{n-1}) + Q_n x_n = 0, \qquad \forall n \in \mathbb{N} = \{1, 2, \dots\}, \tag{1.1}$$

where

$$P_n > 0, \qquad \forall n = 0, 1, 2, \dots. \tag{1.2}$$

The oscillatory behavior of (1.1) has been extensively discussed by several authors; see for example [1-4]. However, they deal with (1.1) under the hypothesis

$$\sum_{i=1}^{\infty} Q_i = \infty. \tag{1.3}$$

Therefore, it is interesting to discuss (1.1) without requiring the hypothesis (1.3). In fact, we shall establish an oscillation theorem for (1.1) when (1.3) is not satisfied. Our results are motivated by those of Yu and Chen [5]. For general oscillations in neutral delay difference equations, we refer to [6].

Let Δ_1, Δ_2, Δ_1^2 be partial difference operators defined as $\Delta_1 u_{m,n} = u_{m+1,n} - u_{m,n}$, $\Delta_2 u_{m,n} = u_{m,n+1} - u_{m,n}$, and $\Delta_1^2 u_{m,n} = \Delta_1(\Delta_1 u_{m,n})$. Basing on the results in Section 2, we shall discuss oscillations of the discrete analogue for the nonlinear wave equation

$$\begin{aligned} &\Delta_2(a_{n-1}\Delta_2 u_{m,n-1}) + b_{n-1}\Delta_2 u_{m,n-1} \\ &-c_n\Delta_1^2(G(m-1, n, u_{m-1,n})u_{m-1,n}) + g(m, n, u_{m,n})u_{m,n} = 0, \qquad \forall m \in \Omega, \quad n \in \mathbb{N}, \\ &u_{0,n} = u_{M+1,n} = 0, \qquad \forall n \in S, \end{aligned} \tag{1.4}$$

where $M \in \mathbb{N}$, $\Omega = \{1, 2, \dots, M\}$, $S = \{0, 1, 2, \dots\}$, $a_n, c_n > 0$, $b_n \in \mathbb{R}$, $\forall n \in \mathbb{N}$, and g and G are continuous functions in $\Omega \times \mathbb{N} \times \mathbb{R}$. The main results are given in Theorems 3.5 and 3.6. In

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addition, we also obtain a sufficient condition such that every nonoscillatory nontrivial solution of (1.4) is bounded.

2. OSCILLATIONS OF SELF-ADJOINT SECOND-ORDER LINEAR EQUATION

In this section, we shall consider the self-adjoint second-order difference equation (1.1). The following result is known [3].

THEOREM [A]. *If*

$$\sum_{i=0}^{\infty} \frac{1}{P_i} = \infty$$

and the condition (1.3) hold, then (1.1) is oscillatory. ∎

Here we shall establish an oscillation theorem for (1.1) when (1.3) is not satisfied. Throughout this paper we say that a nontrivial solution $\{x_n\}_{n=0}^{\infty}$ of the difference equation (1.1) is oscillatory if for every $n \in \mathbb{N}$ there exists $n_1, n_2 \geq n$ such that $x_{n_1} x_{n_2} \leq 0$. Otherwise, it is nonoscillatory. The difference equation (1.1) is said to be oscillatory if it has no nonoscillatory nontrivial solution. Otherwise, it is nonoscillatory.

We shall use the following conditions as assumptions in our theorems.

(A1) There exists a nonnegative integer N such that $P_n \leq 1$ and $Q_n > 0$ for all $n \geq N$.

(A2) If we define $H_n^{(0)} = \sum_{i=n}^{\infty} Q_i$, then there exists a nonnegative integer k such that the sequences $H_n^{(m)} = \sum_{i=n}^{\infty} i Q_i H_i^{(m-1)} < \infty$ for all $m = 1, 2, \ldots, k$ and $\sum_{i=1}^{\infty} i Q_i H_i^{(k)} = \infty$.

LEMMA 2.1. *Assume that (1.2) and (A1) hold. Let x_n be an eventually positive solution of the difference inequality*

$$\Delta(P_{n-1} \Delta x_{n-1}) + Q_n x_n \leq 0, \tag{2.1}$$

and set

$$y_n = P_n \Delta x_n. \tag{2.2}$$

Then we have eventually

$$y_n > 0. \tag{2.3}$$

PROOF. Assume that $x_n > 0$, $\forall n \geq N_1$, for some $N_1 \geq N$. Then we have $\Delta y_{n-1} \leq -Q_n x_n < 0$, $\forall n \geq N_1$. Therefore, if (2.3) does not hold, then there exist $M > 0$ and $N_2 \geq N_1$ such that $\forall n \geq N_2$ we have $y_n < -M$. Thus by (A1), (1.2), and (2.2), we get $\Delta x_n \leq y_n < -M$, $\forall n \geq N_2$. This implies $x_n \to -\infty$ which contradicts the positivity of x_n. ∎

LEMMA 2.2. *Assume that (1.2), (A1), and (A2) hold. Let x_n be an eventually positive solution of the difference inequality (2.1) and let y_n be defined by (2.2). Then we have eventually*

$$y_n < 0. \tag{2.4}$$

PROOF. Assume that

$$x_n > 0, \qquad \forall n \geq N_0, \tag{2.5}$$

for some $N_0 \geq N$. Then by (2.1), (2.2), and (2.5), we have

$$\Delta y_{n-1} \leq -Q_n x_n < 0, \qquad \forall n \geq N_0.$$

Therefore, if the conclusion does not hold, then $y_n > 0$ eventually. By (1.2) and (2.2), we have eventually $\Delta x_n > 0$. Thus there exist $M > 0$ and $N_1 \in \mathbb{N}$ such that $\Delta x_n > 0$ and $x_n \geq M$, $\forall n \geq N_1$. Then

$$\Delta y_{n-1} \leq -Q_n x_n \leq -M Q_n, \qquad \forall n \geq N_1,$$

and so

$$y_{n-1} \geq MH_n^{(0)}, \qquad \forall n \geq N_1. \tag{2.6}$$

From (A1), (1.2), (2.2), and (2.6), we obtain

$$\Delta x_{n-1} \geq MH_n^{(0)}, \qquad \forall n \geq N_1 + 1,$$

which yields

$$x_n \geq M(n - N_1)H_n^{(0)}, \qquad \forall n \geq N_1 + 1.$$

Then by (2.1) and (2.2), it follows that

$$\Delta y_{n-1} \leq -M(n - N_1)Q_n H_n^{(0)}, \qquad \forall n \geq N_1 + 1.$$

Since $(M(n - N_1)Q_n H_n^{(0)})/(nQ_n H_n^{(0)}) = (M(n - N_1)/n) \to M$ as $n \to \infty$ and is increasing, there exists an integer $N_2 \geq N_1 + 1$ such that

$$\Delta y_{n-1} \leq -\frac{M}{2} nQ_n H_n^{(0)}, \qquad \forall n \geq N_2, \tag{2.7}$$

which contradicts the positivity of y_n for $k = 0$.

If $k \neq 0$, by (2.7), we obtain

$$y_{n-1} \geq \frac{M}{2} H_n^{(1)}, \qquad \forall n \geq N_2,$$

and so

$$x_n \geq \frac{M}{2}(n - N_2)H_n^{(1)}, \qquad \forall n \geq N_2 + 1,$$

which, by (2.1) and (2.2), yields

$$\Delta y_{n-1} \leq -\frac{M}{2}(n - N_2)Q_n H_n^{(1)}, \qquad \forall n \geq N_2 + 1.$$

Then there exists an integer$N_3 \geq N_2 + 1$ such that

$$\Delta y_{(n-1)} \leq -\frac{M}{2^2} nQ_n H_n^{(1)}, \qquad \forall n \geq N_3,$$

which contradicts the positivity of y_n for $k = 1$. By induction, we conclude that such k in (A2) does not exist, which contradicts our assumption (A2). ∎

LEMMA 2.3. *Assume that (1.2) and (A1) hold. Let x_n be an eventually negative solution of the difference inequality*

$$\Delta(P_{n-1}\Delta x_{n-1}) + Q_n x_n \geq 0, \tag{2.8}$$

and let y_n be defined by (2.2). Then we have eventually $y_n < 0$. ∎

LEMMA 2.4. *Assume that (1.2), (A1), and (A2) hold. Let x_n be an eventually negative solution of the difference inequality (2.8) and let y_n be defined by equation (2.2). Then we have eventually $y_n > 0$.* ∎

From the above discussion, the following theorem is immediate.

THEOREM 2.5. *Assume that (1.2), (A1), and (A2) hold, then (1.1) is oscillatory.* ∎

EXAMPLE 2.6. The self-adjoint difference equation

$$\Delta^2 x_{n-1} + \frac{1}{n^\alpha} x_n = 0, \qquad \forall n = 1, 2, \ldots \tag{2.9}$$

satisfies all assumptions of Theorem [A] if and only if $\alpha \leq 1$. In fact, condition (1.3) is not satisfied when $\alpha > 1$. But all assumptions of Theorem 2.5 hold when $1 < \alpha < 2$. So by Theorem [A] and Theorem 2.5, equation (2.9) is oscillatory when $\alpha < 2$. ∎

3. OSCILLATIONS OF DISCRETE ANALOGUE FOR NONLINEAR WAVE EQUATION

Following [7], we discuss oscillation of discrete analogue for nonlinear wave equation (1.4). A nontrivial solution $u_{m,n}$ of (1.4) is said to be oscillatory if for every $N \in \mathbb{N}$ there exist $m_1, m_2 \in \Omega$ and $n_1, n_2 \geq N$ such that $u_{m_1,n_1} u_{m_2,n_2} \leq 0$. Otherwise, it is nonoscillatory.

We shall obtain sufficient conditions for the oscillation of a nontrivial solution of (1.4). For this, we first consider a method of finding a linear difference equation such that a nontrivial solution of (1.4) is oscillatory if a nontrivial solution of the linear difference equation is oscillatory. To do this, we need several results which are provided in the following lemmas.

LEMMA 3.1. *Consider the following Sturm-Liouville system:*

$$\begin{aligned} \Delta^2 \phi_{n-1} + \lambda \phi_n &= 0, \\ \phi_0 = 0,\ \phi_{M+1} &= 0, \end{aligned} \qquad \forall\, n \in \Omega. \tag{3.1}$$

Let λ_1 be the least eigenvalue of system (3.1) and $\phi_n^{(1)}$ be the eigenfunction corresponding to λ_1. Then $\lambda_1 > 0$ and $\phi_n^{(1)} > 0$, $\forall\, n \in \Omega$.

PROOF. See [1,8]. ∎

Let $Ly_n = \Delta(a_{n-1}\Delta y_{n-1}) + b_n y_n$. We shall establish the discrete version of Green's formula.

LEMMA 3.2. *Let $\{y_n\}_{n=0}^{M+1}$ and $\{z_n\}_{n=0}^{M+1}$ be sequences. Then*

$$\sum_{n=1}^{M} z_n L y_n - \sum_{n=1}^{M} y_n L z_n = \{a_n \omega[z_n, y_n]\}_{n=0}^{M},$$

where $\omega[z_n, y_n]$ is called the Casoratian of z_n and y_n, and is defined by

$$\omega[z_n, y_n] = \begin{vmatrix} z_n & y_n \\ \Delta z_n & \Delta y_n \end{vmatrix}.$$

PROOF. See [1,3]. ∎

LEMMA 3.3. *Let $N \in \mathbb{N}$ and $\{U_n\}_{n \in S}$ be a sequence such that*

$$U_n \geq 0, \qquad \forall\, n \geq N \tag{3.2}$$

and

$$\Delta(a_{n-1}\Delta U_{n-1}) + b_{n-1}\Delta U_{n-1} + c_n^* U_n \leq 0, \qquad \forall\, n \geq N, \tag{3.3}$$

where

$$a_n, c_n^* > 0, \qquad \forall\, n \in S. \tag{3.4}$$

If we assume

$$b_n < a_n, \qquad \forall\, n \in S, \tag{3.5}$$

then we have

(a) $U_n > 0$, $\forall\, n \geq N$,

(b) *there exists $\{V_n\}_{n=N}^{\infty}$, such that $V_n \geq U_n > 0$, $\forall\, n \geq N$ and*

$$\Delta(a_{n-1}\Delta V_{n-1}) + b_{n-1}\Delta V_{n-1} + c_n^* V_n = 0, \qquad \forall\, n \geq N+1.$$

PROOF. First, we rewrite (3.3) in the form

$$a_n \Delta U_n + (b_{n-1} - a_{n-1})\Delta U_{n-1} + c_n^* U_n \le 0, \qquad \forall n \ge N. \tag{3.6}$$

From (3.4), (3.5), and (3.6), we have $\Delta U_n \le 0$, if $\Delta U_{n-1} \le 0$, $\forall n \ge N$. Then suppose that there exists an integer $N^* \ge N$ such that $\Delta U_{N^*} \le 0$, we conclude that $\{U_n\}_{n\in S}$ is eventually decreasing. Otherwise, $\{U_n\}_{n\in S}$ is eventually increasing. Consequently, from (3.2), we get the desired result (a). To prove (b), we set

$$S_n = \frac{U_{n+1}}{U_n} > 0, \qquad \forall n \ge N. \tag{3.7}$$

Substitution of (3.7) into (3.3) gives

$$a_n(S_n - 1) + (b_{n-1} - a_{n-1})\left(1 - \frac{1}{S_{n-1}}\right) + c_n^* \le 0, \qquad \forall n \ge N. \tag{3.8}$$

Let $\{T_n\}_{n=N}^{\infty}$ be a sequence which satisfies

$$T_N = S_N \tag{3.9}$$

and

$$a_n(T_n - 1) + (b_{n-1} - a_{n-1})\left(1 - \frac{1}{T_{n-1}}\right) + c_n^* = 0, \qquad \forall n \ge N+1. \tag{3.10}$$

From (3.4), (3.5), (3.8), (3.9), and (3.10), we obtain

$$T_N \ge S_n, \qquad \forall n \ge N. \tag{3.11}$$

Now define the new sequence $\{V_n\}_{n=N}^{\infty}$ as follows:

$$V_N = U_N \tag{3.12}$$

and

$$V_{N+n} = U_N \prod_{i=N}^{N+n-1} T_i, \qquad \forall n \in \mathbf{N}. \tag{3.13}$$

It is easy to check that

$$\Delta(a_{n-1}\Delta V_{n-1}) + b_{n-1}\Delta V_{n-1} + c_n^* V_n = 0, \qquad \forall n \ge N+1.$$

Finally, we need to show that

$$V_n \ge U_n > 0, \qquad \forall n \ge N.$$

In fact, it is trivial for $n = N$. For $n \ge N+1$,

$$V_n = U_N \prod_{i=N}^{n-1} T_i \ge U_N \prod_{i=N}^{n-1} S_i = U_n,$$

which follows from (3.7), (3.11), (3.12), and (3.13). ∎

Hereafter we shall require the following assumptions.

(B1) g is nonnegative in $\Omega \times [0,\infty) \times \mathbf{R}$.

(B2) There exists a constant $G_0 > 0$ such that $G(m,n,u) \ge G_0$ in $\Omega \times [0,\infty) \times \mathbf{R}$.

THEOREM 3.4. *Consider the difference equation*

$$\Delta(a_{n-1}\Delta V_{n-1}) + b_{n-1}\Delta V_{n-1} + \lambda_1 G_0 c_n V_n = 0, \qquad \forall n \in \mathbb{N}, \tag{3.14}$$

where a_n, b_n, c_n are defined as in system (1.4) and λ_1 is the least eigenvalue of system (3.1). Assume that (B1) and (B2) hold and

$$b_n < a_n, \qquad \forall n \in S, \tag{3.15}$$

then every nontrivial solution of system (1.4) is oscillatory, if every solution of equation (3.14) is oscillatory.

PROOF. Suppose system (1.4) has an eventually positive nontrivial solution $u_{m,n}$. We set

$$U_n = \sum_{m=1}^{M} u_{m,n}\phi_m^{(1)}, \qquad \forall n \in S, \tag{3.16}$$

where $\phi_m^{(1)}$ is the eigenfunction of system (3.1) corresponding to its least eigenvalue λ_1. If we multiply equation (1.4) by $\phi_m^{(1)}$ and sum from $m = 1$ to M, then the following expression is obtained:

$$\begin{aligned}\sum_{m=1}^{M} \Delta_2(a_{n-1}\Delta_2 u_{m,n-1})\phi_m^{(1)} &+ \sum_{m=1}^{M} b_{n-1}\Delta_2 u_{m,n-1}\phi_m^{(1)} \\ &- \sum_{m=1}^{M} c_n\Delta_1^2(G(m-1,n,u_{m-1,n})u_{m-1,n})\phi_m^{(1)} \\ &+ \sum_{m=1}^{M} g(m,n,u_{m,n})u_{m,n}\phi_m^{(1)} = 0,\end{aligned} \tag{3.17}$$

$\forall m \in \Omega$, and $n \in \mathbb{N}$. By assumption (B2), Lemma 3.2, and the boundary conditions:

$$u_{0,n} = u_{M+1,n} = 0, \qquad \forall n \in S,$$

and

$$\phi_0^{(1)} = \phi_{M+1}^{(1)} = 0,$$

we obtain

$$\begin{aligned}&\sum_{m=1}^{M} \Delta_1^2(G(m-1,n,u_{m-1,n})u_{m-1,n})\phi_m^{(1)} \\ &\quad= \sum_{m=1}^{M} G(m,n,u_{m,n})u_{m,n}\Delta^2\phi_{m-1}^{(1)} + \left\{\omega\left[\phi_m^{(1)}, G(m,n,u_{m,n})u_{m,n}\right]\right\}_{m=0}^{M} \\ &\quad= \sum_{m=1}^{M} G(m,n,u_{m,n})u_{m,n}\Delta^2\phi_{m-1}^{(1)}.\end{aligned} \tag{3.18}$$

Since $\phi_m^{(1)}$ is a solution of system (3.1), we have

$$\Delta^2\phi_{m-1}^{(1)} = -\lambda\phi_m^{(1)}. \tag{3.19}$$

Since $u_{m,n}$ is an eventually positive solution of system (1.4), there exists $N \in \mathbb{N}$ such that

$$u_{m,n} \geq 0, \qquad \forall m \in \Omega \text{ and } n \geq N. \tag{3.20}$$

From (B1), (B2), (3.16), (3.17), (3.18), (3.19), and (3.20), we obtain

$$\begin{aligned}
0 &= \Delta(a_{n-1}\Delta U_{n-1}) + b_{n-1}\Delta U_{n-1} - c_n \sum_{m=1}^{M} G(m,n,u_{m,n})u_{m,n}\Delta^2\phi^{(1)}_{m-1} \\
&\quad + \sum_{m=0}^{M} g(m,n,u_{m,n})u_{m,n}\phi^{(1)}_m \\
&\geq \Delta(a_{n-1}\Delta U_{n-1}) + b_{n-1}\Delta U_{n-1} + \lambda_1 c_n \sum_{m=1}^{M} G(m,n,u_{m,n})u_{m,n}\phi^{(1)}_m \\
&\geq \Delta(a_{n-1}\Delta U_{n-1}) + b_{n-1}\Delta U_{n-1} + \lambda_1 G_0 c_n U_n, \qquad \forall n \geq N.
\end{aligned}$$

By Lemma 3.3, equation (3.14) has an eventually positive solution. Finally, by replacing $u_{m,n}$ by $-u_{m,n}$ if system (1.4) has an eventually negative solution, we obtain that equation (3.14) has an eventually negative solution. ∎

Note that equation (3.14), if $b_n < a_n$, $\forall n \in S$, can be written in the self-adjoint form

$$\Delta(p_{n-1}\Delta V_{n-1}) + q_n V_n = 0, \qquad \forall n \in \mathbb{N},$$

where

$$p_n = a_n \prod_{i=0}^{n-1} \frac{a_i}{a_i - b_i}$$

and

$$q_n = \lambda_1 c_n \prod_{i=0}^{n-1} \frac{a_i}{a_i - b_i}.$$

Therefore, by Theorem [A] and Theorem 2.5, we get the following sufficient conditions for the oscillations of equation (1.4).

THEOREM 3.5. *Assume that (B1), (B2), and (3.5) hold. If*

$$\sum_{n=0}^{\infty} a_n^{-1} \prod_{i=0}^{n-1} \frac{a_i - b_i}{a_i} = \infty$$

and

$$\sum_{n=0}^{\infty} c_n \prod_{i=0}^{n-1} \frac{a_i}{a_i - b_i} = \infty,$$

then every solution of (1.4) is oscillatory. ∎

THEOREM 3.6. *Assume that (B1), (B2), and (3.5) hold. Assume that*

(C1) *there exists a nonnegative integer N such that*

$$a_n \prod_{i=0}^{n-1} \frac{a_i}{a_i - b_i} \leq 1, \qquad \forall n \geq N,$$

and that

(C2)

$$\sum_{n=0}^{\infty} c_n \prod_{i=0}^{n-1} \frac{a_i}{a_i - b_i} < \infty.$$

Furthermore, by defining

$$H_n^{(0)} = \sum_{j=n}^{\infty} c_j \prod_{i=0}^{j-1} \frac{a_i}{a_i - b_i}, \qquad n \in \mathbb{N},$$

we assume that

(C3) *there exists a nonnegative integer k such that the sequences*

$$H_n^{(m)} = \sum_{j=n}^{\infty} j\, c_j \prod_{i=0}^{j-1} \frac{a_i}{a_i - b_i} H_j^{(m-1)} < \infty,$$

for $m = 1, 2, \dots, k$ and

$$\sum_{j=1}^{\infty} j\, c_j \prod_{i=0}^{j-1} \frac{a_i}{a_i - b_i} H_j^{(k)} = \infty.$$

Then every solution of (1.4) is oscillatory. ∎

EXAMPLE 3.7. The system

$$\begin{aligned} \Delta_2^2 u_{m,n-1} - \frac{1}{n^a} \Delta_1^2 u_{m-1,n} + u_{m,n}^3 &= 0, \qquad \forall\, m = 1, \dots, M \text{ and } n \in \mathbb{N} \\ u_{0,n} = u_{M+1,n} &= 0, \qquad \forall\, n = 0, 1, \dots \end{aligned}$$

is oscillatory when $a < 2$. ∎

LEMMA 3.8. *Let $N \in \mathbb{N}$ and $P_n, Q_n > 0$ for $n \geq 0$. If we assume*

$$\sum_{n=0}^{\infty} \frac{1}{P_n} < \infty, \tag{3.21}$$

then every nonoscillatory solution of the difference inequality

$$\Delta(P_{n-1} \Delta U_{n-1}) + Q_n U_n \leq 0, \qquad \forall\, n \geq N, \tag{3.22}$$

is bounded.

PROOF. Let U_n be an eventually positive solution of (3.22). We can assume that there exists an integer $N_1 \in \mathbb{N}$ such that $U_n > 0$, $\forall\, n \geq N_1$. By (3.22), we have

$$\begin{aligned} U_{n+1} &\leq \left(\frac{P_n - Q_n}{P_n}\right) U_n + \frac{P_{n-1}}{P_n}(U_n - U_{n-1}) \\ &\leq U_n + \frac{P_{n-1}}{P_n}(U_n - U_{n-1}), \end{aligned}$$

for $n \geq N$, and hence,

$$P_n(U_{n+1} - U_n) \leq P_{n-1}(U_n - U_{n-1}), \qquad \forall\, n \geq N.$$

By induction, we obtain

$$U_n \leq U_0 + d \sum_{k=0}^{n-1} \frac{1}{P_k}.$$

Then by (3.21), U_n is bounded. ∎

From Lemma 3.8, we immediately obtain the following.

THEOREM 3.9. *Assume that $b_n < a_n$, $\forall\, n \in S$ and*

$$\sum_{n=0}^{\infty} \left(\frac{1}{P_n} \prod_{i=0}^{n-1} \frac{P_i - Q_i}{P_i} \right) < \infty,$$

then every nonoscillatory solution of system (1.4) is bounded. ∎

REFERENCES

1. R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, (1992).
2. S. Chen and L.H. Erbe, Riccati techniques and discrete oscillations, *J. Math. Anal. Appl.* **142**, 468–487, (1989).
3. W.G. Kelley and A.C. Peterson, *Difference Equations, An Introduction with Applications*, Academic Press, New York, (1991).
4. P.J.Y. Wong and R.P. Agarwal, Oscillation theorems and existence of positive monotone solutions for second order nonlinear difference equations, *Mathl. Comput. Modelling* **21** (3), 63–84, (1995).
5. J.S. Yu and M.P. Chen, Oscillation in neutral equation with an "integrally small" coefficient, *Internat. J. Math.& Math. Sci.* **17** (2), 361–368, (1994).
6. M.P. Chen, B.S. Lalli and J.S. Yu, Oscillation in neutral delay difference equations with variable coefficients, *Computers Math. Applic.* **29** (3), 5–11, (1995).
7. H. Uesaka, An oscillatory property of solutions of nonlinear wave equations, In *Proceedings of 4^{th} MSJ International Research Institute on Nonlinear Waves*, Hokkaido University, Sapporo, Japan, (1995).
8. R.E. Mickens, *Difference Equations*, Van Nostrand Reinhold, (1990).