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Convergent dynamics for multistable delayed neural networks

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Abstract

This investigation aims at developing a methodology to establish convergence of dynamics for delayed neural network systems with multiple stable equilibria. The present approach is general and can be applied to several network models. We take the Hopfield-type neural networks with both instantaneous and delayed feedbacks to illustrate the idea. We shall construct the complete dynamical scenario which comprises exactly 2^n stable equilibria and exactly $(3^n - 2^n)$ unstable equilibria for the n-neuron network. In addition, it is shown that every solution of the system converges to one of the equilibria as time tends to infinity. The approach is based on employing the geometrical structure of the network system. Positively invariant sets and componentwise dynamical properties are derived under the geometrical configuration. An iteration scheme is subsequently designed to confirm the convergence of dynamics for the system. Two examples with numerical simulations are arranged to illustrate the present theory.

Mathematics Subject Classification: 34K20, 92B20

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Delay, as occurring in the propagation of action potentials along the axon, the transmission of signal across the synapse and the implementation of artificial neural networks, has been an important concern in the study of neural systems [1–4]. On the other hand, the global dynamics and the effect of time lags upon the dynamics have been interesting subjects in delayed systems [5]. In a delayed system with multiple equilibria, it is appealing to investigate how the basins of attraction for the stable equilibria are affected by delays, and to compare the

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dynamics of the system with those of the corresponding ODEs obtained from setting delays to zero [6]. However, mathematical analysis for the studies in these directions remains to be explored.

In this presentation, we develop a methodology to investigate convergence of dynamics for delayed neural network systems. In particular, we consider the following neural network with time-varying delays:

$$\dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} \beta_{ij} g_j(x_j(t - \tau_{ij}(t))) + J_i, \tag{1.1}$$

where $i = 1, 2, \ldots, n$, $\mu_i > 0$; α_{ij}, $\beta_{ij} \in \mathbb{R}$ denote the instantaneous feedback and delayed feedback connection strength from the ith to the jth unit; the time-dependent lags $\tau_{ij}(t) \geqslant 0$ are bounded continuous functions defined on $[t_0, +\infty)$, for some $t_0 \in \mathbb{R}$; $J_i \in \mathbb{R}$ correspond to the external bias; g_j are single-variable functions to be defined below. System (1.1) reduces to the classical and delayed Hopfield neural networks [7, 8], as $\beta_{ij} = 0$ and $\alpha_{ij} = 0$ for all i, j, respectively. It also represents the cellular neural networks without delays [9] and with delays [3].

There is a large amount of neural network theory in the literatures of applied mathematics, engineering, information science and applied physics, etc. Most of these studies focus on the existence of a unique equilibrium and the global convergence to the equilibrium, see [4, 10–15] and the references therein. On the other hand, 'multistability', a notion to describe coexistence of multiple stable equilibria or cycles, is essential in several applications of neural networks, including pattern recognition and associative memory storage [7, 16–18]. Recently, a systematic methodology on existence of multiple stationary solutions for the Hopfield neural network with or without delays has been reported in [19]. More precisely, the structure of single-neuron equation is employed to construct the existence of 3^n equilibria, 2^n positively invariant sets and basins of attraction for 2^n, among these 3^n, stable equilibria. However, there was no theoretical methodology to capture behaviour for solutions lying outside or crossing these basins, hence the global dynamical picture.

In the classical neural networks without delays and other ODEs, the typical treatment for studying the convergence of dynamics is to construct a Lyapunov function and apply LaSalle's invariant principle. There does not exist a global Lyapunov function or functional for system (1.1) with multiple equilibria, to the best of our knowledge. There does exist a global Lyapunov functional for the delayed Hopfield network with single equilibria and a local Lyapunov functional for the same system with multiple stable equilibria; for example,

$$V(\boldsymbol{x}_t) = \sum_{i=1}^{n} g_i(x_i(t))^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} |\beta_{ij}| \int_{t-\tau_{ij}}^{t} [g_j(x_j(s)) - g_j(\overline{x}_j)]^2 \, \mathrm{d}s,$$

where $\overline{\boldsymbol{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ is the equilibrium or one of the equilibria, cf [19]. There also exists a Lyapunov function $W(\boldsymbol{y}) = \frac{1}{2}\|\boldsymbol{y}\|^2$, for the delayed Hopfield network with a single equilibrium [15].

In this investigation, we develop a new treatment to conclude the convergence of dynamics for (1.1). Under this formulation, certain componentwise dynamical properties are derived and a subsequent iteration scheme is designed to confirm that every solution of the system converges to one of the equilibria as time tends to infinity. With this formulation, we justify that there exist exactly 2^n stable equilibria and exactly $(3^n - 2^n)$ unstable equilibria for (1.1). The conclusion for this existence of exact number of stable and unstable equilibria is new due to distinct treatment. Our results also improve the multistability theory in the cellular neural networks, for example, the one in [21]. The arguments presented for confirming stability of equilibria are nonstandard in delayed equations, as compared with the linearization with computation of the characteristic roots, and the Lyapunov function approach, employed in [22–25].

The present approach can also be applied to other additive networks, including the generalized Cohen–Grossberg's model [26,27], bi-directional associative memory model [28], Lotka–Volterra system [29] and the networks with distributed delay [27,30]. While the main idea of this work is developed to treat the systems with multiple stable equilibria, it can be applied to study monostability as well.

We shall present our main results for (1.1) with typical activation functions $g_i : \mathbb{R} \to \mathbb{R}$ in the following class:

$$\text{Class } \mathcal{A} : \begin{cases} g_i \in C^2, \quad \lim_{t\to+\infty} g_i(\xi) = v_i \in \mathbb{R}, \quad \lim_{t\to-\infty} g_i(\xi) = u_i \in \mathbb{R} \\ \exists\, \sigma_i \in \mathbb{R}, g_i'(\sigma_i) > g_i'(\xi) > 0, \quad \text{for } \xi \neq \sigma_i \quad \text{and} \quad g_i''(\xi)\cdot\xi < 0, \quad \text{for } \xi \neq \sigma_i. \end{cases}$$

These are bounded smooth sigmoidal functions and the commonly adopted ones are $g_i(\xi) = \tanh \xi$ and $g_i(\xi) = 1/[1 + \mathrm{e}^{-\xi/\varepsilon_i}]$ with $\varepsilon_i > 0$. Without loss of generality, we set $\sigma_i = 0$, for all i, throughout the presentation. Extension of the theory to other activation functions, including the piecewise linear ones, will be addressed in section 4. We denote the bounds for the activation functions, the slopes of the activation functions and the time lags by

$$\rho_i := \max\{|u_i|, |v_i|\}, \quad L_i := g_i'(0) \geqslant g_i'(\xi), \qquad \text{for all } \xi \in \mathbb{R} \tag{1.2}$$

$$\tau := \max_{1\leqslant i,j\leqslant n}\{\tau_{ij}\}, \quad \tau_{ij}(t) \leqslant \tau_{ij}, \qquad \text{for all } t \in [t_0, +\infty). \tag{1.3}$$

We start our formulation from the single-neuron equation in section 2. The propositions derived for the single-neuron equation will be used to develop componentwise dynamical properties for the coupled equations (1.1) in section 3. The main theorems of convergence of dynamics and stability of equilibria for the multidimensional system are presented in section 3. Extension of the theorems to other activation functions is arranged in section 4. We demonstrate this theory by two numerical examples in section 5.

2. Scalar equation with time-dependent input

In this section, as a preparation for the main theory in section 3, we consider the following scalar equation with time-dependent external input $w(t)$:

$$\dot{x}(t) = -\mu x(t) + \alpha g(x(t)) + \beta g(x(t-\tau_1(t)) + w(t), \tag{2.1}$$

where $\mu > 0, \alpha > 0$ and $\beta \in \mathbb{R}$; $\tau_1(t)$ is a continuous function with $0 \leqslant \tau_1(t) \leqslant \tau \in \mathbb{R}$, for all $t \geqslant t_0$; $w(t)$ is a bounded continuous function defined for $t \geqslant t_0$; g is an activation function of class $\mathcal{A}$ with $u < g(\xi) < v$, $g'(\xi) \leqslant L := g'(0) = \max\{g'(\eta) : \eta \in \mathbb{R}\}$, for all $\xi \in \mathbb{R}$, cf figure 1(a). Let $\rho = \max\{|u|, |v|\}$. We present the basic formations and propositions in section 2.1. The propositions derived herein lead to componentwise dynamical properties, and subsequently, the dynamical scenario in the whole phase space, for multidimensional system (1.1), in section 3. The proofs for these lemma and propositions are given in section 2.2.

2.1. Formulations and properties

The main result (theorem 2.4) in this section asserts that there exist three disjoint, bounded and closed intervals to which every solution of (2.1) converges, under certain parameter conditions. The assertions are derived by formulating successive sequences of upper and lower bounds for the motions at each advanced time step. The sequences of upper and lower bounds are then shown to contract to their limits, as time evolves.

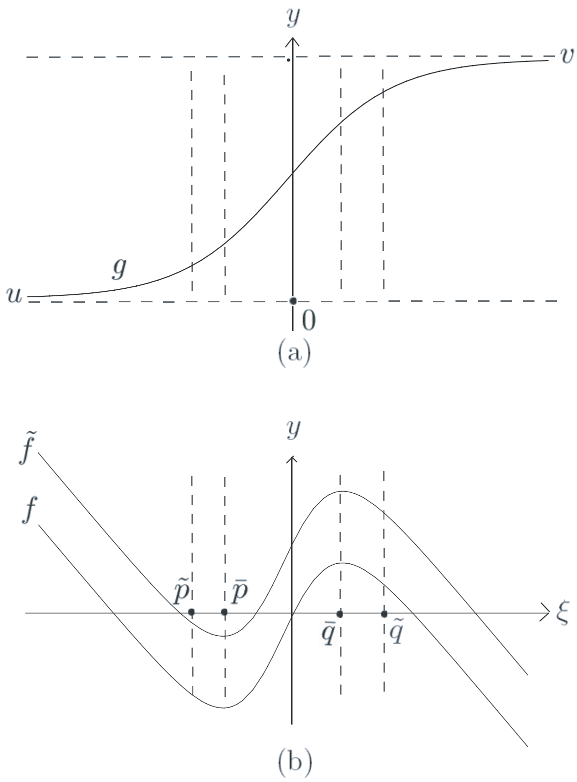

Figure 1. (*a*) The graph of activation function g of class $\mathcal{A}$. (*b*) The configurations for functions $\tilde{f}$, f with their critical points $\overline{p}, \overline{q}$, and the points $\tilde{p}, \tilde{q}$ at which g has designated slopes.

The first two conditions we impose on activation function g and parameters are

Condition (A1): $L > 2\mu/\alpha > 0$,
Condition (A2): $L < \mu/|\beta|$.

Let us define $f(\xi) := -\mu\xi + \alpha g(\xi)$, where g is the same as in (2.1). Then, $\tilde{f}'(\xi) = -\mu + \alpha g'(\xi)$, for any vertical shift $\tilde{f}$ of f. If condition (A1) holds, there exist exactly two points $\overline{p}, \overline{q}$ with $\overline{p} < 0 < \overline{q}$ such that $\tilde{f}'(\overline{p}) = \tilde{f}'(\overline{q}) = 0$; $\tilde{f}'(\xi) > 0$ for $\xi \in (\overline{p}, \overline{q})$; and $\tilde{f}'(\xi) < 0$ for $\xi \in \mathbb{R} - [\overline{p}, \overline{q}]$. Restated, if $g'(0) > \mu/\alpha$, then $\overline{p}$ and $\overline{q}$ are the only two critical points of $\tilde{f}$, and $g'(\overline{p}) = g'(\overline{q}) = \mu/\alpha$, cf figure 1(*b*). In addition, conditions (A1) and (A2) imply $0 < (\mu - L|\beta|)/(\alpha + |\beta|) < \mu/\alpha$. Thus, there always exist two points $\tilde{p}$ and $\tilde{q}$, where $\tilde{p} < \overline{p} < \overline{q} < \tilde{q}$ such that

$$g'(\tilde{p}) = g'(\tilde{q}) = \frac{\mu - L|\beta|}{\alpha + |\beta|}.$$

We shall formulate the desired configuration and properties for equation (2.1) through the following quantities and functions. For $T \geqslant t_0$, let

$$w^{\min}(T) := \inf\{w(t) \mid t \geqslant T\}, \qquad w^{\max}(T) := \sup\{w(t) \mid t \geqslant T\},$$

$$\hat{f}^{(0)}(\xi, T) := -\mu\xi + \alpha g(\xi) + |\beta|\rho + w^{\max}(T),$$

$$\check{f}^{(0)}(\xi, T) := -\mu\xi + \alpha g(\xi) - |\beta|\rho + w^{\min}(T).$$

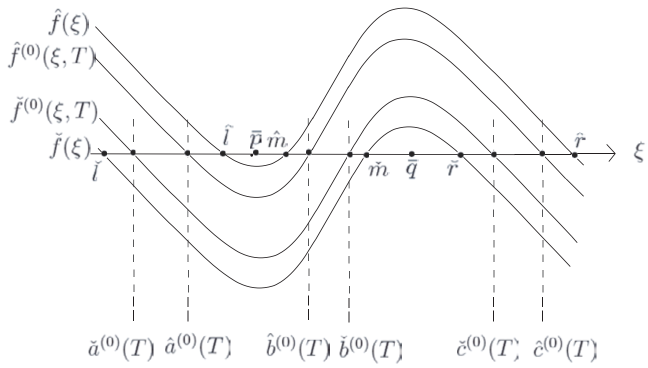

Figure 2. Configurations of functions $\hat{f}$, $\hat{f}^{(0)}$, $\check{f}^{(0)}$ and $\check{f}$, for fixed $T \geqslant t_0$.

For convenience of later use, we denote

$$\hat{f}(\xi) := \hat{f}^{(0)}(\xi, t_0),\ \check{f}(\xi) := \check{f}^{(0)}(\xi, t_0). \tag{2.2}$$

Notably, $\hat{f}$ and $\check{f}$ are also vertical shifts of f. Let us introduce the third condition.

Condition (A3): $\check{f}(\tilde{q}) > 0$, $\hat{f}(\tilde{p}) < 0$.

Under conditions (A1)–(A3), there exist three solutions $\hat{l}$, $\hat{m}$ and $\hat{r}$ (respectively, $\check{l}$, $\check{m}$ and $\check{r}$) of $\hat{f}(\xi) = 0$ (respectively, $\check{f}(\xi) = 0$). Moreover, $\check{l} < \hat{l} < \tilde{p} < \overline{p} < \hat{m} < \check{m} < \overline{q} < \tilde{q} < \check{r} < \hat{r}$. We further impose a slope condition on the middle part of the activation function. This condition actually covers (A1).

Condition (A4): $g'(\xi) > 2\mu/\alpha$, for all $\xi \in [\hat{m}, \check{m}]$.

Let $\check{a}^{(0)}(T)$ (respectively, $\check{b}^{(0)}(T)$, $\check{c}^{(0)}(T)$) be the unique solution of $\check{f}^{(0)}(\cdot, T) = 0$ lying in interval $[\check{l}, \hat{l}]$ (respectively, $[\hat{m}, \check{m}]$, $[\check{r}, \hat{r}]$), and $\hat{a}^{(0)}(T)$ (respectively, $\hat{b}^{(0)}(T)$, $\hat{c}^{(0)}(T)$) be the unique solution of $\hat{f}^{(0)}(\cdot, T) = 0$ lying in $[\check{l}, \hat{l}]$ (respectively, $[\hat{m}, \check{m}]$, $[\check{r}, \hat{r}]$), cf figure 2. The following functions can be defined iteratively for each fixed $T \geqslant t_0$: for $k \in \mathbb{N}$,

$$\hat{f}_{\mathrm{l}}^{(k)}(\xi, T) := \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\hat{a}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta \geqslant 0,\\ -\mu\xi + \alpha g(\xi) + \beta g(\check{a}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta < 0, \end{cases}$$

$$\check{f}_{\mathrm{l}}^{(k)}(\xi, T) := \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\check{a}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta \geqslant 0,\\ -\mu\xi + \alpha g(\xi) + \beta g(\hat{a}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta < 0, \end{cases}$$

$$\hat{f}_{\mathrm{m}}^{(k)}(\xi, T) := \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\check{b}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta \geqslant 0,\\ -\mu\xi + \alpha g(\xi) + \beta g(\hat{b}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta < 0, \end{cases}$$

$$\check{f}_{\mathrm{m}}^{(k)}(\xi, T) := \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\hat{b}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta \geqslant 0,\\ -\mu\xi + \alpha g(\xi) + \beta g(\check{b}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta < 0, \end{cases}$$

$$\hat{f}_{\mathrm{r}}^{(k)}(\xi, T) := \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\hat{c}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta \geqslant 0,\\ -\mu\xi + \alpha g(\xi) + \beta g(\check{c}^{(k-1)}(T)) + w^{\max}(T), & \text{for } \beta < 0, \end{cases}$$

$$\check{f}_{\mathrm{r}}^{(k)}(\xi, T) := \begin{cases} -\mu\xi + \alpha g(\xi) + \beta g(\check{c}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta \geqslant 0,\\ -\mu\xi + \alpha g(\xi) + \beta g(\hat{c}^{(k-1)}(T)) + w^{\min}(T), & \text{for } \beta < 0. \end{cases}$$

These functions are all vertical shifts of f for each fixed T. Herein, $\check{a}^{(k)}(T)$ (respectively, $\check{b}^{(k)}(T)$, $\check{c}^{(k)}(T)$) is the unique solution of $\check{f}_{\mathrm{l}}^{(k)}(\cdot, T) = 0$ (respectively, $\check{f}_{\mathrm{m}}^{(k)}(\cdot, T) = 0$, $\check{f}_{\mathrm{r}}^{(k)}(\cdot, T) = 0$) lying in interval $[\check{l}, \hat{l}]$ (respectively, $[\hat{m}, \check{m}]$, $[\check{r}, \hat{r}]$), and $\hat{a}^{(k)}(T)$ (respectively, $\hat{b}^{(k)}(T)$, $\hat{c}^{(k)}(T)$) is the unique solution of $\hat{f}_{\mathrm{l}}^{(k)}(\cdot, T) = 0$ (respectively, $\hat{f}_{\mathrm{m}}^{(k)}(\cdot, T) = 0$, $\hat{f}_{\mathrm{r}}^{(k)}(\xi, T) = 0$) lying in $[\check{l}, \hat{l}]$ (respectively, $[\hat{m}, \check{m}]$, $[\check{r}, \hat{r}]$). We also define $w^{\min}(\infty) := \lim_{T\to\infty} w^{\min}(T)$, $w^{\max}(\infty) := \lim_{T\to\infty} w^{\max}(T)$.

The following lemma summarizes the properties for zeros of the above-defined sequences of single-variable functions.

Lemma 2.1. *Assume that conditions (A2)–(A4) hold. Then, for each $T \geqslant t_0$, the sequences $\{\check{b}^{(k)}(T)\}_{k\geqslant 0}$, $\{\hat{b}^{(k)}(T)\}_{k\geqslant 0}$ $\{\check{a}^{(k)}(T)\}_{k\geqslant 0}$, $\{\hat{a}^{(k)}(T)\}_{k\geqslant 0}$, $\{\check{c}^{(k)}(T)\}_{k\geqslant 0}$, $\{\hat{c}^{(k)}(T)\}_{k\geqslant 0}$ can be defined iteratively. Moreover,*

- *(i) for any fixed $k \in \mathbb{N} \cup \{0\}$, each of $\hat{b}^{(k)}(T)$, $\check{a}^{(k)}(T)$ and $\check{c}^{(k)}(T)$ is increasing, and each of $\check{b}^{(k)}(T)$, $\hat{a}^{(k)}(T)$, and $\hat{c}^{(k)}(T)$ is decreasing with respect to $T \geqslant t_0$;*
- *(ii) for any $T \geqslant t_0$, there exist $\underline{b}(T), \overline{b}(T), \underline{a}(T), \overline{a}(T), \underline{c}(T), \overline{c}(T) \in \mathbb{R}$ such that $\hat{b}^{(k)}(T) \to \underline{b}(T)$, $\check{a}^{(k)}(T) \to \underline{a}(T)$, and $\check{c}^{(k)}(T) \to \underline{c}(T)$ increasingly, and $\check{b}^{(k)}(T) \to \overline{b}(T)$, $\hat{a}^{(k)}(T) \to \overline{a}(T)$, and $\hat{c}^{(k)}(T) \to \overline{c}(T)$ decreasingly, as $k \to \infty$;*
- *(iii) there exist $\underline{b}, \overline{b}, \underline{a}, \overline{a}, \underline{c}, \overline{c} \in \mathbb{R}$, such that $\underline{b}(T) \to \underline{b}$, $\underline{a}(T) \to \underline{a}$, $\underline{c}(T) \to \underline{c}$ increasingly and $\overline{b}(T) \to \overline{b}$, $\overline{a}(T) \to \overline{a}$, $\overline{c}(T) \to \overline{c}$ decreasingly, as $T \to \infty$;*
- *(iv) $\cap_{T\geqslant t_0}[\underline{b}(T), \overline{b}(T)] = [\underline{b}, \overline{b}]$, $\cap_{T\geqslant t_0}[\underline{a}(T), \overline{a}(T)] = [\underline{a}, \overline{a}]$, $\cap_{T\geqslant t_0}[\underline{c}(T), \overline{c}(T)] = [\underline{c}, \overline{c}]$;*
- *(v) $0 \leqslant \overline{b}(T) - \underline{b}(T) \leqslant [w^{\max}(T) - w^{\min}(T)]/(\mu - |\beta|L)$, $0 \leqslant \overline{a}(T) - \underline{a}(T), \overline{c}(T) - \underline{c}(T) \leqslant [w^{\max}(T) - w^{\min}(T)]/(|\beta|L)$, for any $T \geqslant t_0$, moreover*

$$0 \leqslant d_b := \overline{b} - \underline{b} \leqslant \frac{w^{\max}(\infty) - w^{\min}(\infty)}{\mu - |\beta|L},$$

$$0 \leqslant d_a := \overline{a} - \underline{a}, \qquad d_c := \overline{c} - \underline{c} \leqslant \frac{w^{\max}(\infty) - w^{\min}(\infty)}{|\beta|L}.$$

In the following discussions, for an initial value $\phi \in C([-\tau, 0], \mathbb{R})$, we denote by $x(t) = x(t; t_0; \phi)$ the solution of (2.1) with $x(t_0 + \theta; t_0; \phi) = \phi(\theta)$, for $\theta \in [-\tau, 0]$.

Definition 2.1. *A solution $x(t)$ of (2.1) is said to satisfy property $\mathcal{M}$, $\mathcal{L}$, $\mathcal{R}$, if, respectively,*

for each $k \in \mathbb{N} \cup \{0\}$, $T \geqslant t_0$, $x(t) \in [\hat{b}^k(T), \check{b}^k(T)]$, for all $t \geqslant T + k\tau$,
there exists $s \geqslant t_0$ such that $x(s) \in [\check{l}, \hat{l}]$,
there exists $s \geqslant t_0$ such that $x(s) \in [\check{r}, \hat{r}]$.

Proposition 2.2. *Assume that conditions (A2)–(A4) hold.*

- *(i) If $x(t)$ is a solution of (2.1) and for any fixed $T \geqslant t_0$, $k \in \mathbb{N}$, $x(t) \in [\hat{b}^{k-1}(T), \check{b}^{k-1}(T)]$ for all $t \geqslant T + (k-1)\tau$, then $x(t) \in [\hat{b}^k(T), \check{b}^k(T)]$, for all $t \geqslant T + k\tau$;*
- *(ii) If $x(t)$ is a solution of (2.1) and $x(s) > \check{b}^{(0)}(T)$ (respectively, $x(s) < \hat{b}^{(0)}(T)$), for some $s \geqslant T \geqslant t_0$, then $x(t)$ satisfies property $\mathcal{R}$ (respectively, $\mathcal{L}$);*
- *(iii) If the solution $x(t)$ of (2.1) satisfies property $\mathcal{M}$, then $x(t) \to [\underline{b}(T), \overline{b}(T)]$ as $t \to \infty$, for any $T \geqslant t_0$; subsequently, $x(t) \to [\underline{b}, \overline{b}]$ as $t \to \infty$.*
- *(iv) Each of $[\check{l}, \hat{l}]$ and $[\check{r}, \hat{r}]$ is a positively invariant interval for (2.1). Moreover, if $x(t)$ is a solution of (2.1), which satisfies property $\mathcal{R}$ (respectively, $\mathcal{L}$), then $x(t) \to [\underline{c}, \overline{c}]$ (respectively, $[\underline{a}, \overline{a}]$), as $t \to \infty$.*

Proposition 2.3. *Assume that conditions (A2)–(A4) hold. Every solution $x(t)$ of (2.1) satisfies one of properties $\mathcal{M}$, $\mathcal{L}$, $\mathcal{R}$.*

Proof. Let $x(t)$ be a solution of (2.1) which does not satisfy property $\mathcal{M}$. Then there exist $k \in \mathbb{N} \cup \{0\}$, $T \geqslant t_0$, such that

$$x(t) \in \mathbb{R} - [\hat{b}^k(T), \check{b}^k(T)], \qquad \text{for some } t \geqslant T + k\tau. \tag{2.3}$$

Set $\mathcal{K} := \{(k, T) : k \in \mathbb{N} \cup \{0\}, T \geqslant t_0, \text{ and (2.3) holds}\}$, $k_0 := \min\{k : \text{there exists } T \geqslant t_0$ such that $(k, T) \in \mathcal{K}\}$. There are two possibilities: $k_0 \geqslant 1$ and $k_0 = 0$.

Case (i): If $k_0 \geqslant 1$, then for any $T \geqslant t_0$,

$$x(t) \in [\hat{b}^{k_0-1}(T), \check{b}^{k_0-1}(T)], \qquad \text{for all } t \geqslant T + (k_0 - 1)\tau.$$

It follows from proposition 2.2(i) that $x(t) \in [\hat{b}^{k_0}(T), \check{b}^{k_0}(T)]$, for all $t \geqslant T + k_0\tau$, which is a contradiction to the definition of k_0.

Case (ii): If $k_0 = 0$, then there exist $T \geqslant t_0$ and $t \geqslant T$ such that $x(t) \in \mathbb{R} - [\hat{b}^{(0)}(T), \check{b}^{(0)}(T)]$. $x(t)$ then satisfies property $\mathcal{L}$ or $\mathcal{R}$, according to proposition 2.2(ii). □

Combining proposition 2.2(iii) and (iv) and proposition 2.3, we conclude the main result in this section.

Theorem 2.4. *Assume that conditions (A2)–(A4) hold. Let $x(t)$ be a solution of (2.1). Then $x(t) \to [\underline{a}, \overline{a}]$, or $[\underline{b}, \overline{b}]$, or $[\underline{c}, \overline{c}]$, as $t \to \infty$.*

2.2. *Proofs of lemma and propositions*

We only prove the case of $\beta > 0$, as the arguments for $\beta \leqslant 0$ are similar.

Proof of lemma 2.1. The labelling in the proof corresponds to the one in the statement of lemma 2.1.

(i) Let us show that for any $T \geqslant t_0$, $\check{b}^{(k)}(T)$ and $\hat{b}^{(k)}(T)$ are well-defined for all $k \in \mathbb{N} \cup \{0\}$. Assume that $\hat{b}^{(j-1)}(T)$, $\check{b}^{(j-1)}(T)$ have been defined, for a fixed $T \geqslant t_0$. Notably,

$$\begin{aligned}\hat{f}_{\mathrm{m}}^{(j)}(\xi, T) &= -\mu\xi + \alpha g(\xi) + \beta g(\check{b}^{(j-1)}(T)) + w^{\max}(T)\\ &\leqslant -\mu\xi + \alpha g(\xi) + \beta\rho + w^{\max}(t_0) = \hat{f}(\xi),\\ \check{f}_{\mathrm{m}}^{(j)}(\xi, T) &= -\mu\xi + \alpha g(\xi) - \beta g(\hat{b}^{(j-1)}(T)) + w^{\min}(T)\\ &\geqslant -\mu\xi + \beta g(\xi) - \beta\rho + w^{\min}(t_0) = \check{f}(\xi).\end{aligned}$$

It follows that $\check{f}(\xi) \leqslant \check{f}_{\mathrm{m}}^{(j)}(\xi, T) \leqslant \hat{f}_{\mathrm{m}}^{(j)}(\xi, T) \leqslant \hat{f}(\xi)$, for all $\xi \in \mathbb{R}$. In addition, $\overline{p}$ and $\overline{q}$ are two critical points of $\check{f}(\cdot)$, $\hat{f}(\cdot)$, $\hat{f}_{\mathrm{m}}^{(j)}(\cdot, T)$, and $\check{f}_{\mathrm{m}}^{(j)}(\cdot, T)$, and $g'(\overline{p}) = g'(\overline{q}) = \mu/\alpha$, due to condition (A1). There exists exactly one solution for each of $\hat{f}_{\mathrm{m}}^{(j)}(\cdot, T) = 0$ and $\check{f}_{\mathrm{m}}^{(j)}(\cdot, T) = 0$ in interval $(\hat{m}, \check{m})$. Accordingly, both $\check{b}^{(j)}(T)$ and $\hat{b}^{(j)}(T)$ are well defined. Moreover, it is straightforward to observe that $\hat{b}^{(j)}(T_1) \geqslant \hat{b}^{(j)}(T_2)$ and $\check{b}^{(j)}(T_1) \leqslant \check{b}^{(j)}(T_2)$, due to $\hat{f}_{\mathrm{m}}^{(j)}(\cdot, T_1) \leqslant \hat{f}_{\mathrm{m}}^{(j)}(\cdot, T_2)$ and $\check{f}_{\mathrm{m}}^{(j)}(\cdot, T_1) \geqslant \check{f}_{\mathrm{m}}^{(j)}(\cdot, T_2)$, for any $T_1 \geqslant T_2 \geqslant t_0$. Thus, for each $k \in \mathbb{N} \cup \{0\}$, $\hat{b}^{(k)}(T)$ increases and $\check{b}^{(k)}(T)$ decreases, with respect to T. The arguments for $\check{a}^{(k)}(T)$, $\hat{a}^{(k)}(T)$, $\check{c}^{(k)}(T)$, $\hat{c}^{(k)}(T)$ are similar.

(ii) Let us show that for each $T \geqslant t_0$,

$$\hat{b}^{(k+1)}(T) \geqslant \hat{b}^{(k)}(T); \check{b}^{(k+1)}(T) \leqslant \check{b}^{(k)}(T), \qquad \text{for all } k \geqslant 0. \tag{2.4}$$

Assume that (2.4) holds for some $k = j - 1$. Notably, $\hat{b}^{(j+1)}(T)$ and $\hat{b}^{(j)}(T)$ satisfy $\hat{f}_{\mathrm{m}}^{(j+1)}(\cdot, T) = 0$ and $\hat{f}_{\mathrm{m}}^{(j)}(\cdot, T) = 0$ respectively; i.e.

$$-\mu\hat{b}^{(j+1)}(T) + \alpha g(\hat{b}^{(j+1)}(T)) + \beta g(\check{b}^{(j)}(T)) + w^{\max}(T) = 0, \tag{2.5}$$

$$-\mu\hat{b}^{(j)}(T) + \alpha g(\hat{b}^{(j)}(T)) + \beta g(\check{b}^{(j-1)}(T)) + w^{\max}(T) = 0. \tag{2.6}$$

The difference of (2.5) and (2.6) is

$$\mu[\hat{b}^{(j+1)}(T) - \hat{b}^{(j)}(T)] - \alpha g'(\xi)[\hat{b}^{(j+1)}(T) - \hat{b}^{(j)}(T)] = \beta g'(\zeta)[\check{b}^{(j)}(T) - \check{b}^{(j-1)}(T)], \tag{2.7}$$

where ξ (respectively, ζ) is a number between $\hat{b}^{(j+1)}(T)$ and $\hat{b}^{(j)}(T)$ (respectively, $\hat{b}^{(j)}(T)$ and $\hat{b}^{(j-1)}(T)$). (2.7) then yields

$$\hat{b}^{(j+1)}(T) - \hat{b}^{(j)}(T) = \frac{\beta g'(\zeta)[\check{b}^{(j)}(T) - \check{b}^{(j-1)}(T)]}{\mu - \alpha g'(\xi)} \geqslant 0,$$

due to that $g'(\xi) > \mu/\alpha$ for ξ between $\hat{b}^{(j+1)}(T)$ and $\hat{b}^{(j)}(T)$. Thus, the first part of (2.4) holds for $k = j$. The second part can be proved similarly. It follows that for any $T \geqslant t_0$, $\lim_{k\to\infty} \hat{b}^{(k)}(T) = \underline{b}(T) \in \mathbb{R}$, and $\lim_{k\to\infty} \check{b}^{(k)}(T) = \overline{b}(T) \in \mathbb{R}$, respectively, since both of $\check{b}^{(k)}(T)$ and $\hat{b}^{(k)}(T)$ are bounded monotone sequences. The situations for $\check{a}^{(k)}(T)$, $\hat{a}^{(k)}(T)$, and $\check{c}^{(k)}(T)$, $\hat{c}^{(k)}(T)$ are similar.

(iii) For each $k \in \mathbb{N} \cup \{0\}$, it has been shown in (i) that $\hat{b}^{(k)}(T_2) \leqslant \hat{b}^{(k)}(T_1)$, if $T_1 > T_2 \geqslant t_0$. Thus, $\lim_{k\to\infty} \hat{b}^{(k)}(T_2) \leqslant \lim_{k\to\infty} \hat{b}^{(k)}(T_1)$, i.e. $\underline{b}(T_2) \leqslant \underline{b}(T_1)$. Therefore, $\underline{b}(T) \to \underline{b} \in \mathbb{R}$ increasingly as $T \to \infty$, since $\underline{b}(T)$ is bounded above for all $T \geqslant t_0$. Similarly, $\overline{b}(T) \to \overline{b} \in \mathbb{R}$ decreasingly as $T \to \infty$. Similar proofs apply to $\underline{a}(T) \to \underline{a}$, $\overline{a}(T) \to \overline{a}$, $\underline{c}(T) \to \underline{c}$ and $\overline{c}(T) \to \overline{c}$.

(iv) It is straightforward to see that $\cap_{T\geqslant t_0}[\underline{b}(T), \overline{b}(T)] = [\underline{b}, \overline{b}]$, $\cap_{T\geqslant t_0}[\underline{a}(T), \overline{a}(T)] = [\underline{a}, \overline{a}]$, $\cap_{T\geqslant t_0}[\underline{c}(T), \overline{c}(T)] = [\underline{c}, \overline{c}]$.

(v) It is obvious that $\overline{b}(T) - \underline{b}(T) \geqslant 0$, since $\check{b}^{(k)}(T) > \hat{b}^{(k)}(T)$ for any $k \in \mathbb{N} \cup \{0\}$, and any $T \geqslant t_0$. Next, we justify that $\overline{b}(T) - \underline{b}(T) \leqslant [w^{\max}(T) - w^{\max}(T)]/(\mu - |\beta|L)$, for any $T \geqslant t_0$. For such an assertion, we shall construct a mapping $\Gamma_T : H_T \to H_T$, for each $T \geqslant t_0$, where $H_T := [\hat{b}^{(0)}(T), \check{b}^{(0)}(T)] \times [\hat{b}^{(0)}(T), \check{b}^{(0)}(T)] \cap \{(y_1, y_2) | y_1 \leqslant y_2\} \subset \mathbb{R}^2$ and such a mapping is a contraction, mainly due to $g' > 2\mu/\alpha$, on $[\hat{b}^{(0)}(T), \check{b}^{(0)}(T)]$. The map Γ_T thus admits a unique fixed point $(\underline{b}(T), \overline{b}(T))$. The difference of $\underline{b}(T)$ and $\overline{b}(T)$ can then be estimated to yield the assertion. Let us elaborate. For each $T \geqslant t_0$, we define the following functions:

$$h_T^{\max}(\xi, \gamma) := -\mu\xi + \alpha g(\xi) + \beta g(\gamma) + w^{\max}(T),$$
$$h_T^{\min}(\xi, \gamma) := -\mu\xi + \alpha g(\xi) + \beta g(\gamma) + w^{\min}(T).$$

Notably, $\check{f}_{\mathrm{m}}^{(1)}(\xi, T) \leqslant h_T^{\min}(\xi, \gamma_1) \leqslant h_T^{\max}(\xi, \gamma_2) \leqslant \hat{f}_{\mathrm{m}}^{(1)}(\xi, T)$, if $\hat{b}^{(0)}(T) \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \check{b}^{(0)}(T)$. For $(\xi, \gamma) \in H_T$, we define $\Gamma_T(\xi, \gamma) = (\xi_s, \gamma_s)$, where ξ_s (respectively, γ_s) is the unique point lying in $[\hat{b}^{(0)}(T), \check{b}^{(0)}(T)]$ satisfying $h_T^{\max}(\xi_s, \gamma) = 0$ (respectively, $h_T^{\min}(\gamma_s, \xi) = 0$). Suppose $h_T^{\max}(\xi_s, \gamma) = 0$, $h_T^{\max}(\xi_s', \gamma') = 0$, then we derive

$$\mu(\xi_s' - \xi_s) - \alpha g'(\eta)(\xi_s' - \xi_s) + \beta[g(\gamma) - g(\gamma')] = 0,$$

where η is between ξ_s' and ξ_s. Subsequently, $|\xi_s' - \xi_s| \leqslant |\beta|L|\gamma' - \gamma|/[\alpha(2\mu/\alpha) - \mu] = |\beta|L|\gamma' - \gamma|/\mu$, thanks to $g'(\eta) > 2\mu/\alpha$, for $\eta \in [\hat{b}^{(0)}(T), \check{b}^{(0)}(T)] \subset [\hat{b}^{(0)}(t_0), \check{b}^{(0)}(t_0)] = [\hat{m}, \check{m}]$. Similarly, we can prove that $|\gamma_s' - \gamma_s| \leqslant |\beta|L|\xi - \xi'|/\mu$, if $h_T^{\min}(\gamma_s, \xi) = 0$, $h_T^{\min}(\gamma_s', \xi') = 0$. We thus establish

$$\|\Gamma_T(\xi, \gamma) - \Gamma_T(\xi', \gamma')\|_\infty = \|(\xi_s, \gamma_s) - (\xi_s', \gamma_s')\|_\infty \leqslant \frac{|\beta|L}{\mu} \|(\xi, \gamma) - (\xi', \gamma')\|_\infty.$$

Γ_T is thus a contracting mapping under our condition (A2): $L < \mu/|\beta|$. Thus, there exists a unique fixed point of Γ_T in H_T. Observe that $\Gamma_T^k(\hat{b}^{(0)}(T), \check{b}^{(0)}(T)) = (\hat{b}^{(k)}(T), \check{b}^{(k)}(T))$, which converges to $(\underline{b}(T), \overline{b}(T))$ as $k \to \infty$. Thus $(\underline{b}(T), \overline{b}(T)) \in H_T$ is the fixed point of Γ_T and

$$-\mu\underline{b}(T) + \alpha g(\underline{b}(T)) + \beta g(\overline{b}(T)) + w^{\max}(T) = 0,$$
$$-\mu\overline{b}(T) + \alpha g(\overline{b}(T)) + \beta g(\underline{b}(T)) + w^{\min}(T) = 0.$$

Therefore,

$$\begin{aligned}\overline{b}(T) - \underline{b}(T) &= [w^{\max}(T) - w^{\min}(T)]/[\alpha g'(\xi) - |\beta|g'(\xi) - \mu] \\ &\leqslant [w^{\max}(T) - w^{\min}(T)]/[\mu - |\beta|g'(\xi)] \\ &\leqslant [w^{\max}(T) - w^{\min}(T)]/[\mu - |\beta|L],\end{aligned}$$

due to condition (A4): $g'(\xi) \geqslant 2\mu/\alpha$ for $\xi \in [\underline{b}(T), \overline{b}(T)] \subset [\hat{b}^{(0)}(T), \check{b}^{(0)}(T)] \subset [\hat{m}, \check{m}]$, condition (A2): $\mu - |\beta|L > 0$, and $g'(\xi) \leqslant L$, for all ξ. Moreover, $\overline{b} - \underline{b} \leqslant \overline{b}(T) - \underline{b}(T) \leqslant [w^{\max}(T) - w^{\min}(T)]/[\mu - |\beta|L]$, for any $T \geqslant t_0$. We thus establish

$$0 \leqslant d_b := \overline{b} - \underline{b} \leqslant [w^{\max}(\infty) - w^{\min}(\infty)]/[\mu - |\beta|L].$$

The estimate for $\overline{c}(T) - \underline{c}(T)$ follows from

$$\mu[\overline{c}(T) - \underline{c}(T)] - \alpha g'(\xi)[\overline{c}(T) - \underline{c}(T)] - \beta g'(\xi)[\overline{c}(T) - \underline{c}(T)] + w^{\max}(T) - w^{\min}(T) = 0,$$

for some $\xi \in [\check{c}^{(0)}(T), \hat{c}^{(0)}(T)]$, and

$$\overline{c}(T) - \underline{c}(T) \leqslant \frac{w^{\max}(T) - w^{\min}(T)}{\mu - (\alpha + |\beta|)g'(\tilde{q})} = \frac{w^{\max}(T) - w^{\min}(T)}{|\beta|L}.$$

The estimate for $\overline{a}(T) - \underline{a}(T)$ is similar. The bounds for $\overline{a} - \underline{a}$, and $\overline{c} - \underline{c}$ can then be derived.

Proof of proposition 2.2.

(i) Assume that $x(t) \in [\hat{b}^{(k-1)}(T), \check{b}^{(k-1)}(T)]$, for all $t \geqslant T + (k-1)\tau$. Then it is not difficult to derive that $\check{f}_{\mathrm{m}}^{(k)}(x(t), T) \leqslant \dot{x}(t) \leqslant \hat{f}_{\mathrm{m}}^{(k)}(x(t), T)$ for $t \geqslant T + k\tau$. Therefore, if the assertion does not hold, $x(t)$ eventually leaves $[\hat{b}^{(k-1)}(T), \check{b}^{(k-1)}(T)]$ after $t = T + k\tau$, and yields a contradiction, cf figure 3. For a detailed proof, let us suppose the assertion does not hold, then there exists some $s \geqslant T + k\tau$ such that $x(s) \in [\hat{b}^{(k-1)}(T), \check{b}^{(k-1)}(T)] - [\hat{b}^{(k)}(T), \check{b}^{(k)}(T)]$. Suppose that $x(s) \in (\check{b}^{(k)}(T), \check{b}^{(k-1)}(T)] \neq \emptyset$ (the case $x(s) \in (\hat{b}^{(k-1)}(T), \hat{b}^{(k)}(T)] \neq \emptyset$ can be similarly discussed). Notably, $\check{f}_{\mathrm{m}}^{(k)}(\xi, T) := -\mu\xi + \alpha g(\xi) + \beta g(\hat{b}^{(k-1)}(T)) + w^{\min}(T) \geqslant \check{f}_{\mathrm{m}}^{(k)}(x(s), T) =: h_1 > 0$, for all $\xi \in [x(s), \check{b}^{(k-1)}(T)]$, with respect to the definition of $\check{f}_{\mathrm{m}}^{(k)}(\xi, T)$, cf figure 3. In addition,

$$\begin{aligned}\dot{x}(s) &= -\mu x(s) + \alpha g(x(s)) + \beta g(x(s - \tau_1(s))) + w(s) \\ &\geqslant -\mu x(s) + \alpha g(x(s)) + \beta g(\hat{b}^{(k-1)}(T)) + w^{\min}(T) \\ &= \check{f}_{\mathrm{m}}^{(k)}(x(s), T) = h_1 > 0,\end{aligned}$$

due to $s - \tau_1(s) \geqslant T + (k-1)\tau$. Therefore, $x(t)$ enters $(x(s), \check{b}^{k-1}(T)]$ after $t = s$, and will never go back into $(-\infty, x(s)]$ again. Indeed, if there exists a time $s_1 > s$, such that $x(t) \in (x(s), \check{b}^{(k-1)}(T))$ for all $t \in (s, s_1)$, and $x(s_1) = x(s)$, then,

$$\begin{aligned}x(s_1) - x(s) &= \dot{x}(\tilde{s})(s_1 - s) \\ &= [-\mu x(\tilde{s}) + \alpha g(x(\tilde{s})) + \beta g(x(\tilde{s} - \tau_1(\tilde{s}))) + w(\tilde{s})](s_1 - s) \\ &\geqslant [-\mu x(\tilde{s}) + \alpha g(x(\tilde{s})) + \beta g(\hat{b}^{(k-1)}(T)) + w^{\min}(T)](s_1 - s) \\ &= [\check{f}_{\mathrm{m}}^{(k)}(x(\tilde{s}), T)](s_1 - s) \geqslant h_1 \cdot (s_1 - s) > 0,\end{aligned}$$

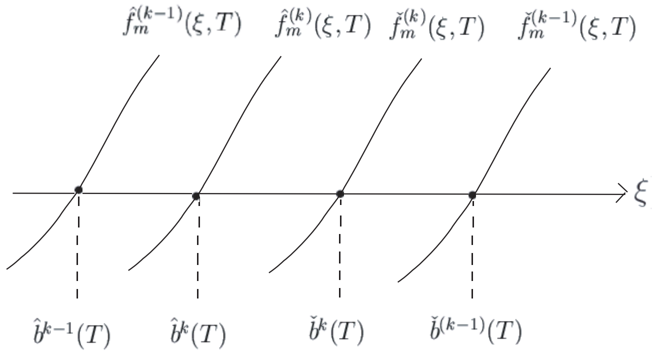

Figure 3. Configuration for the proof of proposition 2.2(i), for some $T \geqslant t_0$.

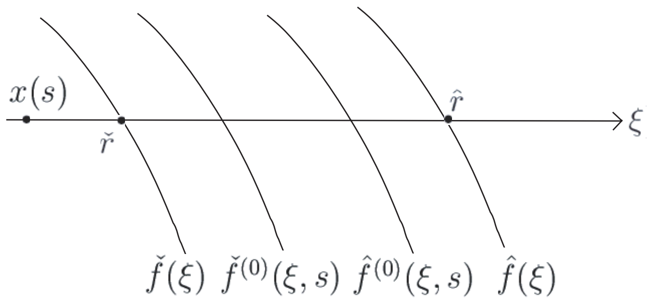

Figure 4. Configuration for the proof of proposition 2.2(ii), for some $s \geqslant T$.

for some $\tilde{s} \in (s, s_1)$, which is a contradiction. Thus, $x(t)$ stays in $[x(s), \check{b}^{k-1}(T)]$ for all $t \geqslant s$ with $\dot{x}(t) \geqslant h_1 > 0$. This is impossible and we conclude that $x(t) \in [\hat{b}^{(k)}(T), \check{b}^{(k)}(T)]$ for all $t \geqslant T + k\tau$.

(ii) We only prove the $\mathcal{R}$ case. This property holds mainly due to $\check{f}^{(0)}(x(t), T) \leqslant \dot{x}(t) \leqslant \hat{f}^{(0)}(x(t), T)$, for $t \geqslant T$. Therefore, if $x(s) \in (\check{b}^{(0)}(T), \infty)$ (respectively, $(-\infty, \hat{b}^{(0)}(T))$) for some $s \geqslant T$, then $x(t)$ eventually enters $[\check{r}, \hat{r}]$ (respectively, $[\check{l}, \hat{l}]$), cf figure 2. Let us give detailed arguments. If $x(s) \in (\check{b}^{(0)}(T), \check{r})$, then $h_0 := \min\{\check{f}^{(0)}(x(s), s), \check{f}^{(0)}(\check{r}, s)\} > 0$, and $\check{f}^{(0)}(\xi, s) \geqslant h_0$, for $\xi \in [x(s), \check{r}]$, as observed from the graph of $\check{f}^{(0)}(\cdot, s)$ in figure 4. In addition,

$$\begin{aligned}
\dot{x}(s) &= -\mu x(s) + \alpha g(x(s)) + \beta g(x(s - \tau_1(s))) + w(s),\\
&\geqslant -\mu x(s) + \alpha g(x(s)) - \beta\rho + w^{\min}(s),\\
&= \check{f}^{(0)}(x(s), s) \geqslant h_0.
\end{aligned}$$

Thus, $x(t)$ is increasing with a positive rate should it remain in $(\check{b}^{(0)}(T), \check{r})$. On the other hand, if $x(s) > \hat{r}$ (figure 4),

$$\begin{aligned}
\dot{x}(s) &= -\mu x(s) + \alpha g(x(s)) + \beta g(x(s - \tau_1(s))) + w(s)\\
&\leqslant -\mu x(s) + \alpha g(x(s)) + \beta\rho + w^{\max}(s)\\
&= \hat{f}^{(0)}(x(s), s) < 0.
\end{aligned}$$

Thus, $x(t)$ eventually enters $[\check{r}, \hat{r}]$.

(iii) Let us show that $x(t) \to [\underline{b}(T), \overline{b}(T)]$, for any $T \geqslant t_0$. Assume otherwise that $x(t)$ does not converge to $[\underline{b}(T), \overline{b}(T)]$ as $t \to \infty$, for some $T \geqslant t_0$. Then, there exist $\varepsilon > 0$ and an increasing time sequence $\{t_n\}$ tending to $+\infty$, such that $x(t_n)$ does not belong to $[\underline{b}(T)-\varepsilon, \overline{b}(T)+\varepsilon]$ for all n. This contradicts that for each $k \in \mathbb{N}\cup\{0\}$, $T > t_0$, $x(t) \in [\hat{b}^k(T), \check{b}^k(T)]$ for all $t \geqslant T+k\tau$, by the assumption of property $\mathcal{M}$, and that $\hat{b}^k(T)$ converges to $\underline{b}(T)$ increasingly, $\check{b}^k(T)$ converges to $\overline{b}(T)$ decreasingly, as $k \to \infty$. Moreover, since $\underline{b}(T)$ tends to $\underline{b}$ increasingly and $\overline{b}(T)$ tends to $\overline{b}$ decreasingly, as $T \to \infty$, we conclude that $x(t) \to [\underline{b}, \overline{b}]$, as $t \to \infty$.

(iv) First, both $[\check{l}, \hat{l}]$ and $[\check{r}, \hat{r}]$ are positively invariant sets for system (2.1) mainly because $\check{f}(x(t)) \leqslant \dot{x}(t) \leqslant \hat{f}(x(t))$ for all $t \geqslant t_0$, cf figure 2. More precisely, assume that there exists $s \geqslant t_0$ such that $x(t) \in [\check{r}, \hat{r}]$ for $t_0 \leqslant t \leqslant s$ and $x(t_1) \notin [\check{r}, \hat{r}]$ for some $t_1 > s$. Let s_1 be the first time after time s such that $x(s_1) = \check{r}$, and $x(t)$ leaves $[\check{r}, \hat{r}]$ after time s_1 and enters $(-\infty, \check{r})$, without loss of generality. Then there exists $s_2 > s_1$ such that $\check{m} < x(t) < \check{r}$ for $t \in (s_1, s_2)$. A contradiction then arises as

$$\begin{aligned} x(s_2) - x(s_1) &= \dot{x}(s_3)(s_2 - s_1) \\ &= [-\mu x(s_3) + \alpha g(x(s_3)) + \beta g(x(s_3 - \tau_1(s_3))) + w(s_3)](s_2 - s_1) \\ &\geqslant \check{f}(x(s_3))(s_2 - s_1) > 0, \end{aligned}$$

for some $s_3 \in (s_1, s_2)$. A similar contradiction occurs if we consider $x(s_1) = \hat{r}$ and $x(t)$ enters $(\hat{r}, \infty)$. The proof for positive invariance of $[\check{l}, \hat{l}]$ is similar.

Next, we assume that $x(t)$ satisfies property $\mathcal{R}$, namely, there exists $s \geqslant t_0$ such that $x(s) \in [\check{r}, \hat{r}]$. We assert that for each $T \geqslant t_0$,

$$x(t) \to [\check{c}^{(k)}(T), \hat{c}^{(k)}(T)], \qquad \text{as } t \to \infty, \qquad \text{for all } k \geqslant 0. \tag{2.8}$$

We justify (2.8) by induction. Let $s_T := \max\{s, T\}$. It can be concluded that if $x(t_1) \in [\check{c}^{(0)}(T), \hat{c}^{(0)}(T)]$ for some $t_1 \geqslant s_T$, then $x(t) \in [\check{c}^{(0)}(T), \hat{c}^{(0)}(T)]$ for all $t \geqslant t_1$, by arguments similar to the previous ones for proving that $[\check{r}, \hat{r}]$ is positively invariant. If $x(t) \in [\check{r}, \check{c}^{(0)}(T))$, for all $t \geqslant s_T$, then

$$\begin{aligned} \dot{x}(t) &= -\mu x(t) + \alpha g(x(t)) + \beta g(x(t - \tau_1(t))) + w(t) \\ &\geqslant -\mu x(t) + \alpha g(x(t)) - \beta\rho + w^{\min}(T) \\ &= \check{f}^{(0)}(x(t), T) > 0, \end{aligned}$$

and yields a contradiction. Similarly, it cannot hold that $x(t) \in (\hat{c}^{(k)}(T), \hat{r}]$, for all $t \geqslant s_T$. Hence, (2.8) holds for $k = 0$. Now, we assume that (2.8) holds for $k = j - 1$, i.e. $x(t) \to [\check{c}^{(j-1)}(T), \hat{c}^{(j-1)}(T)]$, as $t \to \infty$. Let us illustrate that it also holds for $k = j$. Consider a point x_U arbitrarily close to $[\check{c}^{(j)}(T), \hat{c}^{(j)}(T)]$, and assume $x_U \leqslant \check{c}^{(j)}(T)$; there exists a function, say f_U, which is a vertical shift of $\hat{f}_{\mathrm{r}}^{(0)}(\cdot, T)$ and f_U has a unique zero at x_U, cf figure 5. It can be derived that $\dot{x}(t) \geqslant f_U(x(t))$, as t is large enough. Subsequently, it follows that $x(t)$ must become closer to $[\check{c}^{(j)}(T), \hat{c}^{(j)}(T)]$ than to x_U, as $t \to \infty$. Equation (2.8) thus holds for $k = j$. The arguments for $x_U \geqslant \hat{c}^{(j)}(T)$ and $\dot{x}(t) \leqslant f_U(x(t))$ are similar. Let us give detailed arguments. Assume that $x(t)$ does not converge to $[\check{c}^{(j)}(T), \hat{c}^{(j)}(T)]$. Then, without loss of generality, there exist an $\varepsilon > 0$ and a time sequence $\{t_n\}$ with $t_n \geqslant s_T$ and $t_n \to \infty$, as $n \to \infty$, such that

$$x(t_n) \in [\check{r}, \check{c}^{(j)}(T) - \varepsilon); \tag{2.9}$$

moreover, $\check{c}^{(j)}(T) > \check{c}^{(j-1)}(T)$. Notably, $\check{c}^{(j)}(T)$ is the unique solution of the equation $-\mu\xi + \alpha g(\xi) + \beta g(\check{c}^{(j-1)}(T)) + w^{\min}(T) = 0$, which lies in $[\check{r}, \hat{r}]$. Thus, there exist $\delta_\varepsilon > 0$ and $x_U \in [\check{c}^{(j)}(T) - \frac{\varepsilon}{2}, \check{c}^{(j)}(T) + \frac{\varepsilon}{2}]$ such that x_U is the unique solution of

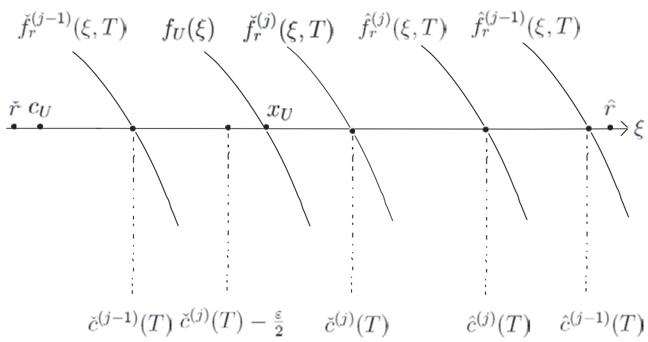

Figure 5. Configuration for the proof of proposition 2.2(iv), with fixed T.

$f_U(\xi) := -\mu\xi + \alpha g(\xi) + \beta g(c_U) + w^{\min}(T) = 0$, where $c_U := \min\{\xi : \xi \in U\}$, $U := [\check{c}^{(j-1)}(T) - \delta_\varepsilon, \check{c}^{(j-1)}(T) + \delta_\varepsilon] \cap [\check{r}, \hat{r}]$, by continuity, cf figure 5. On the other hand, there exists $\tilde{t}$ large enough such that $x(t) \geqslant c_U$, for all $t \geqslant \tilde{t}$, since $x(t)$ converges to $[\check{c}^{(j-1)}(T), \hat{c}^{(j-1)}(T)]$. It follows that

$$\begin{aligned}\dot{x}(t_N) &= -\mu x(t_N) + \alpha g(x(t_N)) + \beta g(x(t_N - \tau_1(t_N))) + w(t_N)\\ &\geqslant -\mu x(t_N) + \alpha g(x(t_N)) + \beta g(c_U) + w^{\min}(T) > 0,\end{aligned}$$

for some $t_N \geqslant \tilde{t} + \tau$, since $x(t_N) < \check{c}^{(j)}(T) - \varepsilon < x_U$. Moreover,

$$\begin{aligned}\dot{x}(t) &= -\mu x(t) + \alpha g(x(t)) + \beta g(x(t - \tau_1(t))) + w(t)\\ &\geqslant -\mu x(t) + \alpha g(x(t)) + \beta g(c_U) + w^{\min}(T) = f_U(x(t)) > 0,\end{aligned}$$

if $t \geqslant t_N$ and $x(t) \in (x(t_N), x_U)$. Therefore, $x(t)$ is increasing until it reaches x_U and never goes back into $[\check{r}, \check{c}^{(j)}(T) - \varepsilon)$. This yields a contradiction to (2.9). We have therefore justified that (2.8) holds. Consequently, $x(t)$ converges to $[\underline{c}(T), \overline{c}(T)]$ for all $T \geqslant t_0$, and thus converges to $[\underline{c}, \overline{c}]$, as $t \to \infty$. The proof for $x(t)$ satisfying property $\mathcal{L}$ and converging to $[\underline{a}, \overline{a}]$ is similar.

3. Multi-dimensional system

In this section, we shall derive the convergence of dynamics and stability of equilibria for the mainly considered system (1.1), by applying the propositions and theorem in section 2 and further analysis. Notably, system (1.1) is dissipative, as observed from the equation that the summation terms in the right-hand side of (1.1) are bounded. Such a property was formally justified in [31]. Thereafter, the solution $\boldsymbol{x}(t; t_0; \phi)$ of (1.1), starting from any $\phi \in C([-\tau, 0], \mathbb{R}^n)$, at $t = t_0$ exists on $[t_0, \infty)$.

3.1. Main results

Let us introduce the following upper and lower bounds for each component of system (1.1):

$$\hat{F}_i(\xi) := -\mu_i\xi + \alpha_{ii} g_i(\xi) + \sum_{j\neq i} |\alpha_{ij}|\rho_j + \sum_{j=1}^{n} |\beta_{ij}|\rho_j + J_i,$$

$$\check{F}_i(\xi) := -\mu_i\xi + \alpha_{ii} g_i(\xi) - \sum_{j\neq i} |\alpha_{ij}|\rho_j - \sum_{j=1}^{n} |\beta_{ij}|\rho_j + J_i,$$

where ρ_i are the bounds for activation functions g_i, defined in (1.2). Note that such a setting for the upper and lower bounds is different from the one in [20].

Recall that L_i is the largest slope of activation function g_i at its inflection point, as defined in (1.2). We consider the following conditions which are the multi-dimensional versions of conditions (A1) and (A2).

Condition (H1): $L_i > 2\mu_i/\alpha_{ii} > 0$, for $i = 1, 2, \cdots, n$,
Condition (H2): $\mu_i > \mu_i - L_i|\beta_{ii}| > \sum_{j\neq i} L_j|\alpha_{ij}| + \sum_{j\neq i} L_j|\beta_{ij}|$, for $i = 1, 2, \ldots, n$.

Notably, condition (H1) implies $\alpha_{ii} > 0$, and the first inequality in condition (H2) is equivalent to $\beta_{ii} \neq 0$, for all i. The discussions on critical points of f, $\hat{f}$ and $\check{f}$ and their vertical shifts in section 2.1 are valid for $\hat{F}_i$, $\check{F}_i$, $i = 1, 2, \ldots, n$, as well as their vertical shifts. Accordingly, under condition (H1), there exist critical points $\overline{p}_i$ and $\overline{q}_i$ of $\hat{F}_i$, $\check{F}_i$, which satisfy $g_i'(\overline{p}_i) = g_i'(\overline{q}_i) = \mu_i/\alpha_{ii}$. In addition, $\hat{F}_i$ and $\check{F}_i$ are strictly increasing in $(-\infty, \overline{p}_i)$, $(\overline{q}_i, \infty)$, and strictly decreasing in $(\overline{p}_i, \overline{q}_i)$, for $i = 1, 2, \ldots, n$. On the other hand,

$$0 < \left[\mu_i - \left(\sum_{j\neq i} L_j|\alpha_{ij}| + \sum_{j=1}^{n} L_j|\beta_{ij}|\right)\right]/(\alpha_{ii} + |\beta_{ii}|) < \mu_i/(\alpha_{ii} + |\beta_{ii}|), \tag{3.1}$$

under conditions (H2). Hence, there always exist exactly two points $\tilde{p}_i$ and $\tilde{q}_i$ with $\tilde{p}_i < \overline{p}_i < \overline{q}_i < \tilde{q}_i$ such that

$$g_i'(\tilde{p}_i) = g_i'(\tilde{q}_i) = \left[\mu_i - \left(\sum_{j\neq i} L_j|\alpha_{ij}| + \sum_{j=1}^{n} L_j|\beta_{ij}|\right)\right]/(\alpha_{ii} + |\beta_{ii}|), \tag{3.2}$$

for $i = 1, 2, \ldots, n$. Next, we introduce

Condition (H3): $\check{F}_i(\tilde{q}_i) > 0$ and $\hat{F}_i(\tilde{p}_i) < 0$, for all $i = 1, 2, \ldots, n$.

Under condition (H3), there exist three solutions $\hat{l}_i^F$, $\hat{m}_i^F$, and $\hat{r}_i^F$ (respectively, $\check{l}_i^F$, $\check{m}_i^F$ and $\check{r}_i^F$) to $\hat{F}_i(\cdot) = 0$ (respectively, $\check{F}_i(\cdot) = 0$), for each $i = 1, 2, \ldots, n$. Moreover, $\check{l}_i^F < \hat{l}_i^F < \tilde{p}_i < \hat{m}_i^F < \check{m}_i^F < \tilde{q}_i < \check{r}_i^F < \hat{r}_i^F$. The following condition is the multi-dimensional version of condition (A4).

Condition (H4): $g_i'(\xi) > 2\mu_i/\alpha_{ii}$ for all $\xi \in [\hat{m}_i^F, \check{m}_i^F]$, $i = 1, 2, \ldots, n$.

Let us introduce the following sets in $\mathbb{R}^n$

$$\Omega_{\lambda_1\lambda_2\cdots\lambda_n} = \Omega_1^{\lambda_1} \times \Omega_2^{\lambda_2} \times \cdots \times \Omega_n^{\lambda_n}, \lambda_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}, \qquad i = 1, 2, \ldots, n,$$

$$\tilde{\Omega}_{\lambda_1\lambda_2\cdots\lambda_n} = \tilde{\Omega}_1^{\lambda_1} \times \tilde{\Omega}_2^{\lambda_2} \times \cdots \times \tilde{\Omega}_n^{\lambda_n}, \ \lambda_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}, \qquad i = 1, 2, \ldots, n,$$

which are defined through the following intervals

$$\Omega_i^{\mathrm{l}} = [\check{l}_i^F, \hat{l}_i^F],\ \Omega_i^{\mathrm{m}} = [\hat{m}_i^F, \check{m}_i^F],\ \Omega_i^{\mathrm{r}} = [\check{r}_i^F, \hat{r}_i^F],$$

$$\tilde{\Omega}_i^{\mathrm{l}} = (-\infty, \hat{m}_i^F),\ \tilde{\Omega}_i^{\mathrm{m}} = \Omega_i^{\mathrm{m}},\ \tilde{\Omega}_i^{\mathrm{r}} = (\check{m}_i^F, \infty).$$

Herein, 'l', 'm', 'r' represent, respectively, left, middle and right. By applying the contraction mapping principle, we derive the existence of 3^n equilibria for system (1.1).

Theorem 3.1. *There exist exactly 3^n equilibria for system (1.1) under conditions (H2)–(H4). Each region $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$ contains exactly one of these 3^n equilibria.*

Proof. We will show that there exists exactly one equilibrium point in each $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$. Consider a fixed $\Omega = \Omega_{\lambda_1\lambda_2\cdots\lambda_n}$. Set $f_i(\xi) := -\mu_i\xi + \alpha_{ii}g_i(\xi)$. For a given $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \Omega$, we define

$$h_i(\xi) := -\mu_i\xi + \alpha_{ii}g_i(\xi) + \sum_{j=1, j\neq i}^{n} \alpha_{ij}g_j(y_j) + \sum_{j=1}^{n} \beta_{ij}g_j(y_j) + J_i,$$

for $\xi \in \mathbb{R}$, $i = 1, 2, \ldots, n$. Note that $\check{F}_i(\xi) \leqslant h_i(\xi) \leqslant \hat{F}_i(\xi)$, and all functions $\check{F}_i$, h_i, $\hat{F}_i$ are vertical shifts of f_i. Thus, there exists a unique solution y_i^* to equation $h_i(\cdot) = 0$, lying in $\Omega_i^{\lambda_i}$. We define a mapping $G_\Omega : \Omega \to \Omega$ by $G_\Omega(\boldsymbol{y}) = \boldsymbol{y}^*$, where $\boldsymbol{y}^* = (y_1^*, y_2^*, \ldots, y_n^*)$. Then G_Ω is continuous and we shall illustrate that it is a contraction map. Assume that $G_\Omega(\boldsymbol{y}) = \boldsymbol{y}^*$, $G_\Omega(\boldsymbol{x}) = \boldsymbol{x}^*$, i.e. for each $i = 1, 2, \ldots, n$

$$-\mu_i y_i^* + \alpha_{ii}g_i(y_i^*) + \sum_{j=1, j\neq i}^{n} \alpha_{ij}g_j(y_j) + \sum_{j=1}^{n} \beta_{ij}g_j(y_j) + J_i = 0,$$

$$-\mu_i x_i^* + \alpha_{ii}g_i(x_i^*) + \sum_{j=1, j\neq i}^{n} \alpha_{ij}g_j(x_j) + \sum_{j=1}^{n} \beta_{ij}g_j(x_j) + J_i = 0.$$

Then

$$(x_i^* - y_i^*)[\mu_i - \alpha_{ii}g_i'(\xi_i^*)] - \sum_{j=1, j\neq i}^{n} \alpha_{ij}g_j'(\eta_j^*)[x_j - y_j] - \sum_{j=1}^{n} \beta_{ij}g_j'(\eta_j^*)[x_j - y_j] = 0, \tag{3.3}$$

where ξ_i^* is some number between x_i^* and y_i^*; η_j^* is some number between x_j and y_j.

(i) If $\lambda_i =$‘m’, then $x_i^*, y_i^*, \xi_i^* \in [\hat{m}_i^F, \check{m}_i^F]$ and $g_i'(\xi_i^*) > 2\mu_i/\alpha_{ii}$, by condition (H4). Hence

$$\begin{aligned}|x_i^* - y_i^*| &= \left| \sum_{j=1, j\neq i}^{n} \alpha_{ij}g_j'(\eta_j^*)(x_j - y_j) + \sum_{j=1}^{n} \beta_{ij}g_j'(\eta_j^*)(x_j - y_j)|/|\alpha_{ii}g_i'(\xi_i^*) - \mu_i \right| \\ &\leqslant \left\{ \left[\sum_{j=1, j\neq i}^{n} L_j|\alpha_{ij}| + \sum_{j=1}^{n} L_j|\beta_{ij}| \right] /\mu_i \right\} \|\boldsymbol{x} - \boldsymbol{y}\|_\infty \\ &=: \tilde{\gamma}_i \|\boldsymbol{x} - \boldsymbol{y}\|_\infty,\end{aligned}$$

and $0 < \tilde{\gamma}_i < 1$, owing to condition (H2).

(ii) If $\lambda_i =$‘r’, then $x_i^*, y_i^* \in [\check{r}_i^F, \hat{r}_i^F]$ and $\xi_i^* > \tilde{q}_i$ (for $\tilde{q}_i < \check{r}_i^F$). Thus, $0 \leqslant g_i'(\xi_i^*) < [\mu_i - (\sum_{j\neq i} L_j|\alpha_{ij}| + \sum_{j=1}^{n} L_j|\beta_{ij}|)]/[\alpha_{ii} + |\beta_{ii}|] < \mu_i/\alpha_{ii}$, as mentioned in (3.1). It follows that

$$\begin{aligned}|\alpha_{ii}g_i'(\xi_i^*) - \mu_i| &= \mu_i - \alpha_{ii}g_i'(\xi_i^*) \\ &> \mu_i - \alpha_{ii}\left[\mu_i - \left(\sum_{j=1, j\neq i}^{n} L_j|\alpha_{ij}| + \sum_{j=1}^{n} L_j|\beta_{ij}| \right)\right]/(\alpha_{ii} + |\beta_{ii}|) \\ &\geqslant \sum_{j=1, j\neq i}^{n} L_j|\alpha_{ij}| + \sum_{j=1}^{n} L_j|\beta_{ij}|.\end{aligned}$$

Subsequently, from (3.3)

$$|x_i^* - y_i^*| \leqslant \left\{\left[\sum_{j=1, j\neq i}^{n} L_j|\alpha_{ij}| + \sum_{j=1}^{n} L_j|\beta_{ij}|\right] / |\alpha_{ii} g_i'(\xi_i^*) - \mu_i|\right\} \|\boldsymbol{x} - \boldsymbol{y}\|_\infty$$
$$=: \gamma_i \|\boldsymbol{x} - \boldsymbol{y}\|_\infty,$$

and $\gamma_i < 1$. The situation for $\lambda_i =$ ‘l’ is similar. Therefore, G_Ω is a contraction map and there exists a unique fixed point $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ of G_Ω, lying in Ω. Restated, for each $i = 1, 2, \ldots n$,

$$-\mu_i \bar{x}_i + \alpha_{ii} g_i(\bar{x}_i) + \sum_{j=1, j\neq i}^{n} \alpha_{ij} g_j(\bar{x}_j) + \sum_{j=1}^{n} \beta_{ij} g_j(\bar{x}_j) + J_i = 0. \tag{3.4}$$

Thus, $\bar{\boldsymbol{x}}$ is a unique equilibrium point of (1.1) lying in Ω.

On the other hand, if $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ is an equilibrium of (1.1), then (3.4) holds. Hence, $\bar{x}_i$ lies in one of $\Omega_i^{\mathrm{l}}, \Omega_i^{\mathrm{m}}, \Omega_i^{\mathrm{r}}$, for each i, and thus $\bar{\boldsymbol{x}}$ coincides with the unique equilibrium lying in $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$, $\lambda_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}$. System (1.1) therefore admits exactly 3^n equilibria. □

We note that the construction of 3^n equilibria in [19, 20] employed Brouwer's fixed point theorem. Therefore, the exactness for the number of equilibria was not concluded therein.

Let us denote by $\bar{\boldsymbol{x}}_{\lambda_1\lambda_2\cdots\lambda_n}$ the equilibrium lying in $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$, $\lambda_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}$. In the following discussions, we consider a fixed initial value $\phi \in C([-\tau, 0], \mathbb{R}^n)$, and the solution $\boldsymbol{x}(t) = \boldsymbol{x}(t; t_0; \phi) = (x_1(t; t_0; \phi), x_2(t; t_0; \phi), \ldots, x_n(t; t_0; \phi))$ to system (1.1), which is evolved from ϕ at $t = t_0$. For each $i = 1, 2, \ldots, n$, we write the ith component of system (1.1) in the following form:

$$\dot{\xi}(t) = -\mu_i \xi(t) + \alpha_{ii} g_i(\xi(t)) + \beta_{ii} g_i(\xi(t - \tau_{ii}(t))) + w_i(t), \tag{3.5}$$

where $w_i(t) = w_i(t; t_0; \phi) := \sum_{j=1, j\neq i}^{n} [\alpha_{ij} g_j(x_j(t)) + \beta_{ij} g_j(x_j(t - \tau_{ij}(t)))] + J_i$ is regarded as a bounded function of t. The notation, lemma 2.1, propositions 2.2, 2.3 and theorem 2.4 can all be adapted to (3.5). In particular, for $i = 1, 2, \ldots, n$, we define

$$\hat{f}_i(\xi) = -\mu_i \xi + \alpha_{ii} g_i(\xi) + |\beta_{ii}|\rho_i + w_i^{\max}(t_0),$$
$$\check{f}_i(\xi) = -\mu_i \xi + \alpha_{ii} g_i(\xi) - |\beta_{ii}|\rho_i + w_i^{\min}(t_0).$$

Under conditions (H1) and (H2), $\hat{f}_i, \check{f}_i$ admit similar properties as $\hat{f}, \check{f}$ in section 2.1. In particular, there exist $\hat{l}_i^f, \hat{m}_i^f, \hat{r}_i^f, \check{l}_i^f, \check{m}_i^f, \check{r}_i^f$ which are the zeros of $\hat{f}_i, \check{f}_i$, respectively, and $\bar{p}_i, \bar{q}_i$ which are both critical points of $\hat{f}_i$ and $\check{f}_i$. Notice that $\hat{F}_i, \check{F}_i$, and $\hat{f}_i, \check{f}_i$ share the same critical points $\bar{p}_i, \bar{q}_i$. According to our setting,

$$\check{F}_i(\xi) \leqslant \check{f}_i(\xi) \leqslant \hat{f}_i(\xi) \leqslant \hat{F}_i(\xi), \qquad \text{for all } \xi \in \mathbb{R}.$$

Therefore, condition (H3) implies that $\check{f}_i(\tilde{q}_i) \geqslant \check{F}_i(\tilde{q}_i) > 0$ and $\hat{f}_i(\tilde{p}_i) \leqslant \hat{F}_i(\tilde{p}_i) < 0$; in addition, $\bar{p}_i < \hat{m}_i^F < \hat{m}_i^f < \check{m}_i^f < \check{m}_i^F < \bar{q}_i$, $\check{l}_i^F < \check{l}_i^f < \hat{l}_i^f < \hat{l}_i^F < \tilde{p}_i$, and $\tilde{q}_i < \check{r}_i^F < \check{r}_i^f < \hat{r}_i^f < \hat{r}_i^F$, where $\tilde{p}_i, \tilde{q}_i$ are defined in (3.2), cf figure 6. Moreover, we note that condition (H4): $g_i'(\xi) > 2\mu_i/\alpha_{ii}$ on $[\hat{m}_i^F, \check{m}_i^F]$ yields $g_i'(\xi) > 2\mu_i/\alpha_{ii}$ on $[\hat{m}_i^f, \check{m}_i^f]$ since $[\hat{m}_i^f, \check{m}_i^f] \subset [\hat{m}_i^F, \check{m}_i^F]$.

According to theorem 2.4, for each $i = 1, 2, \ldots, n$, there exist three disjoint, closed and bounded intervals $[\underline{a}_i, \bar{a}_i]$, $[\underline{b}_i, \bar{b}_i]$ and $[\underline{c}_i, \bar{c}_i]$ and the ith component $x_i(t)$ of the solution converges to one of them. Moreover, by lemma 2.1, we can estimate the lengths of these intervals. Restated, $x_i(t) = x_i(t; t_0; \phi)$, the ith component of solution starting from $\phi \in C([-\tau, 0], \mathbb{R}^n)$, converges to an interval I_i of length d_i, and

$$d_i \leqslant [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/\eta_i, \tag{3.6}$$

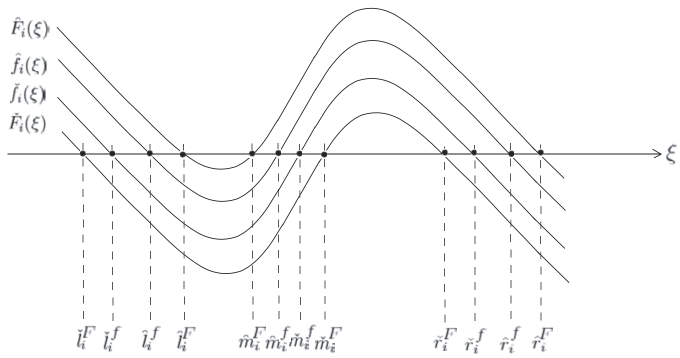

Figure 6. Configurations for functions $\hat{F}_i$, $\hat{f}_i$, $\check{f}_i$, $\check{F}_i$ (with g_i of class $\mathcal{A}$).

where $\eta_i := \min\{\mu_i - L_i|\beta_{ii}|, L_i|\beta_{ii}|\}$, $w_i^{\max}(\infty) = \lim_{T\to\infty} w_i^{\max}(T)$, $w_i^{\min}(\infty) = \lim_{T\to\infty} w_i^{\min}(T)$, $w_i^{\max}(T) := \sup\{w_i(t) \mid t \geqslant T\}$, and $w_i^{\min}(T) := \inf\{w_i(t) \mid t \geqslant T\}$. Notably, in (3.6), the magnitude of d_i depends on the difference between $w_i^{\max}(\infty)$ and $w_i^{\min}(\infty)$ which are terms involving non-i components of the solution and cannot be measured without further elaboration. In the following, we employ an upper bound for $w_i^{\max}(\infty)$ and a lower bound for $w_i^{\min}(\infty)$, which are definite terms, and derive a rough estimate on d_i. From this estimate, we compute more precise upper (respectively, lower) bounds for $w_i^{\max}(\infty)$ (respectively, $w_i^{\min}(\infty)$) through an iterative process. This idea for estimating the magnitude of d_i is illustrated and implemented in the following proposition.

Proposition 3.2. *Assume that conditions (H2)–(H4) hold. For each $i = 1, 2, \ldots, n$, there exists a sequence of intervals $\{I_i^{(k)}\}_{k=0}^{\infty}$ such that for each k, the ith component $x_i(t)$ of every solution $\boldsymbol{x}(t)$ to system (1.1) converges to $I_i^{(k)}$ as $t \to \infty$, and the length $d_i^{(k)}$ of $I_i^{(k)}$ satisfies*

$$d_i^{(k)} \leqslant \left\{ \sum_{j=1}^{i-1} (|\alpha_{ij}| + |\beta_{ij}|) L_j d_j^{(k)} + \sum_{j=i+1}^{n} (|\alpha_{ij}| + |\beta_{ij}|) L_j d_j^{(k-1)} \right\} / \eta_i. \tag{3.7}$$

Proof. We prove the case of $\beta_{ii} > 0$. Let us define $d_i^{(0)} := 2\rho_i / L_i$, for $i = 1, 2, \ldots, n$. First, we illustrate that the assertion holds for $k = 1$ and $i = 1$. Set

$$\check{W}_1^{(1)}(\infty) := -\sum_{j=2}^{n} (|\alpha_{1j}| + |\beta_{1j}|)\rho_j + J_1, \qquad \hat{W}_1^{(1)}(\infty) := \sum_{j=2}^{n} (|\alpha_{1j}| + |\beta_{1j}|)\rho_j + J_1.$$

Notably, $\check{W}_1^{(1)}(\infty) \leqslant w_1^{\min}(\infty) \leqslant w_1^{\max}(\infty) \leqslant \hat{W}_1^{(1)}(\infty)$. Recall $\eta_i := \min\{\mu_i - L_i|\beta_{ii}|, L_i|\beta_{ii}|\}$. We have shown that $x_1(t)$ converges to interval I_1 of length d_1, and

$$\begin{aligned} d_1 &\leqslant [w_1^{\max}(\infty) - w_1^{\min}(\infty)]/\eta_1 \\ &\leqslant [\hat{W}_1^{(1)}(\infty) - \check{W}_1^{(1)}(\infty)]/\eta_1 \\ &= \left[\sum_{j=2}^{n} |\alpha_{1j}| + \sum_{j=2}^{n} |\beta_{1j}| \right] L_j d_j^{(0)} / \eta_1. \end{aligned}$$

We may say that $x_1(t)$ converges to a closed and bounded interval $I_1^{(1)} \supset I_1$, whose length $d_1^{(1)}$ satisfies $d_1^{(1)} \leqslant [\sum_{j=2}^{n} |\alpha_{1j}| + \sum_{j=2}^{n} |\beta_{1j}|] L_j d_j^{(0)} / \eta_1$. Assume that the assertion holds for $k = 1$, $i = 1, 2, \ldots, \ell - 1$, $1 < \ell \leqslant n$ and $x_i(t)$ converges to a closed and bounded interval

$I_i^{(1)} \supset I_i$ of length $d_i^{(1)} \leqslant \big\{ \sum_{j=1}^{i-1}(|\alpha_{ij}| + |\beta_{ij}|)L_j d_j^{(1)} + \sum_{j=i+1}^{n}(|\alpha_{ij}| + |\beta_{ij}|)L_j d_j^{(0)} \big\}/\eta_i$. Let us justify that the assertion also holds for $k = 1$ and $i = \ell$ as follows. Set

$$\check{W}_\ell^{(1)}(\infty) := \sum_{j=1}^{\ell-1} \min_{\xi,\eta\in I_j^{(1)}} \{\alpha_{\ell j} g_j(\xi) + \beta_{\ell j} g_j(\eta)\} - \sum_{j=\ell+1}^{n} (|\alpha_{\ell j}| + |\beta_{\ell j}|)\rho_j + J_\ell,$$

$$\hat{W}_\ell^{(1)}(\infty) := \sum_{j=1}^{\ell-1} \max_{\xi,\eta\in I_j^{(1)}} \{\alpha_{\ell j} g_j(\xi) + \beta_{\ell j} g_j(\eta)\} + \sum_{j=\ell+1}^{n} (|\alpha_{\ell j}| + |\beta_{\ell j}|)\rho_j + J_\ell.$$

It follows that $x_\ell(t)$ converges to an interval $I_\ell^{(1)}$ whose length $d_\ell^{(1)}$ satisfies

$$\begin{aligned} d_\ell^{(1)} &\leqslant [w_\ell^{\max}(\infty) - w_\ell^{\min}(\infty)]/\eta_\ell \\ &\leqslant [\hat{W}_\ell^{(1)}(\infty) - \check{W}_\ell^{(1)}(\infty)]/\eta_\ell \\ &= \left\{ \sum_{j=1}^{\ell-1}(|\alpha_{\ell j}| + |\beta_{\ell j}|)L_j d_j^{(1)} + \sum_{j=\ell+1}^{n} (|\alpha_{\ell j}| + |\beta_{\ell j}|)L_j d_j^{(0)} \right\}/\eta_\ell. \end{aligned}$$

Next, assume that the assertion holds for some $(k-1)$ and all $i = 1, 2, \dots, n$. Namely, $x_i(t)$ converges to a closed and bounded interval $I_i^{(k-1)}$, whose length satisfies $d_i^{(k-1)} \leqslant \big\{ \sum_{j=1}^{i-1}(|\alpha_{ij}| + |\beta_{ij}|)L_j d_j^{(k-1)} + \sum_{j=i+1}^{n}(|\alpha_{ij}| + |\beta_{ij}|)L_j d_j^{(k-2)} \big\}/\eta_i$. Now, let us verify that the assertion holds for k and $i = 1$ as well. Set

$$\check{W}_1^{(k)}(\infty) := \sum_{j=2}^{n} \min_{\xi,\eta\in I_j^{(k-1)}} \{\alpha_{1j} g_j(\xi) + \beta_{1j} g_j(\eta)\} + J_1,$$

$$\hat{W}_1^{(k)}(\infty) := \sum_{j=2}^{n} \max_{\xi,\eta\in I_j^{(k-1)}} \{\alpha_{1j} g_j(\xi) + \beta_{1j} g_j(\eta)\} + J_1.$$

Thus, $x_1(t)$ converges to an interval $I_1^{(k)}$ whose length $d_1^{(k)}$ satisfies

$$\begin{aligned} d_1^{(k)} &\leqslant [w_1^{\max}(\infty) - w_1^{\min}(\infty)]/\eta_1 \\ &\leqslant [\hat{W}_1^{(k)}(\infty) - \check{W}_1^{(k)}(\infty)]/\eta_1 \\ &= \left[\sum_{j=2}^{n} |\alpha_{1j}| + \sum_{j=2}^{n} |\beta_{1j}| \right] L_j d_j^{(k-1)}/\eta_1. \end{aligned}$$

By continuing the above process, we can prove that for each $i = 2, \dots, n$, $x_i(t)$ converges to an interval $I_i^{(k)}$ whose length is $d_i^{(k)} \leqslant \big\{ \sum_{j=1}^{i-1}(|\alpha_{ij}| + |\beta_{ij}|)L_j d_j^{(k)} + \sum_{j=i+1}^{n}(|\alpha_{ij}| + |\beta_{ij}|)L_j d_j^{(k-1)} \big\}/\eta_i$. □

To establish further dynamical properties for system (1.1), we need the following condition which is stronger than condition (H2).

Condition (H2)*: $\eta_i := \min\{\mu_i - L_i|\beta_{ii}|, L_i|\beta_{ii}|\} > \sum_{j\neq i} L_j|\alpha_{ij}| + \sum_{j\neq i} L_j|\beta_{ij}|$, for $i = 1, 2, \dots, n$.

So far, we have considered a single solution to system (1.1), which is evolved from a given ϕ at $t = t_0$. From our previous derivations, it can be shown that every component of the solution converges to a sequence of closed intervals whose lengths $d_i^{(k)}$, $i = 1, 2, \dots, n$, can be controlled by iterative formula (3.7). Next, it will be examined that for each i, $d_i^{(k)}$ converges to zero, as $k \to \infty$, via the Gauss–Seidal iteration approach. Thus, the intervals to which each component of the solution converges degenerate into a single point. Hence the solution converges to a singleton.

Theorem 3.3. *Assume that conditions (H2)*, (H3) and (H4) hold. Then the solution* $\boldsymbol{x}(t) := \boldsymbol{x}(t;t_0;\phi)$ *of (1.1) evolved from any initial value* $\phi \in C([-\tau,0],\mathbb{R}^n)$ *converges to one of the* 3^n *equilibria of the system.*

Proof. By proposition 3.2, for each $i = 1,2,\ldots,n$, we can find an interval sequence $\{I_i^{(k)}\}_{k=0}^{\infty}$ so that $x_i(t)$ converges to $I_i^{(k)}$ whose length satisfies (3.7), for each k. Below, we shall show that for all $i = 1,2,\ldots,n$, $d_i^{(k)}$ converges to zero as k tends to infinity. Set $z_i^{(0)} := d_i^{(0)}$, and for $i = 1,2,\ldots,n$,

$$z_i^{(k)} := \left\{ \sum_{j=1}^{i-1}(|\alpha_{ij}| + |\beta_{ij}|)L_j z_j^{(k)} + \sum_{j=i+1}^{n}(|\alpha_{ij}| + |\beta_{ij}|)L_j z_j^{(k-1)} \right\}/\eta_i, \qquad k \in \mathbb{N},$$

$$\boldsymbol{z}^{(k)} := (z_1^{(k)}, z_2^{(k)}, \ldots, z_n^{(k)}), \qquad k \in \mathbb{N}\cup\{0\}.$$

We observe that $\{z_i^{(k)} \mid i = 1,2,\ldots,n\}$ are just the Gauss–Seidal iterations for solving the linear system

$$\begin{aligned} &(\boldsymbol{ML}+\boldsymbol{E})\boldsymbol{y} = \boldsymbol{0}, \\ &\boldsymbol{M} := [m_{ij}]_{1\leqslant i,j\leqslant n}, \qquad m_{ii} = 0, m_{ij} = -|\alpha_{ij}| - |\beta_{ij}|, \quad \text{for } i \neq j, \\ &\boldsymbol{L} := \mathrm{diag}(L_1, L_2, \ldots, L_n), \qquad \boldsymbol{E} := \mathrm{diag}(\eta_1, \eta_2, \ldots, \eta_n). \end{aligned} \tag{3.8}$$

Notably, $\boldsymbol{ML}+\boldsymbol{E}$ is strictly diagonal-dominant [32,33]; indeed, $\eta_i - \sum_{j\neq i}(|\alpha_{ij}|+|\beta_{ij}|)L_j > 0$, for all $i = 1,2,\cdots,n$, by condition (H2)*. Accordingly, $\boldsymbol{z}^{(k)}$ converges to the unique solution of (3.8), which is zero, as $k \to \infty$.

Below, let us justify the following inequality:

$$0 \leqslant d_i^{(k)} \leqslant z_i^{(k)}, \qquad \text{for } i = 1,2,\ldots,n, k \in \mathbb{N}\cup\{0\}. \tag{3.9}$$

It is obvious that for $i = 1,2,\ldots,n$, $0 \leqslant d_i^{(k)}$, for $k \in \mathbb{N}\cup\{0\}$ and (3.9) holds for $k = 0$. In addition, (3.9) holds for $i = 1$ and $k = 1$ since $d_1^{(1)} \leqslant \left\{\sum_{j=2}^{n}(|\alpha_{1j}| + |\beta_{1j}|)L_j d_j^{(0)}\right\}/\eta_1 \leqslant \left\{\sum_{j=2}^{n}(|\alpha_{1j}| + |\beta_{1j}|)L_j z_j^{(0)}\right\}/\eta_1 = z_1^{(1)}$. We can continue to prove that (3.9) holds for $i = 2,3\ldots,n$ and $k = 1$. Assume that (3.9) holds for all $i = 1,2,\ldots,n$ and $k = \ell$, for some $\ell \geqslant 1$, then (3.9) also holds for $i = 1$, $k = \ell+1$ due to $d_1^{(\ell+1)} \leqslant \left\{\sum_{j=2}^{n}(|\alpha_{\ell j}|+|\beta_{\ell j}|)L_j d_j^{(\ell)}\right\}/\eta_1 \leqslant \left\{\sum_{j=2}^{n}(|\alpha_{\ell j}|+|\beta_{\ell j}|)L_j z_j^{(\ell)}\right\}/\eta_1 = z_1^{(\ell+1)}$. Assume that (3.9) holds for $0 \leqslant k_0 - 1$ and all $i = 1,2,\ldots,n$, and $k = k_0$, $i = 1,\ldots,(\ell-1)$, then

$$\begin{aligned} d_\ell^{(k_0)} &\leqslant \left\{ \sum_{j=1}^{\ell-1}(|\alpha_{\ell j}| + |\beta_{\ell j}|)L_j d_j^{(k_0)} + \sum_{j=\ell+1}^{n}(|\alpha_{\ell j}| + |\beta_{\ell j}|)L_j d_j^{(k_0-1)} \right\}/\eta_\ell \\ &\leqslant \left\{ \sum_{j=1}^{\ell-1}(|\alpha_{\ell j}| + |\beta_{\ell j}|)L_j z_j^{(k_0)} + \sum_{j=\ell+1}^{n}(|\alpha_{\ell j}| + |\beta_{\ell j}|)L_j z_j^{(k_0-1)} \right\}/\eta_\ell \\ &= z_\ell^{(k_0)}. \end{aligned}$$

Hence, for each $i = 1,2,\ldots,n$, $d_i^{(k)}$ converges to zero as k tends to infinity. Therefore, each $x_i(t)$ converges to a single point and $\boldsymbol{x}(t)$ converges to a constant which is an equilibrium, as time tends to infinity. □

The stability of all 3^n equilibria of (1.1) can be concluded in the following theorem.

Theorem 3.4. *Assume that conditions (H2)*, (H3) and (H4) hold. Then, (i) every equilibrium* $\bar{\boldsymbol{x}}_{\lambda_1\lambda_2\cdots\lambda_n}$ *with* $\lambda_i =$ 'l', 'r', *for all* $i = 1,2,\ldots,n$, *is asymptotically stable; (ii) the equilibrium* $\bar{\boldsymbol{x}}_{\mathrm{m}\cdots\mathrm{m}}$ *is unstable; (iii) every equilibrium* $\bar{\boldsymbol{x}}_{\lambda_1\lambda_2\cdots\lambda_n}$ *with* $\lambda_i =$ 'm' *for some* i *and* $\lambda_j =$ 'l', 'r' *for some* j, *is unstable.*

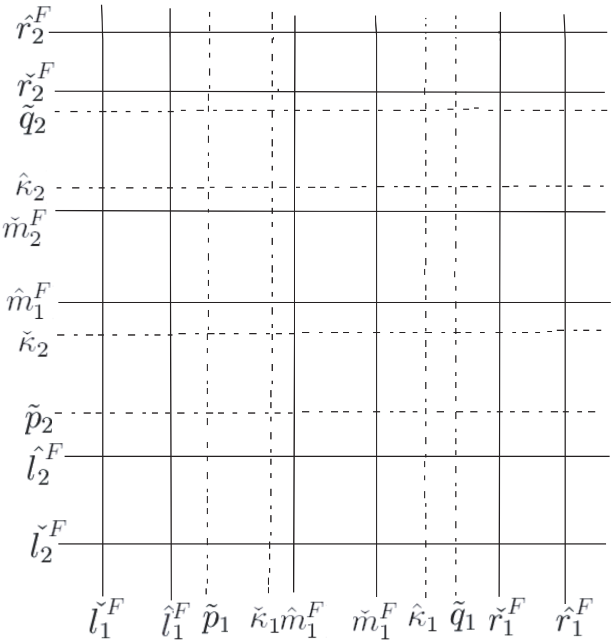

Figure 7. Configuration for the proof of theorem 3.4.

Proof. We prove the case with all $\beta_{ii} > 0$, cf figure 7.

(i) Consider an exterior region $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$, λ_i = 'l' or 'r', $i = 1, 2, \ldots, n$. We show that the equilibrium $\bar{\boldsymbol{x}} := (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ in $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$ is stable. Note that for each i, either $\bar{x}_i \in [\check{r}_i^F, \hat{r}_i^F]$ or $\bar{x}_i \in [\check{l}_i^F, \hat{l}_i^F]$. There exists $\varepsilon_i > 0$ such that $\check{r}_i^F - \varepsilon_i > \tilde{q}_i$ and $\check{l}_i^F + \varepsilon_i < \tilde{p}_i$, due to $\check{r}_i^F > \tilde{q}_i$ and $\check{l}_i^F < \tilde{p}_i$. We shall illustrate that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\boldsymbol{x}_t - \bar{\boldsymbol{x}}\| \leqslant \varepsilon$ for all $t \geqslant t_0$, for any $\phi \in C([-\tau, 0], \mathbb{R}^n)$ with $\|\phi - \bar{\boldsymbol{x}}\| \leqslant \delta$. For $\varepsilon > 0$, we set $\delta := \min\{\varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$. For an initial condition $\phi \in C([-\tau, 0], \mathbb{R}^n)$ with $\|\phi - \bar{\boldsymbol{x}}\| \leqslant \delta$, the solution satisfies $x_i(s) > \tilde{q}_i$ if λ_i = 'r', and $x_i(s) < \tilde{p}_i$, if λ_i = 'l', for all $s \in [t_0 - \tau, t_0]$. It follows from a similar argument as in the proof of proposition 2.2(ii) that $x_i(t) > \tilde{q}_i$ for all $t \in [t_0 - \tau, \infty)$ or $x_i(t) < \tilde{p}_i$, for all $t \in [t_0 - \tau, \infty)$. We define $z_i(t) := x_i(t) - \bar{x}_i$, for $i = 1, 2, \ldots, n$. It follows from (1.1) that

$$\dot{z}_i(t) = -\mu_i z_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j'(\xi_j(t)) z_j(t) + \sum_{j=1}^{n} \beta_{ij} g_j'(\eta_{ij}(t)) z_j(t - \tau_{ij}(t))$$

where $\xi_j(t)$ is between $x_j(t)$ and $\bar{x}_j$, $\eta_{ij}(t)$ is between $x_j(t - \tau_{ij}(t))$ and $\bar{x}_j$, $i, j = 1, 2, \ldots, n$. It can be computed that

$$D_r|z_i(t)| \leqslant -\mu_i |z_i(t)| + \sum_{j=1}^{n} |\alpha_{ij}| g_j'(\xi_j(t)) |z_j(t)| + \sum_{j=1}^{n} |\beta_{ij}| g_j'(\eta_{ij}(t)) |z_j(t - \tau_{ij}(t))|,$$

for $t \geqslant t_0$, where D_r denotes the right-hand derivative. Define $N(t) := \|z_t\| = \max_{1\leqslant i\leqslant n}\{\max_{s\in[t-\tau,t]} |z_i(s)|\}$. We shall show below that

$$D_r N(t) := \lim_{h\to 0^+} \frac{N(t+h) - N(t)}{h} \leqslant 0, \qquad \text{for all } t \geqslant t_0. \tag{3.10}$$

For $t \geqslant t_0$, let $\tilde{I}(t) := \{i : |z_i(t)| \geqslant |z_j(t)|$, for all $j = 1, 2, \ldots, n\}$, and $i(t) := \min\{i \in \tilde{I}(t) : D_r|z_i(t)| \geqslant D_r|z_j(t)|$, for all $j \in \tilde{I}(t)\}$. Consider a fixed $t > t_0$, and denote $i(t)$ by k. If $N(t) = |z_k(t)| > |z_j(t - \tau)|$ for all $j = 1, 2, \ldots, n$, then either $N(t) > |z_j(s)|$ for all $j = 1, 2, \ldots, n$ and all $s \in [t - \tau, t)$ or $N(t) = |z_{i(s)}(s)|$ for some $s \in (t - \tau, t)$. For the former case, it can be derived that

$$\begin{aligned}
D_r|z_k(t)| &\leqslant -\mu_k|z_k(t)| + \sum_{j=1}^{n} |\alpha_{kj}|g_j'(\xi_j(t))|z_j(t)| + \sum_{j=1}^{n} |\beta_{kj}|g_j'(\eta_{kj}(t))|z_j(t - \tau_{kj}(t))| \\
&\leqslant \left[-\mu_k + \alpha_{kk} g_k'(\xi_k(t)) + \sum_{j\neq k} |\alpha_{kj}|g_j'(\xi_j(t)) + \sum_{j=1}^{n} |\beta_{kj}|g_j'(\eta_{kj}(t)) \right] N(t) \\
&< \left[-\mu_k + \alpha_{kk} g_k'(\gamma_k) + \sum_{j\neq k} |\alpha_{kj}|L_j + \sum_{j=1}^{n} |\beta_{kj}|L_j \right] N(t) \\
&\leqslant 0,
\end{aligned}$$

for all $t \geqslant t_0$, where $\gamma_k = \tilde{p}_k$ or $\tilde{q}_k$, recalling that

$$g_k'(\tilde{p}_k) = g_k'(\tilde{q}_k) = \left[\mu_k - \left(\sum_{j\neq i} L_j|\alpha_{kj}| + \sum_{j=1}^{n} L_j|\beta_{kj}| \right) \right] / (\alpha_{kk} + |\beta_{kk}|).$$

Thus,

$$\begin{aligned}
D_r N(t) &= \lim_{h\to 0^+} \frac{N(t+h) - N(t)}{h} \\
&= \lim_{h\to 0^+} \frac{|z_k(t+h)| - |z_k(t)|}{h} \\
&= D_r|z_k(t)| \leqslant 0.
\end{aligned}$$

For the latter case,

$$\begin{aligned}
D_r N(t) &= \lim_{h\to 0^+} \frac{N(t+h) - N(t)}{h} \\
&= \lim_{h\to 0^+} \frac{N(t) - N(t)}{h} = 0.
\end{aligned}$$

For the other cases: $N(t) = |z_i(t - \tau)|$ for some $i \in \{1, 2, \ldots, n\}$; $N(t) = |z_i(s)|$ for some $i \in \{1, 2, \ldots, n\}$ and some $s \in (t - \tau, t)$ with $N(t) > |z_j(t - \tau)|$ and $N(t) > |z_j(t)|$ for all $j = 1, 2, \ldots, n$, (3.10) can also be justified. Hence, $N(t) = \|z_t\| = \|\boldsymbol{x}_t - \bar{\boldsymbol{x}}\| \leqslant N(t_0) = \|z_{t_0}\| = \|\boldsymbol{x}_{t_0} - \bar{\boldsymbol{x}}\| = \|\phi - \bar{\boldsymbol{x}}\|$ for all $t \geqslant t_0$. Therefore, $\bar{\boldsymbol{x}}$ is stable, hence asymptotically stable, in respecting theorem 3.3.

(ii) We shall show that $\bar{\boldsymbol{x}} := \bar{\boldsymbol{x}}_{\mathrm{mm\cdots m}}$ is unstable. We choose an initial value which is close to the equilibrium $\bar{\boldsymbol{x}}$. Then the solution must move away from $\bar{\boldsymbol{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$. Such an assertion holds mainly because if the ith component $x_i(t)$ of the solution remains close to $\bar{x}_i$ for all $i = 1, 2, \ldots, n$, then the magnitude of $g_i'(x_i(t))$ will remain large and yield a contradiction. Notably, for $i = 1, 2, \ldots, n$, $g_i'(\xi) > 2\mu_i/\alpha_{ii}$, for all $\xi \in [\hat{m}_i^F, \check{m}_i^F]$, thus there exist $\hat{\kappa}_i$ and $\check{\kappa}_i$ such that $g'(\check{\kappa}_i) = g'(\hat{\kappa}_i) = 2\mu_i/\alpha_{ii}$, where $\check{\kappa}_i < \hat{m}_i^F < \check{m}_i^F < \hat{\kappa}_i$, for all $i = 1, 2, \ldots, n$. Set $\varepsilon_i := \min\{\hat{\kappa}_i - \check{m}_i^F, \hat{m}_i^F - \check{\kappa}_i\}$, $\varepsilon := \min_{1\leqslant i\leqslant n}\{\varepsilon_i\}/2$. For any $\delta \in (0, \varepsilon)$, we choose the initial condition $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ with $\|\phi - \bar{\boldsymbol{x}}\| < \delta$, $\phi(s) \in \tilde{\Omega}_{\mathrm{mm\cdots m}}$, for all $s \in [-\tau, 0]$, $\|\phi - \bar{\boldsymbol{x}}\| = |\phi_i(0) - \bar{x}_i|$ for some $i \in \{1, 2, \cdots, n\}$ and $\|\phi - \bar{\boldsymbol{x}}\| > |\phi_j(s) - \bar{x}_j|$, for all

$j = 1, 2, \ldots, n$, $s \in [-\tau, 0)$. Now, let us show that there exist $j \in \{1, 2, \ldots, n\}$, and $t_1 > t_0$ such that $x_j(t_1) > \hat{\kappa}_j$ or $x_j(t_1) < \check{\kappa}_j$. Assume otherwise that

$$\check{\kappa}_i \leqslant x_i(t) \leqslant \hat{\kappa}_i, \qquad \text{for all } t \geqslant t_0 - \tau, \quad i = 1, 2, \ldots, n. \tag{3.11}$$

Notice that, under the assumption above, $g_i'(x_i(t)) \geqslant 2\mu_i/\alpha_{ii}$ for all $t \geqslant t_0 - \tau$ and all $i = 1, 2, \ldots, n$. Let $z_i(t) = x_i(t) - \overline{x}_i$, and

$$B(t) := \max_{1\leqslant i\leqslant n} \{ \max_{t_0-\tau\leqslant s\leqslant t} |z_i(s)|\}. \tag{3.12}$$

Then $B(t_0) = \max_{1\leqslant i\leqslant n}\{|z_i(t_0)|\} > 0$ and $B(t) > 0$ for all $t \geqslant t_0$. Let us show that

$$B(t) = \max_{1\leqslant i\leqslant n} \{|z_i(t)|\}, \qquad \text{for all } t \geqslant t_0, \tag{3.13}$$

i.e. at least one component of $(|z_1(s)|, |z_2(s)|, \ldots, |z_n(s)|)$ will reach the value of $B(t)$ at time t. If otherwise, there is a $t > t_0$ so that $B(t) = |z_k(t_2)|$, for some $k \in \{1, 2, \cdots, n\}$ and some $t_2 \in [t_0, t)$, then either $B(t) = z_k(t_2)$ or $B(t) = -z_k(t_2)$. For the former case,

$$\begin{aligned}
\dot{z_k}(t_2) &= -\mu_k z_k(t_2) + \sum_{j=1}^{n} \alpha_{kj} g_j'(\xi_j(t_2)) z_j(t_2) + \sum_{j=1}^{n} \beta_{kj} g_j'(\eta_{kj}(t_2)) z_j(t_2 - \tau_{kj}(t_2)) \\
&\geqslant -\mu_k z_k(t_2) + \alpha_{kk}[2\mu_k/\alpha_{kk}] z_k(t_2) - \sum_{j\neq k} |\alpha_{kj}| g_j'(\xi_j(t_2)) |z_j(t_2)| \\
&\quad - \sum_{j=1}^{n} |\beta_{kj}| g_j'(\eta_{kj}(t_2)) |z_j(t_2 - \tau_{kj}(t_2))| \\
&\geqslant \left[\mu_k - \sum_{j\neq k} |\alpha_{kj}| L_j - \sum_{j=1}^{n} |\beta_{kj}| L_j \right] B(t) > 0,
\end{aligned}$$

owing to condition (H2). For the latter case, we can also show that $\frac{\mathrm{d}(-z_k)}{\mathrm{d}t}(t_2) \geqslant \big[\mu_k - \sum_{j\neq k} |\alpha_{kj}| L_j - \sum_{j=1}^{n} |\beta_{kj}| L_j\big] B(t) > 0$. A contradiction to $B(t) = |z_k(t_2)|$ with $t_2 \in [t_0, t)$ then arises. Thus, (3.13) holds. For any $t \geqslant t_0$, we define $k(t) := \min\{j : |z_j(t)| = B(t)\}$, then

$$\begin{aligned}
D_r B(t) &\geqslant D_r |z_{k(t)}(t)| \\
&\geqslant \left[\mu_{k(t)} - \sum_{j\neq k(t)} |\alpha_{k(t)j}| L_j + \sum_{j=1}^{n} |\beta_{k(t)j}| L_j \right] B(t) \\
&\geqslant \min_{1\leqslant i\leqslant n} \left\{ \mu_i - \sum_{j\neq i} |\alpha_{ij}| L_j + \sum_{j=1}^{n} |\beta_{ij}| L_j \right\} B(t).
\end{aligned}$$

It follows that $B(t)$ grows unboundedly as t tends to infinity, which yields a contradiction to (3.11). We thus conclude that $\overline{\boldsymbol{x}}_{\mathrm{mm}\cdots\mathrm{m}}$ is unstable.

(iii) Consider a mixed region $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$, where $\mathcal{I} := \{i : \lambda_i = \text{'m'}\} \neq \emptyset$ and $\mathcal{E} := \{i : \lambda_i = \text{'l' or 'r'}\} \neq \emptyset$. It will be shown that the equilibrium $\overline{\boldsymbol{x}} := (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ in $\Omega_{\lambda_1\lambda_2\cdots\lambda_n}$ is unstable. We shall choose an initial value which is close to equilibrium $\overline{\boldsymbol{x}}$, then the evolved solution must move away from $\bar{\boldsymbol{x}}$. This is because if the ith component remains close to $\bar{x}_i$ for all $i \in \mathcal{I}$, then the magnitude of $g_i'(x_i(t))$ will remain large for all $i \in \mathcal{I}$. Moreover, it can be seen that the magnitude of $g_j'(x_j(t))$ remains small for all $j \in \mathcal{E}$. In such a situation, there exists some $k \in \mathcal{I}$ such that $x_k(t)$ will move away from $\bar{x}_k$; subsequently a contradiction arises. To be more precise, let us define $\varepsilon_i := \min\{\hat{\kappa}_i - \check{m}_i^F, \hat{m}_i^F - \check{\kappa}_i\}$, for $i \in \mathcal{I}$, and $\varepsilon_j := \bar{x}_j - \tilde{q}_j$ if $\lambda_j =$ 'r', $\varepsilon_j := \tilde{p}_j - \bar{x}_j$ if $\lambda_j =$ 'l', for $j \in \mathcal{E}$, and set $\varepsilon := \min_{1\leqslant i\leqslant n}\{\varepsilon_i\}/2$. For $\delta \in (0, \varepsilon)$, we choose an initial condition ϕ satisfying: $\|\phi - \bar{\boldsymbol{x}}\| < \delta$, and $\phi_j(s) \neq \bar{x}_j$, for some $j \in \mathcal{I}$ and some $s \in [-\tau, 0]$, $\|\phi - \bar{\boldsymbol{x}}\| = |\phi_k(0) - \bar{x}_k|$, for some $k \in \mathcal{I}$ and $\|\phi - \bar{\boldsymbol{x}}\| > |\phi_i(s) - \bar{x}_i|$, for all $i \in \mathcal{E}$ and all $s \in [-\tau, 0]$. Below, let us claim that there exist $j \in \mathcal{I}$, and some $t > t_0$ such that $x_j(t) > \hat{\kappa}_j$ or $x_j(t) < \check{\kappa}_j$. Assume otherwise that $\check{\kappa}_i \leqslant x_i(t) \leqslant \hat{\kappa}_i$, for all $i \in \mathcal{I}$ and $t \geqslant t_0 - \tau$. Note that then $g_i'(x_i(t)) \geqslant 2\mu_i/\alpha_{ii}$, for all $t \geqslant t_0 - \tau$ and all $i \in \mathcal{I}$. Define $B(t)$ as (3.12) and $J(t) := \{j \in \mathcal{I} : |z_j(t)| \geqslant |z_i(t)|, \text{ for all } i \in \mathcal{I}\}$, $j(t) := \min\{\ell \in J(t) : D_r|z_\ell(t)| \geqslant D_r|z_j(t)|, \text{ for all } j \in J(t)\}$. There are two possibilities: $|z_{j(t)}(t)| \geqslant |z_i(t)|$, for all $t \geqslant t_0$, for all $i \in \mathcal{E}$, and $|z_k(t_3)| > |z_{j(t_3)}(t_3)|$, for some $t_3 > t_0$, and some $k \in \mathcal{E}$. For the first one, $B(t) := \max_{i\in\mathcal{I}}\{\max_{t_0-\tau\leqslant s\leqslant t} |z_i(s)|\}$, for all $t \geqslant t_0$. Similarly to the previous discussion in (ii), we can also show that $B(t) = \max_{i\in\mathcal{I}}\{|z_i(t)|\}$. Subsequently, $B(t)$ will blow up and yield a contradiction. For the latter situation, there exists $s_1 \in (t_0, t_3)$ such that $|z_{j(s)}(s)| \geqslant |z_j(s)|$ for all $j \in \mathcal{E}$ and all $s \in [t_0, s_1)$, and there exists $k \in \mathcal{E}$ such that $|z_k(s_1)| = |z_{j(s_1)}(s_1)|$ and $D_r|z_k(s_1)| \geqslant D_r|z_{j(s_1)}(s_1)|$. Thereafter, it can be shown that $B(s) := \max_{i\in\mathcal{I}}\{|z_i(s)|\}$, for all $s \in [t_0, s_1]$ as before. Let us fix s_1 and denote $j(s_1)$ by ℓ. There are four possible subcases: subcase (a): $B(s_1) = z_\ell(s_1) = z_k(s_1) > 0$; subcase (b): $B(s_1) = z_k(s_1) = -z_\ell(s_1) > 0$; subcase (c): $B(s_1) = -z_\ell(s_1) = -z_k(s_1) > 0$; subcase (d): $B(s_1) = z_\ell(s_1) = -z_k(s_1) > 0$. Let us consider subcase (a). Note that $x_\ell(t) \in [\check{\kappa}_\ell, \hat{\kappa}_\ell]$, for all $t \geqslant t_0 - \tau$, and either $x_k(t) > \tilde{q}_k$ or $x_k(t) < \tilde{p}_k$, for all $t \geqslant t_0 - \tau$. We compute that

$$
\begin{aligned}
&D_r|z_\ell(s_1)| - D_r|z_k(s_1)| \\
&\qquad \geqslant (\mu_k - \mu_\ell)B(s_1) + [\alpha_{\ell\ell}g_\ell'(\xi_\ell(s_1)) - \alpha_{kk}g_k'(\xi_k(s_1))]B(s_1) - \sum_{j\neq\ell}|\alpha_{\ell j}|L_jB(s_1) \\
&\qquad\quad - \sum_{j\neq k}|\alpha_{kj}|L_jB(s_1) - \sum_{j=1}^{n}[|\beta_{\ell j}|L_jB(s_1) + |\beta_{kj}|L_jB(s_1)] \\
&\qquad \geqslant \{(\mu_k - \mu_\ell) + \alpha_{\ell\ell}\frac{2\mu_\ell}{\alpha_{\ell\ell}} - \alpha_{kk}\frac{\mu_k - (\sum_{j\neq k}|\alpha_{kj}|L_j + \sum_{j=1}^{n}|\beta_{kj}|L_j)}{\alpha_{kk} + |\beta_{kk}|} \\
&\qquad\quad - \left[\sum_{j\neq\ell}|\alpha_{\ell j}|L_j + \sum_{j\neq k}|\alpha_{kj}|L_j)\right] - \sum_{j=1}^{n}[|\beta_{\ell j}|L_j + |\beta_{kj}|L_j]\}B(s_1) \\
&\qquad \geqslant \left[\mu_\ell - \sum_{j\neq\ell}|\alpha_{\ell j}|L_j - \sum_{j=1}|\beta_{\ell j}|L_j\right]B(s_1) > 0,
\end{aligned}
$$

which yields a contradiction. Other subcases can be similarly discussed. Hence, there exist $k \in \mathcal{I}$, and $t_3 > t_0$ such that $x_k(t_3) > \hat{\kappa}_i$ or $x_k(t_3) < \check{\kappa}_i$ and $|x_k(t_3) - \bar{x}_k| \geqslant \min\{\hat{\kappa}_i - \check{m}_i^F, \hat{m}_i^F - \check{\kappa}_i\} > \varepsilon$. Therefore, there exists $\varepsilon > 0$ such that for any $\delta \in (0, \varepsilon)$, there is an $\phi \in C([-\tau, 0], \mathbb{R}^n)$ with $\|\phi - \bar{\boldsymbol{x}}\| < \delta$ and $\|\boldsymbol{x}_{t_3} - \bar{\boldsymbol{x}}\| > \varepsilon$, for some $t_3 > t_0$. Thereafter, $\bar{\boldsymbol{x}}$ is unstable. $\square$

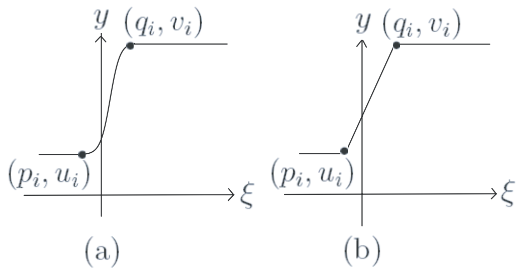

Figure 8. The graph of activation function g_i in (*a*) class $\mathcal{B}$, (*b*) class $\mathcal{C}$.

4. Extensions to other activation functions

All the results in sections 2 and 3 can be extended to the following activation functions which are also commonly employed in the literature:

$$\text{Class } \mathcal{B}: \begin{cases} g_i \in C^2,\ \exists\, p_i < q_i, g_i(\xi) = v_i \in \mathbb{R}, \quad \text{for } \xi \geqslant q_i, \quad g_i(\xi) = u_i \in \mathbb{R}, \quad \text{for } \xi \leqslant p_i, \\ \exists\, p_i < \sigma_i < q_i, g_i'(\sigma_i) > g_i'(\xi) > 0 \quad \text{and} \quad g_i''(\xi) \cdot \xi < 0, \quad \text{for } \xi \in [p_i, q_i]; \end{cases}$$

$$\text{Class } \mathcal{C}: \begin{cases} g_i \in C,\ \exists\, p_i < q_i, g_i(\xi) = v_i \in \mathbb{R}, \quad \text{for } \xi \geqslant q_i, \quad g_i(\xi) = u_i \in \mathbb{R}, \quad \text{for } \xi \leqslant p_i, \\ g_i(\xi) = u_i + (\xi - p_i)[v_i - u_i]/[q_i - p_i], \quad \text{for } p_i \leqslant \xi \leqslant q_i. \end{cases}$$

Class $\mathcal{B}$ are non-decreasing functions with flat parts on two sides. The functions in class $\mathcal{C}$ are piecewise linear; they include the standard output function in cellular neural networks [9, 16]:

$$\overline{g}(\xi) = (|\xi + 1| - |\xi - 1|)/2. \tag{4.1}$$

The graphs for these functions are depicted in figure 8. All the propositions and theorems in section 3 are valid for activation functions g_i of classes $\mathcal{B}$ and $\mathcal{C}$, if we set $\overline{p}_i := p_i$, $\overline{q}_i := q_i$, $\tilde{p}_i := p_i$ and $\tilde{q}_i := q_i$, as class $\mathcal{C}$ is considered.

If only class $\mathcal{C}$ is considered, simpler conditions yield the same conclusion.

Condition $(\mathrm{H1})_{\mathcal{C}}$: $L_i > [\mu_i + \sum_{j \neq i}(L_j|\alpha_{ij}| + L_j|\beta_{ij}|)]/(\alpha_{ii} - |\beta_{ii}|) > 0, i = 1, 2, \ldots, n.$

Condition $(\mathrm{H3})_{\mathcal{C}}$: $\check{F}_i(q_i) > 0$ and $\hat{F}_i(p_i) < 0, i = 1, 2, \ldots, n.$

Theorem 4.1. *All the assertions of theorems 3.1, 3.3 and 3.4 hold for (1.1) with activations of class $\mathcal{C}$ under conditions $(H1)_{\mathcal{C}}$ and $(H3)_{\mathcal{C}}$.*

Proof. Under conditions $(\mathrm{H1})_{\mathcal{C}}$ and $(\mathrm{H3})_{\mathcal{C}}$, the existence of 3^n equilibria for (1.1) can be justified by similar arguments as in [21], where only the standard activation function (4.1) is considered. The stability of equilibria can be verified by similar arguments as in the proof of theorem 3.4. Herein, we only sketch the proof for the convergence of dynamics. The structure of piecewise linearity of the activation function leads to a different approach.

Similarly to lemma 2.1, for each $i = 1, 2, \ldots, n$, we can construct bounded and closed intervals $[\underline{b}_i, \overline{b}_i]$. Moreover,

$$d_{b_i} := \overline{b}_i - \underline{b}_i \leqslant [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/[\alpha_{ii}L_i - \mu_i - |\beta_{ii}|L_i].$$

It can be proved that $[q_i, \infty)$ and $(-\infty, p_i]$ are positively invariant for every ith component equation (3.5), for each $i = 1, 2, \ldots, n$. Thus, $x_i(t)$ either converges to $[\underline{b}_i, \overline{b}_i]$ as t tends to infinity or enters $[q_i, \infty)$ or $(-\infty, p_i]$, for each $i = 1, 2, \ldots, n$. Define

$$\mathcal{P} := \{i \in \{1, 2, \ldots, n\}, \text{ and } x_i(t) \to [\underline{b}_i, \overline{b}_i]\}$$

$$\mathcal{D} := \{i \in \{1, 2, \ldots, n\}, \text{ and } x_i(t) \text{ enters } [q_i, \infty) \text{ or } (-\infty, p_i]\}.$$

We consider the case $\mathcal{D} \neq \emptyset$, as the situation of $\mathcal{D} = \emptyset$ is more straightforward. As in proposition 3.2 and theorem 3.3, it can be proved that the bounded and closed interval $[\underline{b}_i, \overline{b}_i]$ to which $x_i(t)$ converges is indeed a singleton, for every $i \in \mathcal{P}$, under conditions $(\mathrm{H1})_{\mathcal{C}}$ and $(\mathrm{H3})_{\mathcal{C}}$. Subsequently, $g_i(x_i(t))$ converges to some $y_i^\star \in \mathbb{R}$ as t tends to infinity, for all $i = 1, 2, \ldots, n$. Below, let us justify that $x_i(t)$ also converges as t tends to infinity, for all $i = 1, 2, \ldots, n$. Indeed, for each $i = 1, 2, \ldots, n$

$$\lim_{t\to\infty}\left\{\sum_{j=1}^{n}[\alpha_{ij}g_j(x_j(t)) + \beta_{ij}g_j(x_j(t-\tau_{ij}(t)))] + J_i\right\} = \sum_{j=1}^{n}[\alpha_{ij}y_j^\star + \beta_{ij}y_j^\star] + J_i =: h_i$$

Consider a fixed $i \in \{1, 2, \ldots, n\}$. If $h_i > 0$, then there exists $T_i > t_0$ such that $\sum_{j=1}^{n}[\alpha_{ij}g_j(x_j(t)) + \beta_{ij}g_j(x_j(t-\tau_{ij}(t)))] + J_i \geqslant h_i/2$, for all $t \geqslant T_i$. Therefore, $\int_{T_i}^{t} \mathrm{e}^{\mu_i s}\{\sum_{j=1}^{n}[\alpha_{ij}g_j(x_j(s)) + \beta_{ij}g_j(x_j(s-\tau_{ij}(s)))] + J_i\}\,\mathrm{d}s$ increases unboundedly as t increases to ∞. Thus,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\int_{T_i}^{t} \mathrm{e}^{\mu_i s}\{\sum_{j=1}^{n}[\alpha_{ij}g_j(x_j(s)) + \beta_{ij}g_j(x_j(s-\tau_{ij}(s)))] + J_i\}\,\mathrm{d}s}{\mathrm{e}^{\mu_i t}}\\
&\quad= \lim_{t\to\infty}\frac{\mathrm{e}^{\mu_i t}\{\sum_{j=1}^{n}[\alpha_{ij}g_j(x_j(t)) + \beta_{ij}g_j(x_j(t-\tau_{ij}(t)))] + J_i\}}{\mu_i\mathrm{e}^{\mu_i t}}\\
&\quad= h_i/\mu_i.
\end{aligned}$$

Applying the variation of constant formula to (1.1), we derive that for $t > T_i$,

$$\begin{aligned}
x_i(t) &= x_i(t_0)\mathrm{e}^{-\mu_i(t-t_0)} + \mathrm{e}^{-\mu_i t}\int_{t_0}^{T_i}\mathrm{e}^{\mu_i s}\left\{\sum_{j=1}^{n}[\alpha_{ij}g_j(x_j(s)) + \beta_{ij}g_j(x_j(s-\tau_{ij}(s)))] + J_i\right\}\mathrm{d}s\\
&\quad+ \mathrm{e}^{-\mu_i t}\int_{T_i}^{t}\mathrm{e}^{\mu_i s}\left\{\sum_{j=1}^{n}[\alpha_{ij}g_j(x_j(s)) + \beta_{ij}g_j(x_j(s-\tau_{ij}(s)))] + J_i\right\}\mathrm{d}s.
\end{aligned}$$

It follows that $\lim_{t\to\infty} x_i(t) = h_i/\mu_i$, due to

$$\begin{aligned}
&\lim_{t\to\infty} x_i(t_0)\mathrm{e}^{-\mu_i(t-t_0)}\\
&\quad= \lim_{t\to\infty}\mathrm{e}^{-\mu_i t}\int_{t_0}^{T_i}\mathrm{e}^{\mu_i s}\left\{\sum_{j=1}^{n}[\alpha_{ij}g_j(x_j(s)) + \beta_{ij}g_j(x_j(s-\tau_{ij}(s)))] + J_i\right\}\mathrm{d}s = 0.
\end{aligned}$$

If $h_i < 0$, similar arguments yield $\lim_{t\to\infty} x_i(t) = h_i/\mu_i$. For the case $h_i = 0$, from (4.2), for any $\varepsilon > 0$, there exists some $T_i^\varepsilon > t_0$ such that

$$\left|\sum_{j=1}^{n}[\alpha_{ij}g_j(x_j(t)) + \beta_{ij}g_j(x_j(t-\tau_{ij}(t)))] + J_i\right| \leqslant \frac{\varepsilon}{2}, \qquad \text{for all } t \geqslant T_i^\varepsilon.$$

Therefore there exists $Q_i > 0$ such that

$$\left|\sum_{j=1}^{n}[\alpha_{ij}g_j(x_j(t))\beta_{ij}g_j(x_j(t-\tau_{ij}(t)))] + J_i\right| \leqslant Q_i, \qquad \text{for all } t \geqslant t_0.$$

Since $\lim_{t\to\infty}\mathrm{e}^{-\mu_i t}\{\mathrm{e}^{\mu_i t_0}|x_i(t_0)| + \frac{Q_i}{\mu_i}(\mathrm{e}^{\mu_i T_i^\varepsilon} - \mathrm{e}^{\mu_i t_0})\} = 0$, there exists $\tilde{T}_i^\varepsilon > t_0$ such that

$$\mathrm{e}^{-\mu_i t}\left\{\mathrm{e}^{\mu_i t_0}|x_i(t_0)| + \frac{Q_i}{\mu_i}(\mathrm{e}^{\mu_i T_i^\varepsilon} - \mathrm{e}^{\mu_i t_0})\right\} < \varepsilon/2, \qquad \text{for all } t \geqslant \tilde{T}_i^\varepsilon.$$

For $t \geqslant \max\{T_i^\varepsilon, \tilde{T}_i^\varepsilon\}$,

$$\begin{aligned}
|x_i(t)| &\leqslant \mathrm{e}^{-\mu_i t}\left\{\mathrm{e}^{\mu_i t_0}|x_i(t_0)| + \int_{t_0}^{T_i^\varepsilon} \mathrm{e}^{\mu_i s}\left|\sum_{j=1}^n [\alpha_{ij} g_j(x_j(s)) + \beta_{ij} g_j(x_j(s-\tau_{ij}(s)))] + J_i\right| \mathrm{d}s\right. \\
&\quad \left. + \int_{T_i^\varepsilon}^{t} \mathrm{e}^{\mu_i s}\left|\sum_{j=1}^n [\alpha_{ij} g_j(x_j(s)) + \beta_{ij} g_j(x_j(s-\tau_{ij}(s)))] + J_i\right| \mathrm{d}s\right\} \\
&\leqslant \mathrm{e}^{-\mu_i t}\left\{\mathrm{e}^{\mu_i t_0}|x_i(t_0)| + \int_{t_0}^{T_i^\varepsilon} \mathrm{e}^{\mu_i s} Q_i \,\mathrm{d}s + \int_{T_i^\varepsilon}^{t} \mathrm{e}^{\mu_i s}\frac{\varepsilon}{2}\,\mathrm{d}s\right\} \\
&\leqslant \mathrm{e}^{-\mu_i t}\left\{\mathrm{e}^{\mu_i t_0}|x_i(t_0)| + \frac{Q_i}{\mu_i}(\mathrm{e}^{\mu_i T_i^\varepsilon} - \mathrm{e}^{\mu_i t_0})\right\} + \frac{\varepsilon}{2}\mathrm{e}^{-\mu_i t}\int_{T_i^\varepsilon}^{t} \mathrm{e}^{\mu_i s}\,\mathrm{d}s \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2\mu_i}\mathrm{e}^{-\mu_i t}\{\mathrm{e}^{\mu_i t} - \mathrm{e}^{\mu_i T_i^\varepsilon}\} \\
&< \left(\frac{1}{2} + \frac{1}{2\mu_i}\right)\varepsilon.
\end{aligned}$$

It follows that $\lim_{t\to\infty} x_i(t) = 0$, for all $i = 1, 2, \ldots, n$. □

Remark 4.1. System (1.1) with $\mu_i = 1$, $i = 1, 2, \ldots, n$, and the standard output function (4.1) is the cellular neural network which has been intensively studied in the community of electrical engineering and information science. If such a network is considered, conditions $(\mathrm{H1})_\mathcal{C}$ and $(\mathrm{H3})_\mathcal{C}$ reduce to

$$\alpha_{ii} - 1 > \sum_{j\neq i} |\alpha_{ij}| + \sum_{j=1}^n |\beta_{ij}| + |J_i|, \;\; i = 1, 2, \ldots, n. \tag{4.2}$$

For this network, it was proved in [21] that under condition (4.2), there exist exactly 3^n equilibria; in addition, 2^n among them are locally exponential stable and the others are unstable. With the same condition (4.2), our theorem 4.1 not only assures these results, but also further concludes the convergence of dynamics for the system.

5. Numerical illustrations

We give two numerical examples to illustrate the present theory. The activation function in Example 5.1 belongs to class $\mathcal{A}$. Example 5.2 demonstrates the convergence of dynamics for the delayed cellular neural networks with standard activation function which belongs to class $\mathcal{C}$, under condition (4.2).

Example 5.1. Consider the following two-dimensional system with activation functions $g_1(\xi) = g_2(\xi) = \tanh(\xi)$.

$$\begin{aligned}
\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= -x_1(t) + 7g_1(x_1(t)) + 0.1g_2(x_2(t)) - 0.5g_1(x_1(t-1)) + 0.1g_2(x_2(t-1)) - 0.1 \\
\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= -x_2(t) - 0.2g_1(x_1(t)) + 8g_2(x_2(t)) + 0.1g_1(x_1(t-1)) + 0.6g_2(x_2(t-1)).
\end{aligned}$$

Then $\hat{F}_1(\xi) = -\xi + 7g(\xi) + 0.6$, $\check{F}_1(\xi) = -\xi + 7g(\xi) - 0.8$, $\hat{F}_2(\xi) = -\xi + 8g(\xi) + 0.9$, $\check{F}_2(\xi) = -\xi + 8g(\xi) - 0.9$; $\tilde{p}_1 = -2.292\,431\,670$, $\tilde{q}_1 = 2.292\,431\,670$, $\tilde{p}_2 = -2.917\,401\,094$, $\tilde{q}_2 = 2.917\,401\,094$; $\check{m}_1^F = -0.100\,0391\,8992$, $\hat{m}_1^F = 0.134\,267\,9254$, $\hat{m}_2^F = -0.129\,391\,1878$,

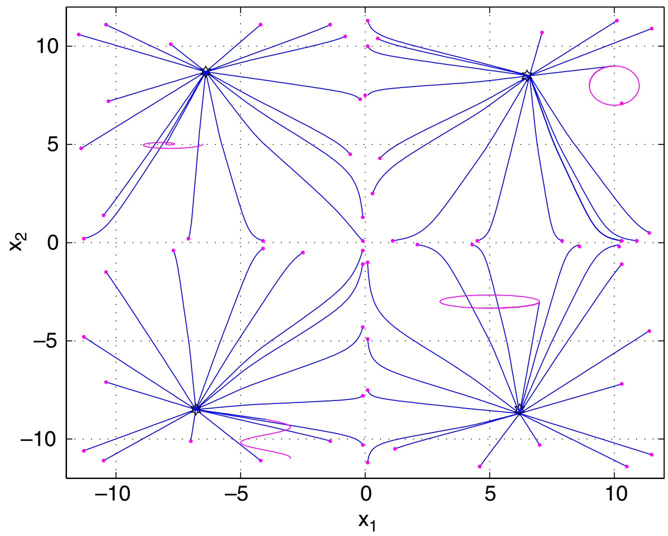

Figure 9. Numerical simulation for example 5.1, with solutions evolved from initial functions at various locations.

$\hat{m}_2^F = 0.129\,391\,1878$; $\check{\kappa}_1 = -1.238\,944\,365$, $\hat{\kappa}_1 = 1.238\,944\,365$, $\check{\kappa}_2 = -1.316\,957\,897$, $\hat{\kappa}_2 = 1.316\,957\,897$. Herein, $\check{\kappa}_1$ and $\hat{\kappa}_1$ are solutions of $g_1'(\cdot) = 2\mu_1/\alpha_{11} = 2/7$; $\check{\kappa}_2$ and $\hat{\kappa}_2$ are solutions of $g_2'(\cdot) = 2\mu_2/\alpha_{22} = 2/8$. It can be justified that conditions (H2)*, (H3) and (H4) hold as follows: condition (H2)* holds since $\min\{\mu_1 - L_1|\beta_{11}|, L_1|\beta_{11}|\} = 0.5 > L_2|\alpha_{12}| + L_2|\beta_{12}| = 0.2$ and $\min\{\mu_2 - L_2|\beta_{22}|, L_2|\beta_{22}|\} = 0.4 > L_1|\alpha_{21}| + L_1|\beta_{21}| = 0.3$; condition (H3) holds since $\check{F}_1(\tilde{q}_1) = 3.766\,139\,610 > 0$, $\hat{F}_1(\tilde{p}_1) = -3.966\,139\,610 < 0$, $\check{F}_2(\tilde{q}_2) = 4.135\,951\,278 > 0$ and $\hat{F}_2(\tilde{p}_2) = -4.135\,951\,278 < 0$; condition (H4) holds since $[\hat{m}_1^F, \check{m}_1^F] \subset [\check{\kappa}_1, \hat{\kappa}_1]$ and $[\hat{m}_2^F, \check{m}_2^F] \subset [\check{\kappa}_2, \hat{\kappa}_2]$; subsequently $g_1'(\xi) > 2\mu_1/\alpha_{11}$ for $\xi \in [\hat{m}_1^F, \check{m}_1^F]$ and $g_2'(\xi) > 2\mu_2/\alpha_{22}$ for $\xi \in [\hat{m}_2^F, \check{m}_2^F]$. Numerical simulations depicted in figures 9 and 10 demonstrate the convergence to four stable equilibria for solutions evolved from various initial conditions at different locations.

Example 5.2. The following system satisfies condition (4.2).

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -x_1(t) + 7\overline{g}_1(x_1(t)) + \overline{g}_2(x_2(t)) + 2\overline{g}_1(x_1(t-1)) - \overline{g}_2(x_2(t-1))$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -x_2(t) - \overline{g}_1(x_1(t)) + 6\overline{g}_2(x_2(t)) - \overline{g}_1(x_1(t-1)) + \overline{g}_2(x_2(t-1)) + 1,$$

where $g_i(\xi) = \overline{g}(\xi)$, $i = 1, 2$, are the standard output functions for the cellular neural network, defined in (4.1). This system satisfies condition (4.2) and admits the convergence of dynamics, according to theorem 4.1 and remark 4.1, as demonstrated numerically in figure 11.

6. Conclusions

We have presented a methodology which combines a geometric formulation and an iteration scheme to establish convergence of dynamics and confirm stability of equilibria for a multistable neural network with time-varying delays. Our approach does not employ Lyapunov-function arguments nor require the symmetry of connection weights. It is valid

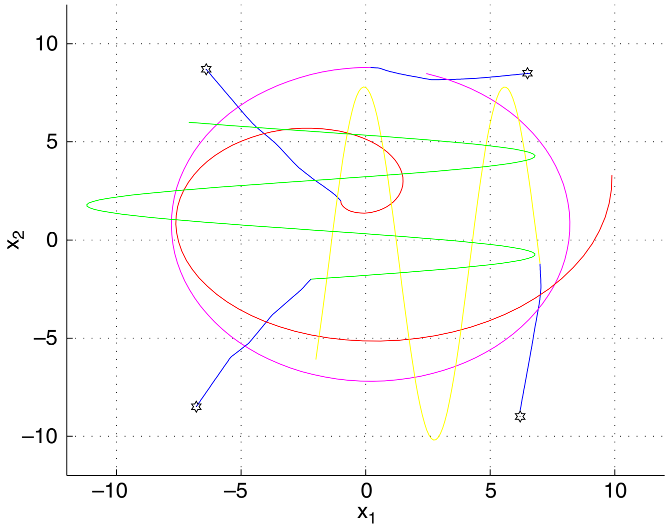

Figure 10. Numerical simulation for example 5.1, with solutions evolved from initial functions crossing the phase space.

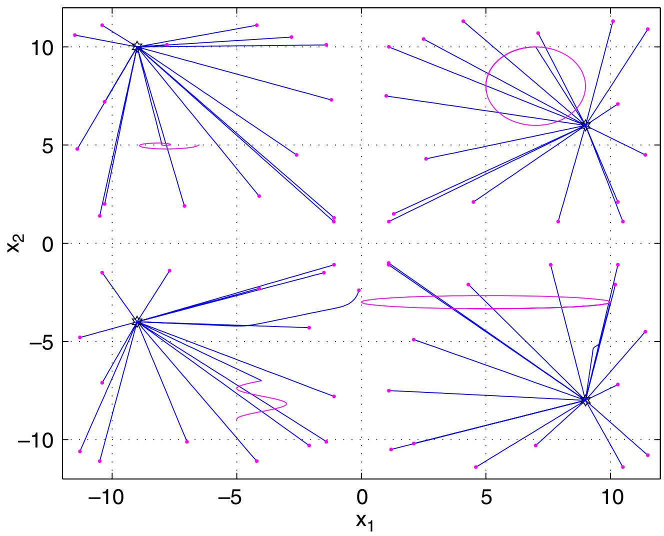

Figure 11. Numerical simulation for example 5.2.

for the commonly used activation functions. Modifications of the formulation can be further developed to derive delay-dependent criteria for the dynamics and to investigate the effect of delay magnitude upon basins of attraction for the stable equilibria.

Indeed, the following delay-dependent criteria for parallel conclusions in this presentation have been derived in [34]:

$$\text{Condition (H1): } \begin{cases} \alpha_{ii} \geqslant 0, \qquad e_i < \mu_i/[(\alpha_{ii}+\beta_{ii})L_i] < 1, \\ \lambda_i \nu_i + (1-\lambda_i)(\mu_i - \nu_i) > \sum_{j=1, j\neq i}^{n} (|\alpha_{ij}| + |\beta_{ij}|) L_j > 0, \end{cases}$$

Condition (H2): $\check{F}_i(q_i) > 0$ and $\hat{F}_i(p_i) < 0$, for $i = 1, 2, \ldots, n$.

Condition (H3): $g_i'(\xi) > [\lambda_i \nu_i + (1 - \lambda_i)(\mu_i - \nu_i) + \mu_i]/(\alpha_{ii} + \beta_{ii})$, $\xi \in [\hat{m}_i^F, \check{m}_i^F]$, for $i = 1, 2, \ldots, n$, where

$e_i := 2|\beta_{ii}|L_i\tau$, $\nu_i := e_i(\alpha_{ii} + \beta_{ii})L_i = 2|\beta_{ii}|\tau(\alpha_{ii} + \beta_{ii})L_i^2$, $\qquad \lambda_i \in [0, 1]$.

The convergence of dynamics for the system holds under these conditions which favour smaller delays. Numerical simulations demonstrate that the dynamics change as the delays become large, for the system with the same parameters.

The study has extended the exploration on how single-neuron structures contribute towards the coherent behaviour of a collection of neurons. The approach is general and can be applied to some ODE systems, delayed equations and other additive neural networks in investigating stability, monostability, multistability, basins of attraction and convergence of dynamics.

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