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A DETERMINISTIC APPROACH FOR SOLVING THE HULL AND WHITE INTEREST RATE MODEL

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Abstract. This work considers the resolution of the Hull and White interest rate model. A deterministic process is adopted to model the random behavior of interest rate variation as a deterministic perturbation. It shows that the interest rate function and the yield function of the Hull and White interest rate model can be obtained by solving a nonlinear semi-infinite programming problem. A relaxed cutting plane algorithm is then proposed for the resulting optimization problem. The features of the proposed method are tested using a set of real data and compared with some commonly used spline fitting methods.

1. INTRODUCTION

The interest rate model plays a central role in the theory of modern economics and finance. In the past studies interest rate models described by stochastic process are widely used. It is usually assumed that the interest rates are sufficient statistics for the stochastic movement of current term structure. An enormous amount of work has been directed towards modeling and estimation of the short term interest rate dynamics. Some single-factor models [3, 5, 26] have been proposed and widely used in practice because of their tractability and their ability to fit reasonably well the dynamics of the short term interest rates. Econometric estimation of these models has also been intensively studied in the literature [3, 6, 20]. Generally, the problem of estimation occurring in nondeterministic systems has been investigated by mean of many stochastic models, beginning with the papers of Wiener[27] and Kalman[14]. Earlier in the 1970s, nonstochastic observation models under uncertainty appeared in [4, 17, 18]. A new approach for optimization of linear dynamical systems under uncertainty was presented in [9] based on the earlier fundamental papers of Gabasov, Kirillova and colleagues[7, 8, 16]. Recently, Kortanek and Medvedev[15] considered the development and application of a new class of models of this type

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for the term structure of interest rates. A deterministic process was introduced to model the random behavior of interest rate variation as a deterministic perturbation which was later investigated by [23], and [24]. Sarychev et al. [22] included a set of original papers based on the mathematics of control systems and applications to finance. Inspired and motivated by the recent research, this work considers the Hull and White interest rate model, which can be described by the following stochastic differential equation.

$$dr(t) = \alpha(\mu(t) - r(t))dt + \sigma dB(t), \tag{1.1}$$

where

$$\mu(t) = \frac{1}{\alpha}\frac{\partial}{\partial t}f(0,t) + f(0,t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-2\alpha t}), \tag{1.2}$$

$f(0,t)$ is the forward interest rate, $r(t)$ is the instance interest rate, $B(t)$ denotes the Brownian motion, σ is the instantaneous standard deviation of the interest rate and the coefficient α, satisfyies

$$0 < \underline{\alpha} \leq \alpha \leq \overline{\alpha}$$

with the pre-assigned bounds $\underline{\alpha}$, and $\overline{\alpha}$.

To solve the Hull and White interest rate model (1.1), the concept of the deterministic perturbations is adopted to deal with the random behavior of interest rate variations. It is shown that the interest rate function and the yield function of the Hull and White interest rate model (1.1) can be obtained by solving a nonlinear semi-infinite programming problem. A relaxed cutting plane algorithm is proposed for solving the resulting optimization problem. In each iteration, we solve a finite optimization problem and add one or some more constraints. The proposed algorithm chooses a point at which the infinite constrains are violated to a degree rather than the violation being maximized. The organization of the rest of this paper is as follows. Section 2 provides some basic definitions to formulate the Brownian motion in the Hull and White interest rate model in terms of the deterministic perturbation. It shows that the Hull and White interest rate model can be solved via a nonlinear semi-infinite programming problem. Solution algorithms are developed in Section 3 for solving the resulting semi-infinite programming problem. The numerical results and comparison to some commonly used spline fitting methods are reported in Section 4. The paper is concluded in Section 5.

2. The Hull and White Interest Rate Model with Impulse Perturbation

As mentioned in the previous section, in this paper a deterministic process is adopted to model the uncertainty in the interest rate behavior. It is assumed that the uncertainty is deterministic, which is depending on the time t. For convenience we denote the nonstochastic uncertainty as an integral function $w(t)$, and $\overline{w}(t)$, $\underline{w}(t)$ are assumed to be the pre-assigned upper and lower bounds of $w(t)$, respectively, i.e.,

$$\underline{w}(t) \leq w(t) \leq \overline{w}(t). \tag{2.1}$$

In this case, the Hull and White interest rate model can be formulated as the following differential equation with uncertainty.

$$dr(t) = \alpha(\mu(t) - r(t))dt + \sigma w(t)dt. \tag{2.2}$$

To simplify the calculation process, we reduce the forward interest rate to the following form [2].

$$\tilde{f}(0,t) \stackrel{\triangle}{=} \tilde{f}_i = c + d\exp(bt), \forall t \in \aleph_i, i = 1, 2, \cdots, N. \quad \text{where } b, c, d \in R. \tag{2.3}$$

Moreover, to specify the perturbation function $w(t)$, here we introduce some notations and definitions. Assume that there are M observed yields, say $\bar{R}_i$, with time to maturity $\mathcal{T}$ corresponding to the $i-$th day of observation, $i = 1, 2, \cdots, M$. Let $\tilde{\aleph} \stackrel{\triangle}{=} \{t_0, t_1, \cdots, t_{M+\mathcal{T}}\}$, where $t_{i-1} < t_i$, and $\aleph_i \stackrel{\triangle}{=} [t_{i-1}, t_i), i = 1, 2, \cdots, M+\mathcal{T}$. For convenient, we denote $M + \mathcal{T} = N$.

Definition 2.1. The Observed Treasury Yield.
The observed Treasury yield is defined as follows.

$$\bar{R}(t \mid \mathcal{T}) \stackrel{\triangle}{=} \bar{R}_i, \forall t \in \aleph_i, i = 1, 2, \cdots, M.$$

Definition 2.2. The Yield Function.
The yield function is defined as the mean value of interest rate of integral, i.e.,

$$y(t \mid \mathcal{T}) \stackrel{\triangle}{=} \frac{1}{\mathcal{T}} \int_t^{t+\mathcal{T}} r(\tau)d\tau, \forall t \in \aleph_i, i = 1, 2, \cdots, M. \tag{2.4}$$

Definition 2.3. The Function of Estimation Error.
The function of estimation error is defined as the difference of the yield function and the observed Treasury yield, i.e.,

$$\xi(t) \stackrel{\triangle}{=} y(t \mid \mathcal{T}) - \overline{R}(t \mid \mathcal{T}), \forall t \in \aleph_i, i = 1, 2, \cdots, M. \tag{2.5}$$

Definition 2.4. The Impulse Perturbation.
Let $w(t) \stackrel{\triangle}{=} w_i(t), \forall t \in \aleph_i, i = 1, 2, \cdots, N$. The impulse perturbation is the class of piecewise constant functions, which is locally constant in connected regions and can be defined as follows:

$$w_i(t) = w_i, \forall t \in \aleph_i, i = 1, 2, \cdots, N, \tag{2.6}$$

where $w_i \in \mathbb{R}$ is a constant and

$$\underline{w}_i \leq w_i \leq \overline{w}_i, i = 1, 2, \cdots, N,$$

with $\underline{w}_i$ and $\overline{w}_i, i = 1, 2, \cdots, N$, are pre-assigned bounds for the perturbations.

Definition 2.5. The Forward Mean Function.
The forward mean function is defined as the mean value of forward rate of integral, i.e.,

$$\begin{aligned} F(c, b, d, t \mid \mathcal{T}) &\triangleq \frac{1}{\mathcal{T}} \int_t^{t+\mathcal{T}} \tilde{f}(0, \tau) d\tau, \\ &= c + \frac{d}{b\mathcal{T}}(\exp(b(t+\mathcal{T})) - \exp(bt)) \\ & \quad b, c, d, \in R, \forall t \in \aleph_i, i = 1, 2, \cdots, M. \end{aligned} \tag{2.7}$$

The solution of the Hull and White interest rate model (2.2) with the impulse perturbation function defined in (2.6) has the form described in Theorem 2.1.

Theorem 2.1. *The instance interest rate function of the Hull and White interest rate model (2.2) is given by*

$$\begin{aligned} r(t) &= \tilde{f}(0, t) + \frac{\sigma^2}{2\alpha^2}\left(1 + 2e^{-2\alpha t} - 2e^{-\alpha t}\right) + \sum_{j=1}^{i-1} \sigma w_j \frac{e^{-\alpha t}}{\alpha}(e^{\alpha t_j} - e^{\alpha t_{j-1}}) \\ &+ \frac{w_i \sigma}{\alpha}(1 - e^{-\alpha(t - t_{i-1})}), \forall t \in \aleph_i, i = 1, 2, \ldots, N. \end{aligned} \tag{2.8}$$

Proof. The proof is given in the appendix A. ■

It is well known that the yield function is one of the most important financial indicators in the theory of modern economics and finance. Substituting (2.8) into (2.4) yields the following result.

Theorem 2.2. *The yield function has the form*

$$\begin{aligned} & y(t|\mathcal{T}) \\ = {} & F(c, b, d, t \mid \mathcal{T}) + \frac{\sigma^2}{2\alpha^2}\left(1 - \frac{e^{-2\alpha(t+\mathcal{T})} - e^{-2\alpha t}}{2\mathcal{T}\alpha} + \frac{2e^{-\alpha(t+\mathcal{T})} - 2e^{-\alpha t}}{\mathcal{T}\alpha}\right) \\ & + \sum_{k=1}^{i-1}\left(\frac{e^{\alpha(t_k - t)} + e^{\alpha(t_{k-1} - t - \mathcal{T})}}{\mathcal{T}\alpha^2} - \frac{e^{\alpha(t_k - t - \mathcal{T})} + e^{\alpha(t_{k-1} - t)}}{\mathcal{T}\alpha^2}\right)\sigma w_k \\ & + (\frac{t_i - t}{\mathcal{T}\alpha} + \frac{e^{\alpha(t_{i-1} - t - \mathcal{T})} - e^{\alpha(t_i - t - \mathcal{T})} - e^{\alpha(t_{i-1} - t)} + 1}{\mathcal{T}\alpha^2})\sigma w_i \\ & + \sum_{k=i+1}^{i+\mathcal{T}-1}(\frac{t_k - t_{k-1}}{\mathcal{T}\alpha} + \frac{e^{\alpha(t_{k-1} - t - \mathcal{T})} - e^{\alpha(t_k - t - \mathcal{T})}}{\mathcal{T}\alpha^2})\sigma w_k \end{aligned} \tag{2.9}$$

$$+(\frac{t+\mathcal{T}-t_{i+\mathcal{T}-1}}{\mathcal{T}\alpha}+\frac{e^{\alpha(t_{i+\mathcal{T}-1}-t-\mathcal{T})}-1}{\mathcal{T}\alpha^2})\sigma w_{i+\mathcal{T}}, \quad t\in\aleph_i, i=1,2,...,M.$$

Proof. The proof is given in the appendix B. ■

To shorten the mathematical formulas in (2.9), the following notations are introduced. Let

$$a_k(\alpha,t|\mathcal{T}) = \begin{cases} \left(\frac{e^{\alpha(t_k-t)}+e^{\alpha(t_{k-1}-t-\mathcal{T})}}{\mathcal{T}\alpha^2}-\frac{e^{\alpha(t_k-t-\mathcal{T})}+e^{\alpha(t_{k-1}-t)}}{\mathcal{T}\alpha^2}\right)\sigma, & \text{if } k<i,\\ \left(\frac{t_i-t}{\mathcal{T}\alpha}+\frac{e^{\alpha(t_{i-1}-t-\mathcal{T})}+1}{\mathcal{T}\alpha^2}-\frac{e^{\alpha(t_i-t-\mathcal{T})}+e^{\alpha(t_{i-1}-t)}}{\mathcal{T}\alpha^2}\right)\sigma, & \text{if } k=i,\\ \left(\frac{t_k-t_{k-1}}{\mathcal{T}\alpha}+\frac{e^{\alpha(t_{k-1}-t-\mathcal{T})}-e^{\alpha(t_k-t-\mathcal{T})}}{\mathcal{T}\alpha^2}\right)\sigma, & \text{if } i<k<i+\mathcal{T},\\ \left(\frac{t+\mathcal{T}-t_{i+\mathcal{T}-1}}{\mathcal{T}\alpha}+\frac{e^{\alpha(t_{i+\mathcal{T}-1}-t-\mathcal{T})}-1}{\mathcal{T}\alpha^2}\right)\sigma, & \text{if } k=i+\mathcal{T}. \end{cases} \tag{2.10}$$

we have

$$\begin{aligned} y(t|\mathcal{T}) &= F(c,b,d,t\mid\mathcal{T})+\frac{\sigma^2}{2\alpha^2}(1-\frac{e^{-2\alpha(t+\mathcal{T})}-e^{-2\alpha t}}{2\mathcal{T}\alpha}\\ &\quad+\frac{2e^{-\alpha(t+\mathcal{T})}-2e^{-\alpha t}}{\mathcal{T}\alpha})+\sum_{k=1}^{i+\mathcal{T}}a_k(\alpha,t|\mathcal{T})w_k,\\ &\quad t\in\aleph_i, i=1,2,\cdots,M. \end{aligned} \tag{2.11}$$

This work considers to find the impulse perturbation $w(t)$ that minimizes the maximum absolute value of the function of estimation errors defined in (2.5). It leads to the following optimization problem.

Problem 1.

$$\begin{aligned} \min \quad & \epsilon\\ \textbf{s.t.} \quad & \bar{R}(t\mid\mathcal{T})\leqslant y(t|\mathcal{T})+\epsilon, \forall t\in\aleph_i, i=1,2,\ldots,M,\\ & \bar{R}(t\mid\mathcal{T})\geqslant y(t|\mathcal{T})-\epsilon, \forall t\in\aleph_i, i=1,2,\ldots,M,\\ & \underline{\alpha}\le\alpha\le\overline{\alpha}, \quad \underline{w}_i\le w_i\le\overline{w}_i, \quad i=1,2,\ldots,N. \end{aligned}$$

Substituting (2.11) into the **Problem 1** leads to the following nonlinear programming problem.

Problem 2.

$$
\begin{aligned}
\min \quad & \epsilon \\
\textbf{s.t.} \quad & \overline{R}_i \leqslant F(c,b,d,t \mid \mathcal{T}) \\
& \quad +\frac{\sigma^2}{2\alpha^2}\left(1-\frac{e^{-2\alpha(t+\mathcal{T})}}{2\mathcal{T}\alpha}-\frac{e^{-2\alpha t}}{2\mathcal{T}\alpha}+\frac{2e^{-\alpha(t+\mathcal{T})}-2e^{-\alpha t}}{\mathcal{T}\alpha}\right) \\
& \quad +\sum_{j=1}^{i+\mathcal{T}} a_j(\alpha,t|\mathcal{T})w_j+\epsilon, \\
& \underline{R}_i \geqslant F(c,b,d,t \mid \mathcal{T}) \\
& \quad +\frac{\sigma^2}{2\alpha^2}\left(1-\frac{e^{-2\alpha(t+\mathcal{T})}-e^{-2\alpha t}}{2\mathcal{T}\alpha}+\frac{2e^{-\alpha(t+\mathcal{T})}-2e^{-\alpha t}}{\mathcal{T}\alpha}\right) \\
& \quad +\sum_{j=1}^{i+\mathcal{T}} a_j(\alpha,t|\mathcal{T})w_j-\epsilon, \forall\ t\in\aleph_i, i=1,2,\ldots,M, \\
& \underline{\alpha}\leq\alpha\leq\overline{\alpha},\ \underline{w_i}\leq w_i\leq\overline{w}_i, i=1,2,\ldots,N.
\end{aligned}
$$

It should be noticed that the **Problem 2** is a semi-infinite programming problem with finite variables, $b, c, d, \alpha, \sigma, \epsilon, w_i, i = 1, 2, \cdots, N$, and infinite many constraints.

3. An Algorithm

There are many semi-infinite programming algorithms [11, 12, 13] available for solving the **Problem 2**. The difficulty lies in how to effectively deal with the infinite number of constrains. Based on a recent review [13], the "cutting plane approach" is an effective one for such application. Following the basic concept of the cutting plane approach, we can easily design an iterative algorithm which adds one or some more constraint at a time for consideration until an optimal solution is identified. To be more specific, at the $k-th$ iteration, given subsets $N_i^k = \{\tau_1^i, \tau_2^i, \cdots, \tau_{p_i^k}^i\}$ and $\aleph_i^k = \{u_1^i, u_2^i, \cdots, u_{q_i^k}^i\}$ of $\aleph_i$, where $p_i^k, q_i^k \geq 1, i = 1, 2, \cdots, M$, we consider the following finite optimization problem.

Program SD^k

$$
\begin{aligned}
& \min\ \phi(b,c,d,\alpha,\sigma,w,\epsilon)=\epsilon \\
& \textbf{s.t.}\ \ \overline{R}_i \leqslant F(c,b,d,\tau_s^i \mid \mathcal{T}) \\
& \quad +\frac{\sigma^2}{2\alpha^2}(1-\frac{e^{-2\alpha(\tau_s^i+\mathcal{T})}}{2\mathcal{T}\alpha}-\frac{e^{-2\alpha(\tau_s^i)}}{2\mathcal{T}\alpha}+\frac{2e^{-\alpha(\tau_s^i+\mathcal{T})}-2e^{-\alpha(\tau_s^i)}}{\mathcal{T}\alpha}) \\
& \quad +\sum_{j=1}^{i+\mathcal{T}} a_j(\alpha,\tau_s^i|\mathcal{T})w_j+\epsilon, s=1,2,\cdots,p_i^k, i=1,2,\cdots,M,
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&\overline{R}_i \geqslant F(c,b,d,u_l^i \mid \mathcal{T}) \\
&\quad + \frac{\sigma^2}{2\alpha^2}\Big(1 - \frac{e^{-2\alpha(u_l^i+\mathcal{T})} - e^{-2\alpha(u_l^i)}}{2\mathcal{T}\alpha} + \frac{2e^{-\alpha(u_l^i+\mathcal{T})} - 2e^{-\alpha(u_l^i)}}{\mathcal{T}\alpha}\Big) \\
&\quad + \sum_{j=1}^{i+\mathcal{T}} a_j(\alpha, u_l^i|\mathcal{T}) w_j - \epsilon,\ \ l = 1,2,\cdots,q_i^k, i = 1,\ldots,M, \\
&\quad \underline{\alpha} \leq \alpha \leq \overline{\alpha},\ \ \underline{w}_i \leq w_i \leq \overline{w}_i, i = 1,2,\ldots,N.
\end{aligned}
$$

Let F^k be the feasible region of Program SD^k. Suppose that $(b^k, c^k, d^k, \alpha^k, \sigma^k, w^k, \epsilon^k)$ is an optimal solution of SD^k. We define the "constraint violation functions" as follows.

$$
\begin{aligned}
&g_i^{k+1}(\tau) \stackrel{\triangle}{=} \overline{R}_i - F(b^k,c^k,d^k,\tau|\mathcal{T}) - \frac{(\sigma^k)^2}{2(\alpha^k)^2}\Big(1 - \frac{2e^{-\alpha(\tau_s^i+\mathcal{T})} - 2e^{-\alpha(\tau_s^i)}}{\mathcal{T}\alpha} \\
&\quad + \frac{2e^{-\alpha^k(\tau+\mathcal{T})} - 2e^{-\alpha^k(\tau)}}{\mathcal{T}\alpha^k}\Big) - \sum_{j=1}^{i+\mathcal{T}} a_j(\alpha^k,\tau|\mathcal{T}) w_j^k - \epsilon^k, \\
&\quad \tau \in \aleph_i, i = 1,2,\cdots,M,
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
&v_i^{k+1}(u) \stackrel{\triangle}{=} F(b^k,c^k,d^k,\tau \mid \mathcal{T}) + \frac{(\sigma^k)^2}{2(\alpha^k)^2}\Big(1 - \frac{e^{-2(\alpha^k)(u+\mathcal{T})} - e^{-2\alpha^k(u)}}{2\mathcal{T}\alpha^k} \\
&\quad + \frac{2e^{-\alpha^k(u+\mathcal{T})} - 2e^{-\alpha^k(u)}}{\mathcal{T}\alpha^k}\Big) + \sum_{j=1}^{i+\mathcal{T}} a_j(\alpha^k,u|\mathcal{T}) w_j^k - \epsilon^k - \overline{R}_i, \\
&\quad u \in \aleph_i, i = 1,\cdots,M.
\end{aligned}
\tag{3.3}
$$

Since $\overline{R}_i, a_j$ are continuous over the compact set $\aleph_i$, the function $g_i^{k+1}(\tau)$ achieves its maximum over $\aleph_i, i = 1,2,\cdots,M$. A similar argument holds for the function $v_i^{k+1}(u), i = 1,2,\cdots,M$. Let $\tau^i_{p_i^k+1}$ and $u^i_{q_i^k+1}$ be such maximizers, $i = 1,2,\cdots,M$, and consider the values of $g_i^{k+1}(\tau^i_{p_i^k+1})$ and $v_i^{k+1}(u^i_{q_i^k+1}), i = 1,2,\cdots,M$. If the values are less than or equal to zero, then $(b^k, c^k, d^k, \alpha^k, \sigma^k, w^k, \epsilon^k)$ becomes a feasible solution of the **Problem 2**, and hence $(b^k, c^k, d^k, \alpha^k, \sigma^k, w^k, \epsilon^k)$ is optimal for the **Problem 2** (because the feasible region F^k of Program SD^k is no smaller than the feasible region of the **Problem 2**). Otherwise, we know that at least $\tau^i_{p_i^k+1} \notin N_i^k$ or $u^i_{q_i^k+1} \notin \aleph_i^k, i = 1,2,\cdots,M$. This background provides a foundation for us to outline a cutting plane algorithm for solving the **Problem 2**.

CPSD Algorithm:

Initialization

Set $k = p_i^k = q_i^k = 1, i = 1,2,\cdots,M$; Choose any $\tau_1^i, u_1^i \in \aleph_i, i = 1,2,\cdots,M$; Set $N_i^1 = \{\tau_1^i\}$ and $\aleph_i^1 = \{u_1^i\}, i = 1,2,\cdots,M$.

Step 1. Solve SD^k and obtain an optimal solution $(b^k, c^k, d^k, \alpha^k, \sigma^k, w^k, \epsilon^k)$.

Step 2. Find a maximizer $\tau^i_{p^k_i+1}$ of $g^{k+1}_i(\tau)$ over $\aleph_i$ and a maximizer $u^i_{q^k_i+1}$ of $v^{k+1}_i(u)$ over $\aleph_i, i = 1, 2, \cdots, M$.

Step 3. If $g^{k+1}_i(\tau^i_{p^k_i+1}) \leq 0$ and $v^{k+1}_i(u^i_{q^k_i+1}) \leq 0, i = 1, 2, \cdots, M$, then stop with $(b^k, c^k, d^k, \alpha^k, \sigma^k, w^k, \epsilon^k)$ being an optimal solution of the **Problem 2**. Otherwise, go to step 4.

Step 4. If $g^{k+1}_i(\tau^i_{p^k_i+1}) > 0$, then set $N^{k+1}_i \leftarrow N^k_i \bigcup \{\tau^i_{p^k_i+1}\}, p^{k+1}_i \leftarrow p^k_i + 1$. Otherwise, set $N^{k+1}_i \leftarrow N^k_i, p^{k+1}_i \leftarrow p^k_i, i = 1, 2, \cdots, M$.

Step 5. If $v^{k+1}_i(u^i_{q^k_i+1}) > 0$, then set $\aleph^{k+1}_i \leftarrow \aleph^k_i \bigcup \{u^i_{q^k_i+1}\}, q^{k+1}_i \leftarrow q^k_i + 1$. Otherwise, set $\aleph^{k+1}_i \leftarrow \aleph^k_i, q^{k+1}_i \leftarrow q^k_i, i = 1, 2, \cdots, M$.

Step 6. Set $k \leftarrow k + 1$ go to Step 1.

When the **Problem 2** has at least one feasible solution, it can be shown without much difficulty that the CPSD algorithm either terminates in a finite number of iterations with an optimal solution or generates a sequence of points $\{(b^k, c^k, d^k, \alpha^k, \sigma^k, w^k, \epsilon^k), k = 1, 2, \cdots\}$, which converges to an optimal solution $(b^*, c^*, d^*, \alpha^*, \sigma^*, w^*, \epsilon^*)$, under some appropriate assumptions. However, for the above cutting plane algorithm, one major computation bottleneck lies in Step 2 of finding maximizers. Ideas of relaxing the requirement of finding global maximizers for different settings can be referred to [10] and [25]. But the required computation work could still be a bottleneck. Here we propose a simple and yet very effective relaxation scheme which chooses points at which the infinite constrains are violated to a degree rather than at which the violation are maximized. The proposed algorithm is stated as follows.

Relaxed CPSD Algorithm:
Let $\delta > 0$ be a prescribed small number.
$Initialization$ Set $k = p^k_i = q^k_i = 1, i = 1, 2, \cdots, M$; Choose any $\tau^i_1, u^i_1 \in \aleph_i, i = 1, 2, \cdots, M$; Set $N^1_i = \{\tau^i_1\}$ and $\aleph^1_i = \{u^i_1\}, i = 1, 2, \cdots, M$.

Step 1. Solve SD^k and obtain an optimal solution $(b^k, c^k, d^k, \alpha^k, \sigma^k, w^k, \epsilon^k)$. Define $g^{k+1}_i(\tau)$ and $v^{k+1}_i(u), i = 1, 2, \cdots, M$, according to (3.2) and (3.3) ,respectively.

Step 2. Find any $\tau^i_{p^k_i+1} \in \aleph_i$ such that $g^{k+1}_i(\tau^i_{p^k_i+1}) > \delta$, and $u^i_{q^k_i+1} \in \aleph_i$ such that $v^{k+1}_i(u^i_{q^k_i+1}) > \delta, i = 1, 2, \cdots, M$.

Step 3. If such $\tau^i_{p^k_i+1}$ and $u^i_{q^k_i+1}$ do not exist, then output $(b^k, c^k, d^k, \alpha^k, \sigma^k, w^k, \epsilon^k)$ as a solution. Otherwise, go to step 4.

Step 4. If such $\tau^i_{p_i^k+1}$ exists, then set $N_i^{k+1} \leftarrow N_i^k \bigcup \{\tau^i_{p_i^k+1}\}, p_i^{k+1} \leftarrow p_i^k + 1$. Otherwise, set $N_i^{k+1} \leftarrow N_i^k, p_i^{k+1} \leftarrow p_i^k, i = 1, 2, \cdots, M$.

Step 5. If such $u^i_{q_i^k+1}$ exists, then set $\aleph_i^{k+1} \leftarrow \aleph_i^k \bigcup \{u^i_{q_i^k+1}\}, q_i^{k+1} \leftarrow q_i^k + 1$. Otherwise, set $\aleph_i^{k+1} \leftarrow \aleph_i^k, q_i^{k+1} \leftarrow q_i^k, i = 1, 2, \cdots, M$.

Step 6. Set $k \leftarrow k + 1$; go to step 1.

Note that in Step 2, since no maximizer is required, the computational work can be greatly reduced. Also note that when δ is chosen to be sufficiently small, if the relaxed algorithm terminates in a finite number of iterations at Step 3, then an optimal solution is indeed obtained, assuming that the original the **Problem 2** is feasible.

4. Numerical Results

In this section, the features of the proposed method are tested using a set of real data and compared with the smoothing spline method [1], the cubic smoothing spline method [21], and the maximum smoothing spline method [19]. The numerical examples and results of the Hull and White interest rate model with impulse perturbations are presented in this section. The observed 3-MONTH TREASURY BILL RATE data of the St. Louis Federal Reserve Bank from 2006-11-17 to 2009-7-17 (140 weeks) is employed for analysis. The initial guesses and bounds of the parameters of the Hull and White interest rate model are listed in Table 1.

Table 1: The initial guesses and bounds of the parameters of the Hull and White interest rate model

0	initial guess	lower bound	upper bound
α	Shown in Tables 2 & 3	0	∞
σ	Shown in Tables 2 & 3	0	∞
b	0.1	$-\infty$	∞
c	0	$-\infty$	∞
d	0	$-\infty$	∞
$w(t)$	0	$-\infty$	∞
ϵ	0	0	∞

In our implementation, the initial values are obtained by solving the **Problem 2** for any arbitrary initial guesses as listed in Table 1. The numerical analysis results for different initial values of α, and σ are shown in Tables 2 and 3, respectively. In Tables 2 and 3, b^*, c^*, d^*,σ^*, α^*, ϵ^* denote the optimal solutions of the **Problem 2,** and Tol is the stopping tolerance value for solving the **Problem 2**. Table 4 compares

Table 2: The numerical analysis results for different α with $\sigma = 0.75$

Case	1	2	3	4	5
α	0.01	0.05	0.1	0.5	0.75
σ	0.75	0.75	0.75	0.75	0.75
b^*	0.763019	0.755906	0.674160	0.774988	0.728731
c^*	0.025725	0.025412	0.018815	0.023034	0.019984
d^*	0.027378	0.027691	0.033969	0.029745	0.032451
α^*	0.034349	0.036236	0.012892	-0.006085	0.005129
σ^*	-0.097582	-0.097569	-0.098571	-0.098782	-0.099933
ϵ^*	0.828525	0.828557	0.827316	0.826682	0.825190
Tol	10^{-7}	10^{-7}	10^{-7}	10^{-7}	10^{-7}

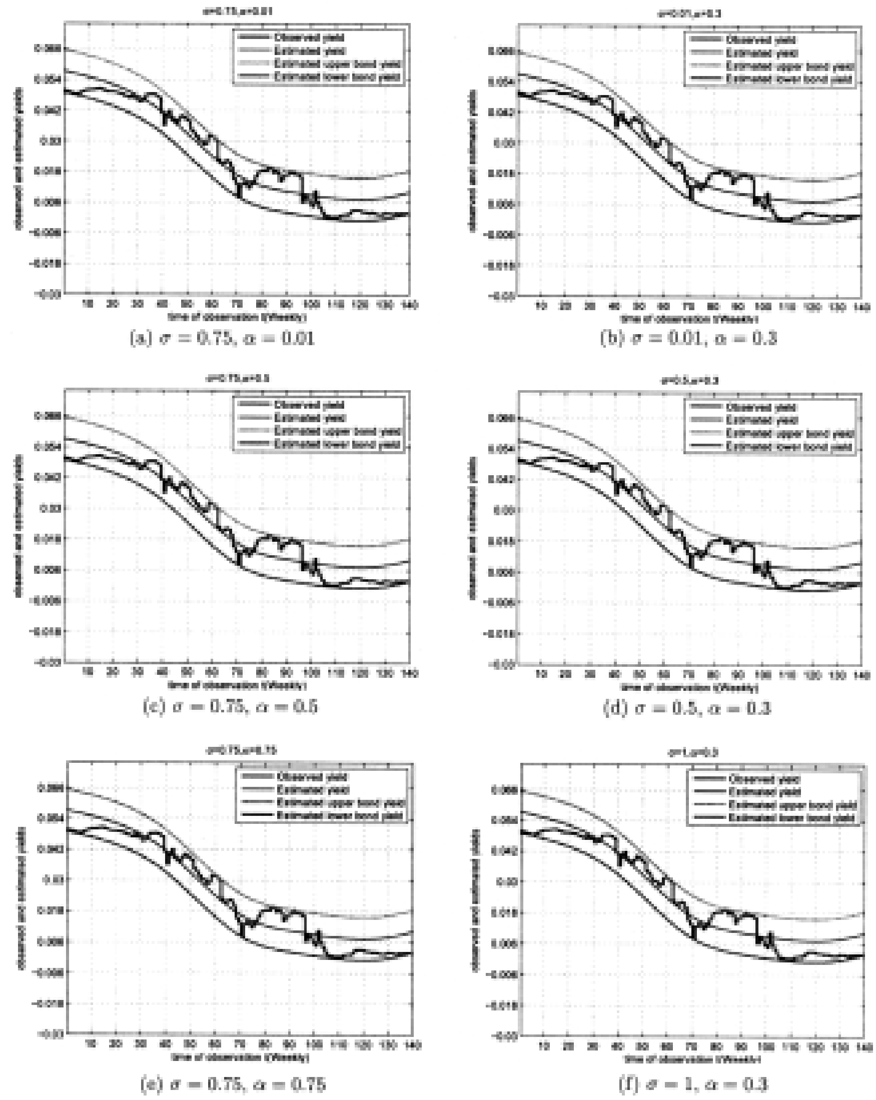
Table 3: The numerical analysis results for different σ with $\alpha = 0.3$

Case	1	2	3	4	5
α	0.3	0.3	0.3	0.3	0.3
σ	0.01	0.1	0.25	0.5	0.75
b^*	0.238864	0.226407	0.220583	0.226314	0.237644
c^*	-0.066543	-0.058622	-0.061317	-0.057974	-0.050042
d^*	0.118422	0.111296	0.113988	0.110650	0.102765
α^*	0.006016	0.069467	0.072010	0.072917	0.083773
σ^*	-0.098581	-0.098299	-0.098312	-0.098336	-0.098342
ϵ^*	0.827870	0.827818	0.827804	0.827783	0.827739
Tol	10^{-7}	10^{-7}	10^{-7}	10^{-7}	10^{-7}

Table 4: The yield curve variance for different methods

	The yield curve variance
Original data	3.5536e-004
Smoothing Spline	3.5066e-004
Cubic Smoothing Spline	3.5130e-004
Maximum Smoothing Spline Method	3.6418e-004
Deterministic Approach($\sigma = 0.5, \alpha = 1$)	3.1656e-004

the yield curve variance for the smoothing spline method, the cubic smoothing spline method, the maximum smoothing spline method, and our approach. The result illustrates that our approach generates the yield function with smaller oscillation. Moreover, Figure 1(a) $\sim$(f) show the estimates of the yield curves for different initial values of α and σ. It should be noted that for a fixed value of α, the yield function with minimal fitting errors can be obtained when σ is chosen to be sufficiently large. Moreover, for a fixed value of σ, when α is selected to be large enough, highly smooth interest rate and yield functions are conducted.



(a) $\sigma = 0.75$, $\alpha = 0.01$ (b) $\sigma = 0.01$, $\alpha = 0.3$

(c) $\sigma = 0.75$, $\alpha = 0.5$ (d) $\sigma = 0.5$, $\alpha = 0.3$

(e) $\sigma = 0.75$, $\alpha = 0.75$ (f) $\sigma = 1$, $\alpha = 0.3$

Fig. 1. Yield curves for different values of α and σ.

5. Conclusion

The Hull and White interest rate model with impulse perturbation is studied. The concept of deterministic perturbation is adopted to deal with the random behavior of interest rate variation. It shows that the interest rate function and the yield function of the Hull and White interest rate model can be obtained by solving a nonlinear semi-infinite programming problem. A relaxed cutting plane algorithm is then proposed for solving the resulting optimization problem. In each iteration, we solve a finite optimization problem and add one or some more constraint. The proposed algorithm chooses a point at which the infinite constrains are violated to a degree rather than at which the violation is maximized. Compared to some commonly used spline fitting methods, our approach essentially generates the yield functions with minimal fitting errors and small oscillations.

Appendix A. Derivation of the Instance Interest Rate Function

Proof of Theorem 1. Multiply both sides with the integrating factor $e^{\alpha t}$ for (2.2) we have

$$e^{\alpha t}\frac{dr(t)}{dt} + \alpha e^{\alpha t} r(t) = \alpha e^{\alpha t}\mu(t) + \sigma e^{\alpha t} w(t)$$

$$\int_{t_0}^{t} d(e^{\alpha\tau} r(\tau)) = \int_{t_0}^{t} \alpha e^{\alpha\tau}\mu(\tau) + \sigma e^{\alpha\tau} w(\tau) d\tau$$

$$e^{\alpha\tau} r(\tau)|_{t_0}^{t} = \int_{t_0}^{t} \alpha e^{\alpha\tau}\mu(\tau) d\tau + \int_{t_0}^{t} \sigma e^{\alpha\tau} w(\tau) d\tau e^{\alpha t} r(t)$$

$$e^{\alpha t} r(t) - e^{\alpha t_0} r(t_0)$$

(A1)
$$\begin{aligned}
&= \int_{t_0}^{t} \alpha e^{\alpha\tau}\left(\frac{1}{\alpha}\frac{\partial}{\partial\tau} f(0,\tau) + f(0,\tau) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-2\alpha\tau})\right) d\tau \\
&\quad + \int_{t_0}^{t} \sigma e^{\alpha\tau} w(\tau) d\tau \\
&= \int_{t_0}^{t} d\left(e^{\alpha\tau} f(0,\tau)\right) + \frac{\sigma^2}{2\alpha}\int_{t_0}^{t}\left(e^{\alpha\tau} - e^{-\alpha\tau}\right) d\tau + \int_{t_0}^{t} \sigma e^{\alpha\tau} w(\tau) d\tau \\
&= e^{\alpha t} f(0,t) - e^{\alpha t_0}\bar{f}(0,t_0) + \frac{\sigma^2}{2\alpha^2}\left(e^{\alpha t} + e^{-\alpha t} - e^{\alpha t_0} - e^{-\alpha t_0}\right) \\
&\quad + \int_{t_0}^{t} \sigma e^{\alpha\tau} w(\tau) d\tau
\end{aligned}$$

$$\begin{aligned}
r(t) &= e^{\alpha(t_0 - t)} r(t_0) + f(0,t) - e^{\alpha(t_0 - t)} r(t_0) \\
&\quad + \frac{\sigma^2}{2\alpha^2}\left(1 + e^{-2\alpha t} - e^{\alpha(t_0 - t)} - e^{-\alpha(t_0 + t)}\right) + \int_{t_0}^{t} \sigma e^{\alpha(\tau - t)} w(\tau) d\tau
\end{aligned}$$

$$\begin{aligned}
r(t) &= f(0,t) + \frac{\sigma^2}{2\alpha^2}\left(1 + e^{-2\alpha t} - e^{\alpha(t_0 - t)} - e^{-\alpha(t_0 + t)}\right) \\
&\quad + \sigma\int_{t_0}^{t_1} e^{\alpha(\tau - t)} w_1(\tau) d\tau + \sigma\int_{t_1}^{t_2} e^{\alpha(\tau - t)} w_2(\tau) d\tau + \ldots \\
&\quad + \sigma\int_{t_{i-2}}^{t_{i-1}} e^{\alpha(\tau - t)} w_{i-1}(\tau) d\tau + \sigma\int_{t_{i-1}}^{t} e^{\alpha(\tau - t)} w_i(\tau) d\tau
\end{aligned}$$

Substitute (2.3) into (A1), we have

$$
\begin{aligned}
r(t) &= \tilde{f}(0,t) + \frac{\sigma^2}{2\alpha^2}\left(1 + e^{-2\alpha t} - e^{\alpha(t_0 - t)} - e^{-\alpha(t_0+t)}\right) \\
&\quad + \sigma \sum_{j=1}^{i-1} w_j \int_{t_{j-1}}^{t_j} e^{\alpha(\tau - t)} d\tau + \sigma w_i \int_{t_{i-1}}^{t} e^{\alpha(\tau - t)} d\tau. \\
&= \tilde{f}(0,t) + \frac{\sigma^2}{2\alpha^2}\left(1 + e^{-2\alpha t} - e^{\alpha(t_0 - t)} - e^{-\alpha(t_0+t)}\right) \\
&\quad + \sum_{j=1}^{i-1} \frac{\sigma w_j}{\alpha}(e^{\alpha(t_j - t)} - e^{\alpha(t_{j-1}-t)}) + \frac{w_i \sigma}{\alpha}(1 - e^{\alpha(t_{i-1}-t)}). \\
&= \tilde{f}(0,t) + \frac{\sigma^2}{2\alpha^2}\left(1 + 2e^{-2\alpha t} - 2e^{-\alpha t}\right) + \sum_{j=1}^{i-1} \sigma w_j \frac{e^{-\alpha t}}{\alpha}(e^{\alpha t_j} - e^{\alpha t_{j-1}}) \\
&\quad + \frac{w_i \sigma}{\alpha}(1 - e^{-\alpha(t - t_{i-1})}), \qquad \forall t \in \aleph_i, i = 1, 2, \ldots, N.
\end{aligned} \tag{A2}
$$

■

Appendix B. Derivation of the Yield Function

Proof of Theorem 2.

$$
\begin{aligned}
& y(t|\mathcal{T}) \\
&= \frac{1}{\mathcal{T}} \int_t^{t+\mathcal{T}} r(\tau) d\tau \\
&= \frac{1}{\mathcal{T}} \Bigg\{ \int_t^{t+\mathcal{T}} \tilde{f}(0,\tau) d\tau + \frac{\sigma^2}{2\alpha^2} \int_t^{t+\mathcal{T}} \left(1 + e^{-2\alpha\tau} - 2e^{-\alpha\tau}\right) d\tau \\
&\quad + \int_t^{t_i} \Big(\sum_{k=1}^{i-1} \sigma w_k \frac{e^{-\alpha\tau}}{\alpha}(e^{\alpha t_k} - e^{\alpha t_{k-1}}) + \sigma w_i \frac{1 - e^{\alpha(t_{i-1}-\tau)}}{\alpha} \Big) d\tau \\
&\quad + \sum_{j=i+1}^{i+\mathcal{T}-1} \int_{t_{j-1}}^{t_j} \Big(\sum_{k=1}^{j-1} \sigma w_k \frac{e^{-\alpha\tau}}{\alpha}(e^{\alpha t_k} - e^{\alpha t_{k-1}}) + \sigma w_j \frac{1 - e^{\alpha(t_{j-1}-\tau)}}{\alpha} \Big) d\tau \\
&\quad + \int_{t_{i+\mathcal{T}-1}}^{t+\mathcal{T}} \Big(\sum_{k=1}^{i+\mathcal{T}-1} \sigma w_k \frac{e^{-\alpha\tau}}{\alpha}(e^{\alpha t_k} - e^{\alpha t_{k-1}}) + \sigma w_{i+\mathcal{T}} \frac{1 - e^{\alpha(t_{i+\mathcal{T}-1}-\tau)}}{\alpha} \Big) d\tau \Bigg\} \\
&= F(c,b,d,t \mid \mathcal{T}) + \frac{1}{\mathcal{T}} \Bigg\{ \frac{\sigma^2}{2\alpha^2} \left(\mathcal{T} - \frac{e^{-2\alpha(t+\mathcal{T})} - e^{-2\alpha t}}{2\alpha} + \frac{2e^{-\alpha(t+\mathcal{T})} - 2e^{-\alpha t}}{\alpha} \right) \\
&\quad + \sum_{k=1}^{i-1} \frac{(e^{\alpha t_k} - e^{\alpha t_{k-1}})}{\alpha} \sigma w_k \int_t^{t+\mathcal{T}} e^{-\alpha\tau} d\tau + \Big(\frac{1}{\alpha} \int_t^{t_i} (1 - e^{\alpha(t_{i-1}-\tau)}) d\tau \\
&\quad + \frac{e^{\alpha t_i} - e^{\alpha t_{i-1}}}{\alpha} \int_{t_i}^{t+\mathcal{T}} e^{-\alpha\tau} d\tau \Big) \sigma w_i + \sum_{k=i+1}^{i+\mathcal{T}-1} \Big(\frac{1}{\alpha} \int_{t_{k-1}}^{t_k} (1 - e^{\alpha(t_{k-1}-\tau)}) d\tau
\end{aligned} \tag{B1}
$$

$$
\begin{aligned}
&+\frac{e^{\alpha t_k}-e^{\alpha t_{k-1}}}{\alpha}\int_{t_k}^{t+\mathcal{T}} e^{-\alpha\tau}d\tau)\sigma w_k+\frac{1}{\alpha}\int_{t_{i+\mathcal{T}-1}}^{t+\mathcal{T}}(1-e^{\alpha(t_{i+\mathcal{T}-1}-\tau)})\sigma w_{i+\mathcal{T}}d\tau\Big\}\\
&= F(c,b,d,t\mid\mathcal{T})+\frac{\sigma^2}{2\alpha^2}\left(1-\frac{e^{-2\alpha(t+\mathcal{T})}-e^{-2\alpha t}}{2\mathcal{T}\alpha}+\frac{2e^{-\alpha(t+\mathcal{T})}-2e^{-\alpha t}}{\mathcal{T}\alpha}\right)\\
&+\sum_{k=1}^{i-1}\left(\frac{e^{\alpha(t_k-t)}+e^{\alpha(t_{k-1}-t-\mathcal{T})}}{\mathcal{T}\alpha^2}-\frac{e^{\alpha(t_k-t-\mathcal{T})}+e^{\alpha(t_{k-1}-t)}}{\mathcal{T}\alpha^2}\right)\sigma w_k\\
&+(\frac{t_i-t}{\mathcal{T}\alpha}+\frac{e^{\alpha(t_{i-1}-t-\mathcal{T})}+1}{\mathcal{T}\alpha^2}-\frac{e^{\alpha(t_i-t-\mathcal{T})}+e^{\alpha(t_{i-1}-t)}}{\mathcal{T}\alpha^2})\sigma w_i\\
&+\sum_{k=i+1}^{i+\mathcal{T}-1}(\frac{t_k-t_{k-1}}{\mathcal{T}\alpha}+\frac{e^{\alpha(t_{k-1}-t-\mathcal{T})}-e^{\alpha(t_k-t-\mathcal{T})}}{\mathcal{T}\alpha^2})\sigma w_k\\
&+\left(\frac{t+\mathcal{T}-t_{i+\mathcal{T}-1}}{\mathcal{T}\alpha}+\frac{e^{\alpha(t_{i+\mathcal{T}-1}-t-\mathcal{T})}-1}{\mathcal{T}\alpha^2}\right)\sigma w_{i+\mathcal{T}},\quad t\in\aleph_i, i=1,2,...,M. \qquad\blacksquare
\end{aligned}
$$

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