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Optimal Multiperiod Asset Allocation: Matching Assets to Liabilities in a Discrete Model

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Abstract

Investment and risk control are becoming increasingly important for financial institutions. Asset allocation provides a fundamental investing principle to manage the risk and return trade-off in financial markets. This article proposes a general formulation of a first approximation of multiperiod asset allocation modeling for institutions that invest to meet the target payment structures of a long-term liability. By addressing the shortcomings of both single-period models and the single-point forecast of the mean variance approach, this article derives explicit formulae for optimal asset allocations, taking into account possible future realizations in a multiperiod discrete time model.

Introduction

Pension funds and life insurance with investment guarantees appear in various pension plans and product designs. These options increase the obligations of the fund owner to participants, making the question of whether the fund assets are sufficient to cover the liability an important issue for both pension fund managers and insurers. Changing environments, including demographic aging and low interest rates in the global financial market, have also created financial crises for pension funds and insurers. In this environment, the application of asset liability management (ALM) techniques to deal with pension fund or insurance guarantees is gaining increasing attention. For example, fund managers often use ALM to determine their asset allocation. In general, these fund managers establish a target portfolio return that they wish to maintain; when their portfolio's actual return drifts from this target, fund managers must rebalance the asset allocation to reduce tracking errors. Therefore, fund managers need to determine when or whether to rebalance the portfolio, based on the trade-off between the return and the risk. If the portfolio is not rebalanced, it may lose risk controls and deviate toward an unintentionally large percentage of higher-risk assets or inefficient allocations.

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Asset allocation, which represents an essential part of the ALM technique, entails several factors, such as deterministic or stochastic, single-period or multiperiod, and anticipative or adaptive. Mean variance (MV) optimization is the most popular quantitative methodology; it is a deterministic, single-period approach. Markowitz's (1952) modern portfolio theory involves a model of investor behavior in an MV framework, with which he shows that the best investment strategy results from maximizing the expected return and reducing a certain level of variance. Portfolio selection in ALM was early proposed by Wise (1984a, b, 1987a, b) and Wilkie (1995). Sherris (1992) produces a more general approach to portfolio optimization and demonstrates that the optimal investment strategy can be found by maximizing the investor's utility function.

The MV approach contributes substantially to the development of the portfolio selection technique, but it suffers two key problems because of its deterministic and single-period character. First, the deterministic approach is a single-point forecast, though the change in the "real optimal asset allocation" according to current market conditions may differ greatly from historical average returns. Because an MV optimization approach depends heavily on predictions of the expected return, which involve volatility and cross-correlation of the assets, the approach can be very sensitive to a single-point forecast. Small changes in the asset forecasts can shift a portfolio to extreme solutions (Black and Litterman, 1991). To solve this problem, the approach would need to include a stochastic environment for determining optimal asset allocations.

In general, the scenario portfolio selection process considers various return scenarios with their corresponding probabilities. Thus, scenario-based maximization or Monte Carlo simulations could enable fund managers to imagine more likely outcomes. Instead of depending on a single-point forecast, which is actually a single scenario, portfolio managers could provide a set of plausible scenarios and use this variety of market conditions to obtain the optimal portfolio selection, using scenario analysis and stochastic optimization (Mulvey and Vladimirou, 1989; Koskosidis and Duarte, 1997; Horneff, Maurer, and Stamos, 2008; MacDonald and Cairns, 2009).

Second, a single-period model may not applicable for long-term investments, and multistage models might provide superior performance (Berger and Mulvey, 1996; Consigli and Dempster, 1998; Klaassen, 1994, 1998; Mulvey and Ziemba, 1995; Mulvey and Shetty, 2004). A single-period strategy implies that the portfolio mix should realign to reflect the constant proportion of assets over the whole period. In contrast, a multiperiod strategy means that the portfolio mix could realign to reflect a different proportion of assets. For a short-term liability, if there are not strong reasons for the expected return and covariance matrix to differ in a future period compared with the current period, the MV approach performs well. However, for a long-term liability, it may not be suitable to retain the same proportions for the whole period. Therefore, it seems worthwhile to extend current research from what is effectively a single-period portfolio selection model to a multiperiod model with periodic reallocations of assets.

Theoretical formulae for portfolio selections for a single period appear in the previous literature (Sherris, 1992; Wise, 1984a, b, 1987a, b; Wilkie, 1985; Sharp and Tint, 1990; Blake, Cairns, and Dowd, 2001, 2003; Horneff et al., 2008; Zhu, 2007). However, problems of asset allocation, analyzed mathematically for a multiperiod model, rarely exist in discrete models because multiperiod asset allocation solutions are much more complicated than those for a single period. Two theoretical approaches to deal with the problems of multiperiod asset allocations use continuous-time models. The first employs stochastic control, also called control theory, as used by Merton (1969, 1971). Various articles deal with asset allocation problems using stochastic control theory (Devolder, Princep, and Fabian, 2003; Menoncin and Scaillet, 2006; Josa-Fombellida and Rincon-Zapatero, 2004, 2006; Delong, Gerrard, and Haberman, 2008; Hainaut and Devolder, 2007; Ngwire and Gerrard, 2007; Emms and Haberman, 2007; Chiu and Li, 2006; Moore and Young, 2006; Yang and Zhang, 2005; Gerrard, Haberman, and Vigna, 2006; Liu and Yang, 2004; Steffensen, 2004; Emms and Haberman, 2008). To find the explicit solution for the value function, this method must solve the nonlinear, partial differential Hamilton–Jacobi–Bellman (HJB) equation, the most difficult task associated with the stochastic optimal control approach.

Alternatively, Cox and Huang (1989, 1991) propose an approach for complete markets that relies on Lagrange multipliers. This martingale method frequently appears in research into the optimal design and asset allocation of a pension fund or life insurance policy (Boulier, Huang, and Taillard, 2001; Deelstra, Grasselli, and Koehl, 2003, 2004; Hainaut and Devolder, 2007; Z. Wang, Xia, and Zhang, 2007; N. Wang, 2007; Yang and Zhang, 2005; Grasselli, 2003); it provides a partial differential equation that can be easier to solve than the HJB equation.

Both methods offer advantage because they are stochastic, adaptive, and multiperiod. However, they generally require well-designed assumptions to obtain closed-form solutions. Thus, it can be difficult to capture real-world features, and the models are not flexible enough to apply to practical problems.

Furthermore, impressive progress in computational methods that enables the solution of large-scale problems efficiently and reliably (Lustig, Mulvey, and Carpenter, 1991; Bixby et al., 1992; Levkovitz and Mitra, 1993; Mulvey and Ziemba, 1995) and increasing numbers of publications about stochastic programming (SP) for ALM (Consigli and Dempster, 1998; Dempster and Thompson, 1999; Dempster, Evstigneev, and Schenk-Hoppe, 2003; Mulvey and Shetty, 2004; Yu, Ji, and Wang, 2003; Huang, Hsieh, and Liu, 2008) suggest a general-purpose modeling framework for capturing real-world features. However, to solve SP problems with an optimization search method, such as genetic algorithm or evolution, requires significant reductions.

Specifically, a stochastic programming model uses an event tree to discern the key random variables, and each node of that event tree leads to multiple successors that model the revelation of information progressively over time. The SP approach determines the optimal decision for each node, given information available at that point. A complete path from the root node to a leaf defines a single realization of a set of random variables. In other words, each path in the multistage approach represents a scenario, with its own optimal asset allocation. In such an adaptive model, unlike an anticipative one, uncertainty information is partially available before the decision, so optimization occurs in a learning environment. To apply SP to financial optimization, the researcher must construct event trees with asset returns but also make the underlying return distributions of the SP approach discrete, with few nodes. If the researcher fails to do so, the computational effort to solve a multistage SP model can exceed acceptable bounds; the decisions expand exponentially with time. Fewer nodes that describe the return distribution, however, could cause approximation error. If the event tree includes more nodes, to reduce the approximation error, the optimization model becomes intractable because of the enormous number of decision variables and significant time required to complete the multistage problems, regardless of the available computational power. The optimization model also cannot always obtain a global optimal solution; rather, different sets of initial variables may produce different, locally optimal solutions.

Moreover, both the anticipative and adaptive models represent special forms of stochastic methods. The anticipative model is static; the decision does not depend in any way on future observations of the environment. We adapt the concept of an anticipative model and develop a first approximation of an analytical solution of multiperiod asset allocation in a discrete model. Although this approach is not an adaptive model—as are the stochastic control, martingale, and SP multistage approaches, for which one scenario determines one optimal asset allocation—it can include a large number of simulations (e.g., 4,000 simulations of asset returns and liabilities over 10 years) and thus find realistic asset returns, as well as possible future realizations, to obtain a first approximation of a multiperiod asset allocation. In other words, with 4,000 simulations, a portfolio manager will face 4,000 equal-probability scenarios of 10 years of asset return predictions. The obtained optimal first approximation of asset allocation minimizes the average tracking error of ALM.

In turn, the main contribution of this article is our construction of a theoretical formulation of optimal asset allocations for a multiperiod asset-liability model, which enables us to overcome the shortcomings of both the single-period model and the single-point forecast of the MV approach. In addition, we address the time-consuming and local minimum solutions that occur when the model includes large simulations of future asset returns. This asset allocation formula is flexible and can be applied to any stochastic investment return model. We adopt stochastic simulation to create a representative set of equal-probability plausible scenarios of future returns by choosing an investment return model. Each simulated scenario addresses a liability in the optimization process.

The rest of this article is organized as follows: In the second section we provide a brief introduction to the concept of ALM and its application for dealing with a long-term liability. We also describe the payment structures for various types of liabilities. The third section features our construction of an asset model and the formulae for optimal asset allocations for multiperiod ALM. In the fourth section, we show how the model achieves risk and return trade-offs for various long-term liabilities, incorporating thousands of simulations of asset returns. The fifth section further addresses optimal asset allocations when short selling and borrowing are prohibited. Finally, we draw key conclusions in the sixth section.

ASSET LIABILITY MATCHING AND GUARANTEE

Asset Liability Matching

Maximizing terminal wealth and asset liability matching are the two most common objective functions for applying asset allocations. Utility functions can help maximize an investor's terminal wealth; they generally assume that an investor is risk averse and seeks to maximize the utility of terminal wealth, using the criteria of optimal consumption and savings (Dhaene et al., 2005; Devolder, Princep, and Fabian, 2003; Moore and Young, 2006; Battocchio and Menoncin, 2004; N. Wang, 2007; Yang and Zhang, 2005; Korn, 2005; Zhu, 2007). Whether the assets are enough to cover liabilities is the more important issue for the pension fund manager and insurer, especially when they provide a guaranteed product. Therefore, fund managers might use the ALM technique to make asset allocation decisions (e.g., Wise, 1984a, b, 1987a, b; Wilkie, 1985; Sherris, 1992, 2006; Delong, Gerrard, and Haberman, 2008). Generally, two criteria evaluate ALM risk: the tracking error of ultimate surplus and the asset liability ratio. The former is the difference between the accumulated asset cash flows and the accumulated liability cash flows at a fixed time horizon. The objective function to incorporate this criterion can be expressed as follows:

$$\text{Min } E[(F(n) - L(n))^2]. \tag{1}$$

The asset–liability ratio aims to reach:

$$\frac{F(n)}{L(n)} \to 1. \tag{2}$$

We further extend Equations (2) to (3) by setting the asset liability ratio as quadratic.

$$\text{Min } E\left[\left(\ln \frac{F(n)}{L(n)}\right)^2\right] = \text{Min } E[(\ln F(n) - \ln L(n))^2]. \tag{3}$$

From a risk management perspective, a fund deficit is more important than a surplus in practice because of insolvency considerations. Compared with Equation (1), Equation (3) penalizes deficits (when $L(n) > F(n)$) more for a given liability, which makes it more focused on downside risks. For example, for a given liability $L(n) = 100$, $F(n) = 20$, and $F(n) = 500$ contribute the same tracking errors in Equation (3), but in Equation (1), the same tracking features occur when $F(n) = 20$ and $F(n) = 180$ for a given liability $L(n) = 100$. That is, Equation (3) provides a suitable objective function for liability matching when we prefer a larger surplus to a deficit.

Guarantees With General Long-Term Liability

We use ALM to address the asset allocation decision for a general long-term liability with an investment guarantee. The nature of fund liabilities depends on the form of the guarantee and relates directly to the asset allocation decision. Two categories of guarantees exist: a multiperiod guarantee and a maturity guarantee. To illustrate the relationships of multiperiod asset matching with various liabilities, we investigate the two categories of guarantee, with their specific payment structures, separately.

Multiple-Period Guarantee. For multiple-period guarantees, we consider four types that are common for long-term liabilities, which we refer to as Types A1–A4, as well as one type of guarantee often found in equity-linked insurance products, which we denote a Type B guarantee.

$$\text{Type A1: } L(n) = (1+r)^n.$$

This annual constant guarantee is the simplest form used in pension funds.

$$\text{Type A2: } L(n) = \prod_{t=1}^{n} \{1 + \text{inflation rate } (t)\}.$$

This Type A2 payment is subject to the retail price index (RPI) guarantee each year. The limited indexation of the pension liability (LPI), introduced in the 1990 Social Security Act in Great Britain, is as follows:

$$\text{LPI}(n) = \prod_{t=1}^{n} \min\{1+r, 1 + \text{inflation rate } (t)\}.$$

Most guarantee programs do not allow for negative growth, so we can extend the LPI liability.

$$\text{Type A3: } L(n) = \text{Max}\left(1, \prod_{t=1}^{n} \min\{1+r, 1 + \text{inflation rate } (t)\}\right).$$

We further extend Type A3 to Type A4 as follow.

$$\text{Type A4: } L(n) = \prod_{t=1}^{n} \max\{1+r, 1 + \text{inflation rate } (t)\}.$$

$$\text{Type B: } L(n) = \prod_{t=1}^{n} \max\{1+r, 1 + b \times \text{equity return rate } (t)\}.$$

Finally, Type B is the payment structure for a life insurance product with an annual guarantee. In the general form for this type of payment structure, payments increase with the index of equity return rate, subject to a participating rate b and an annual guarantee rate r.

Maturity Guarantee. A maturity guarantee commonly applies to unit-linked products in Great Britain. The policyholder receives the maximum amount of the guarantee ($G(n)$) or the actual account value ($F(n)$) at the maturity date (n). If $G(n)$ is greater than $F(n)$, the insurer experiences a guaranteed liability. The guaranteed liability represents the difference between the guaranteed and the actual account value. We denote the maturity guarantee liability for a unit-linked contract as Type C liability, defined as $L(n) = \max\{0, G(n) - F(n)\}$.

DERIVATION OF AN ANALYTIC SOLUTION FOR OPTIMAL ASSET ALLOCATIONS

To establish an analytic solution for the multiperiod asset allocation problem with a closed-form solution, we ignore some practical assumptions, such as transaction costs.

Asset Dynamics

We project the future dynamic of the asset return using Wilkie's (1995) investment model to illustrate the numerical results. We generate multiple scenarios of future returns using Monte Carlo simulations.[1] We also consider the standard asset classes used by pension or insurance funds, such as short-term bonds, consols, index-linked gilts (ILG), and equities. Short-term bonds and consols serve important roles in pension or insurance investment. Equity offers a good asset for wage-related liability, because of its strong returns and inflation protection in the long term. The ILGs, introduced in Great Britain in the early 1980s, have coupon and redemption values linked to the RPI.

In order to evaluate the asset value of the portfolio at different dates, we define

P_{kj} : proportion held in asset type j at the kth rebalance, where $j = 1$ is short-term bonds; $j = 2$ is consols; $j = 3$ is ILGs; and $j = 4$ is equities.

T_k : time at the kth rebalance.

$F(0)$: total initial asset holding.

$F(T_k)$: total asset value at the time of the kth rebalance.

$F(n)$: total asset value at the end of the term.

$Z_j(T_k)$: accumulated return of asset type j at the time of the kth rebalance.

Thus, the value of the total asset at maturity date after r times of rebalancing is

$$\begin{aligned} F(n) &\approx F(T_r)\exp\left[\sum_{j=1}^{4} P_{rj} \times \sum_{S=T_r+1}^{n} \ln\left(\frac{Z_j(S)}{Z_j(S-1)}\right)\right], \\ &= F(0)\prod_{i=1}^{r}\prod_{j=1}^{4}\left(\frac{Z_j(T_{i+1})}{Z_j(T_i)}\right)^{P_{ij}}, \end{aligned} \tag{4}$$

where $T_1 = 0$ and $T_{r+1} = n$.

[1] These scenario sets are large enough to represent highly unlikely market swings adequately. Using plausible equal-probability predictions of future returns for each asset, we can find a set of optimal asset allocations in each period at the beginning of the term and thus accumulate enough funds to afford the payment required at maturity.

Theoretical Formula of Optimal Asset Allocations for Dynamic Hedging

Appropriate diversification and asset allocation of the portfolio provide the most popular strategy for ensuring the asset is as close as possible to the liability at the maturity date. The rebalancing function essentially attempts to align the portfolio's return with the target return. To simplify the calculation of the optimal rebalancing proportions (P_{ij}) and the initial asset holding ($F(0)$), we first define the annual asset returns as $k_{ij} = \frac{Z_j(T_{i+1})}{Z_j(T_i)}$. According to Equation (4), we can express the accumulated asset value at the maturity date as follows,

$$F(n) = F(0) \prod_{i=1}^{r} \prod_{j=1}^{4} (k_{ij})^{P_{ij}}.$$

By taking the logarithm of the asset value at the maturity date, we have

$$\begin{aligned}\ln F(n) &= \ln F(0) + \sum_{i=1}^{r} \sum_{j=1}^{4} P_{ij} \ln k_{ij} \\ &= \ln F(0) + \sum_{i=1}^{r} \left(\sum_{j=1}^{3} P_{ij} \ln k_{ij} + (1 - P_{i1} - P_{i2} - P_{i3}) \ln k_{i4} \right) \\ &\quad \times (P_{i4} = 1 - P_{i1} - P_{i2} - P_{i3}).\end{aligned}$$

We can now rewrite Equation (2) as

$$\text{OBJ} = \text{Min} \left[E \left(\left[\ln F(0) + \sum_{i=1}^{r} \left(\sum_{j=1}^{3} P_{ij} (\ln k_{ij} - \ln k_{i4}) + \ln k_{i4} \right) - \ln L(n) \right]^2 \right) \right]. \quad (5)$$

In attempting to find the initial asset holding and the regular rebalancing proportions on certain dates that will meet the criterion of ALM at maturity, we work out the optimal solution of the P vector, or

$$P = (\ln F(0), P_{11}, P_{12}, \ldots, P_{r3})^T.$$

To obtain the solutions for this vector, we first must arrange the parameters of the P vector from Equation (5) as follows,

$$V_j = \begin{cases} \ln k_{(\frac{j+1}{3})1} - \ln k_{(\frac{j+1}{3})4} & j = 2, 5, \ldots, 3r - 1 \\ \ln k_{(\frac{j}{3})2} - \ln k_{(\frac{j}{3})4} & j = 3, 6, \ldots, 3r \\ \ln k_{(\frac{j-1}{3})3} - \ln k_{(\frac{j-1}{3})4} & j = 4, 7, \ldots, 3r + 1 \end{cases},$$

where $V_1 = 1$, because the parameter of $\ln F(0)$ is a constant.

Next, we must calculate the first and second moments of the asset returns (V_j). To express the necessary formula in a user-friendly manner, we define some notations:

$$\begin{aligned}
\omega(i,j) &= \mathrm{Cov}(V_i, V_j) \quad \text{where} \quad i, j = 1, 2, \ldots, 3r+1,\\
\Omega &= (\omega(i,j))_{i,j=1}^{3r+1},\\
KL(n) &= \ln L(n) - (\ln k_{14} + \ln k_{24} + \cdots + \ln k_{r4}),\\
\omega &= \mathrm{Var}\ (KL(n)),\\
\Psi_0 &= E(KL(n)),\\
\Psi &= \begin{pmatrix} 1 \\ E(V_2) \\ \cdot \\ \cdot \\ E(V_{3r+1}) \end{pmatrix}, \quad \text{and}\\
\Gamma &= \begin{pmatrix} \mathrm{Cov}\ (1,\ KL(n)) \\ \cdot \\ \cdot \\ \mathrm{Cov}\ (V_{3r+1},\ KL(n)) \end{pmatrix}.
\end{aligned}$$

Thus, we can rewrite the objective function of Equation (5) as follows:

$$\mathrm{OBJ} = P^T \Omega P - 2P^T \Gamma + \theta(P^T \Psi - \Psi_0)^2.$$

To obtain the optimal values of the allocations, we take the derivatives of the objective function,

$$\frac{\partial \mathrm{OBJ}}{\partial P} = 2\Omega P - 2\Gamma + 2\theta \Psi \Psi^T P - 2\theta \Psi_0 \Psi,$$

where $\frac{\partial OBJ}{\partial P} = 0$. We then obtain the optimal asset allocation on specific dates with an initial asset holding, as follows:

$$\hat{P} = \begin{pmatrix} \ln F(0) \\ \hat{P}_{11} \\ \hat{P}_{12} \\ \hat{P}_{13} \\ \hat{P}_{21} \\ \cdot \\ \cdot \\ \hat{P}_{r1} \end{pmatrix} = (\Omega + \theta \Psi (\Psi)^T)^{-1} (\theta \Psi_0 \Psi + \Gamma). \tag{6}$$

Note this analytical solution does not consider fat-tailed distributions or multiple possible regimes or correlations that might change over time.

Using Equation (6), we can derive a first approximation of the optimal multiperiod asset allocation at the beginning of the term. However, if future economic conditions change by the next valuation date, our proposed method generates a new set of simulations with different initial values, according to the current conditions. We can revise the optimal asset allocation for the rest of the periods, until maturity, using the same Equation (6) but new simulations of the return predictions. For example, if we consider an investment strategy for a 10-year target payment structure, we find a first approximation of a 10-year optimal asset allocation in the start of the term using Equation (6). If the economic situation has changed 3 years later, we generate new simulations of future 7-year predictions using the current initial conditions. The optimal investment strategies for the rest of the 7-year period at the end of year 3 depend on the revised simulations. Furthermore, we can take account of future observations and calculate the rest periods of optimal multiperiod asset allocations every year again; the solutions of the asset allocations should be similar to those of dynamic methods.

NUMERICAL ILLUSTRATION AND ANALYSIS

In this section, we generate 4,000 ten-year Monte Carlo simulations using the Wilkie (1995) investment model. Therefore, we obtain 4,000 equal-probability scenarios that provide a satisfactory representation of the next 10-year potential return outcomes. With these scenarios, we apply Equation (6) to obtain the optimal asset allocation for various payment structures, as we introduced in the second section. We first investigate the efficiency of ALM with a dynamic hedging approach by comparing two investment strategies: buy and hold versus annual rebalancing. The efficiency comparison uses tracking errors as a measure, and we provide these tracking errors for a Type A liability in Table 1.

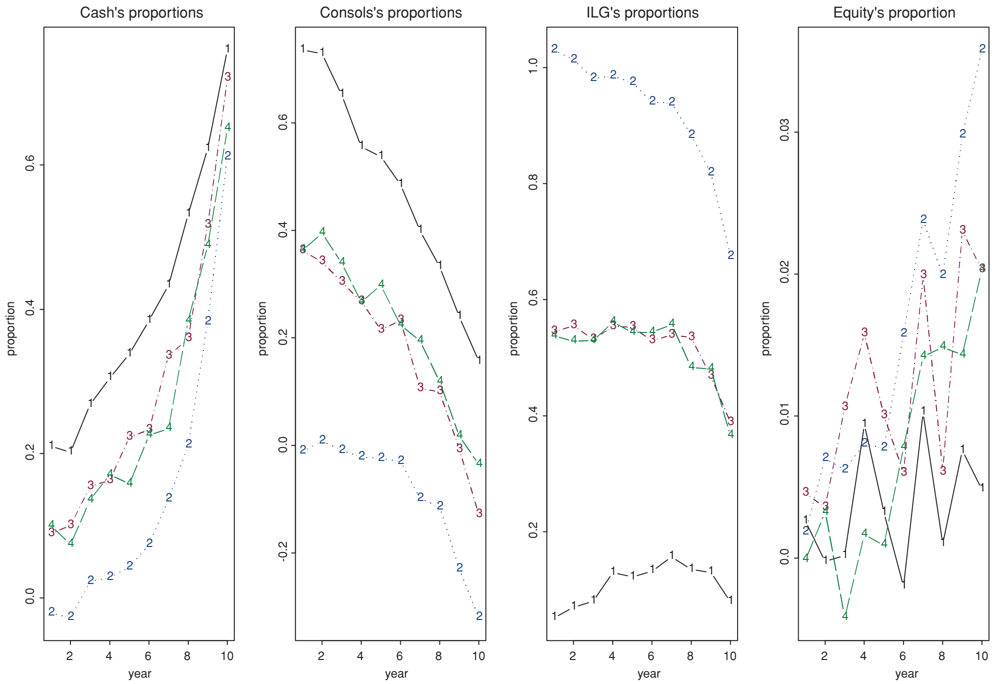
Multiperiod asset allocation investment strategies significantly improve ALM. When we reduce the tracking error for the Type A1 payment structure from 0.006065 to 0.003120, for a reduction rate of approximately 49 percent, the corresponding reduction rates for Types A2, A3, and A4 are 38 percent, 21 percent, and 25 percent, respectively. If the assumptions that drive our simulations are valid, the multiperiod asset allocation approach is better than the single-period approach. In addition, Equation (6) provides effective asset allocation information for the multi-period approach.

TABLE 1
Tracking Errors of Asset Liability Matching, Buy and Hold and Annual Rebalancing Investment Strategies, 10-Year Payment Structures

	A1	A2	A3	A4
Buy and hold	0.006065	0.004333	0.008275	0.007899
Annual rebalancing	0.003120	0.002684	0.006503	0.005926

Note: Tracking error is the difference between the accumulated fund and the target liability.

FIGURE 1
Comparisons of Optimal Asset Allocations Among Liabilities, Types A1–A4

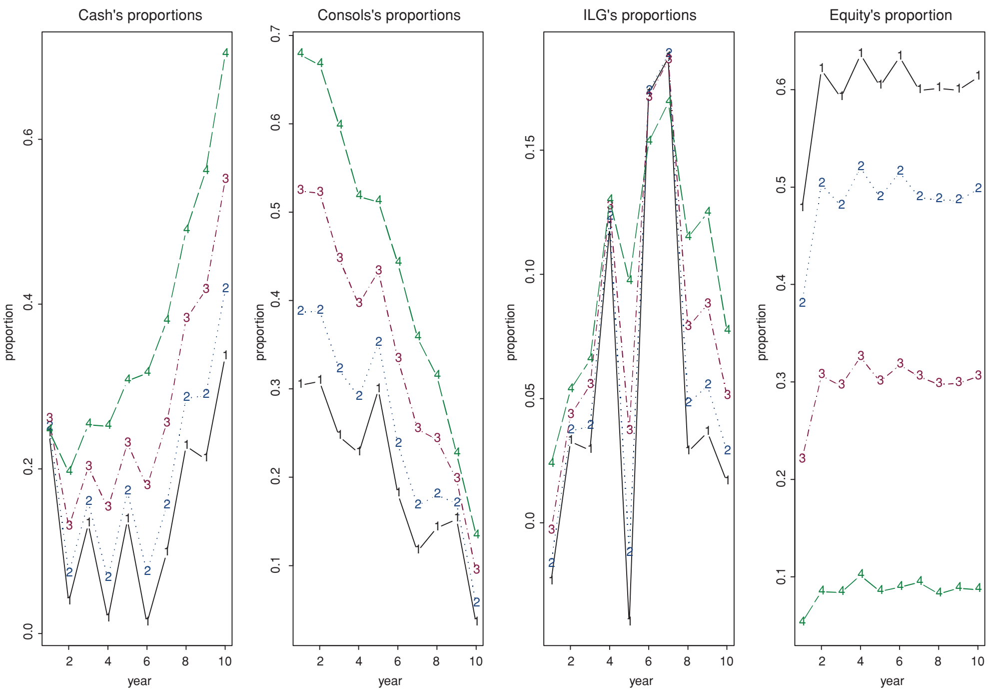


Note: Line 1 is the asset allocation for Type A1, line 2 is the asset allocation for Type A2, and so forth.

The patterns of asset allocations for various types of payment structure appear in Figures 1–3. We first analyze the asset allocation pattern for the Type A1–A4 payment structures. According to Figure 1, a constantly increasing liability (i.e., A1) identifies consols as the most important asset, whereas equity is the least important. For example, at the beginning of the 10-year term, nearly 70 percent is invested in consols, 20 percent in short-term bonds, and the rest in ILGs and equities. This finding is intuitive because the volatility of equity is much higher than that for the rest of the assets. In addition, we gradually switch the proportion of assets from consols to short-term bonds to reduce the liquidity risk. For A2, the inflation-increasing liability, ILGs are the most important assets. Similar to Type A1, the assets switch from ILGs to short-term bonds to meet the liability. At the beginning of the term, almost all asset holdings are invested in ILGs only. Again, these results meet our intuitive expectations because matching entails the selection of assets that most closely resemble the liability cash flows, whereas portfolio selection involves the selection of assets that attain an optimal level of risk.

The payment structures of Types A3 and A4 combine the payment structures of A1 and A2. The asset allocation patterns for A3 and A4 are quite similar, and the investment proportions for each asset fall between those in Types A1 and A2. The main difference is that A4 features a larger initial asset holding than A3, since A4

FIGURE 2
Comparisons of Asset Allocations Among Different Participating Rates



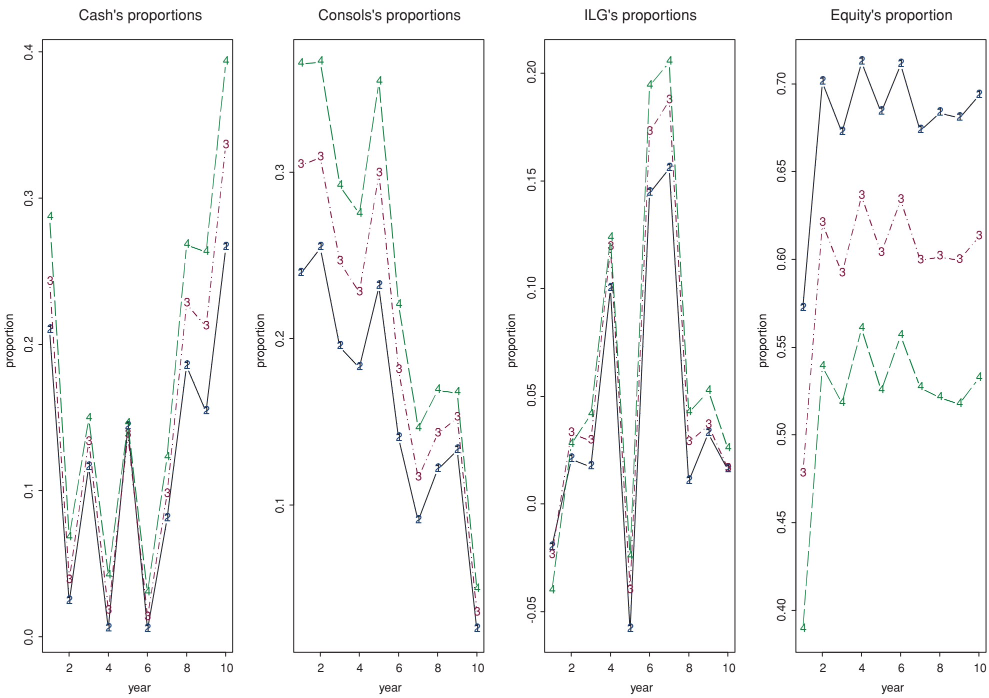
Notes: The graphs indicate 20%, 50%, 80%, and 100%, respectively, with a constant annual guarantee rate of 5% for Type B liability. Lines 1–4 show the asset allocation for a participating rate of 100%, 80%, 50%, and 20%, respectively, with an annual guarantee rate of 5% for Type B liability.

reflects the maximum amount between Type A1 and Type A2, whereas A3 is the minimum.

In Figure 1, for the Type A1 liability, we set the constant increase rate at $r = 5$ percent. If we change the rate of increase from 5 percent to 6 percent, we obtain exactly the same asset allocations but different initial total asset holdings because the optimal asset allocations rely on the distributions of the liability cash flows. Different constant liabilities have the same distributions of cash flows but unique sizes of liabilities. In other words, the optimal asset allocations for any constantly increasing liability are the same; however, a larger constant liability demands the investment of more money.

Type B is the equity-link liability, with participating rate b and annual guarantee rate r. We set the guarantee rate at $r = 5$v and determine how the asset allocation changes with the participating rate b in Figure 2. The higher the participation rate, the higher the proportion held in equities and the lower the proportion held in short-term bonds, consols, and ILG.s. For example, if the participating rate $b = 100\%$, more than 50 percent consists of equities during the term, with 30 percent consols at the beginning, gradually switching to short-term bonds. In this case, ILGs are the

FIGURE 3
Comparisons of Asset Allocations Among Different Annual Guarantee Rates



Notes: The figures depict 20%, 0%, 5%, and 10%, respectively, with a constant participating rate of 80% for Type B liability. Line 1 shows the asset allocations for guarantee rates of 20% with a constant participating rate of 80% for Type B liability, line 2 shows the asset allocations for guarantee rates of 0% with a constant participating rate of 80% for Type B liability, line 3 shows the asset allocations for guarantee rates of 5% with a constant participating rate of 80% for Type B liability, and line 4 shows the asset allocations for guarantee rates of 10% with a constant participating rate of 80% for Type B liability.

least important assets, because liability combines a constant increase $r = 5$ percent and equities. Again, consols and short-term bonds are most important for a constant liability. If the participating rate decreases, the volatility of the liability also declines. In other words, liability is similar to a constant. Therefore, consols and short-term bonds are more important in the portfolio. For example, if the participating rate $b =$ 20 percent and $r = 5$ percent, we invest most of the assets in consols and short-term bonds because most liabilities will be 1.05 during the 4,000 simulations.

In Figure 3, we depict the impact of a different guarantee rate on the asset allocation. If we fix the participating rate ($b = 100$ percent), from Figure 3, we find that the higher the annual guarantee rate, the lower the proportion of equities held, and the higher the proportion of short-term bonds, consols, and ILGs held in the portfolio because there is a higher probability of a constant liability $(1 + r)$. If the participating rate $b = 80$ percent and the annual guarantee rate r increases from -20 percent to 10 percent, the proportion held in short-term bonds, consols, and ILG increases

TABLE 2
Comparisons of Asset Allocations Between Different Economic Scenarios for 10-Year Payment Structures of Type A2 Liability

	Short-Term Bonds		Consols		Index-Linked Gilts		Equities	
Time\Asset	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
1st year	−2.1%	34.5%	−1.1%	−8.4%	103%	54.4%	0.2%	19.6%
2nd year	−2.8%	37.8%	−1%	−14.6%	101%	53.2%	0.7%	18.7%
3rd year	2.2%	43.8%	−1%	−14.5%	98%	52.9%	0.7%	17.8%
4th year	2.8%	46.4%	−2.1%	−14.5%	98.5%	51.7%	0.8%	16.4%
5th year	4.3%	50.1%	−2.4%	−15.7%	97.4%	50.2%	0.8%	14.8%
6th year	7.4%	54.5%	−3%	−16.4%	94%	47.7%	1.6%	12.2%
7th year	13.6%	55.4%	−10%	−15.8%	93.9%	47.8%	2.4%	12.6%
8th year	21.2%	57.2%	−11%	−19.9%	88.3%	46.1%	2%	12.6%
9th year	38.2%	66%	−23%	−24.6%	81.8%	45.9%	3%	12.7%
10th year	61.1%	84.7%	−32.1%	−41.2%	67.4%	57.3%	3.6%	9.3%

between 5 percent and 10 percent, whereas the proportion of equities decreases between 15 percent and 20 percent on average. For example, if the annual guarantee rate is 10 percent, more liabilities are equal to a constant (1.1) during 4,000 simulation liabilities. The portfolio thus consists of close to 40 percent of consols at the beginning of the term and 40 percent of short-term bonds at the end of the term; the proportion of equities decreases from 68 percent to 53 percent during the term.

The numerical results are almost entirely driven by the underlying assumptions about market behavior. Different assumptions would lead to different results, so to investigate changes in the asset allocations due to different underlying assumptions of the investment return model, we perform another set of 4,000 simulations, in which the average of inflation is half of Wilkie's (1995) original assumptions. We refer to these low inflation assumptions as case 2 and compare the result with those based on the original assumption (case 1). The impact of the two different investment model assumptions on the asset allocation, as we show in Table 2, is notable. The portfolio includes more short-term bonds and equities and fewer ILGs in case 2 compared with case 1, because ILGs have smaller returns when inflation is lower. Thus, the portfolio contains fewer ILGs if the inflation rate relates strongly to the return on ILGs.

Asset allocation is a critical issue for ALM, as are asset selections. Table 3 lists the comparisons of asset allocations for four assets (case 1) and three assets (case 3). The asset allocations between cases 1 and 3 are quite different, including more short-term bonds and equities and no ILGs. Case 1 obtains a greater average investment return (i.e., 8.9 percent vs. 8 percent in case 3) and lower tracking error (i.e., 0.2 percent vs. 6 percent in case 3). In other words, without ILGs in the portfolio, tracking error increases by 30 times, and investment return simultaneously falls. Correct asset selections are critical for ALM.

Return and risk measurements are critical issues in the ALM problem. For example, in a defined-benefit pension plan, the cost of maintaining the plan can be evaluated as the

TABLE 3
Comparisons of Asset Allocations Between Different Assets of Portfolio Selections for 10-Year Payment Structures of Type A2 Liability

	Short-Term Bonds		Consols		Index-Linked Gilts		Equity	
Time\Asset	Case 1	Case 3	Case 1	Case 3	Case 1	Case 3	Case 1	Case 3
1st year	−2.1%	64.4%	−1.1%	4.4%	103%	0%	0.2%	31.2%
2nd year	−2.8%	69.9%	−1%	0.6%	101%	0%	0.7%	29.5%
3rd year	2.2%	79.8%	−1%	−10.3%	98%	0%	0.7%	28.6%
4th year	2.8%	92.5%	−2.1%	−20.3%	98.5%	0%	0.8%	27.8%
5th year	4.3%	95%	−2.4%	−22.2%	97.4%	0%	0.8%	27.3%
6th year	7.4%	95.3%	−3%	−23%	94%	0%	1.6%	27.7%
7th year	13.6%	102.8%	−10%	−30.6%	93.9%	0%	2.4%	27.8%
8th year	21.2%	105%	−11%	−31.9%	88.3%	0%	2%	26.8%
9th year	38.2%	110.5%	−23%	−36.4%	81.8%	0%	3%	25.1%
10th year	61.1%	114.7%	−32.1%	−38.7%	67.4%	0%	3.6%	24%

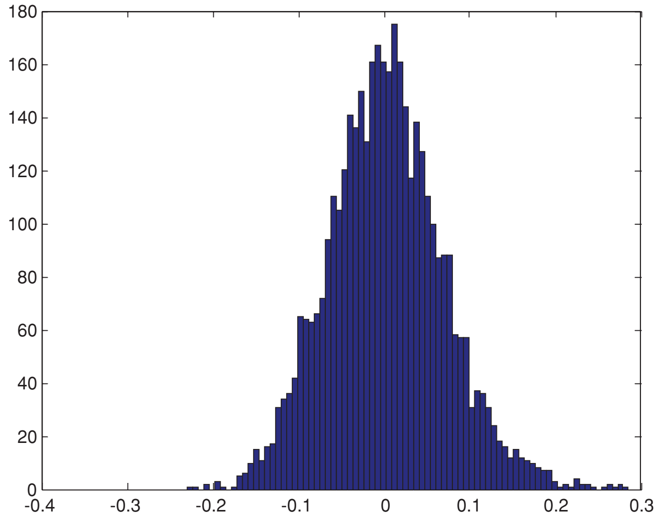
TABLE 4
Residual Information About Type A1 Liability

$\Pr(F(n) > L(n)) = 49.9\%$							
Surplus Residuals				Deficit Residuals			
μ	σ	CVar(95%)	CVar(90%)	μ	σ	CVar(95%)	CVar(90%)
0.0547	0.0457	0.1452	0.1165	−0.0514	0.0388	−0.1252	−0.1063

expected discounted value of contribution. The risk measurement can be probability of making a large contribution, likelihood of a bankruptcy over the planning period, or related worst-case events (Mulvey et al., 2008). Various metrics apply to returns and risks over multiple periods, such as measuring risk as a function of a related probability distribution. Take Type A1 as an example: we obtain an optimal asset allocation through Equation (6), on the basis of a set of 4,000 ten-year simulations. We can then adapt these simulation data to calculate the difference between the accumulated assets and liabilities (residuals) and verify the match to the asset liability. We depict the entire distribution of the residuals in Figure 4, which reveals that the asset liability matching is good in this case; most residuals are close to 0 (tracking error is 0.003). We also find that the tail of the surplus (positive residuals) is longer than that of deficits (negative residuals).

We calculate additional return and risk information in Table 4. Specifically, the probability that accumulated assets will be greater than liabilities is 49.9. As we also see from Table 4, surpluses are slightly greater than deficits, especially for the tail distribution (the CVaR(95%) of surplus is 0.1452, whereas the CVaR(95%) of deficit is −0.1252).

FIGURE 4
Distribution of Residuals ($F(n)-L(n)$) of Type A1 Liability



Notes: Y axis shows the number of simulations, and X axis depicts the value of the residuals ($F(n)-L(n)$).

CONSTRAINTS OF SHORT SELLING AND BORROWING

Because short selling and borrowing might not be allowed in practice, we discuss optimal asset allocations when they are prohibited. We formulate the objective function and constraints as follows.

$$\min_{A(0),p_{ij}} \sum_{k=1}^{m} \{\ln[F^{(k)}(n)] - \ln[L^{(k)}(n)]\}^2,$$
$$\text{subject to} \sum_{j=1}^{4} P_{ij} = 1 \quad \text{and} \quad 0 \leq P_{ij} \leq 1 \quad \text{with} \quad \forall i = 1.2 \ldots r, \quad \forall j = 1 \ldots 4, \tag{7}$$

where $\ln[F^{(k)}(n)] = \ln F(0) + p_{11} \ln k_{11}^{(k)} + p_{12} \ln k_{12}^{(k)} + \ldots\ldots p_{r,4} \ln k_{r,4}^{(k)}$

$$= \ln F(0) + \sum_{i=1}^{r} \sum_{j=1}^{4} P_{ij} \ln k_{ij}^{(k)}.$$

Short shelling and borrowing constraints prevent the closed-form solutions of optimal asset allocations, as in Equation (6). We use optimization software, such as MATLAB, to solve Equation (7). By using MATLAB optimization software, we can obtain exact same solutions as the solutions obtained from Equation (6). We need to short the consols during the period and hold more ILGs and short consols to meet the liability at the maturity date for Type A2. With the constraint $0 \le P_{ij} \le 1$, we find that all the negative values become 0 when we apply optimization software; therefore, the portfolio cannot include so many short bonds near the end of the term because we cannot short the consols.

We adopt optimization software to solve Equation (7) and obtain optimal (global) solutions without demanding too much computing time because we simplify the solution as a quadratic programming problem. This simplification makes it easy to adopt optimization software to obtain global solutions. If we solve Equation (1) using optimization software directly though, we confront two issues. First, it would take much longer to obtain solutions—more than 6 hours if we were to consider large numbers of asset return simulations or many decision variables. Second, we might obtain local optimal solutions, because different "optimal" solutions likely emerge when we set different initial values. However, the solutions for Equations (5) and (7) are very similar. In Equation (7), in contrast to Equation (5), we consider the constraints of prohibitions on short selling and borrowing, such that the main difference entails the negative values obtained from Equation (5) becoming 0 in Equation (7). In turn, Equation (6) provides a set of better (closer) initial values. To save time and still obtain the global solution, we should adopt initial values close to the solution obtained from Equation (6) to search the optimal asset allocation for Equations (7) or (1). That is, the best strategy applies Equation (6) to search optimal asset allocation first, then uses it to obtain a global solution of optimal asset allocation quickly, regardless of the number of decision variables or economic simulation data. However, if the solutions return negative values and the practice of shorting assets is not allowed, we would need to use optimization software to solve Equation (7). We can use the solution obtained from Equation (6) as the initial value to search for the optimal asset allocation.

Conclusions

Existing investigations of the MV approach to a single-period asset liability matching challenge tend to include static portfolio optimization models, such as Markowitz MV allocation, which are short-sighted and can require radical portfolio rebalancing, unless severely constrained by intuition. However, this standard implementation also includes only one period, which means that it cannot capture the multiperiod nature of the problem. We instead adapt an anticipative model and develop an initial approximation of an analytical solution of multiperiod asset allocation in a discrete model. This theoretical formulation of the optimal asset allocation can overcome the shortcomings of both the single-period model and the single-point forecast of the MV approach. In addition, we address the problems of time-consuming and local minimum solutions when the model includes large simulations of future asset returns with theoretical formulae. By adapting large numbers of asset return simulations, our proposed analytical solution can determine realistic asset return features and take into account all possible future realizations to obtain a first approximation of a multiperiod asset allocation. Thus, this model offers the optimal asset allocation without demanding too much computing power and avoids the disadvantages of being highly sensitive to the single-point forecast and the necessities of a normal assumption of returns.

Specifically, we note that among the 651 continental European funds surveyed by Mercer, bonds remain the dominant asset class, reaching €423 billion, though the equity allocation has increased (i.e., 40 percent to 42 percent in 2006). However, in the United Kingdom, equity represents the main asset in pension funds, even as these funds decline (from 68 percent to 2003 to 61 percent in 2007). In the United Kingdom, the average allocation to bonds was 36 percent in 2007. Thus, the debate about the proper level of risk in pension plans persists; some economists argue that far more capital should go to bonds, in a strategy known as liability-driven investing (LDI). Although LDI seems similar to an insurance framework, its liabilities can depend on various factors, including inflation shifts, changes in the workforce, and so on. That means that implementing LDI is expensive. Other institutional investors thus believe equities are the essential ingredient for long-term investors because their returns depend on the economy's long-term growth. As even this brief summary reveals, the debate shows no signs of being resolved soon. In this article, we aim to derive an analytical solution to the ALM problem. We use the Wilkie (1995) investment model to illustrate our numerical results, which reveal that the nature of the guarantee type has a strong relationship with the asset allocation decisions. We know that matching relates to the selection of assets, which most closely resemble liability cash flows, whereas portfolio selection pertains to the selection of assets that will reach an optimal level of risk. Thus, we conclude that consols are the most important asset in the portfolio for constant liability (A1), ILGs are the most important for inflation-increasing liability (A2), and equities are the least important assets for both Type A1 and A2 liabilities because equity volatility is much higher than that for the rest of the assets, and the nature of the liabilities is much different. For any constant liability, we obtain exactly the same asset allocations for each asset, except for different initial investment amounts. In addition, the higher the return we aim for, the higher the proportion of risky assets we must hold. To reduce liquidity risk, we would gradually switch the proportion from risky assets to riskless assets, nearer the maturity date. The numerical results thus are almost entirely driven by underlying assumptions about market behavior. Different assumptions in the investment model certainly will lead to different results. We adopt different model assumptions to investigate the impact of these optimal asset allocations; we also investigate the importance of asset selection and find that it is critical for choosing suitable assets for ALM.

In this research, we have emphasized the derivation of an analytic solution for multiperiod ALM based on simulations. Pension and insurance guarantees have become popular in many countries, partly as a result of pension reforms and customer needs, which increase the importance of ALM for the pension fund and the insurer. This investigation does not address a specific complicated pension or insurance liability; doing so would require the consideration of many more issues, such as transaction costs and contribution strategies. It also can be difficult to cash match over time because future liabilities depend on uncertainties, including inflation, the size and composition of the workforce, and so on. Although we do not discuss a specific pension or insurance liability, our research successfully derives an analytic solution for the multiperiod asset allocation, using combinations of scenario-based and MV approaches in a discrete model. The advantage of the proposed analytic solution is that it is much easier to provide good insights into practice-based issues. Our proposed methodology also offers a viable alternative to practitioners. Several issues remain to be investigated in further studies to support efforts to apply this model to real-world problems, such as incorporating the downside risk in ALM or considering more practical liabilities or assumptions.

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