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On J_m-Hadamard matrices

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Abstract

In this paper, we use the Sylvester's approach to construct another Hadamard matrix, namely a J_m-Hadamard matrix, from a given one. Consequently, we can generate other $2^m - 1$ Hadamard matrices from the constructed J_m-Hadamard matrix. Finally, we also discuss the Kronecker product of an Hadamard matrix and a J_m-Hadamard matrix.

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1. Introduction

An $h \times h$ matrix $H \in \mathbb{M}_{h\times h}(\{\pm 1\})$ is an Hadamard matrix if $HH^{\mathrm{T}} = hI_h$, where I_h is the unit $h \times h$ matrix. As usual, here we have $h = 2$ or $h = 4t,\ t \in \mathbb{N}$. An important problem is to construct other Hadamard matrices from a given one. As it is clear, one gets equivalent Hadamard matrices from the given one by either permuting rows or columns and by multiplying any row or column by -1; for general properties and results on Hadamard matrices, we refer to Dinitz and Stinson [1] and van Lint and Wilson [2]. A recent construction by Marrero [3] allows us to yield three other Hadamard matrices from a given one. The construction goes as follows: let H be any $2t \times 2t$ Hadamard matrix. By using the above

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row and column operations, it is easily seen, that H can be transformed into the following form:

$$H \sim \begin{pmatrix} J & J & A \\ J & -J & B \end{pmatrix} = \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes J \,\middle|\, \begin{matrix} A \\ B \end{matrix} \right),$$

where $\otimes$ is the Kronecker product, $J \in \mathbb{M}_{t\times 1}(\{1\})$ and $A, B \in \mathbb{M}_{t\times(2t-2)}(\{\pm 1\})$.

The main result of such construction asserts that we get three Hadamard matrices by changing A into $-A$ or B into $-B$. More precisely,

$$\begin{pmatrix} J & J & -A \\ J & -J & -B \end{pmatrix}, \quad \begin{pmatrix} J & J & -A \\ J & -J & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} J & J & A \\ J & -J & -B \end{pmatrix}$$

are all Hadamard matrices.

The aim of this paper is to generalize the above construction by replacing the Hadamard matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ with a larger size Hadamard matrix M of order m and by replacing J with a suitably smaller size. The advantage is that it yields another Hadamard matrix of the form:

$$H \sim \left(M \otimes J \,\middle|\, \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \right),$$

where $A_1, A_2, \ldots, A_m \in \mathbb{M}_{t\times(mt-m)}(\{\pm 1\})$. Such Hadamard matrix will be called a J_m-Hadamard matrix (see Definition 2.1); Marrero's Hadamard matrix is an example of a J_2-Hadamard matrix. Note that m is not the order of H but of M.

In this paper, we stress that for $m \geqslant 4$, the J_m construction of an Hadamard matrix H is not always possible (Examples 3.1 and 3.2), whereas any Hadamard matrix is equivalent to a J_2-Hadamard matrix. However, a large class of Hadamard matrices allows us to create J_m-Hadamard matrices, namely the class of Sylvester–Hadamard matrices (Theorem 2.2).

As a final result, we consider the Kronecker product of an Hadamard matrix of order k and a J_m-Hadamard matrix; surprisingly, we obtain an Hadamard matrix equivalent to a J_{km}-Hadamard matrix (Theorem 2.5).

2. J_m-Hadamard matrices

In order to create other Hadamard matrices from a given one, the first step is to transform it into a special form as follows:

Definition 2.1. Let M be an Hadamard matrix of order m. If H is an $mt \times mt$ Hadamard matrix of the form:

$$\left(M \otimes J \,\middle|\, \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \right),$$

then H is called a J_m-Hadamard matrix, where $J \in \mathbb{M}_{t\times 1}(\{1\})$, $A_1, A_2, \ldots, A_m \in \mathbb{M}_{t\times(mt-m)}(\{\pm 1\})$ and $\otimes$ is the Kronecker product.

Here, we again emphasize that m is not the order of H but of M and it is not related to the size t of J. The following is the generalization of Marrero's main result on J_2-Hadamard matrices [3], proposition:

Theorem 2.2. *Let H be a J_m-Hadamard matrix defined as in Definition* 2.1. *Then*

$$\hat{H} = \left(M \otimes J \;\middle|\; \begin{matrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{matrix} \right)$$

is also an Hadamard matrix, where $B_i = A_i$ or $B_i = -A_i$ for $i = 1, 2, \ldots, m$.

Proof. Let

$$M = \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \vdots \\ \mathcal{M}_m \end{pmatrix},$$

where $\mathcal{M}_i$ are the row vectors of M for $i = 1, 2, \ldots, m$. Since M is an Hadamard matrix, then

$$\mathcal{M}_k \mathcal{M}_l^{\mathrm{T}} = \begin{cases} 0 & \text{if } k \neq l, \\ m & \text{if } k = l. \end{cases} \tag{2.1}$$

Because H is a J_m-Hadamard matrix, then, by multilinearity of the Kronecker product, we may write it as follows:

$$H = \left(M \otimes J \;\middle|\; \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \right) = \left(\begin{matrix} \mathcal{M}_1 \otimes J \\ \mathcal{M}_2 \otimes J \\ \vdots \\ \mathcal{M}_m \otimes J \end{matrix} \;\middle|\; \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \right),$$

where every two rows in $\mathcal{M}_i \otimes J$ are equal for $i = 1, 2, \ldots, m$. In fact,

$$\mathcal{M}_i \otimes J = \begin{pmatrix} \mathcal{M}_i \\ \mathcal{M}_i \\ \vdots \\ \mathcal{M}_i \end{pmatrix}_{t\times m}.$$

Rewrite H and $\hat{H}$ in the row vectors form $(\bar{\mathscr{M}}_i \mid \mathscr{A}_i)_{mt\times mt}$ and $(\bar{\mathscr{M}}_i \mid \mathscr{B}_i)_{mt\times mt}$ respectively, where $\bar{\mathscr{M}}_i$, $\mathscr{A}_i$ and $\mathscr{B}_i$ are the ith rows of

$$M \otimes J, \quad \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix},$$

respectively, for $i = 1, 2, \ldots, mt$. Note that $\bar{\mathscr{M}}_i$ may be equal to $\bar{\mathscr{M}}_j$ even if $i \neq j$. The reason is that $\bar{\mathscr{M}}_i$ and $\bar{\mathscr{M}}_j$ may be the rows of some $\mathscr{M}_k \otimes J$. Since H is an Hadamard matrix, then

$$(\bar{\mathscr{M}}_i \mid \mathscr{A}_i)\,(\bar{\mathscr{M}}_j \mid \mathscr{A}_j)^{\mathrm{T}} = \begin{cases} 0 & \text{if } i \neq j, \\ mt & \text{if } i = j, \end{cases}$$

i.e.,

$$\bar{\mathscr{M}}_i \bar{\mathscr{M}}_j^{\mathrm{T}} + \mathscr{A}_i \mathscr{A}_j^{\mathrm{T}} = \begin{cases} 0 & \text{if } i \neq j, \\ mt & \text{if } i = j. \end{cases} \tag{2.2}$$

We claim that $\hat{H}$ is also an Hadamard matrix; i.e.,

$$\bar{\mathscr{M}}_i \bar{\mathscr{M}}_j^{\mathrm{T}} + \mathscr{B}_i \mathscr{B}_j^{\mathrm{T}} = \begin{cases} 0 & \text{if } i \neq j, \\ mt & \text{if } i = j. \end{cases}$$

For $i = j$, no matter $\mathscr{A}_i = \mathscr{B}_i$ or $\mathscr{A}_i = -\mathscr{B}_i$, it is true that

$$\bar{\mathscr{M}}_i \bar{\mathscr{M}}_i^{\mathrm{T}} + \mathscr{B}_i \mathscr{B}_i^{\mathrm{T}} = \bar{\mathscr{M}}_i \bar{\mathscr{M}}_i^{\mathrm{T}} + \mathscr{A}_i \mathscr{A}_i^{\mathrm{T}} = mt.$$

For $i \neq j$, there are two cases.

Case 1: If $\bar{\mathscr{M}}_i$ and $\bar{\mathscr{M}}_j$ are the rows of some $\mathscr{M}_k \otimes J$, then $\bar{\mathscr{M}}_i = \bar{\mathscr{M}}_j = \mathscr{M}_k$. Simultaneously, $\mathscr{B}_i$ and $\mathscr{B}_j$ are the rows of B_k. But this time, we get the case

$$\mathscr{B}_i = \mathscr{A}_i \text{ and } \mathscr{B}_j = \mathscr{A}_j \text{ or } \mathscr{B}_i = -\mathscr{A}_i \text{ and } \mathscr{B}_j = -\mathscr{A}_j.$$

Thus, $\mathscr{B}_i \mathscr{B}_j^{\mathrm{T}} = \mathscr{A}_i \mathbf{A}_j^{\mathrm{T}}$, in any situation. This implies

$$\bar{\mathscr{M}}_i \bar{\mathscr{M}}_j^{\mathrm{T}} + \mathscr{B}_i \mathscr{B}_j^{\mathrm{T}} = \bar{\mathscr{M}}_i \bar{\mathscr{M}}_j^{\mathrm{T}} + \mathscr{A}_i \mathscr{A}_j^{\mathrm{T}} = 0, \text{ by (2.2).}$$

Case 2: If $\bar{\mathscr{M}}_i$ and $\bar{\mathscr{M}}_j$ are the rows of $\mathscr{M}_k \otimes J$ and $\mathscr{M}_l \otimes J$, respectively, for $k \neq l$. In this case, $\bar{\mathscr{M}}_i = \mathscr{M}_k$ and $\bar{\mathscr{M}}_j = \mathscr{M}_l$. Simultaneously, $\mathscr{B}_i$ and $\mathscr{B}_j$ are the rows of B_k and B_l, respectively. Hence $\bar{\mathscr{M}}_i \bar{\mathscr{M}}_j^{\mathrm{T}} = \mathscr{M}_k \mathscr{M}_l^{\mathrm{T}} = 0$, by (2.1). Together with $\bar{\mathscr{M}}_i \bar{\mathscr{M}}_j^{\mathrm{T}} + \mathscr{A}_i \mathscr{A}_j^{\mathrm{T}} = 0$, for $i \neq j$, we have $\mathscr{A}_i \mathscr{A}_j^{\mathrm{T}} = 0$. Now, there are four possibilities:

$$\mathscr{B}_i = \mathscr{A}_i \text{ and } \mathscr{B}_j = \mathscr{A}_j;\ \mathscr{B}_i = \mathscr{A}_i \text{ and } \mathscr{B}_j = -\mathscr{A}_j;$$

$$\mathscr{B}_i = -\mathscr{A}_i \text{ and } \mathscr{B}_j = \mathscr{A}_j;\ \mathscr{B}_i = -\mathscr{A}_i \text{ and } \mathscr{B}_j = -\mathscr{A}_j.$$

Hence $\mathscr{B}_i \mathscr{B}_j^{\mathrm{T}} = \pm \mathscr{A}_i \mathscr{A}_j^{\mathrm{T}} = 0$. This implies $\bar{\mathscr{M}}_i \bar{\mathscr{M}}_j^{\mathrm{T}} + \mathscr{B}_i \mathscr{B}_j^{\mathrm{T}} = 0$ for $i \neq j$. This completes the proof of the theorem. $\square$

From Theorem 2.2, we might create other $2^m - 1$ Hadamard matrices. More precisely, there are only other $2^{m-1} - 1$ Hadamard matrices up to equivalence since we may multiply -1 to a given matrix $\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix}$ to get an equivalent J_m-Hadamard matrix.

The next question is whether there exists a J_m-Hadamard matrix for $m=4t$. The following theorem and corollary tell us how to produce a J_m-Hadamard matrix by use of Sylvester's approach. To this end, let H be a given Hadamard matrix, the Sylvester–Hadamard matrix induced by H is an Hadamard matrix of the form

$$\begin{pmatrix} H & H \\ H & -H \end{pmatrix}.$$

Note that the above Sylvester–Hadamard matrix induced by H is actually equivalent to a J_m-Hadamard matrix (whose proof of this important fact is similar to that of the following theorem):

$$\begin{pmatrix} H & H \\ H & -H \end{pmatrix} \sim \left(H \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| H \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right).$$

Theorem 2.3. *If H is a J_m-Hadamard matrix, then the Sylvester–Hadamard matrix induced by H is equivalent to a J_m-Hadamard matrix.*

Proof. Because H is a J_m-Hadamard matrix, then, by multilinearity of the Kronecker product, we may write it as follows:

$$H = \left(M \otimes J \,\middle|\, \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \right) = \left(\begin{matrix} \mathscr{M}_1 \otimes J \\ \mathscr{M}_2 \otimes J \\ \vdots \\ \mathscr{M}_m \otimes J \end{matrix} \,\middle|\, \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \right),$$

where $\mathscr{M}_i$ are the row vectors of M for $i = 1, 2, \ldots, m$.

After suitably permuting rows, the Sylvester–Hadamard matrix is equivalent to

$$\left(\begin{matrix} \mathscr{M}_1 \otimes J \\ \mathscr{M}_1 \otimes J \\ \mathscr{M}_2 \otimes J \\ \mathscr{M}_2 \otimes J \\ \vdots \\ \mathscr{M}_m \otimes J \\ \mathscr{M}_m \otimes J \end{matrix} \,\middle|\, \begin{matrix} A_1 & \mathscr{M}_1 \otimes J & A_1 \\ A_1 & -\mathscr{M}_1 \otimes J & A_1 \\ A_2 & \mathscr{M}_2 \otimes J & A_2 \\ A_2 & -\mathscr{M}_2 \otimes J & A_2 \\ \vdots & \vdots & \vdots \\ A_m & \mathscr{M}_m \otimes J & A_m \\ A_m & -\mathscr{M}_m \otimes J & A_m \end{matrix} \right) = \left(\begin{matrix} \mathscr{M}_1 \otimes \hat{J} \\ \mathscr{M}_2 \otimes \hat{J} \\ \vdots \\ \mathscr{M}_m \otimes \hat{J} \end{matrix} \,\middle|\, \begin{matrix} A_1 & \mathscr{M}_1 \otimes J & A_1 \\ A_1 & -\mathscr{M}_1 \otimes J & A_1 \\ A_2 & \mathscr{M}_2 \otimes J & A_2 \\ A_2 & -\mathscr{M}_2 \otimes J & A_2 \\ \vdots & \vdots & \vdots \\ A_m & \mathscr{M}_m \otimes J & A_m \\ A_m & -\mathscr{M}_m \otimes J & A_m \end{matrix} \right),$$

where

$$\hat{J} = \begin{pmatrix} J \\ J \end{pmatrix}.$$

This is a J_m-Hadamard matrix and we complete the proof. □

As indicated above, Sylvester's approach allows us to yield the following.

Corollary 2.4. *If there is an Hadamard matrix of order m, then we may get a Sylvester–Hadamard matrix equivalent to a J_m-Hadamard matrix. This construction allows us also to yield an Hadamard matrix which is at the same time equivalent to a J_{m2^t}-Hadamard matrix for all $t \in \mathbb{N}$. In particular, if we choose the initial Hadamard matrix*

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

then there is an Hadamard matrix which is at the same time equivalent to a J_{2^t}-Hadamard matrix for all $t \in \mathbb{N}$.

Proof. Use the same approach as in the proof of Theorem 2.3. □

As it is well known, the Kronecker product of two Hadamard matrices K of order k and H of order h is also an Hadamard matrix of order kh. As it is expected, we have the following result for J_m-Hadamard matrices.

Theorem 2.5. *If K is an Hadamard matrix of order k and H is a J_m-Hadamard matrix, then the Kronecker product of K and H is equivalent to a J_{km}-Hadamard matrix.*

Proof. Let $\mathscr{K}_i$ be the ith row vector of K for $i=1, 2, \ldots, k$ and H be of the form $(M \otimes J \,|\, A)$, where

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}.$$

Then

$$K \otimes H = \begin{pmatrix} \mathscr{K}_1 \otimes (M \otimes J \,|A) \\ \mathscr{K}_2 \otimes (M \otimes J \,|A) \\ \vdots \\ \mathscr{K}_k \otimes (M \otimes J \,|A) \end{pmatrix}.$$

After suitably permuting columns,

$$K \otimes H \sim \left(\begin{array}{c|c} \mathscr{K}_1 \otimes (M \otimes J) & \mathscr{K}_1 \otimes A \\ \mathscr{K}_2 \otimes (M \otimes J) & \mathscr{K}_2 \otimes A \\ \vdots & \vdots \\ \mathscr{K}_k \otimes (M \otimes J) & \mathscr{K}_k \otimes A \end{array}\right) = ((K \otimes M) \otimes J \,|K \otimes A).$$

Since $K \otimes M$ is an Hadamard matrix of order km, then $K \otimes H$ is equivalent to a J_{km}-Hadamard matrix and the proof is completed. □

3. Counterexamples

A normalized Hadamard matrix is an Hadamard matrix with the first row and the first column having entries all 1. In the following two examples, we make use the following fact about normalized Hadamard matrices (see [4, Theorem 10.9, p. 429]): If H is a normalized Hadamard matrix of order $n > 2$, then $n = 4m$ for some m. Moreover, each row (column) except the first has exactly $2m$ 1's and $2m$ -1's, and for any two rows (columns) other than the first, there are exactly m positions in which both rows (columns) have 1's.

Example 3.1. Every Hadamard matrix of order 12 is not a J_4-Hadamard matrix.

Proof. Without loss of generality, we may assume that H be a normalized 12×12 Hadamard matrix. If H is a J_4-Hadamard matrix, then

$$H = \left(M \otimes J \;\middle|\; \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \right),$$

where M is an Hadamard matrix of order 4, $J \in \mathbb{M}_{3\times 1}(\{1\})$ and $A_i \in \mathbb{M}_{3\times 8}(\{\pm 1\})$ for $i = 1, 2, 3, 4$. Since every Hadamard matrix of order 4 is easily known to be equivalent to

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

hence H must be equivalent to the J_4-Hadamard matrix of the form

$$\tilde{H} = \begin{pmatrix} J & J & J & J & A_1 \\ J & J & -J & -J & A_2 \\ J & -J & J & -J & A_3 \\ J & -J & -J & J & A_4 \end{pmatrix} = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & A_1 \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & -1 & -1 & \\ 1 & 1 & -1 & -1 & A_2 \\ 1 & 1 & -1 & -1 & \\ 1 & -1 & 1 & -1 & \\ 1 & -1 & 1 & -1 & A_3 \\ 1 & -1 & 1 & -1 & \\ 1 & -1 & -1 & 1 & \\ 1 & -1 & -1 & 1 & A_4 \\ 1 & -1 & -1 & 1 & \end{array}\right).$$

However, $\tilde{H}$ is not an Hadamard matrix, since there are at least four 1's at the same positions between the second row and the third row contradicting to the fact mentioned above: there are exactly $\frac{12}{4}$ 1's at the same positions in both rows except the first. Thus H is not a J_4-Hadamard matrix. $\square$

Example 3.2. *Every Hadamard matrix of order* 20 *is not a* J_4*-Hadamard matrix*.

Proof. We assume that H be a normalized J_4-Hadamard matrix with the form as in Example 3.1 with $J \in \mathbb{M}_{5\times 1}(\{1\})$ and $A_i \in \mathbb{M}_{5\times 16}(\{\pm 1\})$ for $i = 1, 2, 3, 4$. We will use the same argument as above to derive a contradiction by counting the number of 1's at the second, the third, the fourth and the fifth row. As before, we know that there are exactly ten 1's at each row and $\frac{20}{4}$ 1's at the same position between any two different rows except the first row. By arranging the 1's as forward as possible, so H, with the first five rows written down, is of the following form:

$$H = \begin{pmatrix} J & J & J & J & A_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & & & & & & & & & & & & & & & \\ 1 & 1 & 1 & 1 & & & & & & & & & & & & & & & & \\ 1 & 1 & 1 & 1 & & & & & & & & & & & & & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & & & & & & & & & & & & & \end{pmatrix}.$$

Considering the last ten columns, to fill in the ten 1's in the third row, we need five positions in last ten columns. With the same argument, to fill in the ten 1's in the fourth row, we need at least four positions in the last ten columns differ from the positions already taken in the third row. Finally, in the fifth row, we need at least three positions in the last ten columns differ from the positions already taken in the third and the fourth rows. This means that we need in total at least $5+4+3 = 12$ positions to fill in the 1's in the last ten columns which is impossible. Therefore, we conclude that every Hadamard matrix of order 20 is not a J_4-Hadamard matrix.

By the same way as above, all Hadamard matrices of order 24 and 40 are not J_8-Hadamard matrices, and so on. This fact leads to the following questions:

4. Some open questions

1. As also mentioned in [3], it is well known that every Hadamard matrix is equivalent to a J_2-Hadamard matrix. Given any Hadamard matrix, is it equivalent to a J_m-Hadamard matrix for some $m \geqslant 4$?
2. The above Examples 3.1 and 3.2 show that any Hadamard matrices of order 12 and 20 are not J_4-Hadamard matrix. These examples seem to provide counterexamples to Question 1, if the following is true: a J_8-Hadamard matrix is equivalent to a J_4-Hadamard matrix.

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