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GLOBAL SYNCHRONIZATION AND ASYMPTOTIC PHASES FOR A RING OF IDENTICAL CELLS WITH DELAYED COUPLING*

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Abstract. We consider a neural network which consists of a ring of identical neurons coupled with their nearest neighbors and is subject to self-feedback delay and transmission delay. We present an iteration scheme to analyze the synchronization and asymptotic phases for the system. Delay-independent, delay-dependent, and scale-dependent criteria are formulated for the global synchronization and global convergence. Under this setting, the possible asymptotic dynamics include convergence to single equilibrium, multiple equilibria, and synchronous oscillation. The study aims at elucidating the effects from the scale of network, self-decay, self-feedback strength, coupling strength, and delay magnitudes upon synchrony, convergent dynamics, and oscillation of the network. The disparity between the contents of synchrony induced by distinct factors is investigated. Two different types of multistable dynamics are distinguished. Moreover, oscillation and desynchronization induced by delays are addressed. We answer two conjectures in the literature.

Key words. neural network, delay, synchronization, oscillation, multistability

AMS subject classifications. 92B20, 34K20, 34K18, 92C20

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1. Introduction.

A population of connected neurons can generate intriguingly rich collective dynamics. For instance, coherent rhythms which play important roles in various cognitive activities are ubiquitous in nervous systems [20]. More specifically, in many regions of the brain, synchronization activity has been observed and implicated as a correlate of behavior and cognition [31]. It is known that synchronization encourages strengthening of mutual connections among neurons. In fact, synchronization is a common and elementary phenomenon in many biological and physical systems [4, 25, 29].

Coupled neural networks, as reductions from biological neuronal models or artificial motifs, manifest a variety of collective dynamics. These collective behaviors are determined by individual neuron properties, connection strength, nonlinear coupling functions, network structure, and transmission time lags. Among these factors, delay, as occurring in the propagation of action potentials along the axon, the transmission of signals across the synapse, has been an important concern in the study of neural systems.

As coherent rhythm is essential in neuronal activity, it is interesting to study what factors contribute toward synchronization and, once a neural network is synchronized, what synchronous phase is occurring. On the other hand, if the parameters or the delay magnitudes vary, the system may lose synchrony to different dynamics [8, 9]. Therefore, in addition to synchronization, it is also important to investigate the synchronous phases and their transitions. However, mathematical methodology

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and analytic theory on these issues for models under nonlinear and delayed coupling remain to be further developed.

In this investigation, we consider a neural network which comprises a ring of identical elements with nearest-neighbor coupling under a transmission delay. The individual element is determined by a scalar equation with a linear decay and nonlinear delayed feedback. This network is then modeled by a system of nonlinear delay-differential equations

$$\dot{x}_i(t) = -\mu x_i(t) + \alpha g_I(x_i(t-\tau_I)) + \beta[g_T(x_{i-1}(t-\tau_T)) + g_T(x_{i+1}(t-\tau_T))], \tag{1.1}$$

with i mod N. Herein, N is the scale of the coupled network; $\mu \geq 0$ means self-decay rate; α and β are, respectively, the synaptic strength of *self-feedback* and (nearest-neighbor) *coupling* with corresponding delays $\tau_I \geq 0$ and $\tau_T \geq 0$; g_I and g_T are the activation functions of the following class:

$$\begin{cases} g \in C^2;\ \lim_{\xi\to+\infty} g(\xi) = v \in \mathbb{R},\ \lim_{\xi\to-\infty} g(\xi) = u \in \mathbb{R}; \\ g(0) = 0;\ L := g'(0) > g'(\xi) > 0,\ \text{and } g''(\xi)\cdot\xi < 0,\ \text{for } \xi \neq 0. \end{cases} \tag{1.2}$$

A special case is $g_I = g_T = \tanh$. We say that the self-feedback (resp., coupling) is *inhibitory* if $\alpha < 0$ (resp., $\beta < 0$) and *excitatory* if $\alpha > 0$ (resp., $\beta > 0$), and the self-feedback (resp., coupling) strength is strong/weak if the magnitude of α (resp., β) is large/small. To simplify the presentation, we shall consider system (1.1) with $v = L = 1$, $u = -1$, and $g_I = g_T =: g$. Basically there is no qualitative difference between the cases $g_I = g_T$ and $g_I \neq g_T$ in our analysis. We shall focus on the effect from parameters μ, α, β, delays τ_I, τ_T, and the characteristic of g upon the dynamics of (1.1).

System (1.1), with its nearest-neighbor coupling and internal and transmission delays, represents a basic structure in neural network models. It belongs to the cellular neural type of networks [7, 26]. Let us denote the synchronous set by

$$\mathcal{S} := \{(\phi_1, \ldots, \phi_N) \in \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}^N) : \phi_i = \phi_j \text{ for all } i, j = 1, \ldots, N\}, \tag{1.3}$$

where $\tau_{\max} := \max\{\tau_I, \tau_T\}$. We say that a solution of (1.1) is *synchronous* if it lies in $\mathcal{S}$ completely; a solution is *asymptotically synchronous* if its ω-limit set lies in $\mathcal{S}$. The coupled network (1.1) is said to attain *global synchronization* if every solution is asymptotically synchronous.

System (1.1) and other similar delayed neural networks have been studied extensively. Wu and coworkers [21, 32] extended equivariant Hopf bifurcation [12] to differential equations with time delays and showed that phase-locked oscillations, mirror-reflecting waves, and standing waves may bifurcate simultaneously from the trivial solution at some critical values of the delay. The theory was applied to the Hopfield-type neural network therein. Since then, standard and equivariant bifurcations have been widely studied in delayed neural networks. For instance, standard Hopf bifurcation theory and symmetric bifurcation theory were applied to study the emergence of spatio-temporal patterns for systems analogous to (1.1) but with a single delay, i.e., $\tau_I = \tau_T$; cf. [14, 24, 33]. The direction and stability of the bifurcated periodic solutions have been analyzed in [15] by normal form theory and center manifold theory [10]. As two distinct delays raise the complexity of analysis, Campbell et al. studied systems with two delays in a series of papers. Standard Hopf bifurcation, which gives rise to synchronous periodic solutions, has been studied for (1.1) with $N = 3$ in [1] and general N in [35]. The bifurcation and stability of nontrivial asynchronous oscillations from the trivial equilibrium for (1.1) were analyzed under the D_N equivariant framework for $N = 3$ in [1] and general N in [2].

The above-mentioned research on delayed neural networks largely focuses on exploring various patterns of oscillations which are bifurcated from the trivial equilibrium as the parameters or the delay magnitudes vary. The analysis therein unfolds from the stability region of the trivial equilibrium, which is computed from the characteristic equation for the linearized system. For system (1.1), with two delays, the bifurcation diagram is rather complicated; the stability region was delineated on a two dimensional plane of parameter and delay, holding the other parameters and delay fixed [1, 35]. The combined effect from the self-feedback strength, the coupling strength, and the corresponding delays upon the collective dynamics is not apparent in those works.

The parameter and delay ranges for the stability of the trivial equilibrium were analyzed for (1.1) with $N = 3$ in [1] and general N in [35]. The dynamics of the stable synchronous equilibrium can be regarded as local synchronization for system (1.1) with trivial asymptotic phases. It is thus appealing to explore a more general regime for synchronization of system (1.1) with nontrivial asymptotic phases. While most of the above-mentioned works depict local behaviors around the trivial equilibrium, there do exist other nontrivial equilibrium points for system (1.1). Thus, to delineate the complete dynamical scenario of the system, one needs a new analytical approach.

Mathematical tools for investigating global dynamics of coupled systems and delayed equations are quite limited. One common approach is the method of Lyapunov function. Global stability for the trivial solution of (1.1) with $\mu = 1$ and general N under condition $|\alpha| + 2|\beta| < 1$ was obtained in [35]. On the other hand, it was concluded in [1] that system (1.1) with $\mu = 1$ and $N = 3$ attains the global synchronization if $|\alpha| + |\beta| < 1$. It is obvious that under $|\alpha| + 2|\beta| < 1$, the system admits trivial synchronization. Indeed, the conclusion from Lyapunov function approach, as adopted in [1, 35], often reduces to the situation that every solution converges asymptotically to a unique synchronous equilibrium point. Moreover, it is not always possible to construct Lyapunov functions for nonlinear systems, and even if this construction is possible, typically only delay-independent criteria can be derived. Therefore, it is interesting to develop a new methodology to investigate synchronization with nontrivial synchronous phases for nonlinear coupled systems.

Regarding coherent rhythms, synchronous periodic solutions bifurcated from the trivial equilibrium of (1.1) have been studied in [23, 30, 35]. However, a mathematical result on evolution toward the synchronous set $\mathcal{S}$ has not yet been reported. In fact, the bifurcation analysis was performed under the restriction to set $\mathcal{S}$ in [30]. On the other hand, we also hope to derive concrete criteria for the emergence of synchronous oscillation to confirm that our global synchronization setting can accommodate the local bifurcation analysis.

Recently, a new approach for studying global convergence to multiple equilibria for the Hopfield-type neural network was reported in [28]. Therein, an iterative scheme is developed to construct delay-independent criteria for the globally convergent dynamics. In this presentation, we shall improve and extend the formulation to derive delay-independent, delay-dependent, and scale-dependent criteria for the global synchronization and global convergence of system (1.1). As delay can induce asynchrony, delay-dependent criteria for synchronization are important in knowing when and how synchrony is sustained in systems modeled with delay. We note that the formulation, and hence the conclusion, in [28] relies on strong positive instantaneous self-feedback and weak coupling among connected neurons. That is, the established convergent dynamics are consequences of the intrinsic dynamics for individual neurons. In this investigation, the globally synchronized dynamics for system (1.1) are actually promoted by sufficiently strong interaction between coupling neurons. We remark that effective application of this iteration scheme relies on ingenious designs of upper and lower dynamics according to the targeted behaviors.

Our approach, sprouted from analyzing evolutions toward the synchronous set $\mathcal{S}$, does lead to global synchronization with several possible nontrivial asymptotic behaviors including oscillations. In addition, we answer one conjecture posed in [1] on global synchronization under a delay-dependent criterion which was formulated from local bifurcation analysis. On the other hand, in [35], it was conjectured that when N, the scale of the network, is odd, (1.1) can be synchronized if $|\alpha| + 2|\cos((N-1)\pi/N)||\beta| < 1$. The inequality indicates that the synchronization favors a smaller size of network. The criteria we derive echo this indication with theoretical evidence. We also illustrate through an example that the synchronization is attained for system (1.1) with $N = 3$, but not for $N = 4$, under the same parameters and delays.

Multistability, describing the coexistence of multiple stable patterns, has been an essential concern in several applications of neural networks, including pattern recognition and associative memory, decision making, digital section, and analog amplification [7, 11, 16]. The recent results on multistability in Hopfield-type neural networks [3, 5, 6, 28, 36] are associated with the multistability induced by strong excitatory self-feedback. The analysis therein can be extended to establish the coexistence of 3^N synchronous and asynchronous equilibria with 2^N among them being stable for system (1.1) if the self-feedback is excitatory and its strength α is sufficiently stronger than the coupling strength β. On the other hand, there is a second type of multistability which comprises 3 synchronous equilibria for (1.1) and neural networks of a similar type [13, 27, 33]. In [13], monotone dynamics theory is employed to analyze this second type of multistability and establish the "generic" convergence to 3 synchronous equilibria in a unidirectional excitatory ring of four identical neurons. Therein, the "excitatory coupling" is crucial for the network to generate an eventually strongly monotone semiflow. Wu, Faria, and Huang [33] conjectured that the generic dynamics for system (1.1) with $N = 3$, $\mu = 1$, and $\tau_I = \tau_T$ are convergence to two stable synchronous equilibria if $|\alpha - \beta| < 1$ and $\alpha + 2\beta > 1$. This conjecture was not resolved if $\alpha < 0$ or $\beta < 0$, due to the standard ordering in that region being invalid. Our iteration approach not only recasts the result in [33] but also overcomes the restriction from the monotone dynamics arguments and establishes the "global" convergence for the network which admits the multistability of the second type. The present result implicates that the second type of multistability for system (1.1) is generated by "strong excitatory coupling" among neurons.

Motivated by the above-mentioned unsolved problems in the literature and an attempt to elucidate a more complete dynamical scenario for system (1.1), the aims of this investigation are to derive combined effects from μ, α, β, τ_I, τ_T, N, g upon global synchronization of system (1.1), distinguish the differences between synchrony for (1.1) with and without delays, establish the convergence of dynamics for (1.1) which admits multistability induced by "strong excitatory coupling," and construct concrete criteria for the emergence of synchronous and asynchronous oscillations induced by delays.

The presentation is organized as follows. In section 2, we study two types of scalar equations with time-dependent input, which provide a basis for investigating the global dynamics of the coupled system (1.1). In section 3, we focus on (1.1) of scale $N = 3$; global synchronization and convergence to three synchronous equilibria of (1.1) are investigated in subsections 3.1 and 3.2, respectively. Hopf bifurcation induced by the transmission delay at the trivial equilibrium is studied in subsection 3.3. The investigations in section 3 can be carried over to system (1.1) of scale $N > 3$. In particular, in section 4, via a Gauss–Seidel argument, we modify the analysis in subsection 3.1 to establish the global synchronization for (1.1) of scale $N \geq 3$. We present three numerical illustrations in section 5.

2. Scalar equations with time-dependent input.

We shall consider two types of scalar equations in this section. The first one is deduced from (1.1) in considering synchronization and is discussed in subsection 2.1. The second one, presented in subsection 2.2, is concerned with convergent dynamics for the multidimensional system (1.1). The proofs for Lemmas 2.3 and 2.9 and Propositions 2.4–2.6 and 2.10 will be arranged in subsection 2.3 for fluency of the presentation.

The asymptotic behaviors of these scalar equations will be captured through sequential controls by upper and lower dynamics under some mild or moderate conditions. The functions governing these upper and lower dynamics shall be suitably designed according to the targeted behavior of the scalar equation.

2.1. For synchronization.

In this subsection, we introduce the scalar equation associated with the synchronization for (1.1). Let t_0 be the initial time, and let $x(t)$ and $y(t)$ be C^1 scalar functions which are eventually attracted by some closed and bounded interval $\mathcal{Q}$; namely, $x(t)$ and $y(t)$ remain in $\mathcal{Q}$, for all time $t \geq \tilde{t}_0$, for some $\tilde{t}_0 \geq t_0$. Let $w(t)$ be a bounded continuous function defined for $t \geq t_0$. Assume that $z(t) = x(t) - y(t)$ satisfies the following scalar delay-differential equation:

$$\dot{z}(t) = -\mu z(t) - \Sigma_{i=1}^2 \gamma_i [g(x(t-\tau_i)) - g(y(t-\tau_i))] + w(t), \quad t \geq t_0, \tag{2.1}$$

where $\mu > 0$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $\tau_1, \tau_2 \geq 0$, and g is of class (1.2). We denote

$$\tau := \max\{\tau_1, \tau_2\}, \quad \check{L} := \min\{g'(\xi) : \xi \in \mathcal{Q}\}. \tag{2.2}$$

The main result (Proposition 2.4) derived for (2.1) asserts that there exists a (τ_1, τ_2)-dependent bounded and closed interval containing zero to which every solution of (2.1) converges under some (τ_1, τ_2)-dependent condition. We shall also derive τ_2-independent and (τ_1, τ_2)-independent results in Propositions 2.5 and 2.6, respectively, through modified formulations.

Let us introduce some notation. For $\gamma \in \mathbb{R}$, set

$$\hat{\gamma} := \begin{cases} \gamma, & \gamma \geq 0, \\ \gamma \check{L}, & \gamma < 0, \end{cases} \quad \check{\gamma} := \begin{cases} \gamma \check{L}, & \gamma \geq 0, \\ \gamma, & \gamma < 0. \end{cases} \tag{2.3}$$

Herein, $\hat{\gamma}$ (resp., $\check{\gamma}$) expresses the largest (resp., smallest) possible value of $\gamma g'(\xi)$ for $\xi \in \mathcal{Q}$, as the maximal value of $g'(\xi)$ was set as $L = 1$ in section 1. Obviously, $\widehat{(-\gamma)} = -\check{\gamma}$, $\widecheck{(-\gamma)} = -\hat{\gamma}$, and $\hat{\gamma} \geq \check{\gamma}$. We assume $\Sigma_{i=1}^2 \check{\gamma}_i > 0$; hence $\Sigma_{i=1}^2 \hat{\gamma}_i > 0$ and $\Sigma_{i=1}^2 |\gamma_i| \neq 0$. For $T \geq t_0$, we denote $|w|^{\max}(T) := \sup\{|w(t)| : t \geq T\}$.

The following two scalar functions lead to upper and lower bounds, respectively, for the dynamics of (2.1):

$$\hat{h}(\xi) := \begin{cases} -\mu\xi + 2\Sigma_{i=1}^2 |\gamma_i| + |w|^{\max}(t_0) & \text{if } \xi \geq 0, \\ -(\mu + \Sigma_{i=1}^2 \hat{\gamma}_i)\xi + 2\Sigma_{i=1}^2 |\gamma_i| + |w|^{\max}(t_0) & \text{if } \xi < 0, \end{cases}$$

$$\check{h}(\xi) := \begin{cases} -(\mu + \Sigma_{i=1}^2 \hat{\gamma}_i)\xi - 2\Sigma_{i=1}^2 |\gamma_i| - |w|^{\max}(t_0) & \text{if } \xi \geq 0, \\ -\mu\xi - 2\Sigma_{i=1}^2 |\gamma_i| - |w|^{\max}(t_0) & \text{if } \xi < 0. \end{cases}$$

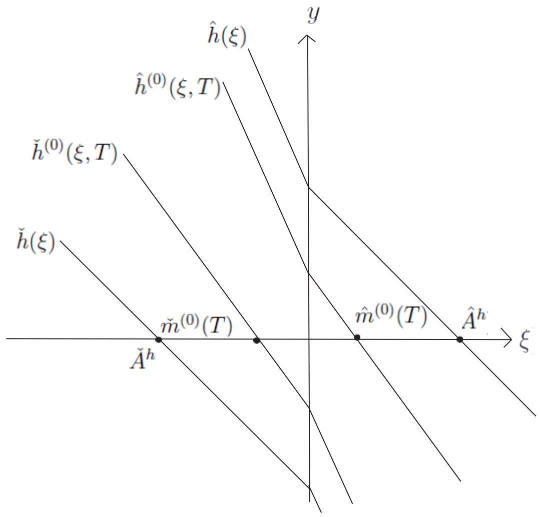

FIG. 1. *Configurations of functions $\hat{h}$, $\check{h}$, $\hat{h}^{(0)}(\cdot,T)$, and $\check{h}^{(0)}(\cdot,T)$.*

Obviously, $\hat{h}(\xi) \geq \check{h}(\xi)$ and $\hat{h}(\xi) = -\check{h}(-\xi)$. The piecewise linear functions $\hat{h}$ and $\check{h}$ are decreasing and have unique zeros at $\hat{A}^h$ and $\check{A}^h$, respectively, where $\hat{A}^h = [2\Sigma_{i=1}^2|\gamma_i| + |w|^{\max}(t_0)]/\mu \geq 0$ and $\check{A}^h = -\hat{A}^h \leq 0$; cf. Figure 1. Assume that $z(t)$ satisfies (2.1). Note that $x(t)$ and $y(t)$ in (2.1) remain in compact set $\mathcal{Q}$ for all $t \geq \tilde{t}_0$. As seen from (2.1) and the setting of g, there exists an $\varepsilon_h > 0$ such that

$$\check{h}(z(t)) + \varepsilon_h \leq \dot{z}(t) \leq \hat{h}(z(t)) - \varepsilon_h \quad \text{for all } t \geq \tilde{t}_0. \tag{2.4}$$

In other words, $\hat{h} - \varepsilon_h$ and $\check{h} + \varepsilon_h$ provide preliminary upper and lower bounds for the dynamics of (2.1). There are certainly other upper and lower bounds for (2.1); however, the design of $\hat{h}$ and $\check{h}$ is closely linked to the consecutive formulation of the sequential upper and lower dynamics for (2.1) to be introduced below. We deduce the following lemma with $x(t)$, $y(t)$, $\mathcal{Q}$, and $\tilde{t}_0$ being provided a priori in considering (2.1).

LEMMA 2.1. *Assume that $\Sigma_{i=1}^2\check{\gamma}_i > 0$. If $z(t)$ satisfies* (2.1)*, then* (2.4) *holds; subsequently, there exists a $T_{x,y} \geq \tilde{t}_0 + \tau$ such that $z(t) \in [\check{A}^h, \hat{A}^h]$ for all $t \geq T_{x,y} - \tau$. Moreover,*

$$\check{h}(\hat{A}^h) < \dot{z}(t) < \hat{h}(\check{A}^h) \quad \textit{for all } t \geq T_{x,y} - \tau.$$

Remark 2.1. If $\Sigma_{i=1}^2\hat{\gamma}_i > 0$, then $\check{h}(\hat{A}^h) = -(2 + \Sigma_{i=1}^2\hat{\gamma}_i/\mu)(2\Sigma_{i=1}^2|\gamma_i| + |w|^{\max}(t_0)) < 0$, $\hat{h}(\check{A}^h) = (2 + \Sigma_{i=1}^2\hat{\gamma}_i/\mu)(2\Sigma_{i=1}^2|\gamma_i| + |w|^{\max}(t_0)) > 0$. Moreover, $x(t)$ and $y(t)$ lie in $\mathcal{Q}$ for all $t \geq T_{x,y} - \tau$, where $T_{x,y}$ is given in Lemma 2.1.

Now, let us consider the following condition for (2.1).

Condition (H1): $\Sigma_{i=1}^2\check{\gamma}_i > 0$, $\Sigma_{i=1}^2(\tau_i|\gamma_i|) < 2\Sigma_{i=1}^2|\gamma_i|/[(2+\Sigma_{i=1}^2\hat{\gamma}_i/\mu)(2\Sigma_{i=1}^2|\gamma_i| + |w|^{\max}(t_0))]$.

The latter inequality in condition (H1) favors small delays τ_1, τ_2. From Remark 2.1, it can be computed directly that $\Sigma_{i=1}^2(\tau_i|\gamma_i|)\hat{h}(\check{A}^h) < 2\Sigma_{i=1}^2|\gamma_i|$ under condition (H1). Subsequently, there exists an $\varepsilon_0 > 0$ with $\varepsilon_0 < \varepsilon_h$ such that

$$\Sigma_{i=1}^2(\tau_i|\gamma_i|)\hat{h}(\check{A}^h) + \varepsilon_0 < 2\Sigma_{i=1}^2|\gamma_i|. \tag{2.5}$$

Below, we will define iteratively two sequences of scalar functions which can be regarded as sequential upper and lower bounds for the dynamics of (2.1) as time proceeds. This iterative construction will be used to capture the asymptotic behavior for solutions of (2.1). For each $T \geq t_0$, we introduce the following functions:

$$\hat{h}^{(0)}(\xi,T) = \begin{cases} -(\mu + \Sigma_{i=1}^2 \check{\gamma}_i)\xi + \Sigma_{i=1}^2(\tau_i|\gamma_i|)\hat{h}(\check{A}^h) + |w|^{\max}(T) + \varepsilon_0 & \text{for } \xi \geq 0, \\ -(\mu + \Sigma_{i=1}^2 \hat{\gamma}_i)\xi + \Sigma_{i=1}^2(\tau_i|\gamma_i|)\hat{h}(\check{A}^h) + |w|^{\max}(T) + \varepsilon_0 & \text{for } \xi < 0, \end{cases}$$

$$\check{h}^{(0)}(\xi,T) = -\hat{h}^{(0)}(-\xi,T).$$

The idea for the formulation of $\check{h}^{(0)}(\xi,T)$ and $\hat{h}^{(0)}(\xi,T)$, which involve the signs of ξ and γ_i, will be seen in the following discussion.

Now, with the design of functions $\hat{h}$, $\check{h}$, $\hat{h}^{(0)}$, and $\check{h}^{(0)}$, under condition (H1), we obtain for $T \geq t_0$,

$$\check{h}(\xi) < \check{h}^{(0)}(\xi,T) \leq \hat{h}^{(0)}(\xi,T) < \hat{h}(\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{2.6}$$

Thanks to (2.6), we let $\check{m}^{(0)}(T)$ (resp., $\hat{m}^{(0)}(T)$) be the unique solution of $\check{h}^{(0)}(\cdot,T) = 0$ (resp., $\hat{h}^{(0)}(\cdot,T) = 0$) lying in interval $[\check{A}^h, \hat{A}^h]$; cf. Figure 1. Note that $\check{m}^{(0)}(T) = -\hat{m}^{(0)}(T) < 0$, and if $T \geq T_{x,y}$, then $x(t-\tau_i)$ and $y(t-\tau_i) \in \mathcal{Q}$ for all $t \geq T \geq T_{x,y}$, $i = 1, 2$. Therefore, for $t \geq T \geq T_{x,y}$,

$$\begin{aligned} \dot{z}(t) &= -\mu z(t) - \Sigma_{i=1}^2 \gamma_i g'(\zeta_i) z(t-\tau_i) + w(t) \\ &= -\mu z(t) - \Sigma_{i=1}^2 \gamma_i g'(\zeta_i)[z(t) - \dot{z}(s_i)\tau_i] + w(t) \end{aligned}$$

for some $\zeta_i \in \mathcal{Q}$ and $s_i \geq t-\tau \geq T_{x,y}-\tau$. If $z(t) \geq 0$, then $\dot{z}(t) \leq -\mu z(t) - \Sigma_{i=1}^2 \check{\gamma}_i z(t) + (\Sigma_{i=1}^2 \tau_i|\gamma_i|)\hat{h}(\check{A}^h) + |w|^{\max}(T) = \hat{h}^{(0)}(z(t),T) - \varepsilon_0$ by Lemma 2.1. If $z(t) < 0$, then $\dot{z}(t) \leq -\mu z(t) - \Sigma_{i=1}^2 \hat{\gamma}_i z(t) + (\Sigma_{i=1}^2 \tau_i|\gamma_i|)\hat{h}(\check{A}^h) + |w|^{\max}(T) = \hat{h}^{(0)}(z(t),T) - \varepsilon_0$. With similar arguments for the lower bound of $\dot{z}$, we thus conclude that for each $T \geq T_{x,y}$,

$$\check{h}^{(0)}(z(t),T) + \varepsilon_0 \leq \dot{z}(t) \leq \hat{h}^{(0)}(z(t),T) - \varepsilon_0 \quad \text{for all } t \geq T \tag{2.7}$$

if $\Sigma_{i=1}^2 \check{\gamma}_i > 0$. This explains the formulation of $\check{h}^{(0)}, \hat{h}^{(0)}$ and then the formulation of $\check{h}, \hat{h}$, under which (2.6) holds.

LEMMA 2.2. *For $T \geq T_{x,y}$,* (2.7) *holds under condition* (H1). *Consequently, $z(t)$ eventually enters and stays afterward in $[\check{m}^{(0)}(T), \hat{m}^{(0)}(T)] = [-\hat{m}^{(0)}(T), \hat{m}^{(0)}(T)]$.*

Notice that interval $[\check{m}^{(0)}(T), \hat{m}^{(0)}(T)]$ in Lemma 2.2 is contained in interval $[\check{A}^h, \hat{A}^h]$ in Lemma 2.1. Now, let $\{\varepsilon_k\}_{k=1}^\infty$ be a decreasing sequence with $\varepsilon_1 < \varepsilon_0$ and $\varepsilon_k \to 0$ as $k \to \infty$. Similar to the construction of $\check{h}^{(0)}(\cdot,T)$ and $\hat{h}^{(0)}(\cdot,T)$, we define the following functions iteratively. For $k \in \mathbb{N}$ and $T \geq t_0$,

$$\hat{h}^{(k)}(\xi,T) := \begin{cases} -(\mu + \Sigma_{i=1}^2 \check{\gamma}_i)\xi + \Sigma_{i=1}^2(\tau_i|\gamma_i|)\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T),T) \\ \qquad + |w|^{\max}(T) + \varepsilon_k, & \xi \geq 0, \\ -(\mu + \Sigma_{i=1}^2 \hat{\gamma}_i)\xi + \Sigma_{i=1}^2(\tau_i|\gamma_i|)\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T),T) \\ \qquad + |w|^{\max}(T) + \varepsilon_k, & \xi < 0, \end{cases}$$

$$\check{h}^{(k)}(\xi,T) := -\hat{h}^{(k)}(-\xi,T),$$

where $\check{m}^{(k)}(T)$ (resp., $\hat{m}^{(k)}(T)$) is the unique solution of $\check{h}^{(k)}(\cdot,T) = 0$ (resp., $\hat{h}^{(k)}(\cdot,T) = 0$). Observe that $\check{m}^{(k)}(T) = -\hat{m}^{(k)}(T) \leq 0$. Let us define

$$|w|^{\max}(\infty) := \lim_{T\to\infty} |w|^{\max}(T).$$

Referring index k to the kth iteration, these $\hat{h}^{(k)}(\cdot,T)$ and $\hat{h}^{(k)}(\cdot,T)$ are formulated to provide further delicate upper and lower bounds, respectively, for dynamics of (2.1), as k and T increase. More precisely, $\hat{h}^{(k)}(\cdot,T)$ (resp., $\check{h}^{(k)}(\cdot,T)$) decreases (resp., increases) with respect to k and T; accordingly, $[\check{m}^{(k)}(T),\hat{m}^{(k)}(T)] = [-\hat{m}^{(k)}(T),\hat{m}^{(k)}(T)]$ shrinks to some interval, say $[-m_p,m_p]$, as $k\to\infty$, $T\to\infty$. Moreover, it can be shown that if $z(t)$ satisfies (2.1), then for each $T\geq T_{x,y}$ and $k\in\mathbf{N}$, $z(t)$ converges to $[-\hat{m}^{(k)}(T),\hat{m}^{(k)}(T)]$ as $t\to\infty$. Consequently, $z(t)$ converges to interval $[-m_p,m_p]$ as $t\to\infty$. We summarize these properties in the following lemma and proposition.

Lemma 2.3. *Assume that condition* (H1) *holds. Then, for each $T\geq t_0$, the sequences $\{\hat{m}^{(k)}(T)\}_{k\geq 0}$ can be defined iteratively. Moreover, the following hold:*

(i) *for any fixed $k\in\mathbb{N}\cup\{0\}$, $\hat{m}^{(k)}(T)$ is decreasing with respect to $T\geq t_0$;*

(ii) *for any $T\geq t_0$, there exists $m(T)\geq 0$ such that $\hat{m}^{(k)}(T)\to m(T)$ decreasingly as $k\to\infty$;*

(iii) *there exists $m_p\geq 0$ such that $m(T)\to m_p$ decreasingly as $T\to\infty$;*

(iv) $0\leq m(T)=|w|^{\max}(T)/[\mu+\Sigma_{i=1}^2\check{\gamma}_i-\Sigma_{i=1}^2(\tau_i|\gamma_i|)(2\mu+\Sigma_{i=1}^2\hat{\gamma}_i+\Sigma_{i=1}^2\check{\gamma}_i)]$ *for any $T\geq t_0$;*

(v) $\cap_{T\geq t_0}[-m(T),m(T)]=[-m_p,m_p]$, *and*

$$0\leq m_p\leq\frac{|w|^{\max}(\infty)}{\mu+\Sigma_{i=1}^2\check{\gamma}_i-\Sigma_{i=1}^2(\tau_i|\gamma_i|)(2\mu+\Sigma_{i=1}^2\hat{\gamma}_i+\Sigma_{i=1}^2\check{\gamma}_i)}.$$

Proposition 2.4. *If $z(t)$ satisfies* (2.1), *then $z(t)$ converges to interval $[-m_p, m_p]$ as $t\to\infty$ under condition* (H1).

Through the modified setting of $\check{h}^{(k)},\hat{h}^{(k)}$ (given in subsection 2.3), we can also establish τ_2-independent and (τ_1,τ_2)-independent conclusions.

Proposition 2.5. *If $z(t)$ satisfies* (2.1), *then $z(t)$ converges to an interval $[-m_q,m_q]$ as $t\to\infty$ under condition* (H2): $\gamma_1>0$ *and* $\tau_1\gamma_1(2+\gamma_1/\mu)(2\Sigma_{i=1}^2|\gamma_i|+|w|^{\max}(t_0))<2\Sigma_{i=1}^2|\gamma_i|(1-|\gamma_2|/\mu)-|\gamma_2||w|^{\max}(t_0)/\mu$*; moreover,*

$$0\leq m_q\leq\frac{|w|^{\max}(\infty)}{\mu+\gamma_1\check{L}-|\gamma_2|-\tau_1\gamma_1(2\mu+\gamma_1+\gamma_1\check{L})}.$$

Proposition 2.6. *If $z(t)$ satisfies* (2.1), *then $z(t)$ converges to an interval $[-m_r,m_r]$ as $t\to\infty$ under condition* (H3): $\Sigma_{i=1}^2|\gamma_i|<\mu-|w|^{\max}(t_0)/2$*; moreover,*

$$0\leq m_r\leq |w|^{\max}(\infty)/(\mu-\Sigma_{i=1}^2|\gamma_i|).$$

2.2. For convergent dynamics. For convergent dynamics of (1.1), we consider the following scalar delay-differential equation with time-dependent external input $E(t)$:

$$\dot{x}(t)=-\mu x(t)+\Sigma_{i=1}^2\gamma_i g(x(t-\tau_i))+E(t), \tag{2.8}$$

where $\mu>0,\gamma_1,\gamma_2\in\mathbb{R}$; $\tau_1,\tau_2\geq 0$; $E(t)$ is a bounded continuous function defined for $t\geq t_0$, and $E(t)\to 0$ as $t\to\infty$; g is an activation function of class (1.2). We also denote $\tau:=\max\{\tau_1,\tau_2\}$.

The main result (Proposition 2.10) in this subsection asserts that there exist three points and every solution of (2.8) converges to one of them. First, we consider the special form of (2.8) with $E(t)$ being identically zero:

$$\dot{x}(t)=-\mu x(t)+\Sigma_{i=1}^2\gamma_i g(x(t-\tau_i)). \tag{2.9}$$

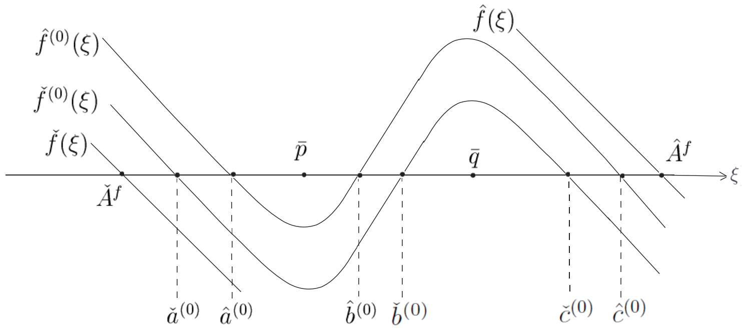

FIG. 2. *Configurations of functions* $\check{f}, \hat{f}$, $\check{f}^{(0)}$, *and* $\hat{f}^{(0)}$.

By employing arguments parallel to the ones in section 2.1, we shall define iteratively sequences of upper and lower functions for the dynamics of (2.9) as time proceeds, to capture the asymptotic behavior of (2.9). We outline only the main process.

Throughout this subsection, we assume that $\Sigma_{i=1}^2 \gamma_i > 0$; hence $\Sigma_{i=1}^2 |\gamma_i| > 0$. First, we define

$$\hat{f}(\xi) := -\mu\xi + 2\Sigma_{i=1}^2|\gamma_i|, \quad \check{f}(\xi) := -\mu\xi - 2\Sigma_{i=1}^2|\gamma_i|.$$

The decreasing linear functions $\hat{f}, \check{f}$ have unique zeros at $\hat{A}^f$ and $\check{A}^f$, respectively, where $\hat{A}^f := 2\Sigma_{i=1}^2|\gamma_i|/\mu$, $\check{A}^f := -2\Sigma_{i=1}^2|\gamma_i|/\mu$; cf. Figure 2. Let $x(t) = x(t; t_0, \phi)$ be the solution of (2.9) evolved from initial value ϕ at $t = t_0$. We have the following upper and lower bounds for the dynamics of (2.9):

$$\check{f}(x(t)) + \Sigma_{i=1}^2|\gamma_i| \le \dot{x}(t) \le \hat{f}(x(t)) - \Sigma_{i=1}^2|\gamma_i| \quad \text{for } t \ge t_0. \tag{2.10}$$

Let us summarize.

LEMMA 2.7. *Assume that* $x(t) = x(t; t_0, \phi)$ *is a solution of* (2.9)*; then* (2.10) *holds. Consequently, for any initial value* ϕ, *there exists some* $T_\phi \ge t_0$ *so that* $x(t)$ *lies in* $[\check{A}^f, \hat{A}^f]$, *and hence* $-3\Sigma_{i=1}^2|\gamma_i| \le \dot{x}(t) \le 3\Sigma_{i=1}^2|\gamma_i|$ *for all* $t \ge T_\phi - \tau$.

Let us define $f(\xi) := -\mu\xi + \Sigma_{i=1}^2\gamma_i g(\xi)$; then $\tilde{f}'(\xi) = -\mu + \Sigma_{i=1}^2\gamma_i g'(\xi)$ for any vertical shift $\tilde{f}$ of f. If $\Sigma_{i=1}^2\gamma_i > \mu$, there exist exactly two points $\bar{p}$, $\bar{q}$ with $\bar{p} < 0 < \bar{q}$ such that $\tilde{f}'(\bar{p}) = \tilde{f}'(\bar{q}) = 0$, $\tilde{f}'(\xi) > 0$ for $\xi \in (\bar{p}, \bar{q})$, and $\tilde{f}'(\xi) < 0$ for $\xi \in \mathbb{R} \setminus [\bar{p}, \bar{q}]$. Restated, if $\Sigma_{i=1}^2\gamma_i > \mu$, then $\bar{p}$ and $\bar{q}$ are the only two critical points of $\tilde{f}$, and

$$g'(\bar{p}) = g'(\bar{q}) = \mu/\Sigma_{i=1}^2\gamma_i. \tag{2.11}$$

Now, we introduce the condition for convergence to three equilibrium points.

Condition (G1): $\Sigma_{i=1}^2\gamma_i > \mu$, $\Sigma_{i=1}^2|\gamma_i|\tau_i < \min\{1/3, [\Sigma_{i=1}^2\gamma_i g(\bar{q}) - \mu\bar{q}]/(3\Sigma_{i=1}^2|\gamma_i|), [\mu\bar{p} - \Sigma_{i=1}^2\gamma_i g(\bar{p})]/(3\Sigma_{i=1}^2|\gamma_i|)\}$.

Basically, condition (G1) requires that $\Sigma_{i=1}^2\gamma_i > \mu$ and delays τ_i, $i = 1, 2$, be small; in particular, if $\Sigma_{i=1}^2|\gamma_i|\tau_i < 1/3$, then $(\Sigma_{i=1}^2|\gamma_i|\tau_i)(3\Sigma_{i=1}^2|\gamma_i|) < \Sigma_{i=1}^2|\gamma_i|$. Thus, there exists an $\varepsilon_0 > 0$ such that

$$\begin{aligned} &(\Sigma_{i=1}^2|\gamma_i|\tau_i)(3\Sigma_{i=1}^2|\gamma_i|) + \varepsilon_0 < \Sigma_{i=1}^2|\gamma_i|, \\ &\Sigma_{i=1}^2|\gamma_i|\tau_i < \min\left\{\frac{\Sigma_{i=1}^2\gamma_i g(\bar{q}) - \mu\bar{q} - \varepsilon_0}{3\Sigma_{i=1}^2|\gamma_i|}, \ \frac{\mu\bar{p} - \Sigma_{i=1}^2\gamma_i g(\bar{p}) - \varepsilon_0}{3\Sigma_{i=1}^2|\gamma_i|}\right\}. \end{aligned} \tag{2.12}$$

We further introduce the following vertical shifts of f:

$$\begin{aligned} \hat{f}^{(0)}(\xi) &:= -\mu\xi + \Sigma_{i=1}^2\gamma_i g(\xi) + (\Sigma_{i=1}^2|\gamma_i|\tau_i)(3\Sigma_{i=1}^2|\gamma_i|) + \varepsilon_0, \\ \check{f}^{(0)}(\xi) &:= -\mu\xi + \Sigma_{i=1}^2\gamma_i g(\xi) - (\Sigma_{i=1}^2|\gamma_i|\tau_i)(3\Sigma_{i=1}^2|\gamma_i|) - \varepsilon_0. \end{aligned}$$

From (2.12), it yields that

$$\check{f}(\xi) < \check{f}^{(0)}(\xi) \le \hat{f}^{(0)}(\xi) < \hat{f}(\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{2.13}$$

In addition, under condition (G1), $\bar{p}$ and $\bar{q}$ are well defined with $\check{f}^{(0)}(\bar{q}) > 0$, $\hat{f}^{(0)}(\bar{p}) < 0$. Accordingly, there exist three solutions $\check{a}^{(0)}$, $\check{b}^{(0)}$, $\check{c}^{(0)}$ (resp., $\hat{a}^{(0)}$, $\hat{b}^{(0)}$, $\hat{c}^{(0)}$) to $\check{f}^{(0)}(\cdot) = 0$ (resp., $\hat{f}^{(0)}(\cdot) = 0$), with $\check{A}^f \le \check{a}^{(0)} \le \hat{a}^{(0)} < \bar{p} < \hat{b}^{(0)} \le \check{b}^{(0)} < \bar{q} < \check{c}^{(0)} \le \hat{c}^{(0)} \le \hat{A}^f$; cf. Figure 2.

According to Lemma 2.7, by arguments similar to the ones for (2.7), we can derive that

$$\check{f}^{(0)}(x(t)) + \varepsilon_0 \le \dot{x}(t) \le \hat{f}^{(0)}(x(t)) - \varepsilon_0 \quad \text{for } t \ge T_\phi. \tag{2.14}$$

We thus conclude the following proposition.

PROPOSITION 2.8. *Inequality* (2.14) *holds under condition* (G1). *Subsequently, if* $x(t)$ *lies in* $[\check{A}^f, \check{a}^{(0)}]$ *or* $[\hat{a}^{(0)}, \hat{b}^{(0)})$ *(resp.,* $(\check{b}^{(0)}, \check{c}^{(0)}]$ *or* $[\hat{c}^{(0)}, \hat{A}^f]$*) for some* $t \ge T_\phi$*, then* $x(t)$ *eventually enters into* $[\check{a}^{(0)}, \hat{a}^{(0)}]$ *(resp.,* $[\check{c}^{(0)}, \hat{c}^{(0)}]$*). In addition, once* $x(t)$ *enters into* $[\check{a}^{(0)}, \hat{a}^{(0)}]$ *or* $[\check{c}^{(0)}, \hat{c}^{(0)}]$ *at time* $t \ge T_\phi$*, it remains in the interval thereafter.*

Proposition 2.8 depicts a trichotomy for the behavior of $x(t)$; i.e., $x(t)$ either remains in $[\hat{b}^{(0)}, \check{b}^{(0)}]$ or is attracted to $[\check{a}^{(0)}, \hat{a}^{(0)}]$ or $[\check{c}^{(0)}, \hat{c}^{(0)}]$ eventually. Accordingly, we can define certain functions iteratively to establish finer upper and lower bounds for the dynamics of (2.9). Note that the formulation for such functions depends on the sign of γ_i. For simplicity, we present only the case $\gamma_i \ge 0$, $i = 1, 2$; the formulation can be adapted to negative γ_i. First, let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a decreasing sequence with $\varepsilon_1 < \varepsilon_0$ and $\varepsilon_k \to 0$ as $k \to \infty$. Those functions are defined as

$$\begin{aligned}
\hat{f}_{\mathrm{l}}^{(k)}(\xi) &:= -\mu\xi + \Sigma_{i=1}^2 \gamma_i g(\xi) - (\Sigma_{i=1}^2 \gamma_i \tau_i) \check{f}_{\mathrm{l}}^{(k-1)}(\hat{a}^{(k-1)}) + \varepsilon_k, \\
\check{f}_{\mathrm{l}}^{(k)}(\xi) &:= -\mu\xi + \Sigma_{i=1}^2 \gamma_i g(\xi) - (\Sigma_{i=1}^2 \gamma_i \tau_i) \hat{f}_{\mathrm{l}}^{(k-1)}(\check{a}^{(k-1)}) - \varepsilon_k, \\
\hat{f}_{\mathrm{m}}^{(k)}(\xi) &:= -\mu\xi + \Sigma_{i=1}^2 \gamma_i g(\xi) - (\Sigma_{i=1}^2 \gamma_i \tau_i) \check{f}_{\mathrm{m}}^{(k-1)}(\hat{b}^{(k-1)}) + \varepsilon_k, \\
\check{f}_{\mathrm{m}}^{(k)}(\xi) &:= -\mu\xi + \Sigma_{i=1}^2 \gamma_i g(\xi) - (\Sigma_{i=1}^2 \gamma_i \tau_i) \hat{f}_{\mathrm{m}}^{(k-1)}(\check{b}^{(k-1)}) - \varepsilon_k, \\
\hat{f}_{\mathrm{r}}^{(k)}(\xi) &:= -\mu\xi + \Sigma_{i=1}^2 \gamma_i g(\xi) - (\Sigma_{i=1}^2 \gamma_i \tau_i) \check{f}_{\mathrm{r}}^{(k-1)}(\hat{c}^{(k-1)}) + \varepsilon_k, \\
\check{f}_{\mathrm{r}}^{(k)}(\xi) &:= -\mu\xi + \Sigma_{i=1}^2 \gamma_i g(\xi) - (\Sigma_{i=1}^2 \gamma_i \tau_i) \hat{f}_{\mathrm{r}}^{(k-1)}(\check{c}^{(k-1)}) - \varepsilon_k,
\end{aligned}$$

where $k \in \mathbb{N}$, $\check{f}_{\mathrm{l}}^{(0)} = \check{f}_{\mathrm{m}}^{(0)} = \check{f}_{\mathrm{r}}^{(0)} := \check{f}^{(0)}$; $\hat{f}_{\mathrm{l}}^{(0)} = \hat{f}_{\mathrm{m}}^{(0)} = \hat{f}_{\mathrm{r}}^{(0)} := \hat{f}^{(0)}$; $\check{a}^{(k)}$ (resp., $\check{b}^{(k)}$, $\check{c}^{(k)}$) is the unique solution of $\check{f}_{\mathrm{l}}^{(k)}(\cdot) = 0$ (resp., $\check{f}_{\mathrm{m}}^{(k)}(\cdot) = 0$, $\check{f}_{\mathrm{r}}^{(k)}(\cdot) = 0$) lying in interval $[\check{a}^{(0)}, \hat{a}^{(0)}]$ (resp., $[\hat{b}^{(0)}, \check{b}^{(0)}]$, $[\check{c}^{(0)}, \hat{c}^{(0)}]$); $\hat{a}^{(k)}$ (resp., $\hat{b}^{(k)}$, $\hat{c}^{(k)}$) is the unique solution of $\hat{f}_{\mathrm{l}}^{(k)}(\cdot) = 0$ (resp., $\hat{f}_{\mathrm{m}}^{(k)}(\cdot) = 0$, $\hat{f}_{\mathrm{r}}^{(k)}(\cdot) = 0$) lying in interval $[\check{a}^{(0)}, \hat{a}^{(0)}]$ (resp., $[\hat{b}^{(0)}, \check{b}^{(0)}]$, $[\check{c}^{(0)}, \hat{c}^{(0)}]$). The following lemma summarizes the properties for zeros of the above-defined sequences of single-variable functions.

LEMMA 2.9. *Assume that condition* (G1) *holds; then sequences* $\{\check{a}^{(k)}\}_{k\ge0}$, $\{\hat{a}^{(k)}\}_{k\ge0}$, $\{\check{b}^{(k)}\}_{k\ge0}$, $\{\hat{b}^{(k)}\}_{k\ge0}$, $\{\check{c}^{(k)}\}_{k\ge0}$, $\{\hat{c}^{(k)}\}_{k\ge0}$ *can be defined iteratively. There exist* $\underline{a}$, $\underline{b}$, $\underline{c}$, $\overline{a}$, $\overline{b}$, $\overline{c} \in \mathbb{R}$ *such that* $\check{a}^{(k)} \to \underline{a}$, $\hat{b}^{(k)} \to \underline{b}$, $\check{c}^{(k)} \to \underline{c}$ *increasingly, and* $\hat{a}^{(k)} \to \overline{a}$, $\check{b}^{(k)} \to \overline{b}$, $\hat{c}^{(k)} \to \overline{c}$ *decreasingly, as* $k \to \infty$*. Moreover,* $\overline{a} = \underline{a} =: a < 0$, $\overline{b} = \underline{b} = 0$, $\overline{c} = \underline{c} =: c > 0$.

Now, we shall depict the trichotomy for the dynamics of (2.9) at successive time steps. Restated, every solution $x(t)$ of (2.9) may remain in $[\hat{b}^{(k)}, \check{b}^{(k)}]$ for all k; otherwise, $x(t)$ is either attracted by $[\check{a}^{(k)}, \hat{a}^{(k)}]$ or $[\check{c}^{(k)}, \hat{c}^{(k)}]$ for all k. By Lemma 2.9,

$[\check{a}^{(k)}, \hat{a}^{(k)}]$, $[\hat{b}^{(k)}, \check{b}^{(k)}]$, and $[\check{c}^{(k)}, \hat{c}^{(k)}]$ shrink into singletons a, 0, and c, respectively, as $k \to \infty$. Accordingly, every solution $x(t)$ of (2.9) converges to the set $\{a, 0, c\}$ as $t \to \infty$. As the derivation is concerned with the asymptotic behavior of all solutions to the equation, it is not difficult to modify the arguments and extend the result to (2.8). We summarize these arguments.

PROPOSITION 2.10. *Assume that condition* (G1) *holds. Let $x(t)$ be a solution of* (2.8) *or* (2.9). *Then $x(t) \to a$ or 0 or c as $t \to \infty$.*

Notably, by modifying the formulation for Proposition 2.10, we can also derive τ_1-independent and τ_2-independent results.

Remark 2.2. The condition $\Sigma_{i=1}^2 \gamma_i > \mu$ in Proposition 2.10 plays a dominant role for the convergence to multiple points for solutions of (2.8) and (2.9). Indeed, if $\Sigma_{i=1}^2 \gamma_i$ is small (smaller than μ basically) instead, then (2.8) and (2.9) will admit the convergence to the origin if the delays are small.

2.3. Proofs of lemmas and propositions. We arrange the proofs of Lemmas 2.3 and 2.9 and Propositions 2.4–2.6 and 2.10 in this subsection. The definitions of upper and lower functions $\hat{h}^{(k)}$, $\check{h}^{(k)}$, $\hat{f}_\varsigma^{(k)}$, $\check{f}_\varsigma^{(k)}$, $\varsigma =$ l, m, r, involve their previous $(k-1)$-step and time delay τ_i and are much more complicated than the setting in [28]. The justifications for the assertions in these lemmas and propositions thus require new techniques.

Proof of Lemma 2.3. It is straightforward to verify assertions (i)–(iii); we sketch only the arguments. First, by mathematical induction, we can show that

$$\check{h}^{(k-1)}(\xi, T) \le \check{h}^{(k)}(\xi, T) \le \hat{h}^{(k)}(\xi, T) \le \hat{h}^{(k-1)}(\xi, T) \tag{2.15}$$

for all $\xi \in \mathbb{R}$, $k \in \mathbb{N}$. Accordingly, we have $\check{h}(\xi) < \check{h}^{(k)}(\xi, T) \le \hat{h}^{(k)}(\xi, T) < \hat{h}(\xi)$ for all $\xi \in \mathbb{R}$, $k \in \mathbb{N} \cup \{0\}$, with the help of (2.6). As $\check{h}^{(k)}(\cdot, T)$ and $\hat{h}^{(k)}(\cdot, T)$ are vertical shifts of $\check{h}(\cdot)$ and $\hat{h}(\cdot)$, respectively, both $\check{m}^{(k)}(T)$ and $\hat{m}^{(k)}(T)$ are well defined for all $k \in \mathbb{N} \cup \{0\}$ under condition (H1). Note that $\check{h}^{(k)}(\xi, T) = -\hat{h}^{(k)}(-\xi, T)$ and the term $|w|^{\max}(T)$ in $\check{h}^{(k)}(\cdot, T)$ is decreasing with respect to T. Accordingly, $\hat{m}^{(k)}(T) = -\check{m}^{(k)}(T)$ is decreasing with respect to both T and k, and thus assertions (i)–(iii) follow. Obviously, the assertion (v) follows from assertion (iv).

Let us consider item (iv). It is obvious that $m(T) \ge 0$ for all $T \ge t_0$. We now estimate $m(T)$. For any fixed $T \ge t_0$, $\{\hat{h}^{(k)}(\cdot, T)|_{[\check{A}^h, \hat{A}^h]}\}_{k \ge 1}$ are uniformly bounded and equicontinuous; in addition, $\hat{h}^{(k)}(\cdot, T)$ decreases with respect to k. There exists a continuous function $\hat{h}^{(\infty)}(\cdot, T)$ defined on $[\check{A}^h, \hat{A}^h]$ such that

$$\hat{h}^{(k)}(\cdot, T) \downarrow \hat{h}^{(\infty)}(\cdot, T) \text{ uniformly on } [\check{A}^h, \hat{A}^h] \text{ as } k \to \infty \tag{2.16}$$

by the Ascoli–Azela theorem. As $\check{h}^{(k)}(\xi, T) = -\hat{h}^{(k)}(-\xi, T)$, we derive

$$\check{h}^{(k)}(\xi, T) \uparrow \check{h}^{(\infty)}(\xi, T) := -\hat{h}^{(\infty)}(-\xi, T) \text{ uniformly for } \xi \in [\check{A}^h, \hat{A}^h] \text{ as } k \to \infty.$$

With (2.16), $\hat{m}^{(k)}(T) \to m(T)$, and the continuity of $\hat{h}^{(k)}$ and $\hat{h}^{(\infty)}$, we derive the following properties for $\hat{h}^{(\infty)}(\cdot, T)$:

$$\text{(P1): } \hat{h}^{(\infty)}(\xi, T) = \begin{cases} -(\mu + \Sigma_{i=1}^2 \check{\gamma}_i)\xi + \Sigma_{i=1}^2 (\tau_i |\gamma_i|)\hat{h}^{(\infty)}(-m(T), T) \\ \quad + |w|^{\max}(T), & \xi \ge 0, \\ -(\mu + \Sigma_{i=1}^2 \hat{\gamma}_i)\xi + \Sigma_{i=1}^2 (\tau_i |\gamma_i|)\hat{h}^{(\infty)}(-m(T), T) \\ \quad + |w|^{\max}(T), & \xi < 0, \end{cases}$$

(P2): $\hat{h}^{(\infty)}(m(T), T) = 0$.

According to properties (P1) and (P2), $\hat{h}^{(\infty)}(\cdot,T)$ is a strictly decreasing function and has a unique zero at $m(T)$. Due to $\hat{h}^{(\infty)}(-m(T),T) = (\mu+\Sigma_{i=1}^{2}\hat{\gamma}_i)m(T) + \Sigma_{i=1}^{2}(\tau_i|\gamma_i|)\hat{h}^{(\infty)}(-m(T),T) + |w|^{\max}(T)$, we derive

$$0 \le \hat{h}^{(\infty)}(-m(T),T) = \frac{(\mu+\Sigma_{i=1}^{2}\hat{\gamma}_i)m(T) + |w|^{\max}(T)}{1-\Sigma_{i=1}^{2}(\tau_i|\gamma_i|)}.$$

Consequently, for $\xi \ge 0$,

$$\begin{aligned}\hat{h}^{(\infty)}(\xi,T) &= -(\mu+\Sigma_{i=1}^{2}\check{\gamma}_i)\xi \\ &\quad + \frac{\Sigma_{i=1}^{2}(\tau_i|\gamma_i|)[(\mu+\Sigma_{i=1}^{2}\hat{\gamma}_i)m(T)+|w|^{\max}(T)]}{1-\Sigma_{i=1}^{2}(\tau_i|\gamma_i|)} + |w|^{\max}(T).\end{aligned}$$

From $\hat{h}^{(\infty)}(m(T),T)=0$, we obtain

$$m(T) = |w|^{\max}(T)/[\mu+\Sigma_{i=1}^{2}\check{\gamma}_i - \Sigma_{i=1}^{2}(\tau_i|\gamma_i|)(2\mu+\Sigma_{i=1}^{2}\hat{\gamma}_i+\Sigma_{i=1}^{2}\check{\gamma}_i)].$$

Proof of Proposition 2.4. Let $z(t)$ be a solution to (2.1). By the spirit in concluding (2.7) and Lemma 2.2 it is not difficult to verify by induction that for arbitrarily fixed $T \ge T_{x,y}$, and $n\in\mathbb{N}$, there exists an increasing sequence $\{T_k\}_{k=0}^{n}$ with $T_{k+1}\ge T_k+\tau$, for $k=0,1,\dots,n-1$ and $T_0 \ge T+\tau$, such that

$$\begin{cases} \check{h}^{(k)}(z(t),T)+\varepsilon_k \le \dot{z}(t) \le \hat{h}^{(k)}(z(t),T)-\varepsilon_k \text{ for } t\ge T_k+\tau,\ k=0,1,\dots,n-1,\\ z(t)\in[\check{m}^{(k)}(T),\hat{m}^{(k)}(T)] \text{ for } t\ge T_{k+1},\ k=0,1,\dots,n-1.\end{cases}$$

This then leads to that for each $T\ge T_{x,y}$, $z(t)$ converges to $[-m(T),m(T)]$ as $t\to\infty$. Subsequently, $z(t)$ converges to $[-m_p,m_p]$ as $t\to\infty$.

Proof of Proposition 2.5. The proof resembles the one for Proposition 2.4 by recomposing the upper and lower formulations:

$$\hat{h}(\xi) := \begin{cases} -\mu\xi + 2\Sigma_{i=1}^{2}|\gamma_i| + |w|^{\max}(t_0), & \xi\ge 0,\\ -(\mu+\gamma_1)\xi + 2\Sigma_{i=1}^{2}|\gamma_i| + |w|^{\max}(t_0), & \xi<0;\end{cases}$$

$$\hat{h}^{(0)}(\xi,T) := \begin{cases} -(\mu+\gamma_1\check{L})\xi + \tau_1\gamma_1\hat{h}(\check{A}^h) + |\gamma_2|\hat{A}^h + |w|^{\max}(T)+\varepsilon_0, & \xi\ge 0,\\ -(\mu+\gamma_1)\xi + \tau_1\gamma_1\hat{h}(\check{A}^h) + |\gamma_2|\hat{A}^h + |w|^{\max}(T)+\varepsilon_0, & \xi<0;\end{cases}$$

$$\hat{h}^{(k)}(\xi,T) := \begin{cases} -(\mu+\gamma_1\check{L})\xi + \tau_1\gamma_1\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T),T) + |\gamma_2|\hat{m}^{(k-1)}(T) & \\ \qquad + |w|^{\max}(T)+\varepsilon_k, & \xi\ge 0,\\ -(\mu+\gamma_1)\xi + \tau_1\gamma_1\hat{h}^{(k-1)}(\check{m}^{(k-1)}(T),T) + |\gamma_2|\hat{m}^{(k-1)}(T) & \\ \qquad + |w|^{\max}(T)+\varepsilon_k, & \xi<0;\end{cases}$$

$$\check{h}(\xi) := -\hat{h}(-\xi),\quad \check{h}^{(0)}(\xi,T) := -\hat{h}^{(0)}(-\xi,T),\quad \check{h}^{(k)}(\xi,T) := -\hat{h}^{(k)}(-\xi,T).$$

Proof of Proposition 2.6. The proof resembles the one for Proposition 2.4 by recomposing the formulation for the upper and lower functions:

$$\begin{aligned}&\hat{h}(\xi) := -\mu\xi + 2\Sigma_{i=1}^2|\gamma_i| + |w|^{\max}(t_0),\\ &\hat{h}^{(0)}(\xi,T) := -\mu\xi + \Sigma_{i=1}^2|\gamma_i|\hat{A}^h + |w|^{\max}(T),\\ &\hat{h}^{(k)}(\xi,T) := -\mu\xi + \Sigma_{i=1}^2|\gamma_i|\hat{m}^{(k-1)}(T) + |w|^{\max}(T),\\ &\check{h}(\xi) := -\hat{h}(-\xi), \quad \check{h}^{(0)}(\xi,T) := -\hat{h}^{(0)}(-\xi,T), \quad \check{h}^{(k)}(\xi,T) := -\hat{h}^{(k)}(-\xi,T).\end{aligned}$$

Proof of Lemma 2.9. By arguments similar to those in Lemma 2.3, we have $\check{f}^{(0)}(\cdot) \le \check{f}_{\rm m}^{(k)}(\cdot) \le \hat{f}_{\rm m}^{(k)}(\cdot) \le \hat{f}^{(0)}(\cdot)$ for all $k \in \mathbb{N}\cup\{0\}$. Accordingly, $\check{f}_{\rm m}^{(k)}(\bar{q}) > 0$ $\hat{f}_{\rm m}^{(k)}(\bar{p}) < 0$, and both $\check{b}^{(k)}$ and $\hat{b}^{(k)}$ are well defined for all $k \in \mathbb{N}\cup\{0\}$ under condition (G1). Moreover, $\hat{b}^{(k+1)} \ge \hat{b}^{(k)}$, $\check{b}^{(k+1)} \le \check{b}^{(k)}$ for all $k \ge 0$. Thus, $\lim_{k\to\infty}\hat{b}^{(k)} = \underline{b}$, and $\lim_{k\to\infty}\check{b}^{(k)} = \overline{b}$, for some $\underline{b}$ and $\overline{b}$, since $\{\check{b}^{(k)}\}_{k\ge0}$ and $\{\hat{b}^{(k)}\}_{k\ge0}$ are both bounded monotone sequences. The arguments for $\check{a}^{(k)}$, $\hat{a}^{(k)}$, $\check{c}^{(k)}$, $\hat{c}^{(k)}$ are similar.

Now, let us justify that $\overline{b} = \underline{b}$. Note that there exists a continuous function $\hat{f}_{\rm m}^{(\infty)}$ defined on $[\hat{b}^{(0)},\check{b}^{(0)}]$ such that $\hat{f}_{\rm m}^{(k)}(\cdot) \downarrow \hat{f}_{\rm m}^{(\infty)}(\cdot)$ uniformly on $[\hat{b}^{(0)},\check{b}^{(0)}]$. Similarly, there exists a continuous function $\check{f}_{\rm m}^{(\infty)}$ defined on $[\hat{b}^{(0)},\check{b}^{(0)}]$ such that $\check{f}_{\rm m}^{(k)}(\cdot) \uparrow \check{f}_{\rm m}^{(\infty)}(\cdot)$ uniformly on $[\hat{b}^{(0)},\check{b}^{(0)}]$. It is obvious that $\check{f}_{\rm m}^{(\infty)}(\xi) \le \hat{f}_{\rm m}^{(\infty)}(\xi)$ for all $\xi \in [\hat{b}^{(0)},\check{b}^{(0)}]$. Moreover, the following properties can be derived:

(P3): $\hat{f}_{\rm m}^{(\infty)}(\xi) = -\mu\xi + \Sigma_{i=1}^2\gamma_i g(\xi) - (\Sigma_{i=1}^2\gamma_i\tau_i)\check{f}_{\rm m}^{(\infty)}(\underline{b})$, $\check{f}_{\rm m}^{(\infty)}(\xi) = -\mu\xi + \Sigma_{i=1}^2\gamma_i g(\xi) - (\Sigma_{i=1}^2\gamma_i\tau_i)\hat{f}_{\rm m}^{(\infty)}(\overline{b})$ for $\xi \in [\hat{b}^{(0)},\check{b}^{(0)}]$;

(P4): $\hat{f}_{\rm m}^{(\infty)}(\underline{b}) = 0$, $\check{f}_{\rm m}^{(\infty)}(\overline{b}) = 0$.

(P4) can be rewritten as

$$\begin{cases} -\mu\underline{b} + \Sigma_{i=1}^2\gamma_i g(\underline{b}) - (\Sigma_{i=1}^2\gamma_i\tau_i)\check{f}_{\rm m}^{(\infty)}(\underline{b}) = 0,\\ -\mu\overline{b} + \Sigma_{i=1}^2\gamma_i g(\overline{b}) - (\Sigma_{i=1}^2\gamma_i\tau_i)\hat{f}_{\rm m}^{(\infty)}(\overline{b}) = 0\end{cases}$$

according to (P3). Hence, we obtain

$$\begin{cases} \mu(-1+\Sigma_{i=1}^2\gamma_i\tau_i)\underline{b} - \mu(\Sigma_{i=1}^2\gamma_i\tau_i)\overline{b} + (\Sigma_{i=1}^2\gamma_i)(1-\Sigma_{i=1}^2\gamma_i\tau_i)g(\underline{b})\\ +(\Sigma_{i=1}^2\gamma_i)(\Sigma_{i=1}^2\gamma_i\tau_i)g(\overline{b}) = 0,\\ \mu(-1+\Sigma_{i=1}^2\gamma_i\tau_i)\overline{b} - \mu(\Sigma_{i=1}^2\gamma_i\tau_i)\underline{b} + (\Sigma_{i=1}^2\gamma_i)(1-\Sigma_{i=1}^2\gamma_i\tau_i)g(\overline{b})\\ +(\Sigma_{i=1}^2\gamma_i)(\Sigma_{i=1}^2\gamma_i\tau_i)g(\underline{b}) = 0.\end{cases} \tag{2.17}$$

The difference of the two equalities in (2.17) is

$$\mu(1-2\Sigma_{i=1}^2\gamma_i\tau_i)(\overline{b}-\underline{b}) - (1-2\Sigma_{i=1}^2\gamma_i\tau_i)(\Sigma_{i=1}^2\gamma_i)[g(\overline{b})-g(\underline{b})] = 0.$$

It follows that, for some $\zeta \in \mathbb{R}$,

$$\mu(1-2\Sigma_{i=1}^2\gamma_i\tau_i)(\overline{b}-\underline{b}) - (1-2\Sigma_{i=1}^2\gamma_i\tau_i)(\Sigma_{i=1}^2\gamma_i)g'(\zeta)(\overline{b}-\underline{b}) = 0.$$

Note that $g'(\zeta) \le 1$ and $1-2\Sigma_{i=1}^2\gamma_i\tau_i > 0$, due to condition (G1). Thus

$$(1-2\Sigma_{i=1}^2\gamma_i\tau_i)(\mu-\Sigma_{i=1}^2\gamma_i)(\overline{b}-\underline{b}) \le 0.$$

Hence $\overline{b} \le \underline{b}$, and subsequently $\overline{b} = \underline{b} = 0$, as $\lim_{k\to\infty}\hat{b}^{(k)} = \underline{b}$ and $\lim_{k\to\infty}\check{b}^{(k)} = \overline{b}$ and $\check{b}^{(k)} \le 0$ and $\hat{b}^{(k)} \ge 0$ for all k. The assertion for $\overline{a} = \underline{a}$ and $\overline{c} = \underline{c}$ can be derived similarly.

Proof of Proposition 2.10. We justify the assertion by the following five steps. Let $x(t) = x(t; t_0; \phi)$ be a solution of (2.9).

(I) Taking the spirit of controlling $\dot{z}(t)$ in Lemma 2.2, we can show that for arbitrary $k \geq 1$, (i) if $x(t) \in [\check{a}^{(j)}, \hat{a}^{(j)}]$ (resp., $[\check{c}^{(j)}, \hat{c}^{(j)}]$) for $t \geq T_\phi + j\tau$, $j = 0, \ldots, k-1$, then, for $t \geq T_\phi + k\tau$,

$$\check{f}_{\mathrm{l}}^{(k)}(x(t))+\varepsilon_k \leq \dot{x}(t) \leq \hat{f}_{\mathrm{l}}^{(k)}(x(t))-\varepsilon_k \text{ (resp., } \check{f}_{\mathrm{r}}^{(k)}(x(t))+\varepsilon_k \leq \dot{x}(t) \leq \hat{f}_{\mathrm{r}}^{(k)}(x(t))-\varepsilon_k);$$

consequently, $x(t)$ enters interval $[\check{a}^{(k)}, \hat{a}^{(k)}]$ (resp., $[\check{c}^{(k)}, \hat{c}^{(k)}]$) eventually and then remains in the interval thereafter; (ii) if $x(t) \in [\hat{b}^{(j)}, \check{b}^{(j)}]$ for $t \geq T_\phi + j\tau$, $j = 0, \ldots, k-1$, then

$$\check{f}_{\mathrm{m}}^{(k)}(x(t)) + \varepsilon_k \leq \dot{x}(t) \leq \hat{f}_{\mathrm{m}}^{(k)}(x(t)) - \varepsilon_k,$$

and $x(t) \in [\hat{b}^{(k)}, \check{b}^{(k)}]$, for $t \geq T_\phi + k\tau$. Next, we consider the following properties named $\mathcal{M}$, $\mathcal{L}$, $\mathcal{R}$ for solutions $x(t)$ of (2.9):

$\mathcal{M}$: for each $k \in \mathbb{N} \cup \{0\}$, $x(t) \in [\hat{b}^{(k)}, \check{b}^{(k)}]$ for all $t \geq T_\phi + k\tau$;

$\mathcal{L}$: there exists $s \geq T_\phi$ such that $x(s) \in [\check{a}^{(0)}, \hat{a}^{(0)}]$;

$\mathcal{R}$: there exists $s \geq T_\phi$ such that $x(s) \in [\check{c}^{(0)}, \hat{c}^{(0)}]$.

(II) By Proposition 2.8, if there exists $s \geq T_\phi$ such that $x(s) > \check{b}^{(0)}$ (resp., $x(s) < \hat{b}^{(0)}$), then $x(t)$ satisfies property $\mathcal{R}$ (resp., $\mathcal{L}$).

(III) By the definition of property $\mathcal{M}$, it is not difficult to show that if solution $x(t)$ of (2.9) satisfies property $\mathcal{M}$, then $x(t) \to [\underline{b}, \overline{b}]$ as $t \to \infty$.

(IV) Assume that $x(t)$ satisfies property $\mathcal{R}$. By arguments similar to those in the proof of Proposition 2.4, we can show that for arbitrary $n \in \mathbb{N}$, there exists an increasing sequence $\{T_k\}_{k=0}^n$ with $T_{k+1} \geq T_k + \tau$ for $k = 0, 1, \ldots, n-1$, and $T_0 \geq T_\phi$, such that

$$\begin{cases} \check{f}_{\mathrm{r}}^{(k)}(x(t)) + \varepsilon_k \leq \dot{x}(t) \leq \hat{f}_{\mathrm{r}}^{(k)}(x(t)) - \varepsilon_k \\ \qquad \text{for all } t \geq T_k + \tau, \text{ and } k = 0, 1, \ldots, n-1; \\ x(t) \in [\check{c}^{(k)}, \hat{c}^{(k)}] \text{ for all } t \geq T_{k+1}, \text{ and } k = 0, 1, \ldots, n-1. \end{cases}$$

Accordingly, $x(t) \to [\underline{c}, \overline{c}]$ as $t \to \infty$. Similarly, we can show that if $x(t)$ satisfies property $\mathcal{L}$, then $x(t) \to [\underline{a}, \overline{a}]$ as $t \to \infty$.

(V) That every solution $x(t)$ of (2.9) satisfies one of properties $\mathcal{M}$, $\mathcal{L}$, $\mathcal{R}$ follows from arguments similar to those in the proof of Proposition 2.3 in [28].

From (III)–(V) and Lemma 2.9, we conclude that $x(t)$ converges to an element of $\{a, 0, c\}$ as $t \to \infty$.

3. Dynamics of (1.1) with $N = 3$. In this section, we focus on (1.1) of scale $N = 3$ to establish the synchronization and convergence to multiple synchronous equilibria of the network. They are presented in subsections 3.1 and 3.2, respectively. Moreover, in subsection 3.3, delay Hopf bifurcation theory is employed to conclude the existence of nontrivial synchronous and asynchronous oscillations (standing waves) induced by transmission delay τ_T. Notably, system (1.1) is a dissipative system; hence a solution evolved from any initial condition $\phi \in \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}^3)$ exists for all time $t \geq t_0$.

3.1. Global synchronization. We shall derive criteria for the global synchronization of (1.1); namely, $x_i(t) - x_{i+1}(t) \to 0$ as $t \to \infty$, $i = 1, 2$, for every solution $(x_1(t), x_2(t), x_3(t))$ of (1.1). To this end, we consider the following differential-difference system obtained by subtracting x_{i+1}-component from x_i-component in

(1.1):

$$\begin{aligned}\dot{z}_i(t) = -\mu z_i(t) &+ \alpha[g(x_i(t-\tau_I)) - g(x_{i+1}(t-\tau_I))]\\ &- \beta[g(x_i(t-\tau_T)) - g(x_{i+1}(t-\tau_T))],\end{aligned} \tag{3.1}$$

where $z_i(t) := x_i(t) - x_{i+1}(t)$, $i = 1, 2$. We note that system (1.1) achieves global synchronization if $z_i(t) \to 0$, as $t \to \infty$, for all $i = 1, 2$ and for every $z_i(t)$ satisfying (3.1). Obviously, each component of (3.1) satisfies (2.1) with $\gamma_1 = -\alpha$, $\gamma_2 = \beta$, $\tau_1 = \tau_I$, $\tau_2 = \tau_T$, and $w(t) = 0$. Moreover, $x_i(t)$ and $x_{i+1}(t)$ are eventually attracted by $[-(|\alpha|+2|\beta|)/\mu, (|\alpha|+2|\beta|)/\mu]$, as seen from the equation for x_i and x_{i+1} in (1.1). We denote

$$\hat{\alpha} := \begin{cases} \alpha, & \alpha \geq 0,\\ \alpha\tilde{L}, & \alpha < 0,\end{cases} \quad \check{\alpha} := \begin{cases} \alpha\tilde{L}, & \alpha \geq 0,\\ \alpha, & \alpha < 0,\end{cases} \tag{3.2}$$

$$\hat{\beta} := \begin{cases} \beta, & \beta \geq 0,\\ \beta\tilde{L}, & \beta < 0,\end{cases} \quad \check{\beta} := \begin{cases} \beta\tilde{L}, & \beta \geq 0,\\ \beta, & \beta < 0,\end{cases} \tag{3.3}$$

where

$$\tilde{L} := \min\{g'(\xi) : \xi \in [-(|\alpha|+2|\beta|)/\mu, (|\alpha|+2|\beta|)/\mu]\}. \tag{3.4}$$

Now, let us introduce four different conditions for synchronization of network (1.1).

Condition (S1): $-\hat{\alpha} + \check{\beta} > 0$, $\tau_I|\alpha| + \tau_T|\beta| < \mu/(2\mu - \check{\alpha} + \hat{\beta})$; restated,

$$\begin{cases} \beta > (1/\tilde{L})\alpha \text{ and } \tau_I|\alpha| + \tau_T|\beta| < \mu/(2\mu - \alpha\tilde{L} + \beta) & \text{if } \alpha > 0, \beta \geq 0,\\ \tau_I|\alpha| + \tau_T|\beta| < \mu/(2\mu - \alpha + \beta) < 1/2 & \text{if } \alpha \leq 0, \beta \geq 0,\\ \beta > \tilde{L}\alpha \text{ and } \tau_I|\alpha| + \tau_T|\beta| < \mu/(2\mu - \alpha + \beta\tilde{L}) & \text{if } \alpha \leq 0, \beta < 0.\end{cases}$$

Condition (S2): $\alpha < 0$, $|\beta| < \mu$, $\tau_I < (\mu - |\beta|)/[\alpha(\alpha - 2\mu)]$.

Condition (S3): $\beta > 0$, $|\alpha| < \mu$, $\tau_T < (\mu - |\alpha|)/[\beta(\beta + 2\mu)]$.

Condition (S4): $|\alpha| + |\beta| < \mu$.

It can be verified that each ith component of (3.1) satisfies condition (H1) (resp., (H2), (H3)) under condition (S1) (resp., (S2), (S4)), $i = 1, 2$. According to Lemma 2.3 and Proposition 2.4 (resp., Propositions 2.5 and 2.6) with $w(t) = 0$, we conclude that $z_i(t) \to 0$, as $t \to \infty$, for every $z_i(t)$ satisfying (3.1), $i = 1, 2$. Therefore, network (1.1) can be synchronized under condition (S1) or (S2) or (S4). On the other hand, each ith component of (3.1) can also be regarded in the form of (2.1) with $\gamma_1 = \beta$, $\gamma_2 = -\alpha$, $\tau_1 = \tau_T$, $\tau_2 = \tau_I$, and $w(t) = 0$. Thus every ith component of (3.1) satisfies condition (H2) under condition (S3); hence $z_i(t) \to 0$, as $t \to \infty$, for every z_i satisfying (3.1). Accordingly, (1.1) can be synchronized under condition (S3). We thus conclude the following result.

THEOREM 3.1. *System* (1.1) *with* $N = 3$ *achieves global synchronization under one of conditions* (S1)–(S4).

Observe that condition (S4) is delay-independent; condition (S3) is τ_I-independent; condition (S2) is τ_T-independent; (S1) is (τ_I, τ_T)-dependent. In conditions (S1)–(S3), the inequalities involving delays τ_I, τ_T all hold if τ_I and/or τ_T are small enough. The parameters α, β which satisfy the inequalities uninvolved with delays in conditions (S1)–(S4) are depicted in Figures 3(a)–(d), respectively. In particular, let us interpret the region in Figure 3(a). First, notice that the term $\tilde{L}$ in condition (S1)

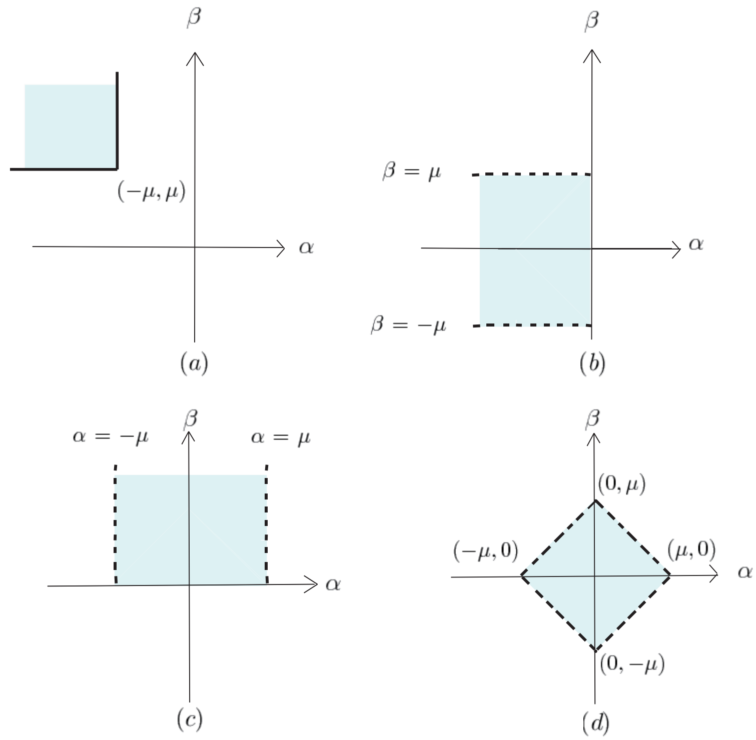

FIG. 3. *The region of* (α, β) *that admits synchronization* (a) *while* τ_T *and* τ_T *are small,* (b) *in spite of* τ_T *while* τ_I *is small,* (c) *in spite of* τ_I *while* τ_T *is small, and* (d) *in spite of* τ_I *and* τ_T.

actually depends on μ, α, β; cf. (3.4). The parameters α, β satisfying condition (S1) may lie in the first, second, or third quadrant of the (α, β)-plane. Indeed, condition (S1) is always satisfied if β is positive and α is negative; i.e., the second quadrant of the (α, β)-plane and τ_I, τ_T are small. However, in general, those parameters (α, β) satisfying condition (S1) and lying in the first (resp., third) quadrant actually also lie in the parameter region depicted in Figure 3(c) (resp., (b)); cf. Figure 4. Note that Figure 3(b) (resp., (c)) corresponds to condition (S2) (resp., (S3)), which provides the τ_T-independent (resp., τ_I-independent) result. Therefore, precise reading of the parameter region for the (τ_I, τ_T)-dependent result under condition (S1) is to subtract the parameter regions satisfying condition (S2) or (S3) from the second quadrant of the (α, β)-plane as depicted in Figure 3(a).

Remark 3.1. (i) These parameter regimes indicate that if the self-feedback strength α is strong, then the self-feedback has to be inhibitory to synchronize system (1.1); on the other hand, if the coupling strength β is strong, then the coupling has to be excitatory for the synchronization of system (1.1).

(ii) Extracting from the results of Theorem 3.1, it can be observed that large self-decay, inhibitory self-feedback (with small τ_I), and excitatory coupling (with small τ_T) are advantageous for (1.1) to be synchronized.

(iii) It will be shown that as $|\alpha|$ (resp., $|\beta|$) gets large, delay τ_I (resp., τ_T) can generate asynchrony; cf. Theorem 3.8, Remark 3.4, and a numerical illustration in Example 5.2.

If we consider, in particular, $\tau_I = \tau_T$ for (1.1), then each ith component of (3.1)

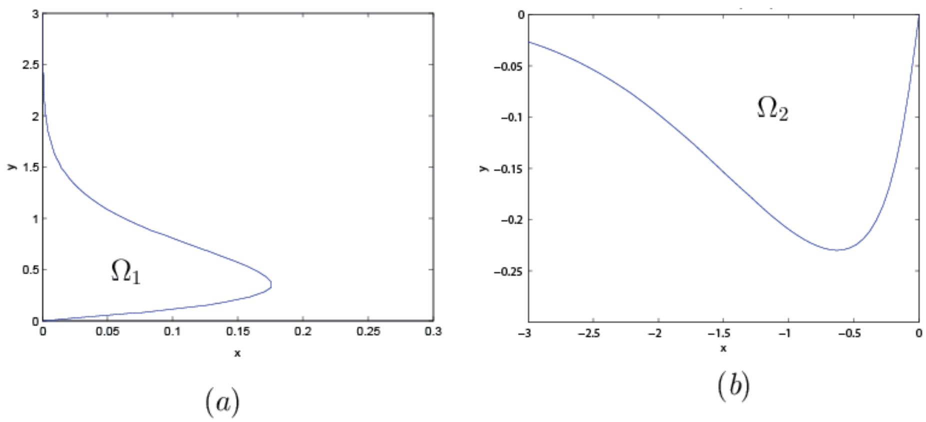

FIG. 4. (a) $\Omega_1 := \{(\alpha, \beta) : \alpha > 0, \beta \geq 0$ *and* $\beta > (1/\tilde{L})\alpha\}$. (b) $\Omega_2 := \{(\alpha, \beta) : \alpha \leq 0, \beta < 0$ and $\beta > \tilde{L}\alpha\}$ *as* $g(\xi) = \tanh(\xi)$ *and* $\mu = 1$.

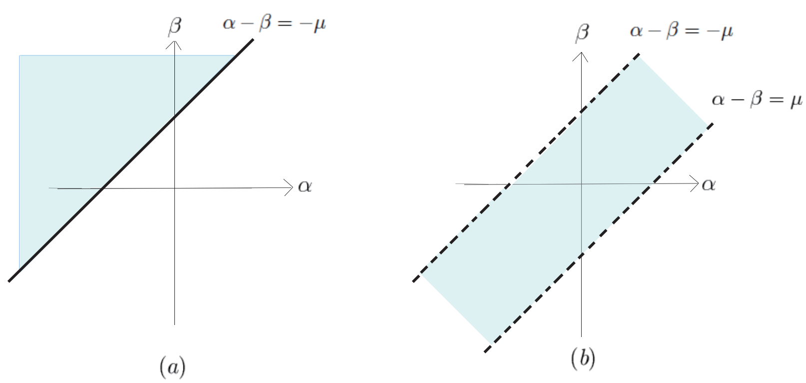

FIG. 5. *System* (1.1) *with* $\tau_I = \tau_T$ *attains synchronization if* (α, β) *lies in the shaded region in* (a) *and* $\tau_I = \tau_T$ *is small,* (b) *in spite of delays.*

satisfies (2.1) with $\gamma_1 = -(\alpha - \beta)$, $\tau_1 = \tau_I$, $\gamma_2 = 0$, and $w(t) = 0$. We thus derive the following result.

THEOREM 3.2. *System* (1.1) *with* $N = 3$ *and* $\tau_I = \tau_T$ *attains global synchronization under one of the following conditions:*

(i) $\alpha - \beta \leq -\mu$ *and* $\tau_I = \tau_T < \mu/[(\beta - \alpha)(2\mu - \alpha + \beta)]$;

(ii) $|\alpha - \beta| < \mu$.

The parameter conditions in Theorem 3.2 are depicted in Figures 5(a)–(b), respectively. Notice that the union of these regions is larger than the union of the ones in Figures 3(a)–(d). This implicates that the stronger result is obtained if $\tau_I = \tau_T$.

Remark 3.2. The result in Theorem 3.2 indicates that system (1.1) without delays ($\tau_I = \tau_T = 0$) can be synchronized if $\beta - \alpha > \mu$. This can be interpreted as sufficiently strong inhibitory self-feedback or that excitatory coupling can synchronize system (1.1) without delays. It is then natural to ask whether the same factors can also synchronize system (1.1) with delays. We shall see from Theorem 3.8 and Remark 3.4 that this depends on the delay size. Indeed, once the self-feedback strength α (resp., coupling strength β) is sufficiently stronger than coupling strength (resp., self-feedback strength), then synchrony for network (1.1) with nonzero delays (τ_I and τ_T may be distinct) can be lost and nontrivial asynchronous oscillations are bifurcated from the origin at delay magnitude τ_I (resp., τ_T) near bifurcation values (there are infinitely many such values). This highlights the difference between the effect from the self-feedback or coupling upon the synchronization of the coupled network with delays and without delays.

The only global synchronization result in the literature was established under the delay-independent criterion $|\alpha| + |\beta| < 1$, for $N = 3$, in [1]. Some delay-dependent and delay-independent conditions for the stability of the trivial equilibrium, which is obviously synchronous, were derived in [1, 35]: basically, both magnitudes of α and β need to be small, or if α is negative, the magnitude of β has to be dominated by that of α and if the magnitude of α is large, the corresponding delay τ_I is required to be small. The dynamics of a stable synchronous equilibrium can be regarded as local synchronization. Our analysis has extended the synchronization for system (1.1) to a range wider than the stable region for the origin, for example, as β is positive and of large magnitude, as addressed in Theorem 3.1. It was conjectured in [1] that if $|\beta| < |1 - \alpha|$ and $0 \le \tau_I < \tau_S^{(1)}$ for some $\tau_S^{(1)}$, or $|\beta| < |1 - \alpha|/2$ and $0 \le \tau_I < \tau_S^{(2)}$ for some $\tau_S^{(2)}$, then (1.1) can be synchronized for all $\tau_T \ge 0$. Herein, $\tau_S^{(i)}$, $i = 1, 2$, are quantities from bifurcation analysis. Roughly speaking, these conditions require that $|\alpha|$ be relatively larger than $|\beta|$ and τ_I be small enough. Our Theorem 3.1 under condition (S2) answers this conjecture (under the assumption of small τ_I) and indicates that "inhibitory" self-feedback strength ($\alpha < 0$, $|\alpha|$ large) is crucial for the synchrony of system (1.1). The following example demonstrates that the conjecture is incorrect if $\alpha > 0$.

Example 3.1. Consider system (1.1) with $N = 3$, parameters $\mu = 1$, $\alpha = 6$, $\beta = -2$, and delays $\tau_I = 0$, $\tau_T = 5$. These parameters and delays satisfy the above condition of the conjecture, but there exist some solutions which converge to an asynchronous equilibrium.

3.2. Convergence to multiple synchronous equilibria. In this subsection, we shall investigate the stability of nontrivial synchronous equilibria $\mathbf{x}^{\pm}$ of (1.1) and derive criteria for the global convergence to these equilibria. Let us consider the parameter regions

$$D_1 := \{(\alpha, \beta) : \alpha - \beta \le -\mu \text{ and } \alpha + 2\beta > \mu\},$$
$$D_2 := \{(\alpha, \beta) : |\alpha - \beta| < \mu \text{ and } \alpha + 2\beta > \mu\},$$

which are depicted in Figure 6.

THEOREM 3.3. *System* (1.1) *with* $N = 3$ *has exactly three equilibria* $(0, 0, 0)$, $\mathbf{x}^+ := (u^+, u^+, u^+)$, *and* $\mathbf{x}^- := (u^-, u^-, u^-)$ *with* $u^+ > 0$ *and* $u^- < 0$ *if* $(\alpha, \beta) \in D_1 \cup D_2$. *If* $(\alpha, \beta) \in D_1$ *and* $\alpha \ge 0$, $\beta \ge 0$, *or* $(\alpha, \beta) \in D_2$, *then* $\mathbf{x}^{\pm}$ *is stable in spite of delays.*

Proof. The existence of equilibria for $(\alpha, \beta) \in D_1 \cup D_2$ and the stability of $\mathbf{x}^{\pm}$ for $(\alpha, \beta) \in D_2$ can be established by arguments similar to those in [33]. It remains to verify the stability of $\mathbf{x}^{\pm}$ for $(\alpha, \beta) \in D_1$, and $\alpha \ge 0$ and $\beta \ge 0$. We merely verify the case of $\mathbf{x}^+$; the case for $\mathbf{x}^-$ is similar. The linearization of (1.1) about $\mathbf{x}^+$ is given by

$$\dot{v}_i(t) = -\mu v_i(t) + \alpha g'(u^+) v_i(t - \tau_I) + \beta g'(u^+)[v_{i-1}(t - \tau_T) + v_{i+1}(t - \tau_T)],\ i = 1, 2, 3.$$

Thus the characteristic equation is

$$\Delta_1(\lambda)\Delta_2^2(\lambda) = 0,$$

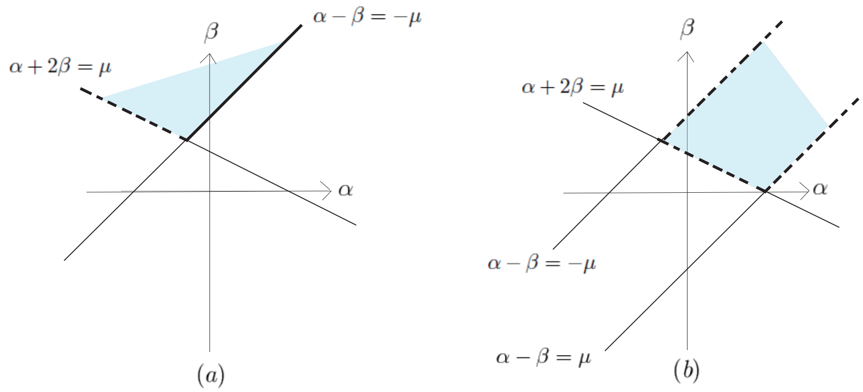

FIG. 6. *System* (1.1) *with parameters in regions* (a) D_1 *and* (b) D_2 *admits exactly three synchronous equilibria.*

where $\Delta_1(\lambda) := \mu+\lambda-\alpha g'(u^+)e^{-\lambda\tau_I}-2\beta g'(u^+)e^{-\lambda\tau_T}$, $\Delta_2(\lambda) := \mu+\lambda-\alpha g'(u^+)e^{-\lambda\tau_I}+\beta g'(u^+)e^{-\lambda\tau_T}$. We substitute $\lambda = \nu + iw$, $\nu, w \in \mathbb{R}$, into $\Delta_1(\lambda) = 0$ and collect the real and imaginary parts to obtain

$$\nu + \mu = g'(u^+)[\alpha e^{-\nu\tau_I}\cos(\tau_I w) + 2\beta e^{-\nu\tau_T}\cos(\tau_T w)],$$
$$w = g'(u^+)[-\alpha e^{-\nu\tau_I}\sin(\tau_I w) - 2\beta e^{-\nu\tau_T}\sin(\tau_T w)].$$

Summing up squares of these equations gives $I_1(\nu) = I_2(\nu)$, where $I_1(\nu) = (\nu+\mu)^2 + w^2$, $I_2(\nu) = [g'(u^+)]^2[\alpha^2 e^{-2\nu\tau_I} + 4\beta^2 e^{-2\nu\tau_T} + 4\alpha\beta e^{-\nu(\tau_I+\tau_T)}\cos((\tau_I-\tau_T)w)]$. Note that u^+ satisfies the stationary equation $-\mu x+(\alpha+2\beta)g(x) = 0$ which admits exactly three zeros e_1, e_2, and 0, where $e_1 < p^* < 0$, $0 < q^* < e_2$, and

$$g'(p^*) = g'(q^*) = \mu/(\alpha+2\beta). \tag{3.5}$$

Obviously, $u^+ = e_2$; hence $g'(u^+) < \mu/(\alpha+2\beta)$. If $\nu \geq 0$, then a contradiction occurs since $I_2(\nu) < [\mu/(\alpha+2\beta)]^2[\alpha^2+4\beta^2+4\alpha\beta] = \mu^2 \leq I_1(\nu)$. Therefore, $\nu < 0$. If we substitute $\lambda = \nu + iw$ into $\Delta_2(\lambda) = 0$, it can also be verified that $\nu < 0$ by similar arguments. The proof is thus completed. ∎

THEOREM 3.4. *System* (1.1) *with* $N = 3$ *admits exactly three equilibria* $(0,0,0)$, $\mathbf{x}^+ := (u^+,u^+,u^+)$, *and* $\mathbf{x}^- := (u^-,u^-,u^-)$, *and every solution of the system converges to one of these equilibria under one of the following conditions:*

(i) $\alpha \leq 0$, $\beta \geq 0$, $\alpha+2\beta > \mu$, $|\alpha|\tau_I+|\beta|\tau_T < \mu/(2\mu-\alpha+\beta)$, *and* $|\alpha|\tau_I+2|\beta|\tau_T < \tau^*$;

(ii) $\beta > 0$, $|\alpha| < \mu$, $\alpha+2\beta > \mu$, $\tau_T < (\mu-|\alpha|)/[\beta(\beta+2/\mu)]$, *and* $|\alpha|\tau_I + 2|\beta|\tau_T < \tau^*$, *where* $\tau^* := \min\{1/3, [(\alpha+2\beta)g(q^*)-\mu q^*]/[3(|\alpha|+2|\beta|)], [\mu p^*-(\alpha+2\beta)g(p^*)]/[3(|\alpha|+2|\beta|)]\}$, *and* p^*, q^* *are defined in* (3.5).

Proof. We prove only the first case; the other case can be treated similarly. We arrange (1.1) into the form

$$\dot{x_i}(t) = -\mu x_i(t) + \alpha f(x_i(t-\tau_I)) + 2\beta g(x_i(t-\tau_T)) + E_i(t), \tag{3.6}$$

where $E_i(t) = \beta[g(x_{i-1}(t-\tau_T)) + g(x_{i+1}(t-\tau_T)) - 2g(x_i(t-\tau_T))]$. Owing to $\alpha \leq 0$, $\beta \geq 0$, and $\alpha+2\beta > \mu$, (1.1) has exactly three equilibria $(0,0,0)$, $\mathbf{x}^+$, and $\mathbf{x}^-$ by Theorem 3.3. Moreover, since $\alpha \leq 0$, $\beta \geq 0$, and $|\alpha|\tau_I+|\beta|\tau_T < \mu/(2\mu-\alpha+\beta) < 1/2$, the system achieves global synchronization according to Theorem 3.1. Therefore,

$E_i(t) \to 0$ as $t \to \infty$. Obviously, each component of (3.6) satisfies (2.8) with $\gamma_1 = \alpha$, $\gamma_2 = 2\beta$, $\tau_1 = \tau_I$, and $\tau_2 = \tau_T$. Note that $\bar{p} = p^*$, $\bar{q} = q^*$ as $\gamma_1 = \alpha$, $\gamma_2 = 2\beta$. Under condition (i), every ith component of (3.6) satisfies condition (G1). According to Proposition 2.10, every ith component $x_i(t)$ satisfying (3.6) converges to an element of $\{u^+, 0, u^-\}$. The assertion is thus verified. □

If $\tau_I = \tau_T$ in (1.1) is considered in particular, we can further derive the following theorem.

THEOREM 3.5. *System* (1.1) *with* $\tau_I = \tau_T$ *admits exactly three equilibria* $(0,0,0)$, $\mathbf{x}^+ := (u^+, u^+, u^+)$, *and* $\mathbf{x}^- := (u^-, u^-, u^-)$, *and every solution of the system converges to one of these equilibria under one of the following conditions:*

(i) $(\alpha, \beta) \in D_1$ *and* $\tau_I = \tau_T < \{\tilde{\tau}, \mu/[(\beta - \alpha)(2\mu - \alpha + \beta)]\}$;

(ii) $(\alpha, \beta) \in D_2$ *and* $\tau_I = \tau_T < \tilde{\tau}$, *where*

$$\tilde{\tau} := \frac{\min\{1/3, [(\alpha + 2\beta)g(p^*) - \mu p^*]/[3(\alpha + 2\beta)], [\mu q^* - (\alpha + 2\beta)g(q^*)]/[3(\alpha + 2\beta)]\}}{\alpha + 2\beta}.$$

System (1.1) of scale $N = 3$ with $\mu = 1$ and $\tau_I = \tau_T$ was considered in [33]. Therein, it was shown that the system achieves synchronization in spite of delay if $|\alpha - \beta| < 1$, and the system has three equilibria $(0,0,0)$, $\mathbf{x}^+$, and $\mathbf{x}^-$ if $(\alpha, \beta) \in D_1 \cup D_2$. Our Theorems 3.2–3.5 improve and extend these results. Note that Theorems 3.3 and 3.4 apply to system (1.1) with independent τ_I, τ_T. It was conjectured in [33] that almost every solution converges to either $\mathbf{x}^+$ or $\mathbf{x}^-$ (generic convergence) for system (1.1) if $(\alpha, \beta) \in D_2$. Our Theorem 3.5 answers this conjecture by concluding that every solution converges to one of the synchronous equilibria $(0,0,0)$, $\mathbf{x}^+$, $\mathbf{x}^-$ (global convergence) if $(\alpha, \beta) \in D_1 \cup D_2$ and the time lag is small. Note that as $\alpha < 0$ or $\beta < 0$, standard ordering is invalid in applying the monotone dynamics theory, adopted in [33], to conclude the convergent dynamics.

Remark 3.3. (i) In Theorems 3.4 and 3.5, $\alpha + 2\beta > \mu$ and other conditions yield the existence of multiple equilibria for (1.1). Indeed, if $\alpha + 2\beta$ is positive and small or negative instead, we can modify Theorems 3.4 and 3.5 to conclude that the system achieves global convergence to the origin if delays τ_I and τ_T are small; cf. Remark 2.2.

(ii) As mentioned in Remark 3.1(ii), inhibitory self-feedback and excitatory coupling are advantageous for (1.1) to be synchronized. However, there exists a qualitative difference between the situations of strong inhibitory self-feedback and strong excitatory coupling. Roughly speaking, if the self-feedback is inhibitory (resp., coupling is excitatory) and sufficiently strong, then $\alpha + 2\beta$ is small (resp., large); consequently, system (1.1) achieves global convergence to a single equilibrium (resp., multiple equilibria) when delays are small.

(iii) There is a distinction between multistability of (1.1) induced from strong excitatory coupling and strong excitatory self-feedback. An extension of the investigations in [5, 6, 28] leads to the convergence to 3^N synchronous and asynchronous equilibria if the self-feedback strength is excitatory and sufficiently stronger than the coupling strength. Thus, "strong excitatory self-feedback"-induced multistability of (1.1) comprises coexistence of synchronous and asynchronous equilibria, whereas "strong excitatory coupling"-induced multistability of (1.1) consists of multiple synchronous equilibria.

3.3. Bifurcations and oscillations. The aim of this subsection is to show that synchronous oscillation exists under our global synchronization framework. In addition, we are interested in seeing what self-feedback strength α, coupling strength β, and delays τ_I, τ_T are responsible for the synchronous oscillation. To focus on these effects, we set $\mu = 1$ in this subsection. As standing waves often emerge when synchrony is lost, we shall also discuss the existence of such asynchronous oscillations. Let us denote by $(1.1)_0$ system (1.1) with odd activation functions g in (1.2); i.e., g also satisfies $g(-\xi) = -g(\xi)$ for all $\xi \in \mathbb{R}$. A solution $(x_1(t), x_2(t), x_3(t))$ is said to be in the form of standing waves for system $(1.1)_0$ if two of the components are of opposite sign and the other equal to zero; i.e.,

$$x_i(t) = -x_j(t), \quad x_k(t) \equiv 0,$$

and $(i, j, k) = (1, 2, 3)$ or its permutation; cf. [19].

We shall employ Hopf bifurcation theory to analyze the existence of nontrivial synchronous solutions for (1.1) and standing wave solutions for $(1.1)_0$, induced by transmission delay τ_T. Similar discussions can proceed for bifurcation induced by self-feedback delay τ_I. Standard and equivariant Hopf bifurcation theories have been applied to investigate oscillations for (1.1) in [2, 1, 35]. However, our goal is to establish concrete criteria for the oscillations which can be accommodated in our global synchronization setting.

According to the coupling topology of system (1.1),

$$\mathcal{S} := \{(\phi, \phi, \phi) : \phi \in C([-\tau_{\max}, 0]; \mathbb{R})\}$$

is positively invariant under the flow generated by system (1.1). On the other hand, $\mathcal{S}$ and $\mathcal{A}_\sigma$ are both positively invariant under the flow generated by system $(1.1)_0$, where

$$\mathcal{A}_\sigma := \{(\phi_1, \phi_2, \phi_3) : \phi_i = 0, \phi_j = -\phi_k \in C([-\tau_{\max}, 0]; \mathbb{R}), (i, j, k) = \sigma(1, 2, 3)\},$$

and $\sigma(1, 2, 3)$ is a permutation of index $(1, 2, 3)$. Hence, it allows us to consider system $(1.1)_+$ (resp., $(1.1)_-^\sigma$), which is a restriction of (1.1) on $\mathcal{S}$ (resp., $(1.1)_0$ on $\mathcal{A}_\sigma$), and consider only evolutions from points in S (resp., $\mathcal{A}_\sigma$). First, let us focus on system $(1.1)_+$. Every component of $(1.1)_+$ satisfies

$$\dot{y}(t) = -y(t) + \alpha g(y(t - \tau_I)) + 2\beta g(y(t - \tau_T)). \tag{3.7}$$

The linearized system at the origin of (3.7) is

$$\dot{v}(t) = -v(t) + \alpha v(t - \tau_I) + 2\beta v(t - \tau_T). \tag{3.8}$$

Thus the characteristic equation for (3.8) is

$$\Delta_+(\lambda) := (1 + \lambda - \alpha e^{-\lambda\tau_I} - 2\beta e^{-\lambda\tau_T}) = 0. \tag{3.9}$$

We substitute $\lambda = iw$ into $\Delta_+(\lambda) = 0$ and collect the real and imaginary parts to yield

$$\begin{cases} 2\beta\cos(\tau_T w) = 1 - \alpha\cos(\tau_I w), \\ 2\beta\sin(\tau_T w) = -w - \alpha\sin(\tau_I w). \end{cases} \tag{3.10}$$

Summing up the squares of equations (3.10) gives

$$Q(w) = 4\beta^2, \tag{3.11}$$

where $Q(w) := w^2 + 2\alpha\sin(\tau_I w)w - 2\alpha\cos(\tau_I w) + \alpha^2 + 1$. Note that the positive solution w_+ of (3.11) corresponds to one pair of purely imaginary roots $\pm iw_+$ of

(3.9). Obviously, $Q(w) \leq \tilde{Q}(w)$ for all $w \geq 0$, where $\tilde{Q}(w) := w^2 + 2|\alpha|w + (1 + |\alpha|)^2$ is an increasing function for $w \geq 0$. A direct computation gives $Q'(w) = [2 + 2\tau_I \alpha \cos(\tau_I w)]w + 2\alpha(1 + \tau_I)\sin(\tau_I w)$. Then, $Q'(w) \geq P(w)$ for all $w \geq 0$, where $P(w) = (2 - 2\tau_I|\alpha|)w - 2|\alpha|(1 + \tau_I)$. Obviously, if $\tau_I|\alpha| < 1$, then $P(w) > 0$ for all $w \geq \tilde{w} := |\alpha|(1 + \tau_I)/(1 - \tau_I|\alpha|) \geq 0$. Therefore, $Q(w)$ is increasing on $[\tilde{w}, \infty)$. Now, let us introduce the condition for the existence of purely imaginary roots of $\Delta_+(\lambda) = 0$.

Condition $(\mathrm{B1})_+$: $\tau_I|\alpha| < 1$ and $\tilde{Q}(\tilde{w}) < 4\beta^2$; i.e.,

$$4\beta^2 > (1 + |\alpha|)^2 + |\alpha|^2 \left[\left(\frac{1 + \tau_I}{1 - \tau_I|\alpha|} \right)^2 + 2 \left(\frac{1 + \tau_I}{1 - \tau_I|\alpha|} \right) \right].$$

Notice that $Q(w) \leq \tilde{Q}(w)$ for all $w \geq 0$; $\tilde{Q}(w)$ is increasing for all $w \geq 0$; and $Q(w)$ is increasing on $[\tilde{w}, \infty)$. Subsequently, (3.11) admits exactly one positive zero, say ω_+^*, under condition $(\mathrm{B1})_+$. We thus conclude the following lemma.

LEMMA 3.6. *There exists exactly one pair of purely imaginary roots, $\pm i\omega_+^*$, for characteristic equation* (3.9) *under condition* $(\mathrm{B1})_+$. *Herein, ω_+^* is the unique positive zero to* (3.11).

On the other hand, every nontrivial component of $(1.1)_-^\sigma$ satisfies

$$\dot{y}(t) = -y(t) + \alpha g(y(t - \tau_I)) - \beta g(y(t - \tau_T)). \tag{3.12}$$

The characteristic equation for the linearization of (3.12) is

$$\Delta_-(\lambda) := (1 + \lambda - \alpha e^{-\lambda\tau_I} + \beta e^{-\lambda\tau_T}) = 0. \tag{3.13}$$

By comparing (3.9) and (3.13), we obtain the condition for the existence of purely imaginary roots for $\Delta_-(\lambda) = 0$.

Condition $(\mathrm{B1})_-$: $\tau_I|\alpha| < 1$ and $\tilde{Q}(\tilde{w}) < \beta^2$; i.e.,

$$\beta^2 > (1 + |\alpha|)^2 + |\alpha|^2 \left[\left(\frac{1 + \tau_I}{1 - \tau_I|\alpha|} \right)^2 + 2 \left(\frac{1 + \tau_I}{1 - \tau_I|\alpha|} \right) \right].$$

LEMMA 3.7. *There exists exactly one pair of purely imaginary roots, $\pm i\omega_-^*$, for characteristic equation* (3.13) *under condition* $(\mathrm{B1})_-$.

To find the value of τ_T at which $\pm i\omega_\pm^*$ are the purely imaginary roots of $\Delta_\pm(\cdot) = 0$, we divide the second equation by the first of (3.10). Then

$$\tan(\tau_T w) = S(w)/C(w),$$
$$S(w) := -w - \alpha\sin(\tau_I w), \quad C(w) := 1 - \alpha\cos(\tau_I w).$$

Let us define, for $k \in \mathbb{Z}$,

$$\eta_k^+ := \frac{1}{\omega_+^*} \begin{cases} 3\pi/2 + 2k\pi & \text{if } \beta C(\omega_+^*) = 0,\ \beta S(\omega_+^*) < 0, \\ \pi/2 + 2k\pi & \text{if } \beta C(\omega_+^*) = 0,\ \beta S(\omega_+^*) > 0, \\ \tan^{-1}(S(\omega_+^*)/C(\omega_+^*)) + 2k\pi & \text{if } \beta C(\omega_+^*) > 0, \\ \tan^{-1}(S(\omega_+^*)/C(\omega_+^*)) + (2k+1)\pi & \text{if } \beta C(\omega_+^*) < 0. \end{cases}$$

Similarly, we can define

$$\eta_k^- := \frac{1}{\omega_-^*} \begin{cases} \pi/2 + 2k\pi & \text{if } \beta C(\omega_-^*) = 0,\ \beta S(\omega_-^*) < 0, \\ 3\pi/2 + 2k\pi & \text{if } \beta C(\omega_-^*) = 0,\ \beta S(\omega_-^*) > 0, \\ \tan^{-1}(S(\omega_-^*)/C(\omega_-^*)) + (2k+1)\pi & \text{if } \beta C(\omega_-^*) > 0, \\ \tan^{-1}(S(\omega_-^*)/C(\omega_-^*)) + 2k\pi & \text{if } \beta C(\omega_-^*) < 0. \end{cases}$$

Herein, $\Delta_{\pm}(\cdot) = 0$ has exactly one pair of purely imaginary roots, $\pm\omega_{\pm}^*$, at the bifurcation value $\tau_T = \eta_k^{\pm}$. In particular, we shall consider the case that $\eta_k^{\pm}$ is positive in the following discussions. We need the following condition for transversality.

Condition (B2)$_{\pm}$: $[\mathbf{R}(\omega_{\pm}^*, \eta_k^{\pm})]^2 + [\mathbf{I}(\omega_{\pm}^*, \eta_k^{\pm})]^2 \neq 0$, and $\Lambda(\omega_{\pm}^*) \neq 0$, where

$$\mathbf{R}(\omega, \tau_T) := 1 + \tau_T + \alpha(\tau_I - \tau_T)\cos(\tau_I w), \mathbf{I}(\omega, \tau_T) := \tau_T w - \alpha(\tau_I - \tau_T)\sin(\tau_I w),$$
$$\Lambda(\omega) := [1 + \alpha\tau_I \cos(\tau_I \omega)]\omega^2 + \alpha(1 + \tau_I)\sin(\tau_I \omega)\omega.$$

THEOREM 3.8. *Assume that α, β, τ_I satisfy condition* (B1)$_+$ *(resp.,* (B1)$_-$*) and* (B2)$_+$ *(resp.,* (B2)$_-$*) holds for an $\eta_k^+ > 0$ (resp., $\eta_k^- > 0$) for some k. Then Hopf bifurcation occurs at $\tau_T = \eta_k^+$ (resp., $\tau_T = \eta_k^-$), and a nontrivial synchronous periodic solution (resp., standing wave solution) is bifurcated from the zero solution of* (1.1) *(resp.,* (1.1)$_0$*).*

Proof. Under the assumptions, it suffices to justify the transversality to apply Hopf bifurcation theory [17]. We prove only the first case. The other can be verified similarly. First, we derive that

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\Delta_+(\lambda)|_{\lambda=i\omega_+^*, \tau_T=\eta_k^+} \\
&= \{1 + \alpha\tau_I e^{-\lambda\tau_I} + 2\beta\tau_T e^{-\lambda\tau_T}\}|_{\lambda=i\omega_+^*, \tau_T=\eta_k^+} \\
&= \{1 + \alpha\tau_I e^{-\lambda\tau_I} + \tau_T(1 + \lambda - \alpha e^{-\lambda\tau_I})\}|_{\lambda=i\omega_+^*, \tau_T=\eta_k^+} \\
&= \mathbf{R}(\omega_+^*, \eta_k^+) + i\mathbf{I}(\omega_+^*, \eta_k^+).
\end{aligned}$$

Thus, $\frac{\partial}{\partial\lambda}\Delta_+(\lambda)|_{\lambda=i\omega_+^*, \tau_T=\eta_k^+} \neq 0$ under condition (B2)$_+$; hence there exist some $\delta > 0$ and a smooth function $\lambda : (\eta_k^+ - \delta, \eta_k^+ + \delta) \to \mathbb{C}$ such that $\Delta_+(\lambda(\tau_T)) = 0$ and $\lambda(\eta_k^+) = i\omega_+^*$. Differentiating $\Delta_+(\lambda(\tau_T)) = 0$ with respect to τ_T at $\tau_T = \eta_k^+$, we obtain

$$\lambda'(\eta_k^+) = \frac{-2\beta e^{-\lambda\tau_T}\lambda}{1 + \alpha\tau_I e^{-\lambda\tau_I} + 2\beta\tau_T e^{-\lambda\tau_T}}|_{\lambda=i\omega_+^*, \tau_T=\eta_k^+} = \frac{Q_1 + iQ_2}{\mathbf{R}(\omega_+^*, \eta_k^+) + i\mathbf{I}(\omega_+^*, \eta_k^+)},$$

where $Q_1 = (\omega_+^*)^2 + \alpha\sin(\tau_I\omega_+^*)\omega_+^*$, $Q_2 = -\omega_+^* + \alpha\cos(\tau_I\omega_+^*)\omega_+^*$ and hence

$$\mathbf{Re}\lambda'(\eta_k^+) = \Lambda(\omega_+^*)/\{[\mathbf{R}(\omega_+^*, \eta_k^+)]^2 + [\mathbf{I}(\omega_+^*, \eta_k^+)]^2\} \neq 0$$

under condition (B2)$_+$. ∎

Theorem 3.8 presents a concrete criterion for the emergence of synchronous and asynchronous oscillations for system (1.1). The combination of Theorems 3.1 and 3.8 gives rise to global synchronization which accommodates synchronous oscillation for system (1.1). The stability and direction of Hopf bifurcation can be computed by an algorithm using the center manifold theorem and normal form method [18]. The criticality of the synchronous periodic orbit induced by τ_T from Hopf bifurcation for (1.1) with $g_I = g_T = \tanh$ has been shown to be determined by

$$N_c := -[\alpha(\tau_T - \tau_I)(w\sin(w\tau_I) - \cos(w\tau_I)) + \tau_T(1 + w^2) + 1], \tag{3.14}$$

where $\tau_T = \eta_k^+$, $w = \omega_+^*$, in [35]. The bifurcation is supercritical (resp., subcritical) and yields a stable (resp., unstable) limit cycle if $N_c < 0$ (resp., $N_c > 0$). We shall illustrate the periodic orbit obtained by Theorem 3.8 with a combination of the computation of N_c at the first bifurcation value in an example in section 5. We also remark that the bifurcation analysis in [30] was performed on $\mathcal{S}$ directly, i.e., on the reduced equation (3.7). The result therein can be upgraded to conclude the dynamics of the original system (1.1) if combined with our global synchronization framework.

Remark 3.4. (i) In Theorem 3.8, condition $(\mathrm{B1})_\pm$ plays the dominant role since condition $(\mathrm{B2})_\pm$ holds generically. Basically, condition $(\mathrm{B1})_\pm$ requires that τ_I be small and $|\beta|$ be relatively larger than $|\alpha|$. Notably, the restriction on the magnitude of τ_I can be relaxed. Observe that the function $Q(w)$ in (3.11) is dominated by the leading term w^2, as w is large. Therefore, if $|\beta|$ is sufficiently large, there exists exactly one positive zero for (3.11) given arbitrarily fixed τ_I and α. Accordingly, large $|\beta|$ is advantageous for Hopf bifurcation, and hence synchronous oscillations induced by transmission delay τ_T, to take place. A similar observation also holds for asynchronous oscillations.

(ii) Obviously, $(\mathrm{B1})_+$ is weaker than $(\mathrm{B1})_-$. We thus see that the τ_T-induced synchronous oscillations appear ahead of the asynchronous oscillations along the way of increasing $|\beta|$.

(iii) Similar formulations and arguments show that large $|\alpha|$ is advantageous to the occurrence of synchronous or asynchronous oscillations induced by transmission delay τ_I. In contrast to (ii), the synchronous oscillations appear behind the asynchronous oscillations along the way of increasing $|\alpha|$.

4. Extension to (1.1) with $N \geq 3$. In this section, we shall discuss the synchronization for system (1.1) of general scale $N \geq 3$. The difference between the synchrony for system (1.1) of scales $N = 3$ and $N > 3$ will be addressed in Remark 4.1(ii). By arguments similar to those in section 3.2, the convergence to multiple synchronous equilibria for (1.1) of scale $N \geq 3$ can also be established.

First, let us introduce the following conditions for global synchronization. These conditions can be regarded as the N-scale versions of conditions (S1)–(S4), respectively, in section 3.

Condition $(\mathrm{S1})^*$: $-\hat{\alpha} + \check{\beta} > 0$, $\mu - \hat{\alpha} + \check{\beta} > (N-3)|\beta|$, $\tau_I|\alpha| + \tau_T|\beta| < \tau_N^{(1)}$; more precisely,

$$\begin{cases} \beta > (1/\tilde{L})\alpha, \mu - \alpha + \beta\tilde{L} - (N-3)|\beta| > 0 & \\ \qquad \text{and } \tau_I|\alpha| + \tau_T|\beta| < \tau_N^{(1)} & \text{if } \alpha > 0, \beta \geq 0, \\ \mu - \alpha\tilde{L} + \beta\tilde{L} - (N-3)|\beta| > 0 & \\ \qquad \text{and } \tau_I|\alpha| + \tau_T|\beta| < \tau_N^{(1)} & \text{if } \alpha \leq 0, \beta \geq 0, \\ \beta > \tilde{L}\alpha, \mu - \alpha\tilde{L} + \beta - (N-3)|\beta| > 0 & \\ \qquad \text{and } \tau_I|\alpha| + \tau_T|\beta| < \tau_N^{(1)} & \text{if } \alpha \leq 0, \beta < 0, \end{cases}$$

where

$$\tau_N^{(1)} := \min\left\{ \frac{(|\alpha| + |\beta|)\mu}{(2\mu - \check{\alpha} + \hat{\beta})[|\alpha| + (N-2)|\beta|]}, \frac{\mu - \hat{\alpha} + \check{\beta} - (N-3)|\beta|}{2\mu - \check{\alpha} - \hat{\alpha} + \hat{\beta} + \check{\beta}} \right\}.$$

Condition $(\mathrm{S2})^*$: $\alpha < 0$, $\mu - \alpha\tilde{L} > 2|\beta|$, and $\tau_I < \tau_N^{(2)}$, where

$$\tau_N^{(2)} := \min\left\{ \frac{\mu}{(2\mu - \alpha)(|\alpha| + 2|\beta|)}, \frac{\mu - \alpha\tilde{L} - 2|\beta|}{\alpha(\alpha + \alpha\tilde{L} - 2\mu)} \right\}.$$

Condition $(\mathrm{S3})^*$: $\beta > 0$, $(|\alpha|+|\beta|)(\mu-|\alpha|) > (N-3)|\alpha\beta|$, $\mu+\beta\tilde{L}-|\alpha| > (N-3)|\beta|$, and $\tau_T < \tau_N^{(3)}$, where

$$\tau_N^{(3)} := \min\left\{ \frac{(|\alpha|+|\beta|)(\mu-|\alpha|)-(N-3)|\alpha\beta|}{\beta(2\mu+\beta)[|\alpha|+(N-2)|\beta|]}, \frac{\mu+\beta\tilde{L}-|\alpha|-(N-3)|\beta|}{\beta(\beta+\beta\tilde{L}+2\mu)} \right\}.$$

Condition (S4)*: $|\alpha|+2|\beta|<\mu$.

THEOREM 4.1. *System* (1.1) *of scale* $N \geq 3$ *achieves global synchronization if one of conditions* (S1)*–(S4)* *holds.*

Proof. The arguments for the τ_T-dependent results under condition (S1)* or (S3)* are different from the ones for the τ_T-independent results under condition (S2)* or (S4)*. First, let us consider the case that condition (S1)* holds. The differential-difference system of (1.1) can be written as

$$\begin{aligned} \dot{z}_i(t) &= -\mu z_i(t) + \alpha[g(x_i(t-\tau_I)) - g(x_{i+1}(t-\tau_I))] \\ &\quad - \beta[g(x_i(t-\tau_T)) - g(x_{i+1}(t-\tau_T))] + w_i(t), \quad i=1,\ldots,N, \end{aligned} \tag{4.1}$$

where $z_i(t) := x_i(t) - x_{i+1}(t)$, $w_i(t) = -\beta\Sigma_{j\in J_i}[g(x_j(t-\tau_T)) - g(x_{j+1}(t-\tau_T))]$, and $J_i := \{1,\ldots,N\} \setminus \{i, i-1, i+1 \ (\mathrm{mod}\ N)\}$. The obvious difference of (4.1) from (3.1) which corresponds to $N=3$ is the additional term $w_i(t)$. Each ith equation of (4.1) is of the form (2.1) with $\gamma_1 = -\alpha$, $\gamma_2 = \beta$, $\tau_1 = \tau_I$, $\tau_2 = \tau_T$ and satisfies condition (H1). According to Proposition 2.4, every ith component $z_i(t)$ of (4.1) converges to some interval $[-\rho_i, \rho_i] =: I_i$ as $t \to \infty$; moreover,

$$0 \leq \rho_i \leq |w_i|^{\max}(\infty)/\eta,$$

where $\eta := \mu - \hat{\alpha} + \check{\beta} - (\tau_I|\alpha| + \tau_T|\beta|)(2\mu - \check{\alpha} - \hat{\alpha} + \hat{\beta} + \check{\beta})$, and $\hat{\alpha}, \check{\alpha}, \hat{\beta}, \check{\beta}$ are as defined in (3.2), (3.3). We shall show that all ρ_i are equal to zero; consequently, $z_i(t)$ converges to zero, and the assertion thus follows.

We can construct, for each i, a sequence $\{\rho_i^{(k)}\}_{k=0}^{\infty}$ with $\rho_i^{(k)} \geq \rho_i$ for all k and where $z_i(t)$ converges to $[-\rho_i^{(k)}, \rho_i^{(k)}]$ as $t \to \infty$ for each k. The constructed $\rho_i^{(k)}$ shall satisfy

$$\begin{aligned} &\rho_i^{(0)} := 2(N-3)|\beta|/\eta, \quad i=1,\ldots,N, \\ &\rho_1^{(k)} = \Sigma_{j=3}^{N-1}|\beta|\rho_j^{(k-1)}/\eta, \quad \rho_N^{(k)} = \Sigma_{j=2}^{N-2}|\beta|\rho_j^{(k)}/\eta, \\ &\rho_i^{(k)} = (\Sigma_{j=1}^{i-2}|\beta|\rho_j^{(k)} + \Sigma_{j=i+2}^{N}|\beta|\rho_j^{(k-1)})/\eta, \quad k \geq 1,\ i=2,\ldots,N-1. \end{aligned}$$

The construction is similar to Proposition 3.2 in [28] and is sketched as follows. First, $\rho_i \leq \rho_i^{(0)}$ for all $i=1,\ldots,N$, due to $|g| \leq 1$; hence, $w_i(\cdot)$ is bounded by $2(N-3)|\beta|$. Also recall that $g' \leq 1$ and $z_j = x_j - x_{j+1}$. If such $\rho_i^{(k)}$, for $k=1,\ldots,\ n-1,\ i=1,\ldots,N$, and $k=n,\ i=1,\ldots,\ \ell-1 < N$, have been defined, then $|w_\ell(t)| = |-\beta\Sigma_{j\in J_\ell}[g(x_j(t-\tau_T)) - g(x_{j+1}(t-\tau_T))]| \leq \beta\Sigma_{j\in J_\ell}|z_j(t-\tau_T)|$. Hence,

$$0 \leq \rho_\ell \leq |w_\ell|^{\max}(\infty)/\eta \leq (\Sigma_{j=1}^{\ell-2}|\beta|\rho_j^{(n)} + \Sigma_{j=\ell+2}^{N}|\beta|\rho_j^{(n-1)})/\eta.$$

We observe that $\{\rho_i^{(k)} \mid i=1,2,\ldots,N\}$ is exactly the Gauss–Seidel iteration for solving the linear system $\mathbf{Mx} = \mathbf{0}$, where $\mathbf{M} := \eta I_N + \mathrm{circ}(0,0,-|\beta|,\ldots,-|\beta|,0)$, and "circ" denotes circular matrix; cf. [32]. Notably, $\mathbf{M}$ is strictly diagonal dominant [34] under condition (S1)*, which yields $\eta - (N-3)|\beta| > 0$. Accordingly, $(\rho_1^{(k)},\ldots,\rho_N^{(k)})$

converges to the unique solution of $\mathbf{Mx} = \mathbf{0}$, which is zero, as $k \to \infty$. Thus, for each i, sequence $\{\rho_i^{(k)}\}$ converges to zero as $k \to \infty$. Consequently, every component of the solution to (4.1), and hence the solution itself, converges to zero.

For the case that condition (S3)* holds, each component of (4.1) is of the form (2.1) with $\gamma_1 = \beta$, $\gamma_2 = -\alpha$, $\tau_1 = \tau_T$, $\tau_2 = \tau_I$ and satisfies condition (H2). The assertion then follows from Proposition 2.5.

Now, let us justify the case of condition (S2)*. Note that the differential-difference system derived from (1.1) can be put in a form different from (4.1):

$$\dot{z}_i(t) = -z_i(t) + \alpha[g(x_i(t-\tau_I)) - g(x_{i+1}(t-\tau_I))] + w_i(t), \quad i = 1, \dots, N, \tag{4.2}$$

where $w_i(t) = \beta[g(x_{i-1}(t-\tau_T)) - g(x_i(t-\tau_T)) + g(x_{i+1}(t-\tau_T)) - g(x_{i+2}(t-\tau_T))]$. Then (4.2) is of the form (2.1) with $\gamma_1 = -\alpha$, $\gamma_2 = 0$, $\tau_1 = \tau_I$ and satisfies condition (H2) under condition (S2)*. According to Proposition 2.5, every z_i of (4.2) converges to some interval $[-\tilde{\rho}_i, \tilde{\rho}_i]$; moreover, $0 \le \tilde{\rho}_i \le |w_i|^{\max}(\infty)/\tilde{\eta}$, where $\tilde{\eta} := 1 - \alpha\tilde{L} + \tau_I\alpha(2-\alpha-\alpha\tilde{L})$. The proof then follows processes parallel to the ones under condition (S1)* and hence is omitted.

For the case of condition (S4)*, each z_i of (4.2) can be regarded in the form (2.1) with $\gamma_1 = -\alpha$, $\gamma_2 = 0$, $\tau_1 = \tau_T$ and satisfies condition (H3). The assertion holds by Proposition 2.6. ∎

Remark 4.1. (i) The inequalities uninvolved with delays in conditions (S1)*–(S4)* favor smaller N (the scale of network). We further observe that for large N, both (S3)* and (S4)* require the magnitudes of α and β to be small relative to μ; on the other hand, large μ and negative α of large magnitude are advantageous for conditions (S1)* and (S2)*. Therefore, basically, large μ, small magnitudes of α and β, or negative α of large magnitude are advantageous for the synchronization of system (1.1), as N is large, according to Theorem 4.1.

(ii) If $N = 3$, the differential-difference equation derived from (1.1) is nearly a decoupled system; cf. (3.1). If $N > 3$, the differential-difference equation derived from (1.1) is a coupled system; cf. (4.1) or (4.2). Such a distinction between the structure of differential-difference equations is the major reason for the disparity of synchrony for (1.1) of scales $N = 3$ and $N > 3$. In fact, Example 5.3 will illustrate that under the same parameters, (1.1) can be synchronized globally as $N = 3$ but not as $N > 3$.

For system (1.1) of scale $N > 3$, Hopf bifurcation for synchronous periodic solutions can also be analyzed through the reduced system $(1.1)_+$ under our global synchronization framework. Asynchronous oscillations can be studied under the equivariant bifurcation framework [1, 2].

5. Numerical examples. We present three examples in this section. In Example 5.1, we illustrate the dynamics of synchronous oscillation. Example 5.2 demonstrates a transition from the convergence of multiple synchronous equilibria to the coexistence of two stable synchronous equilibria and an asynchronous oscillation as transmission delay τ_T increases. Example 5.3 shows that (1.1) of scale $N = 3$ with certain parameters attains global synchronization, but (1.1) of scale $N = 4$ with the same parameters admits asynchronous asymptotic behavior.

Example 5.1. Consider (1.1) with $\mu = 1$, $\alpha = -0.099$, $\beta = -0.9$, $\tau_I = 0.001$, $\tau_T = 1.6$, $N = 3$. The parameter $(\alpha, \beta) = (-0.099, -0.9)$ lies in Figure 3(d), and condition (S4) is met; hence the system can be synchronized in spite of time delays τ_I and τ_T according to Theorem 3.1. In addition, the parameters and delays satisfy the condition of Theorem 3.8; therefore, there exists a nontrivial synchronous periodic solution induced by τ_T near the first bifurcation value $\eta_0^+ \approx 1.562530143$. This bifurcation is

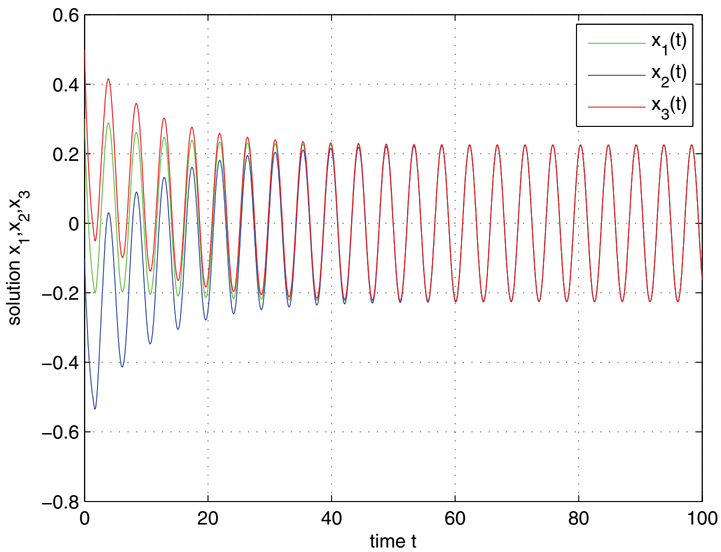

FIG. 7. *An orbit of* (1.1) *with* $\mu = 1$, $\alpha = -0.099$, $\beta = -0.9$, $\tau_I = 0.001$, *and* $\tau_T = 1.6$, *evolved from* $\phi(t) = (0.3, -0.1, 0.5)$, *approaches a synchronous limit cycle.*

supercritical due to $N_c \approx -5.892496067$, which is negative, where N_c is defined in (3.14). Figure 7 illustrates that the solution of (1.1) tends to a synchronous periodic orbit as $t \to \infty$; in the panel, three different colors represent the evolutions of three components x_1, x_2, x_3. We note that the system exhibits convergence to the trivial equilibrium if taking τ_T smaller than η_0^+ instead.

Example 5.2. Consider (1.1) with $\mu = 1$, $N = 3$, $\alpha = 0.9$, $\beta = 2$, $\tau_I = 0.01$, $\tau_T = 0.001, 0.9$. If $\tau_T = 0.001$, the system satisfies condition (S3) and hence achieves global synchronization. Moreover, the system satisfies the assumptions of Theorems 3.3 and 3.4(ii); hence it achieves global convergence to three synchronous equilibria where the nontrivial ones are stable. Figure 8(a) illustrates that the solutions away from the origin and plotted in blue converge to nontrivial stable equilibria; and the solution around the origin and plotted in red converges to the origin. Evolution for each component of the solution which converges to zero is illustrated in Figure 8(b). If taking $\tau_T = 0.9$ near the bifurcation value $\eta_0^- \approx 0.818$ instead, by Theorems 3.3 and 3.8, the nontrivial equilibria remain stable, but an asynchronous periodic solution is bifurcated from the origin. Figure 8(c) illustrates the coexistence of the asynchronous periodic oscillation around the origin and two stable synchronous equilibria. Evolution of each component of this oscillation is illustrated in Figure 8(d).

Example 5.3. Consider (1.1) with $\mu = 1$, $\alpha = 0$, $\beta = 0.99$, $\tau_I = 0.01$, $\tau_T = 10$. Such a system satisfies condition (S4) and hence can be synchronized as $N = 3$ according to Theorem 3.1. In addition, the synchronous phase contains at least two stable equilibria, according to Theorem 3.3, since this (α, β) lies in region D_2. However, Figure 9 illustrates that as $N = 4$, there exists an asynchronous limit cycle.

6. Conclusions. This investigation presented a methodology for studying global synchronization and asymptotic dynamics for a delayed neural network. Through

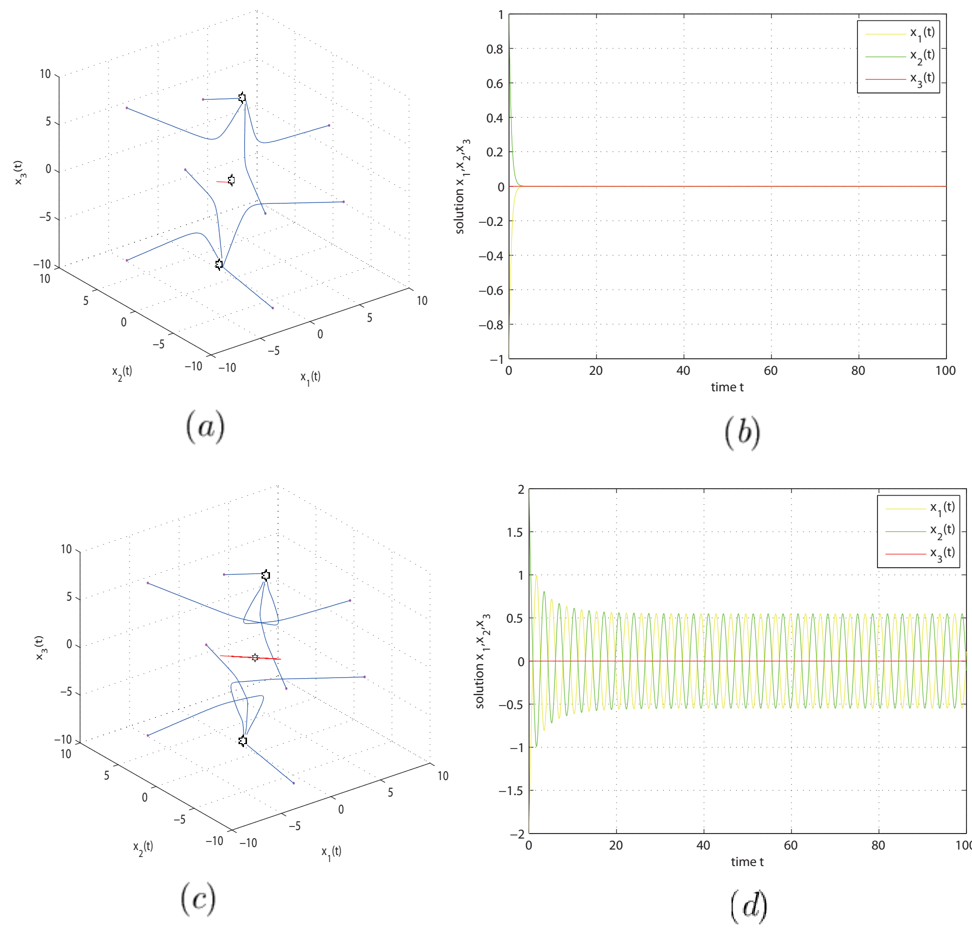

FIG. 8. *System* (1.1) *with* $\mu = 1$, $\alpha = 0.9$, $\beta = 2$, *and* $\tau_I = 0.01$. (a) *Solutions of* (1.1) *with* $\tau_T = 0.001$ *evolved from various initial values converge to one of the three equilibria.* (b) *Evolution of three components of the solution around the origin and plotted in red in* (a). (c) *Coexistence of the asynchronous periodic oscillation around the origin and plotted in red and two stable synchronous equilibria for* (1.1) *with* $\tau_T = 0.9$. (d) *The evolution for three components of the oscillation around the origin and plotted in red in* (c). *The solutions around the origin and plotted in red in* (a) *and* (c) *are both evolved from initial value* $(-2, 2, 0)$.

studying the differential-difference equation obtained from subtracting each component from its neighboring component of the system, we established delay-independent, delay-dependent, and scale-dependent criteria for the synchronization of the network. To elucidate the synchronous phases corresponding to different parameters and delay sizes, we investigated multistability and bifurcation which yields oscillations for the system. We also analyzed the existence of standing wave solutions which often occur when synchrony is lost, i.e., as synchronization yields to asynchronous oscillations.

We summarize the chief findings on the collective dynamics of the considered neural network (1.1). Items (iii)–(vi) apply to the case $N = 3$ in particular.

(i) Small scale of the network, large self-decay, inhibitory self-feedback, and excitatory coupling are advantageous for the synchronization of (1.1), and the corresponding delay τ_I (resp., τ_T) is required to be small if $|\alpha|$ (resp., $|\beta|$) is large; cf. Remarks 3.1

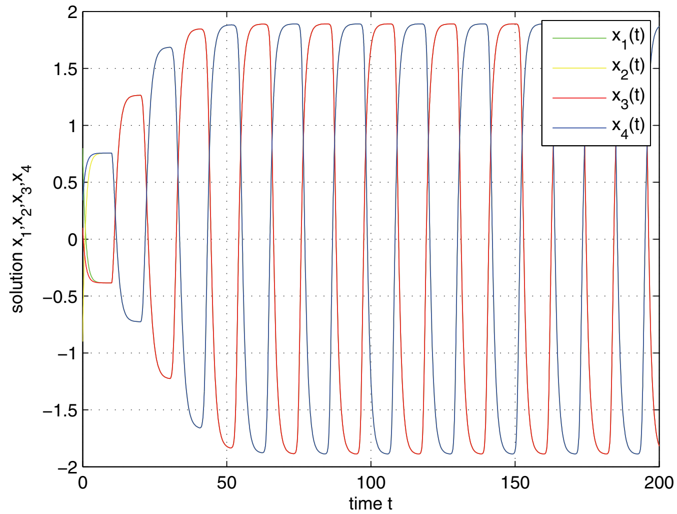

FIG. 9. *Asynchronous limit cycle of* (1.1) *with* $\mu = 1$, $\alpha = 0$, $\beta = 0.99$, $\tau_I = 0.01$, $\tau_T = 10$, *and* $N = 4$. *The orbit is evolved from* $(0.8, -0.9, 0.1, 0.34)$.

and 4.1.

(ii) The synchronization of network (1.1) also depends on the scale of the network. There exists a notable distinction in synchronization between systems (1.1) of scales $N = 3$ and $N > 3$; cf. Example 5.3.

(iii) Sufficiently strong inhibitory self-feedback or excitatory coupling can always synchronize (1.1) if the network is without delays, but it may fail to do so if the network is with delay of substantial magnitude; cf. Remark 3.2.

(iv) Inhibitory self-feedback and excitatory coupling lead to distinct synchronous phases. Basically, strong inhibitory self-feedback promotes convergence to the origin, while strong excitatory coupling leads to the convergence to nontrivial synchronous equilibria, as delays are small; cf. Remark 3.3.

(v) "Strong excitatory self-feedback"-induced multistability admits the coexistence of synchronous and asynchronous equilibria, whereas "strong excitatory coupling"-induced multistability admits the existence of synchronous equilibria; cf. Remark 3.3.

(vi) The delay τ_I (resp., τ_T) can lead to the emergence of synchronous or asynchronous nontrivial oscillations if the self-feedback strength (resp., coupling) is strong. The synchronous and asynchronous oscillations occur in succession as the strength $|\beta|$ or $|\alpha|$ increases; cf. Remark 3.4.

The present methodology can be extended to treat coupled systems with coupling structure admitting $\mathbf{Z}_n$-symmetry. The associated differential-difference equations for synchronization and convergent dynamics can be similarly formulated for systems with this symmetry. The analysis for the asymptotic behaviors of the differential-difference equations relies on constructing elaborate upper and lower dynamics. New idea for establishing the dynamical properties may be needed if the upper and lower dynamics themselves are complicated. This approach can also be extended to treat systems comprising subnetworks or subsystems such as a model on somitogenesis [22].

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