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Elucidating Asymmetric Volatility in Asset Returns and Optimizing Portfolio Choice Using Time-Changed Lévy Processes

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摘　要

本研究顯著地發展時間轉換 Lévy 過程在最適投資組合的運用性。在連續 Lévy 過程模型設定下，槓桿效果直接地產生跨期波動度不對稱避險需求，而波動度回饋效果則透過槓桿效果間接地發生影響。另外，關於無窮跳躍 Lévy 過程模型設定部分，槓桿效果仍扮演重要的影響角色，而波動度回饋效果僅在短期投資決策中發生作用。最後，在本研究所提出之一般化隨機波動度不對稱資產報酬動態模型下，得出在無窮跳躍的資產動態模型設定下，擴散項仍為重要的決定項。

關鍵詞：最適投資組合、隨機波動度、時間轉換 Lévy 過程、槓桿效果、波動度回饋效果、波動度不對稱

Abstract

This study significantly extends the applicability of time-changed Lévy processes to the portfolio optimization. The leverage effect directly induces the *intertemporal asymmetric volatility hedging demand*, while the volatility feedback effect exerts a minor influence via the leverage effect under the pure-continuous time-changed Lévy process. Furthermore, the leverage effect still plays a major role while the volatility feedback effect just works over the short-term investment horizon under the infinite-jump Lévy process. Based on the proposed general stochastic asymmetric volatility asset return model, we conclude that the diffusion term is an essential determinant of financial modeling for index dynamics given infinite-activity jump structure.

Key words: Optimal portfolio choice, stochastic volatility, time-changed Lévy processes, leverage effect, volatility feedback effect, asymmetric volatility

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1. Introduction

The focus of asset price modeling has shifted to a framework based on non-Gaussian distribution to alleviate problems in underestimating the frequency and magnitude of extreme events, namely, crashes and booms. Particularly, many early empirical evidences have fundamentally shaken assumptions made by the diffusion model, for instance, Mandelbrot (1963) and Fama (1965).

Among these non-Gaussian specifications, asymmetric volatility and fat tail are relevant considerations with regard to asset allocation decisions. Campbell *et al.* (1997, Chapter 12) explained asymmetric volatility in terms of the leverage and volatility feedback effects. The leverage effect proposed by Black (1976) claimed that a negative equity return reduces the leverage firm value and thus increases the risk of holding equity, thus increasing volatility risk. Additionally, the volatility feedback effect proposed by Campbell and Hentschel (1992), Bekaert and Wu (2000) and Wu (2001) advocated that it should be satisfied if return volatility behavior involves persistent clustering, in which case a shock in either direction enhances the anticipated increase in volatility and increase the required rate of return for holding stocks, and furthermore, reduces the asset price to enable higher future returns. Kraus and Litzenberger (1976) demonstrated that investors with the power utility favor positive over negative skewness. Therefore, the leverage and volatility feedback effects significantly influence optimal portfolio choices when asset return has asymmetric volatility.

The finance literature has extensively explored the setting of Lévy density, for instance, Variance Gamma (VG) was presented by Madan and Seneta (1990) and extended to skewness by Madan *et al.* (1998), CGMY was adapted by Geman *et al.* (2001) and Carr *et al.* (2002), and exponential dampened power law was proposed by Wu (2006). While the literature documents evidence supporting the superior fitting ability of Lévy processes, room remains for Lévy processes related to financial modeling.

Time-changed Lévy process is widely utilized due to its probabilistic tractability. Furthermore, asset price can be considered the outcome of interaction among several economic variables. The Lévy process accelerated by an increasing stochastic time is cautiously selected to match the features existing in different financial markets (e.g. Mo and Wu (2007), Carr *et al.* (2003) and Carr and Wu (2007)).

Carr *et al.* (2003) discussed stochastic volatility in relation to the pure-continuous asset price model in three homogeneous Lévy processes, including normal inverse Gaussian (NIG) presented by Barndorff-Nielsen (1998), VG, and CGMY, in the form of a stochastic time change independent of the original Lévy processes. Besides stochastic volatility, the instantaneous rate of stochastic time change, a solution to the CIR mean-reverting square root stochastic process, is offered to promote volatility clustering, but without the leverage effect.

Cvitanić *et al.* (2008) denoted risky asset price dynamics as a pure-jump stochastic process in which underlying uncertainty is described via the state-dependent Lévy density for a VG model. The variation of state-dependent Lévy density is fully captured by the state variable followed in the form of a CIR mean-reverting square root stochastic process to investigate the portfolio optimization for investors facing higher moments. However, the research of Cvitanić *et al.* did not directly invoke stochastic time changes; rather they simply randomized the intensity of the jump structure, namely, transforming the constant Lévy density into a varying one. Hence some interesting findings may be sacrificed.

In comparison to those of Carr *et al.* (2003), present study enhances a

Brownian motion with drift subordinated by a pure-continuous increasing stochastic process, an integral of a solution to the CIR stochastic process, such that the time-changed Lévy process is associated with the state variable. In contrast with Cvitanić *et al*. (2008), we provide another infinite-jump asset return model obtained by directly applying a one-sided jump process to randomize the clock in which a Brownian motion with drift is run.

However, few studies have investigated the implications of asymmetric volatility, particularly for leverage and volatility feedback effects, in relation to optimal portfolio choices. Research on the performance of time-charged Lévy processes in optimal portfolio choice still remains immature. This study attempts to fill this gap and enrich the literature.

The primary contributions of this work are as follows: First, this study proposes two distinct exponential time-changed Lévy processes with asymmetric volatility for risky assets. Second, this study numerically examines the economic implications of leverage effect and volatility feedback effect for optimizing portfolio. Finally, we adopt the perspective of econometric analysis to apply the proposed general stochastic asymmetric volatility asset return model by calibrating them to S&P500 index returns. To resolve the difficulties in getting an analytical expression for probability density function, this study employs *spectral GMM estimation* (Chacko and Viceira (2003)) to estimate the parameters of the general asset return model.

The rest of the paper is organized as follows: Section II reviews some essential results of Lévy processes and time-changed Lévy processes, and further proposes two distinct exponential time-changed Lévy processes with asymmetric volatility for risky assets. Section III presents a rigorous formulation of the problems associated with optimizing portfolio choice. Section IV provides some results for optimal portfolio weights together with some relevant numerical examples to investigate the implications of asymmetric volatility. To understand whether the diffusion term provides the critical effect for financial modeling, Section V assesses the asymmetric volatility to explore the proposed general asset return model. Section VI presents conclusions.

2. Time-Changed Lévy Processes

2.1 Fundamental Properties of Lévy Process

Let $(\Omega, \mathbb{F}, (F_s)_{t<s<T}, \mathbb{P})$ denote a filtered complete probability space representing the underlying economic uncertainty. A process $X = (X(t), t \geq 0)$ with values in $\mathbb{R}$ such that $X(0) = 0$ (almost surely) is termed a Lévy process if it is right continuous with left limits almost surely and its increments are independent and time-homogeneous.

From the Lévy-Khintchine formula, the characteristic function of real-valued Lévy process $X(t)$ has the form

$$\Phi_{X_t}(z) \equiv E\left[e^{izX_t}\right] = e^{-t\psi_x(z)} \qquad t \geq 0,$$

where $i = \sqrt{-1}$, $E[\cdot]$ denotes the expectation operator under the measure $\mathbb{P}$ and the characteristic exponent $\Psi_x(z)$, $z \in \mathbb{R}$, is given by

$$\Psi_x(z) \equiv -i\mu z + \frac{1}{2}z^2 \Sigma + \int_{\mathbb{R}-\{0\}} (1 - e^{izx} + izx 1_{|x|\leq 1}) \Pi(x)\, dx$$

$\Pi(x)dx$ is termed the Lévy measure. The characteristic triplet $(\mu, \Sigma, \Pi(x))$ denotes the Lévy triplet, where μ is the constant drift, Σ denotes the constant diffusion coefficient and $\Pi(x)$ describes the arrival rate for jumps of size x.

2.2 Stochastic Time Changes for Lévy Processes

The mapping $t \rightarrow T(t)$ can be regarded as the stochastic time change. Intuitively, the original time t denotes calendar time and the random clock $T(t)$ represents business time.

Clark (1973) was the first researcher to propose stochastically altering the calendar time in the finance literature. Geman and Ané (1996) and Ané and Geman (2000) subsequently further elucidated this concept. A growing number of recent publications and empirical evidences have confirmed the positive contribution of time-changed Lévy processes by extracting and capturing features of asset returns in financial markets. Geman (2002) argued that pure-jump Lévy processes, for instance, CGMY and the hyperbolic motion by Eberlein and Keller (1995) and Barndorff-Nielsen (1998), possess better fit than classical diffusion or jump-diffusion models. Recently, Mendoza *et al.* (2008) proposed time-changed Markov processes designed for defaultable stocks.

2.3 Time-Changed Asset Price Processes with Asymmetric Volatility

2.3.1 Pure-Continuous Asset Dynamic Process

This study denotes the process for the risky asset price by $S(t)$. To model asset price dynamics, the Brownian motion with drift $X_1(t)$ and the stochastic time change $T_1(t)$ are included as:

$$S(t) = S(0)\exp\left[X_1(T_1(t)\right],$$

where the process $X_1(t)$ is as follows[1]:

$$X_1(t) = \theta_1 t + W(t)$$

θ_1 denotes a constant, $S(0)$ represents the initial asset price, $W(t)$ is a standard Brownian motion, and continuous integrated stochastic time change $T_1(t)$ is given by

$$T_1(t) = \int_0^t v_1(s)\,ds \tag{1}$$

[1] Volatility of Brownian motion with drift is captured by the stochastic time change process $T_1(t)$. Hence the volatility term could be omitted.

where $\nu_1(t)$ is considered the state variable of the pure-continuous asset price dynamics, which is unobservable and is a positive quantity.

Following Carr *et al.* (2003), this study assumes a continuous stochastic time change and specifies it using the instantaneous activity rate following a CIR mean-reverting square root stochastic process:

$$d\nu_1(t) = \kappa_1(\eta_1 - \nu_1(t))dt + \delta_1\sqrt{\nu_1(t)}\,dB_1(t)$$

where $B_1(t)$ denotes a standard Brownian motion independent of another Brownian motion $W(t)$ [2].

We replace calendar time t of the Brownian motion with drift with the stochastic time change $T_1(t)$ to yield the following expression:

$$d\,X_1(T_1(t)) = \theta_1 d\,T_1(t) + d\,W(T_1(t)) \tag{2}$$

The driving noise term $W(T_1(t))$ of the Eqn. (2) can be transformed into another Brownian motion along the lines proposed by Karatzas and Shreve (1991, p.174) and Mo and Wu (2007), such that

$$d\,W\big(T_1(t)\big) = \sqrt{\nu_1(t)}\,dB^*(t)$$

Hence the percentage return is given as:

$$\frac{dS(t)}{S(t)} = \mu_1^*\nu_1(t)\,dt + \sqrt{\nu_1(t)}dB^*(t) \tag{3}$$

where $\mu_1^* = \theta_1 + 1/2$. This study assumes that $dB^*(t)$ and $dB_1(t)$ are two Brownian motions with constant correlation ρ_1 such that $E[dB^*(t)\,dB_1(t)] = \rho_1 dt$, thus introducing leverage effect into the model.

The introduction of the CIR dynamic process can help capture the

2 Furthermore, the parameter η_1 is the long-term mean of state variable, while κ_1 controls the speed with which $\nu_1(t)$ returns to its long-term mean and the path variability is determined by δ_1.

volatility clustering as the required condition for volatility feedback effect, because the instantaneous activity rate $\nu_1(t)$ can be considered the instantaneous variance of the Brownian motion.

Since the integrated stochastic time change is continuous, the asset return model of Eqn. (3) is also continuous (Geman *et al.* (2001)).

2.3.2 Infinite-jump Asset Dynamic Process

This study further introduces another way of representing asset prices:

$$S(t) = S(0)\exp\left[Y(t)\right] \tag{4}$$

Economic uncertainty is described by the stochastic process $Y(t)$ satisfying

$$Y(t) = Z(t) + \rho_2 T_2(t) \tag{5}$$

where $T_2(t)$ denotes a stochastic time change, $Z(t) = X_2(T_2(t))$, $X_2(t) = \theta_2 t + W(t)$, θ_2 is a constant drift term, and $W(t)$ represents a standard Brownian motion. The coefficient ρ_2 allows for correlation between the Lévy asset return and changes in volatility[3].

The arrival rate of jumps of every positive size x in the pure-jump stochastic time change $T_2(t)$ is expressed as follows:

$$\pi(x) = \frac{ce^{-\lambda x}}{x^{1+\alpha}} 1_{x>0}$$

with the parameter $\alpha \in \left]0,1\right]$, $c, \lambda \in \mathbb{R}^{+}$. The condition $\alpha \in \left]0,1\right]$ is induced by the requirement that the stochastic time change $T_2(t)$ is the subordinating process.

The parameter c simultaneously controls the intensity of jumps of every

3 For simplicity, we assume that $X_2(t)$ and $T_2(t)$ are independent. The stochastic volatility and higher moments are generated by the stochastic time change.

positive size x and transforms the time scale of the dynamic processes. The idea of stochastic intensity of jumps seems to better capture the varying jump structure (Carr *et al.* (2002)), and is further utilized for optimizing portfolio choice by Cvitanić *et al.* (2008).

Unlike previous studies, rather than indirectly introducing the state variable into the Lévy density resulting from a stochastic time change, this study employs a more intuitive approach, which directly introduces the state variable into the stochastic time change describing the change in economic information over time, that is,

$$\Pi\left(v_2(t), x\right) = v_2(t)\,\pi^*(x)$$

where $\pi^*(x) = \dfrac{e^{-\lambda x}}{x^{1+\alpha}} 1_{x>0}$, and $v_2(t)$ follows CIR dynamic process:

$$dv_2(t) = \kappa_2\left(\eta_2 - v_2(t)\right)dt + \delta_2\sqrt{v_2(t)}\,dB_2(t)$$

The introduction of the CIR dynamic process can help capture the volatility clustering as the required condition for volatility feedback effect, because the instantaneous activity rate $v_2(t)$ alters the intensity of all positive jumps simultaneously. Furthermore, to account for the leverage effect in the asset price dynamics, the second component $\rho_2 T_2(t)$ in Eqn. (5) is included.

Revisiting the Eqn. (5), this study provides the new Lévy process $Z(t)$, which possesses the following Lévy density:

$$\Pi^{Z}\left(v_2(t), x\right) = v_2(t)\,\Pi^{Z*}(x)$$

where

$$\Pi^{Z*}(x) = \frac{F(\alpha,\lambda,\theta_2)}{|x|^{\alpha+\frac{1}{2}}} e^{\theta_2 x} K_{\alpha+\frac{1}{2}}\left(|x|\sqrt{\theta_2^2 + 2\lambda}\right),$$

$$F(\alpha,\lambda,\theta_2)=\frac{2}{\sqrt{2\pi}}\left(\theta_2^2+2\lambda\right)^{\frac{\alpha}{2}+\frac{1}{4}}$$

and K denotes the second kind of modified Bessel function.

Similarly, the Lévy process $Z(t)$ has zero diffusion and the drift term is as follows:

$$\mu^Z=\nu_2(t)\int_{|x|\le 1}x\Pi^{Z*}(x)\,dx$$

The process $Y(t)$ can be established by combining two stochastic processes, $Z(t)$ and $\rho_2 T_2(t)$, which are assumed to be independent. Finally, the Lévy triplet $(\mu_2, \Sigma, \Pi(\nu_2(t), x)$ for the resulting process $Y(t)$ is given by:

$$\Pi\left(\nu_2(t), x\right)=\nu_2(t)\,\Pi^*(x),$$

$$\mu_2=\mu_2^*\nu_2(t),$$

$$\Sigma=0$$

wher $\Pi^*(x)=\Pi^{Z*}(x)+\rho_2\pi^*(x)$ and $\mu_2^*=\int_{|x|\le 1}x\Pi^{Z*}(x)\,dx-\int_{[-\sqrt{2},-1]\cup[1,\sqrt{2}]}x\Pi^*(x)dx$. Percentage return is given by

$$\frac{dS(t)}{S(t)}=\mu_2^*\nu_2(t)\,dt+\int_{-\infty}^{+\infty}\left(e^x-1\right)N\left(\nu_2(t), dx, dt\right)$$

where $N(\nu_2(t), dx, dt)$ denotes the Poisson random measure on $\Omega\times\mathbb{R}\times\mathbb{R}_+$.[4]

When the parameter c of the Lévy density of the stochastic time change is replaced by the instantaneous activity rate $\nu_2(t)$, the introduction to the CIR dynamic process can help include the volatility feedback effect. Additionally, coefficient ρ_2 considers the presence of the leverage effect.

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4 Its Lévy density, $\Pi(\nu_2(t), x)$, captures the varying arrival rate of jumps depending on the information of state variable, which represents the current variation related to trading activity or market volatility.

3. Investment Opportunity Set and Investor Preferences

This section describes the economic environment in which an investor with CRRA utility makes portfolio choices involving two tradable assets: one riskfree asset and one risky asset. We describe the stochastic process for the riskfree asset $P(t)$:

$$dP(t) = rP(t)\,dt$$

where r represents the constant instantaneous riskfree interest rate. The stochastic process for the risky asset $S(t)$ has been presented previously, as follows:

Model 1: Pure-continuous asset dynamic process

$$\frac{dS(t)}{S(t)} = \mu_1^* v_1(t)\,dt + \sqrt{v_1(t)}\,dB^*(t) \tag{6}$$

Model 2: Infinite-jump asset dynamic process

$$\frac{dS(t)}{S(t)} = \mu_2^* v_2(t)dt + \int_{-\infty}^{+\infty} \left(e^x - 1\right) N\left(v_2(t), dx, dt\right) \tag{7}$$

Given the investment opportunity, the investor continuously chooses to invest a fraction $\omega(s)$ of his funds in the risky asset at each time $s, t \le s \le T$, and attempts to maximize the expected utility from terminal wealth with CRRA utility.

The aim of the investor is to maximize his expected utility:

$$\underset{\{\omega(s),\, t \le s \le T\}}{Max} \quad E_t\left[U\left(W_i(T)\right)\right]$$

subject to intertemporal budget constraints, which depend on the model choice:

$$\frac{dW_i(T)}{W_i(T)} = \omega(t)\frac{dS_i(t)}{S_i(t)} + \left(1-\omega(t)\right)\frac{dP(t)}{P(t)} \qquad i = 1, 2.$$

$W_i(T)$ represents investor's wealth dynamics under the different model i, $i = 1, 2$. The investor is endowed with positive initial wealth $W_i(0)$.

4. Optimal Portfolio Choice

4.1 Pure-Continuous Asset Dynamic Process

The budget constraint is given by:

$$dW_1(T) = W_1(T)\left(r + \omega_1(t)\left(\theta_1 \nu_1(t) + \frac{1}{2}\sigma_1^2 \nu_1(t) - r\right)\right) + W_1(T)\omega_1(t)\sqrt{\nu_1(t)}\,dB^*(t).$$

Following the standard procedure of Merton (1971)[5], the value function is defined as

$$J(W_1, \nu_1, t) = \underset{\{\omega_1(s), t \le s \le T\}}{\text{Max}} \quad E_t\left[U(W_1(T))\right].$$

Using the principle of dynamic programming for jump-diffusion processes, the following Hamilton-Jacobi-Bellman equation (HJB equation) is obtained,

$$0 = \underset{\{\omega_1\}}{\text{Max}} \left\{ \begin{array}{l} \frac{1}{2}\delta_1^2 \nu_1 J_{\nu_1\nu_1} + \frac{1}{2}\omega_1^2 W_1^2 \nu_1 J_{W_1W_1} + W_1\omega_1\delta_1\nu_1\rho_1 J_{W_1\nu_1} \\ + \kappa_1(\eta_1 - \nu_1)J_{\nu_1} + \left(r + \omega_1\left(\theta_1\nu_1 + \frac{1}{2}\nu_1 - r\right)\right)W_1 J_{W_1} + J_t \end{array} \right\} \tag{8}$$

where J_{W_1}, J_{ν_1} and J_t denote the first partial derivatives of $J(W_1, \nu_1, t)$, and so on for higher derivatives. This equation is solved by guessing (then verifying) that the solution to the value function has the following form:

5 More details about technical conditions of HJB equation can be found in Øksendal and Sulem (2005).

$$J(W_1,\nu_1,t)=\frac{1}{1-\gamma}W_1^{1-\gamma}e^{\overline{A(t)}+\overline{B(t)}\upsilon_1} \tag{9}$$

Differentiating the above with respect to ω_1, we obtains the following result:

Proposition 1: The optimal portfolio weight $\omega_1^*(t)$ in the presence of asymmetric volatility under the pure-continuous stochastic time-changed asset return model in Eqn. (6) is given by

$$\omega_1^*(t)=\frac{\left(\theta_1+\frac{1}{2}\right)\nu_1(t)-r}{r\nu_1(t)}+\frac{\delta_1\rho_1\overline{B(t)}}{r}, \qquad \text{for all } t>0 \tag{10}$$

$\overline{A(t)}$ and $\overline{B(t)}$ satisfy the following system of ordinary differential equations:

$$\overline{B(t)}'+\frac{1}{2}\delta_1^2\overline{B(t)^2}+((1-\gamma)\omega_1\delta_1\rho_1-\kappa_1)\overline{B(t)}+(1-\gamma)\omega_1\left(\theta_1+\frac{1}{2}(1-\omega_1\gamma)\right)=0 \tag{11}$$

$$\overline{A(t)'}+\kappa_1\eta_1\overline{B(t)}+(1-\gamma)(1-\omega_1)r=0. \tag{12}$$

When the asset return dynamics ignore asymmetric volatility and the stochastic time change is omitted, that is, $\nu_1(t)=1$ and $\rho_1=0$, the investor chooses the following portfolio weight as specified by Merton (1971).

Corollary 1: The optimal portfolio weight $\omega_0^*(t)$ in the absence of asymmetric volatility under the pure-diffusion stock return model as Merton (1971) is given by

$$\omega_0^*(t)=\frac{\left(\theta_1+\frac{1}{2}\right)-r}{r}, \qquad \text{for all } t>0 \tag{13}$$

Using the proposition 1, the optimal portfolio choice is determined via two strategies. The first strategy in Eqn. (10) is the mean-variance portfolio or myopic strategy related to the state variable $\nu_1(t)$ which describes the current economic information. The second strategy is the *intertemporal asymmetric volatility hedging demand* which directly depends on the leverage effect via the correlation coefficient ρ_1, and is indirectly affected by the volatility feedback effect via $\overline{B(t)}$. When correlation ρ_1 is nonzero, the investor can hedge expected utility against asymmetric volatility risk by taking the second holding. Additionally, when the correlation ρ_1 is zero, namely, ignoring the leverage effect, the intertemporal hedging portfolio strategy immediately disappears regardless of the volatility feedback effect embedded by $\overline{B(t)}$.

Similar to those of Liu *et al.* (2003), "market timing" exists for optimizing portfolio choice in relation to asymmetric volatility through the dependence on $\overline{B(t)}$.

4.1.1 Numerical Examples

The quantitative analyses for the optimal portfolio choice are performed using the following selected parameter values: the state variable $v_1(t)$ is set to be 0.05, r=0.02, θ_1=2.5 and δ=0.08.

Fig.1 illustrates how the leverage effect influences demand for risky assets for the purpose of hedging against the risk caused by changes in investment opportunities. For the numerical example, other parameters should be set besides the parameter ρ_1. The relative risk aversion coefficient γ equals 5, while κ equals 15. This work views the optimal portfolio weight as a function of investor horizon measure in years for four different values of the correlation coefficient.

Several important features are illustrated. First, the leverage effect is closely related to the optimal portfolio weights for risky assets. When the leverage effect is positive, that is, $\rho_1 < 0$, demand for risky assets increases with respect to the magnitude of the leverage effect, $|\rho_1|$.

Figure 1: Optimal portfolio weights for the negative correlation coefficient ρ_1 (there exists leverage effect).

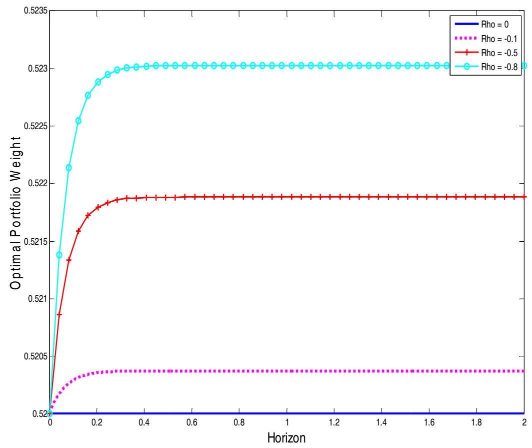


Fig.2 plots the optimal portfolio weight as a function of the volatility of the state variable (δ_1) for various values of correlation coefficient (ρ_1). Larger volatility of the state variable enhances the motivation to hold the intertemporal asymmetric volatility hedging demand against the risk of the varying investment opportunity given the positive leverage effect.

Figure 2: Optimal portfolio weights with respect to the different levels of leverage effect (ρ_1) and volatility of the state variable (δ_1).

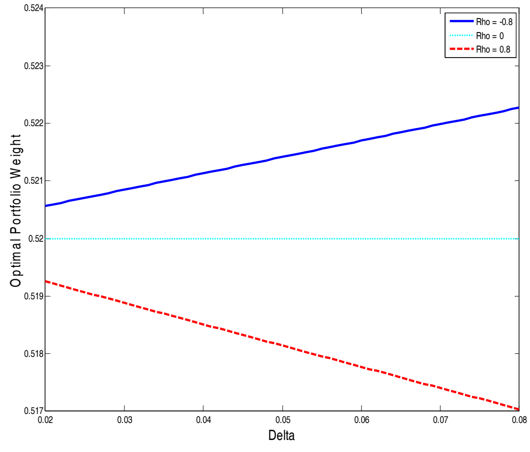
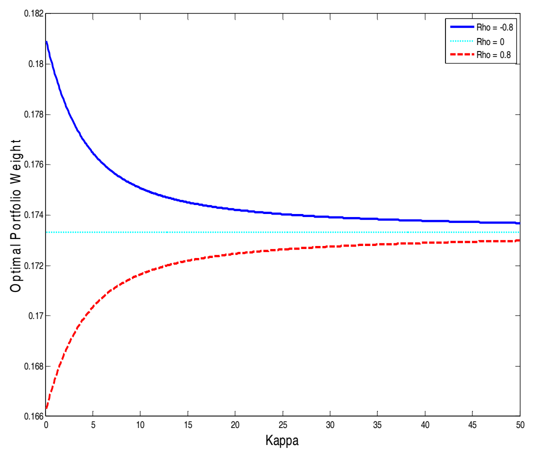


Fig.3 further presents another viewpoint. Optimal portfolio weight is a deceasing function of κ_1 when the economy exhibits a positive leverage effect. Hence demand for risky assets is proportional to the volatility feedback effect. This result is consistent with the previous discussion. The leverage and volatility feedback effects on optimal portfolio choices can be implemented separately in the present example. The case of large κ_1 (e.g. $\kappa_1 = 50$) is regarded as the model without the volatility feedback effect. As reflected in Fig.3, we observe that the leverage effect induces intertemporal hedging demand when the volatility feedback effect is ineffective.

Figure 3: Optimal portfolio weights with respect to the different levels of volatility feedback effect (κ_1) and leverage effect (ρ_1).



From the above numerical examples, the leverage effect mainly induces the intertemporal hedging demand. The volatility feedback effect works indirectly via the leverage effect and then exerts only a minor influence on asset holding.

4.2 Infinite-jump Asset Dynamic Process

As the preceding section, we propose the following result:

Proposition 2: The optimal portfolio weight $\omega_2^*(t)$ in the presence of asymmetric volatility under the infinite-jump stochastic time-changed asset return model in Eqn. (7) is given by

$$\left(\nu_2(t)\mu_2^* - r\right) + \int_{-\infty}^{+\infty} \left(1 + \omega_2^*(t)(e^x - 1)\right)^{-r} (e^x - 1)\nu_2(t)\Pi^*(x)dx = 0, \text{ for all } t > 0 \quad (14)$$

Proposition 2 indicates that the optimal portfolio weight $\omega_2^*(t)$ for risky assets depends on the state variable $\nu_2(t)$ because of the implicit relationship shown in Eqn. (14). In contrast to the findings of Benth *et al.* (2001), namely that optimal portfolio choice is a fixed fraction of wealth over time, the finding of Benth *et al.* can be considered our special case in the situation where the stochastic time change (implicitly including volatility feedback effect) and leverage effect are eliminated. In contrast to Cvitanić *et al.* (2008), present study builds on previous research ensuring agreement with more stylized facts by directly transforming the calendar time into the business time, and allowing for the dependence between the time-changed Lévy process and the state variable.

For simplicity this study applies the *conditional cumulant exponent* K_t, as previously applied by Cvitanić *et al.* (2008), which is defined as

$$K_t(u) = \int \left(e^{ux} - 1\right) \Pi\left(\nu_2(t), dx\right)$$

The unconditional version is

$$K(u) = \int \left(e^{ux} - 1\right) \Pi^*(dx)$$

The instantaneous variance of percentage returns $\dfrac{dS(t)}{S(t)}$ is obtained by

$$\sigma^2(t) = \left(K(2) - 2K(1)\right)\nu_2(t) .$$

Next, we present the following proposition:

Proposition 3: Assume that the ratio $(\mu_2^*\nu_2(t) - r)/\sigma^2(t)$ is a constant under the Model 2 in Eqn. (7), that is, $(\mu_2^*\nu_2(t) - r)/\sigma^2(t) = \xi$. Then the optimal portfolio weight $\omega_2^*(t)$ is independent of state variable, and satisfies

$$\xi\big(K(2)-2K(1)\big)+\int_{-\infty}^{+\infty}\big(1+\omega_2^*(t)(e^x-1)\big)^{-\gamma}(e^x-1)\,\Pi^*(x)dx=0,\quad \text{for all } t>0 \tag{15}$$

Proof: Refer to Cvitanić *et al.* (2008).

From the Eqn. (15), $\omega_2^*(t)$ is determined without the state variable information, which describes the randomness of the economic environment. The independence of the optimal portfolio weight from $v_2(t)$ stems from the fact that stochastic risk premium, $\mu_2^* v_2(t)-r$, is proportional to the instantaneous rate of variance $\sigma^2(t)$ and the randomness of the Lévy density under the present study. Similar results are presented in the finance literature, such as Liu *et al.* (2003) and Cvitanić *et al.* (2008). More importantly, the results claim that the channel for influencing the asymmetric volatility based on volatility feedback effect is ignored. The leverage effect becomes the only cause for the intertemporal asymmetric volatility hedging demand, thus implying that the leverage effect plays a major role for portfolio optimization in this situation.

4.2.1 Reduced Time-Changed Lévy Process

To extend VG model to more general cases, Carr *et al.* provided the following Lévy density with parameters *C*, *G*, *M*, and *Y*:

$$k(x)=\begin{cases}\dfrac{Ce^{-Mx}}{x^{1+Y}}1_{x>0}\\ \dfrac{Ce^{-G|x|}}{|x|^{1+Y}}1_{x<0}\end{cases}$$

where $C>0$, $G\geq 0$, $M\geq 0$, and $Y<2$. Based on the derivation of Madan and Yor (2008), the stochastic time change $T_2(t)$ related to CGMY is absolutely continuous with respect to one-sided stable $Y/2$ subordinator with the following Lévy density:

$$\iota(x)=\frac{\Lambda h(x)}{x^{1+\frac{Y}{2}}}1_{x>0}$$

$$h(x)=e^{-\frac{(B^2-A^2)x}{2}}E\left[e^{-\frac{B^2x}{2}\frac{\gamma_{Y/2}}{\gamma_{1/2}}}\right]$$

$$A=\frac{G-M}{2},\quad B=\frac{G+M}{2}$$

$$\Lambda=\frac{C\Gamma\left(\frac{Y}{4}\right)\Gamma\left(1-\frac{Y}{4}\right)}{2\Gamma\left(1+\frac{Y}{2}\right)}$$

where $\gamma_{Y/2}$, $\gamma_{1/2}$ are two independent gamma variates with unit scale parameters and shape parameter $Y/2$, $1/2$ respectively. Next, the above expectation term can be evaluated, as follows:

$$E\left[e^{-\frac{B^2x}{2}\frac{\gamma_{Y/2}}{\gamma_{1/2}}}\right]=\frac{\Gamma\left(\frac{Y+1}{2}\right)}{\Gamma(Y)\Gamma\left(\frac{1}{2}\right)}2^{Y}\left(\frac{B^2x}{2}\right)^{\frac{Y}{2}}I\left(Y,B^2x,\frac{B^2x}{2}\right)$$

where

$$I(\varepsilon,o,\varsigma)=(2\varsigma)^{-\varepsilon/2}\Gamma(\varepsilon)h_{-\varepsilon}\left(\frac{o}{\sqrt{2\varsigma}}\right)$$

and $h_{-\varepsilon}(z)$ is the Hermite function with parameter $-\varepsilon$.

This study defines a state-dependent Lévy density for the stochastic time change to reduce the general case, as follows:

$$\Pi\left(v_2(t),x\right)=v_2(t)\pi^*(x)$$

where

$$\pi^*(x) = \frac{\Lambda^* \cdot E\left[e^{-\frac{B^2 x}{2}\frac{\gamma_{Y/2}}{\gamma_{1/2}}}\right] e^{-\frac{G \cdot M}{2}x}}{x^{1+\frac{Y}{2}}} 1_{x>0}$$

and $\Lambda^* = \dfrac{\Gamma\left(\frac{Y}{4}\right)\Gamma\left(1-\frac{Y}{4}\right)}{2\Gamma\left(1+\frac{Y}{2}\right)}$. The $\Pi^{Z*}(x)$ can then be replaced by reduced GMY (state-dependent) Lévy density such as:

$$\Pi^{Z*}(x) = \begin{cases} \dfrac{e^{-Mx}}{x^{1+Y}} 1_{x>0} \\ \dfrac{e^{-G|x|}}{|x|^{1+Y}} 1_{x<0} \end{cases}$$

and $\Pi^*(x) = \Pi^{Z*}(x) + \rho_2 \pi^*(x)$. For $G < M$, the rate of exponential decay on the right of the Lévy density is larger than the left, leading to the fact that large negative realizations are more likely to appear than large positive realizations, namely, negative skewness.

From Wu (2006) and Madan and Yor (2008), this work immediately determines that

$$K(u) = \Gamma(-Y)\left\{\left[(M-u)^Y - M^Y + (G+u)^Y - G^Y\right] + \left[2g^Y \cos(qY) - M^Y - G^Y\right]\right\},$$

$$g = \sqrt{GM - 2\rho_2 u}$$

$$q = \tan^{-1}\left(\frac{\sqrt{-2\rho_2 u - \left(\frac{G-M}{2}\right)^2}}{\left(\frac{G+M}{2}\right)}\right)$$

We then obtain the following results:

Corollary 2: When the time-changed Brownian motion with drift follows the CGMY model derived by Madan and Yor (2008), the optimal portfolio weight $\omega_2(t)$ in Eqn. (14) reduces to

$$\int_{-\infty}^{+\infty}\left(1+\omega_2^*(t)(e^x-1)\right)^{-\gamma}(e^x-1)\nu_2(t)\Pi^*(x)dx=r,\quad \text{for all } t>0.$$

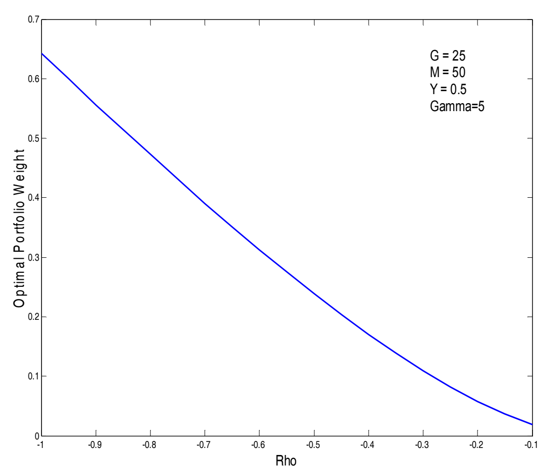
Then

$$\left\{e^{-\kappa t}\nu_2(0)+\eta(1-e^{-\kappa t})\right\}\int_{-\infty}^{+\infty}\left(1+\omega_2^*(t)(e^x-1)\right)^{-\gamma}(e^x-1)\Pi^*(x)dx=r \qquad (16)$$

4.2.2 Numerical Examples

This subsection illustrates the implications of asymmetric volatility for optimizing portfolio under the Proposition 3 and the reduced form of the infinite-jump variation dynamic process.

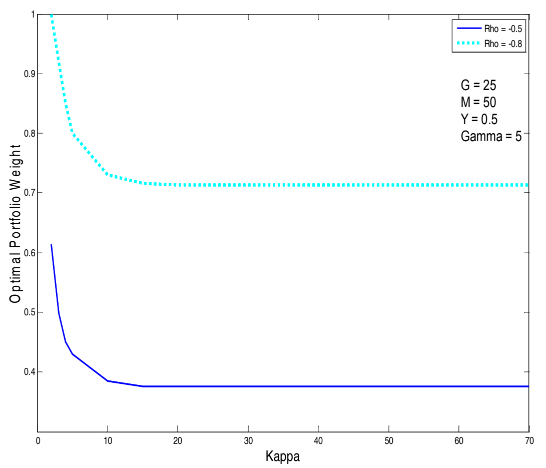
Figure 4: Optimal portfolio weights for the reduced unsymmetrical infinite-jump time-changed Lévy process (GMY model) with respect to the different levels of leverage effect (ρ_2).



Figs.4 shows the decreasing curve of the optimal portfolio weight with respect to the leverage effect, based on the unsymmetrical time-changed Lévy process. When the leverage effect exists, hedging demand for risky assets increases with respect to the magnitude of the leverage effect, $|\rho_2|$. These results appear compatible with the pure-continuous variation case, despite signs of skewness in distributions of asset returns.

Fig.5 shows that optimal portfolio weight is inversely proportional to the value of κ_2 under the unsymmetrical GMY model. Decreasing κ_2 does induce the intertemporal hedging demand. Consequently, the volatility feedback effect motivates hedging demand for risky assets due to the volatility clustering embedded in the stochastic time change.

Figure 5: Optimal portfolio weights for the reduced unsymmetrical infinite-jump time-changed Lévy process (GMY model) with respect to the different levels of volatility feedback effect (κ_2).



According to the numerical examples and discussions, the leverage effect also induces the intertemporal hedging demand. The volatility feedback effect just works over the short-term investment horizon. Restated, the leverage effect plays a major role for portfolio optimization.

5. Empirical Results.

5.1 The General Stochastic Asymmetric Volatility Model

We denote the asset spot price at time t as $S(t)$. The general stochastic asymmetric volatility model is proposed as follows:

$$S(t) = S(0)\exp\left[W\left(T_1(t)\right) + X_2\left(T_2(t)\right) + \rho_2 T_2(t)\right] \tag{17}$$

$$X_2(t) = \theta_2 t + W_2(t), \quad T_1(t) = \int_0^t \nu_1(s)ds$$

$T_2(t)$: State-dependent subordinating Lévy process,

where $T_1(t)$, $T_2(t)$ denote the distinct stochastic time changes as mentioned before, which are mutually independent.

Formally, this work permits the separate treatment of the pure-continuous and infinite-jump time-changed Lévy processes to enable the generation of stochastic asymmetric volatility via both components.

5.2 Data and Model Parameter Estimation

This study uses the S&P 500 index data from January 1, 1980 to June 30, 2008 at two different frequencies, daily and weekly, to estimate the proposed general stochastic asymmetric volatility asset return model, based on the spectral GMM estimation proposed by Chacko and Viceira (2003).

Numerous studies have estimated the parameters of the stochastic volatility model through the characteristic function methods. Particularly, spectral GMM estimation is particularly suitable for time-changed Lévy processes because of its direct use of characteristic function without inversion to recover the density function. Generally, few estimation methods using the traditional maximum likelihood are easy to be use because no analytical expression is known for the density function of the time-changed Lévy process, as the model presented here. Therefore, to respond to the above difficulty, this study employs *spectral GMM* to estimate the parameters of the general stochastic asymmetric volatility asset return model.

The characteristic functions of the pure-continuous and infinite-jump stochastic time-changed asset price models are prioritized in calibrating the general stochastic asymmetric volatility asset return model.

Proposition 4: If the asset percentage return is introduced as Eqn. (6), then the characteristic function $\phi_{\log S(t)}\left(z, \tau; \Theta_1, \log S(t)\right)$ satisfies the following analytic expression:

$$\log \phi_{\log S(t)}\left(z, \tau; \Theta_1, \log S(t)\right) = iz \log S(t) + B(z, \tau; \Theta_1) + \frac{2\kappa_1 \eta_1}{\delta_1^2} \log\left(\frac{2\kappa_1}{2\kappa_1 - \delta_1^2 A(z, \tau; \Theta_1)}\right) \quad (18)$$

where

$$\Theta_1 = \{\theta_1, \rho_1, \kappa_1, \delta_1, \eta_1\}$$

$$A(z, \tau; \Theta_1) = \frac{2}{\delta_1^2}\left[\frac{u_1 u_2 e^{u_1 \tau} - u_1 u_2 e^{u_2 \tau}}{u_1 e^{u_2 \tau} - u_2 e^{u_1 \tau}}\right]$$

$$B(z, \tau; \Theta_1) = \frac{2\kappa_1 \eta_1}{\delta_1^2} \log\left(\frac{u_2 - u_1}{u_2 e^{u_1 \tau} - u_1 e^{u_2 \tau}}\right)$$

$$u_1 = \rho_1 \delta_1 iz - \kappa_1 + \sqrt{(\rho_1 \delta_1 iz - \kappa_1)^2 - \delta_1^2 iz\,(iz + 2\theta_1)}$$

$$u_2 = \rho_1 \delta_1 iz - \kappa_1 + \sqrt{(\rho_1 \delta_1 iz - \kappa_1)^2 - \delta_1^2 iz\,(iz + 2\theta_1)}$$

Proposition 5: If the asset percentage return is introduced as Eqn. (7), then the characteristic function $\phi_{\log S(t)}\left(z, \tau; \Theta_2, \log S(t)\right)$ satisfies the following analytic expression:

$$\log \phi_{\log S(t)}\left(z,\tau;\Theta_2,\log S(t)\right) = iz\log S(t) + \frac{2\kappa_2\eta_2}{\delta_2^2}\log\left(\frac{2\kappa_2}{2\kappa_2-\delta_2^2 C(z,\tau;\Theta_2)}\right) \quad (19)$$

where

$$\Theta_2 = \{\alpha,\lambda,\theta_2,\rho_2,\kappa_2,\delta_2,\eta_2\}$$

$$C(z,\tau;\Theta_2) = \tau\Gamma(-\alpha)\left\{\left[\lambda-\left(\theta_2 iz-\frac{1}{2}z^2\right)\right]^\alpha + (\lambda-\rho_2 iz)^\alpha - 2\lambda^\alpha\right\}$$

The results of Propositions 4 and 5 easily lead to the following expression regarding the conditional characteristic function of the general time-changed asset price dynamics with asymmetric volatility.

Proposition 6: If the asset price dynamics is given as Eqn. (17), then the characteristic function $\phi_{\log S(t)}\left(z,\tau;\Theta,\log S(t)\right)$ satisfies the following analytic expression:

$$\begin{aligned}\log \phi_{\log S(t)}\left(z,\tau;\Theta_2,\log S(t)\right) &= iz\log S(t) + \frac{2\kappa_2\eta_2}{\delta_2^2}\log\left(\frac{2\kappa_2}{2\kappa_2-\delta_2^2 C(z,\tau;\Theta)}\right) \\ &\quad + B(z,\tau;\theta) + \frac{2\kappa_1\eta_1}{\delta_1^2}\log\left(\frac{2\kappa_1}{2\kappa_1-\delta_1^2 A(z,\tau;\Theta)}\right)\end{aligned} \quad (20)$$

where

$$\Theta = \{\alpha,\lambda,\theta_2,\rho_2,\kappa_2,\delta_2,\eta_2,\rho_1,\kappa_1,\delta_1,\eta_1\}$$

$$A(z,\tau;\Theta) = \frac{2}{\delta_1^2}\left[\frac{u_1 u_2 e^{u_1\tau} - u_1 u_2 e^{u_2\tau}}{u_1 e^{u_2\tau} - u_2 e^{u_1\tau}}\right]$$

$$B(z,\tau;\Theta) = \frac{2\kappa_1\eta_1}{\delta_1^2}\log\left(\frac{u_2-u_1}{u_2 e^{u_1\tau} - u_1 e^{u_2\tau}}\right)$$

$$C(z,\tau;\Theta) = \tau\Gamma(-\alpha)\left\{\left[\lambda-\left(\theta_2 iz-\frac{1}{2}z^2\right)\right]^\alpha + (\lambda-\rho_2 iz)^\alpha - 2\lambda^\alpha\right\}$$

$$u_1 = \rho_1 \delta_1 iz - \kappa_1 + \sqrt{(\rho_1 \delta_1 iz - \kappa_1)^2 - \delta_1^2 iz(iz)}$$

$$u_2 = \rho_1 \delta_1 iz - \kappa_1 + \sqrt{(\rho_1 \delta_1 iz - \kappa_1)^2 - \delta_1^2 iz(iz)}$$

With the closed-form conditional characteristic functions as described above, spectral GMM estimation can be applied in the proposed general stochastic asymmetric volatility asset return model.

Table 1: The spectral GMM Estimates of the General Continuous-time Time-changed Asymmetric volatility Model

Model parameters	Daily Data			Weekly Data		
	Estimates	SE	p-value	Estimates	SE	p-value
α	0.811855**	0.036300	0.000000	0.942102**	0.014586	0.000000
λ	82.615909**	30.721756	0.007179	66.147332**	15.646848	0.000025
θ_2	0.767493**	0.367373	0.036729	0.754793	0.461611	0.102236
ρ_1	-0.348467**	0.004590	0.000000	-0.434598**	0.133070	0.001116
ρ_2	0.829387**	0.371334	0.025543	0.772264*	0.412224	0.061210
κ_1	63.253463**	0.538977	0.000000	48.152096**	20.409658	0.018440
κ_2	74.069214**	10.143394	0.000000	101.865132**	27.397853	0.000208
δ_1	0.539658**	0.058467	0.000000	0.880970**	0.152962	0.000000
δ_2	0.446344**	0.075333	0.000000	0.421849**	0.082872	0.000000
η_1	0.013523**	0.003640	0.000204	0.057222*	0.033777	0.090460
η_2	0.078087**	0.022087	0.000410	0.073922**	0.025903	0.004381

* denote an estimate that is significant at 95% level.
** denote an estimate that is significant at 90% level.

Table 1 lists the model parameter estimates, together with their standard errors (SE) and p-values. This table clearly indicates that stochastic asymmetric volatility and infinite-jump structure need to be included to describe the asset return. The coefficients for the pure-continuous and infinite-jump time-changed components are statistically significant in daily frequency and are almost statistically significant in weekly frequency. This work evidently rejects the assumption that index dynamics lack a diffusion term, an assumption that may be present in the dynamics of individual assets. The estimation results demonstrate that the diffusion term is an essential determinant of financial modeling in situations where involve infinite-activity jump structure affecting asset returns.

The empirical findings of this study resemble those of Huang and Wu (2004) in that the diffusion term is necessary for the time-changed Lévy process model when generating dependence on the diffusive state variable. Meanwhile, our findings promote the belief that the diffusion term in the asset return model is still required if one state variable follows a pure-jump process and the correlation between asset return and changes in volatility is established.

6. Concluding Remarks

This investigation proposes two distinct exponential time-changed Lévy processes involving asymmetric volatility, the pure-continuous and infinite-jump asset dynamic processes. Essentially, this work can be considered a combination of the extension of mean-reverting stochastic volatility model developed by Carr *et al.* (2003) and the concept of state-dependent Lévy density proposed by Cvitanić *et al.* (2008).

Regarding economic implications, we first conclude that based on the pure-continuous time-changed Lévy process the leverage effect directly induces the *intertemporal asymmetric volatility hedging demand* while the volatility feedback effect works indirectly via the leverage effect and then exerts a minor influence on asset holding. Simply put, the leverage effect theoretically dominates the volatility feedback effect when engaging in situations involving asset price modeling with asymmetric volatility.

Based on the infinite-jump time-changed Lévy process, we then conclude that the leverage effect also induces the intertemporal hedging demand. Otherwise, the volatility feedback effect just works over the short-term investment horizon.

Empirically, the general asset return model concludes that the diffusion term is an essential determinant when modeling index dynamics with infinite-activity jump structure. The results of present study are inconsistent with those of Carr *et al.* (2002), suggesting that the introduction of asymmetric volatility into asset return dynamics is probably one of causes of the contradiction, even though it is recently believed that the index return processes appear to have efficiently diversified away any diffusion risk. Furthermore, individual asset return may involve a diffusion component. Consequently, present study also has implications to enhance the findings of Carr and Wu (2004) from a viewpoint of theoretical modeling of asset returns.

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