Pricing the Deflation Protection Option in TIPS Using an HJM Model with Inflation- and Interest-Rate Jumps

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Much known about Treasury inflation-protected securities (TIPS) is related to the hedge they offer against inflation, but little is known about their protection against deflation—in the form of a deflation protection option (DPO). In this article, a pricing framework that builds on a Heath–Jarrow–Morton forward-rate economy with codependent inflation- and interest-rate jumps is derived to value this embedded DPO. The model prices for TIPS resulting from this pricing framework are found to most closely fit the 10-year notes issued following the 2008 crisis. Considering these notes accounted for over 70% of the total TIPS-market trading activity, this result underscores the importance of properly assessing DPO value in times of deflationary fears compounded by rising real yields, negligence of which may well be liable for the post-crisis mispricing in TIPS.

Treasury inflation-protected securities (TIPS) are bonds issued by the U.S. Treasury. As instruments that are constructed to facilitate efficient hedging of inflation risk, the protection that they offer against deflation is comparatively less well known.[1] The principal amount upon which the semiannual coupon payments are based is adjusted for inflation, which is measured by the changes in the Consumer Price Index (CPI-U). Specifically, the principal amount is determined by the greater of its original value at par or its inflation-adjusted value at maturity. Based on a currency option analogy for inflation (Amin and Jarrow [1991]; Jarrow and Yildirim [2003]), an embedded European put option written on the current CPI-U is immediately retrieved, with the initial CPI-U acting as the option's strike price. As such, whether or not TIPS holders receive the original principal at par (as a deflation protection) or the inflation-adjusted principal at maturity (as an inflation protection) is translated into an option exercise decision that is contingent on the status of the option's moneyness. This embedded put option, commonly termed the *deflation protection option* (DPO), underlies the protection against deflation offered by TIPS in the form of a floor on the bond's principal payment.

The seminal paper by Jarrow and Yildirim [2003], in which the existence of the DPO was clearly indicated, chose not to assign the option with any value because deflation was once dismissed as improbable for the U.S. inflationary environment prior to 1997. Moreover, because of the DPO's inheritance from TIPS of the codependent characteristics between inflation and interest rates,

[1] As a type of U.S. Treasury security, TIPS account for a market size of over $866 billion in issuance. Other inflation-linked markets of noticeable popularity include U.K. Index-linked Gilts ($549 billion) and the French OATi/OAT€i market ($235 billion) as of April 2012.

the required modeling effort is never easy and entails extending Jarrow and Yildirim's (JY) Heath–Jarrow–Morton (HJM) framework accordingly. In this respect, we are certainly not the first in the literature to have attempted to meet such modeling necessity. The innovative works of Hinnerich [2008] and Chiang, Li, and Chen [2016] are examples of extended HJM economies that belong to this category. Although we focus on the valuation of the DPO embedded in TIPS, Hinnerich's [2008] model is applied to the pricing of inflation-indexed swaps and swaptions. Relative to our HJM forward-rate framework under codependent inflation- and interest-rate jumps, the Markov-modulated HJM framework of Chiang, Li, and Chen [2016] demonstrates how one can, in general, incorporate interest-rate regime shifts and inflation jumps into the JY model.

The pricing framework in this article thus provides several new perspectives on TIPS valuation that the extant literature has yet to fully profile.[2] First, to the best of our knowledge, this article is the first extended HJM framework with autonomous jumps that is tailored specifically to the valuation of the DPO embedded in TIPS. In contrast, the existing approaches to assessing the DPO value tend to be based on modeling the yields only (see, e.g., Adrian and Wu [2010]; Christensen, Lopez, and Rudebusch [2012]; Haubrich, Pennacchi, and Ritchken [2012]; Fleckenstein, Longstaff, and Lustig [2014]; and Grishchenko, Vanden, and Zhang [2016]). In this article, not only is the role of inflation rate inseparable from the pricing framework, but its joint impacts with interest-rate uncertainties are also explicitly considered. The interplay between inflation- and interest-rate uncertainties that is commonly observed in reality—where the benefits of one are often found to be offset by those of the other—is thus reflected in our model setting.

Second, in depicting the co-dependent structure between inflation and interest rates under the JY model, we follow Das and Uppal [2004] to allow the jump amplitudes of the nominal and forward rates and the CPI-U to correlate with one another. Yet in our case, strict concordant co-movements among the jump amplitudes need not be assumed. In this context, our pricing framework, which we refer to as the JY model with correlated jumps (JY-CJ), permits two degenerate cases: (1) the JY model with independent jumps (JY-IJ) and (2) the original JY model without jump risk.

In addition, although there is no current consensus on whether the exact nature of jump risk is systematic or idiosyncratic, this study does not discriminate against assertions of either case. In fact, our pricing framework is general enough to allow for the adaptation of different jump-risk specifications. In particular, we show how the DPO pricing formula is capable of adapting to (1) Merton's pricing measure, when jump risk is assumed to be idiosyncratic and diversifiable, and (2) a modified Esscher measure similar to that of Ballotta [2005], in which jump risk is assumed to be systematic/nondiversifiable. Relative to Ballotta [2005], the modified Esscher measure that we derive incorporates correlated (instead of independent) jump amplitudes, and our DPO pricing formula retains a Black–Scholes type of analytic form that facilitates practical implementation.

Using the daily prices of U.S. Treasury and TIPS bonds from September 2008 to March 2016, this study employs the expectation-maximization (EM) algorithm of Dempster, Laird, and Rubin [1977] to estimate latent parameters for the system of correlated jump diffusion processes. We are certainly aware of other estimation techniques available to deal with models involving latent parameters. The Markov chain Monte Carlo model of Eraker, Johannes, and Polson [2003], for example, although equally applicable, involves much higher computational cost than the point estimation required by the EM algorithm. Likelihood ratio tests (LRT) are used to select the best jump-risk specification among the JY, JY-IJ, and JY-CJ models. To analyze pricing performance and, in particular, to nail down the impact of correlated jump risks on the DPO value and hence TIPS prices, the JY, JY-IJ, and JY-CJ models are examined under the Merton and Esscher measures, respectively.

Our key findings can be summarized as follows. First, the results of the LRT statistics indicate that the best jump-risk specification among the three is the JY-CJ model. These results directly support our rationale for considering inflation- and interest-rate uncertainties as correlated jump diffusions. Second, we find that, under a systematic jump-risk setting, TIPS pricing that

[2] Jarrow and Yildirim [2003] derived a pricing model for inflation-linked Treasury bonds under the Heath, Jarrow, and Morton [1992] framework. They assumed a Brownian motion with constant drift for the nominal and the real instantaneous forward rates and a geometric Brownian motion for the CPI-U. Mercurio [2005] adopted the London Interbank Offered Rate market model of Brace, Gatarek, and Musiela [1997] with deterministic interest rate volatilities to price year-on-year inflation-linked swaps.

incorporates the DPO value outperforms pricing that does not. Under the Esscher measure, both the JY-IJ and JY-CJ models exhibit significant gains in pricing performance relative to the JY model. This result indicates that a systematic jump-risk setting is more appropriate for the pricing of TIPS. Their market quotes, even under normal market conditions, seem to already embody a certain level of the associated jump-risk premiums.[3] Analogous to such encapsulation for the market prices of correlated jump risks is the set of martingale conditions that this study provides. Third, the pricing performance of the JY-CJ model (under the Esscher measure) is particularly strong for the 10-year TIPS issued following the 2008 crisis. Most interestingly, we find the DPO values to be time varying, to be small in absolute terms, and to exhibit an interesting pattern especially for the late-2008 to early-2009 crisis period. This finding is consistent with that of Grishchenko, Vanden, and Zhang [2016], and it suggests that the proper assessment of the DPO value, particularly in times of deflationary fears coupled with rising real yields, is indispensable to the accurate pricing of TIPS.

The rest of this article is organized as follows. The next section introduces our new pricing framework and its theoretical components, including the underlying assumptions, the martingale conditions, and the closed-form solutions for the DPO value. The following section presents the empirical components, including interest rate calibration, parameter estimation based on the expected maximization algorithm, and pricing performance analysis under different choices of risk-neutral probability measures. The last section concludes. All the technical proofs are collected in the Appendix.

THE MODEL

An HJM Economy with Correlated Inflation- and Interest-Rate Jump Risks

We construct our pricing model on a filtered probability space $\{\Omega, \mathbb{F}, \mathbb{P}, \{\mathbb{F}(t)\}_{t=0}^{T}\}$ generated by three jump diffusion processes with diffusion (correlated Brownian) components, $\{W_N(t), W_R(t), W_I(t)\}$, and jump (compound Poisson) components, $\{Y_{N,k}, Y_{R,k}, \ln Y_{I,k}\}$. N and R denote the nominal and real interest rates, respectively, and I is the inflation rate/index; Ω is the set of all the possible outcomes; $\mathbb{F}$ denotes the σ-field of subsets of Ω; $\mathbb{P}$ represents the physical probability measure; and $\{\mathbb{F}(t)\}_{t=0}^{T}$ denotes the sequence of filtrations jointly generated by the correlated Brownian motions and the compound Poisson process at time t. Correlation coefficients among the Brownian motions are denoted by ρ_{NR}, ρ_{RI}, ρ_{IN}.

The arrival of abnormal information, for the inflation, and the nominal and real interest rates, $M(t)$, is modeled as a Poisson process with jump intensity λt, which is defined as the expected number of jumps occurring over a time interval $(0, t]$; that is, $M(t) \sim$ Poisson (λt). Jump amplitudes for the inflation and the nominal and real interest rates are denoted by $\{Y_{N,k}, Y_{R,k}, \ln Y_{I,k}\}$ and assumed to be normally, independently and identically distributed random variables with mean $(\theta_N, \theta_R, \theta_I)$ and variance $(\nu_N^2, \nu_R^2, \nu_I^2)$. Correlation coefficients between the jump amplitudes are denoted by ϕ_{NR}, ϕ_{RI}, and ϕ_{IN}. The compound Poisson process is assumed to be independent of the Brownian motions. Assumption 1 depicts an economy of correlated inflation- and interest-rate uncertainties.

Assumption 1. *Under the physical probability measure $\mathbb{P}$, the nominal forward rate, $f_N(t, T)$, the real instantaneous forward rates, $f_R(t, T)$, and the inflation rate, $I(t)$, are modeled by a system of jump diffusion processes defined as follows:*

$$df_N(t,T) = \alpha_N(t,T)dt + \sigma_N(t,T)dW_N(t) + d\sum_{k=1}^{M(t)} Y_{N,k} \quad (1)$$

$$df_R(t,T) = \alpha_R(t,T)dt + \sigma_R(t,T)dW_R(t) + d\sum_{k=1}^{M(t)} Y_{R,k} \quad (2)$$

$$\frac{dI(t)}{I(t-)} = \left[\mu_I(t) - \lambda\left(e^{\theta_I + \frac{\nu_I^2}{2}} - 1\right)\right]dt + \sigma_I(t)dW_I(t) + d\left(\sum_{k=1}^{M(t)} Y_{I,k} - 1\right) \quad (3)$$

where $\alpha_N(t, T)$ and $\alpha_R(t, T)$ are the expected growth rates for the nominal and real instantaneous forward rates, and $\sigma_D(t, T)$ and $\sigma_F(t, T)$ are the associated volatility functions. The drift term $\mu_I(t) - \lambda\left(e^{\theta_I + \frac{\nu_I^2}{2}} - 1\right)$ for the inflation index $I(t)$ has

[3] These jump-risk premiums may inevitably be related to deflationary expectations and market illiquidity. Pastor and Stambaugh [2003] and Amiram, Cserna, and Levy [2015], for example, showed that the jump volatility component increases the priced liquidity risk and has a positive and statistically significant effect for various measures of liquidity risk.

two components: the instantaneous mean level, $\mu_I(t)$*, and a convexity adjustment,* $\lambda\left(e^{\theta_I+\frac{\nu_I^2}{2}}-1\right)$*, which sets the growth rate of the inflation equal to* $\mu_I(t)$ *under the* $\mathbb{P}$ *measure; that is,* $E_{t-1}\left[\frac{I(t)}{I(t-1)}\right]=e^{\mu_I(t)}$*. The instantaneous volatility function for the inflation index is denoted by* $\sigma_I(t)$*.*

The following lemma, which follows directly from the Itô–Doeblin formula, depicts the dynamics of nominal and inflation-linked zero-coupon bonds in a market where the interest rates and the inflation index evolve correlatively according to Equations 1–3.

Lemma 1. *Given the forward-rate specifications of Assumption 1, the dynamic processes of the nominal and the inflation-linked zero-coupon bond prices under the physical measure* $\mathbb{P}$ *are given by*

$$\frac{dP_N(t,T)}{P_N(t-,T)}=\left[r_N(t)-\hat{\alpha}_N(t,T)+\frac{1}{2}\hat{\sigma}_N^2(t,T)\right]dt-\hat{\sigma}_N(t,T)dW_N(t)+d\left(\sum_{k=1}^{M(t)}\tilde{Y}_{N,k}-1\right) \quad (4)$$

$$\frac{dI(t)P_R(t,T)}{I(t-)P_R(t-,T)}=\begin{bmatrix} r_R(t)-\hat{\alpha}_R(t,T)+\frac{1}{2}\hat{\sigma}_R^2(t,T) \\ -\rho_{RI}\hat{\sigma}_R(t,T)\sigma_I(t)+\mu_I-\lambda\left(e^{\theta_I+\frac{\nu_I^2}{2}}-1\right)\end{bmatrix}dt-\hat{\sigma}_R(t,T)dW_R(t)+\sigma_I(t)dW_I(t)+d\left(\sum_{k=1}^{M(t)}\tilde{Y}_{RI,k}-1\right) \quad (5)$$

where $r_x(t)=f_x(t,t)$*,* $x\in\{\mathrm{N},\mathrm{R}\}$*, denotes the instantaneous nominal or real short rate with drift,* $\hat{\alpha}_x(t,T)=\int_t^T\alpha_x(t,s)ds$*, and diffusion,* $\hat{\sigma}_x(t,T)=\int_t^T\sigma_x(t,s)ds$*. Specifically,* $\sigma_x(t,T)$ *is based on a one-factor model* (Jarrow and Yildirim [2003]) *with exponentially decaying volatility:* $\sigma_x(t,T)=a_xe^{-b_x(T-t)}$*, for some constant numbers* a_x *and* b_x*. The jump amplitude for the nominal zero-coupon bond,* $\tilde{Y}_{N,k}$*, is defined by its log transform:*

$$\ln\tilde{Y}_{N,k}:=-Y_{N,k}(T-t)\sim\mathrm{Normal}[-\theta_N(T-t),\nu_N^2(T-t)^2]$$

and the jump amplitude for the inflation-linked zero-coupon bond, $\tilde{Y}_{RI,k}$*, is defined by*

$$\ln\tilde{Y}_{RI,k}:=-Y_{R,k}(T-t)+\ln Y_{I,k}\sim\mathrm{Normal}\begin{bmatrix}-\theta_R(T-t)+\theta_I,\nu_R^2(T-t)^2 \\ -2\phi_{RI}\nu_R(T-t)\nu_I+\nu_I^2\end{bmatrix}$$

Martingale Conditions

In the following, we derive the set of martingale conditions under which a market with correlated inflation and interest rates is arbitrage free. Essential to devising the martingale conditions for TIPS pricing under risk neutrality is the identification of a market-price-of-risk process that determines the required jump-risk premiums. Furthermore, central to the Radon–Nikodým derivative that associates the physical measure to its risk neutral counterpart, the uniqueness of the market price of risk will determine the completeness of a market.

Proposition 1. *Under the risk-neutral* $\mathbb{Q}$ *measure, the dynamic processes of the discounted nominal zero-coupon bond and the discounted inflation-linked zero-coupon bond are martingales if and only if the following conditions are met:*

$$\hat{\alpha}_N(t,T)=\left[h_{N,1}(1+\rho_{NR}+\rho_{IN})+\frac{1}{2}\right]\hat{\sigma}_N^2(t,T)+\lambda^{\mathbb{Q}}\left(e^{\theta_N^{\mathbb{Q}}+\frac{\nu_N^2(T-t)^2}{2}}-1\right) \quad (6)$$

$$\hat{\alpha}_R(t,T)=\left[h_{R,1}(1+\rho_{RI}+\rho_{NR})+\frac{1}{2}\right]\hat{\sigma}_R^2(t,T)-\rho_{RI}\hat{\sigma}_R(t,T)\sigma_I(t)+\lambda^{\mathbb{Q}}\begin{pmatrix}e^{\theta_R^{\mathbb{Q}}+\theta_I^{\mathbb{Q}}+\frac{\nu_R^2(T-t)^2+2\phi_{RI}\nu_R(T-t)\nu_I+\nu_I^2}{2}} \\ -e^{\theta_I^{\mathbb{Q}}+\frac{\nu_I^2}{2}}\end{pmatrix} \quad (7)$$

$$\mu_I(t)=r_N(t)-r_R(t)-h_{I,1}(1+\rho_{RI}+\rho_{IN})\sigma_I^2(t)+\lambda\left(e^{\theta_I+\frac{\nu_I^2}{2}}-1\right)-\lambda^{\mathbb{Q}}\left(e^{\theta_I^{\mathbb{Q}}+\frac{\nu_I^2}{2}}-1\right) \quad (8)$$

where $\lambda^{\mathbb{Q}}$ *represents the jump intensity;* $(\theta_N^{\mathbb{Q}},\theta_R^{\mathbb{Q}},\theta_I^{\mathbb{Q}})$ *and* $(\theta_N,\theta_R,\theta_I)$ *denote the mean for the normally distributed jump amplitudes under the* $\mathbb{Q}$ *and* $\mathbb{P}$ *measures, respectively; and* (ν_N,ν_R,ν_I) *denotes the standard deviations, which remain intact under the change of measure. That is, we have*

$$
\begin{aligned}
\lambda^{\mathbb{Q}} = \lambda \exp\Bigg[& \frac{(h_{N,2}^2 \nu_N^2 + h_{R,2}^2 \nu_R^2 + 2h_{N,2}h_{R,2}\phi_{NR}\nu_N\nu_R)(T-t)^2}{2} \\
& + (h_{R,2}h_{I,2}\phi_{RI}\nu_R\nu_I + h_{I,2}h_{N,2}\phi_{IN}\nu_I\nu_N - h_{N,2}\theta_N \\
& - h_{N,2}\theta_N - h_{R,2}\theta_R)(T-t) + h_{I,2}\theta_I + \frac{h_{I,2}^2\nu_I^2}{2} \Bigg] \\
\theta_N^{\mathbb{Q}} = & -\theta_N(T-t) + h_{N,2}\nu_N(T-t)[\nu_N(T-t) \\
& + \phi_{NR}\nu_R(T-t) + \phi_{IN}\nu_I] \\
\theta_R^{\mathbb{Q}} = & -\theta_R(T-t) + h_{R,2}\nu_R(T-t)[\nu_R(T-t) \\
& + \phi_{NR}\nu_N(T-t) + \phi_{RI}\nu_I] \\
\theta_I^{\mathbb{Q}} = & \ \theta_I + h_{I,2}\nu_I[\nu_I + \phi_{RI}\nu_R(T-t) \\
& + \phi_{IN}\nu_N(T-t)]
\end{aligned}
$$

where $(h_{N,1}, h_{R,1}, h_{I,1})$ *and* $(h_{N,2}, h_{R,2}, h_{I,2})$ *are the Esscher parameters* (see Appendix A) *associated with the diffusion (Brownian) components and the jump (Poisson) components, respectively.*

Note that deriving the $\mathbb{Q}$ measure entails rescaling the drifts of the associated stochastic processes by the set of Esscher parameters, $(h_{N,1}, h_{R,1}, h_{I,1})$ and $(h_{N,2}, h_{R,2}, h_{I,2})$. Equations 6–8 thus determine, respectively, the risk-adjusted drifts of the nominal forward rate, the real forward rate, and the inflation rate under the risk-neutral $\mathbb{Q}$ measure.

Several interesting observations can be made about these results. First, encapsulated by Equations 6–8 is the market price of risk for correlated inflation- and interest-rate jump uncertainties. Again, one can clearly identify the associated diffusion (Brownian) components and the jump (Poisson) components therein. For example, in Equation 6, the total risk premium consists of the risk premium for the diffusion component, $-h_{N,1}(1+\rho_{NR}+\rho_{IN})\hat{\sigma}_N^2(t,T)$, and the jump component, $-\lambda^{\mathbb{Q}}\left(e^{\theta_N^{\mathbb{Q}}+\frac{\nu_N^2(T-t)^2}{2}}-1\right)$. The same applies to Equation 7, where the total risk premium breaks down into the risk premium for the diffusion component, $-h_{R,1}(1+\rho_{RI}+\rho_{NR})\hat{\sigma}_R^2(t,T) - h_{I,1}(1+\rho_{RI}+\rho_{IN})\sigma_I^2(t)$, and the risk premium for the jump component,

$$-\lambda^{\mathbb{Q}}\left(e^{\theta_R^{\mathbb{Q}}+\theta_I^{\mathbb{Q}}+\frac{\nu_R^2(T-t)^2+2\phi_{RI}\nu_R(T-t)\nu_I+\nu_I^2}{2}}-1\right).$$

Second, and most interestingly from Proposition 1, Equation 8 depicts a modified Fisher equation that captures departures from the Fisher hypothesis, where $-h_{I,1}(1+\rho_{RI}+\rho_{IN})\sigma_I^2(t) + \lambda\left(e^{\theta_I+\frac{\nu_I^2}{2}}-1\right) - \lambda^{\mathbb{Q}}\left(e^{\theta_I^{\mathbb{Q}}+\frac{\nu_I^2}{2}}-1\right)$ denotes the required risk premiums for investors confronting the co-occurrence of inflation- and interest-rate jump uncertainties. In this respect, Equation 8 echoes the no-arbitrage term structure framework of D'Amico, Kim, and Wei [2014] in interpreting the information content of TIPS prices. Consistent with our economic reasoning, they too argued that the Fisher hypothesis has neglected the potential correlation effects between inflation and interest rates.

The Deflation Protection Option Value

Consider a TIPS coupon bearing bond issued at time $t_0 \leq t$ with real coupon payments, c; the original principal at par, $F(t_0)$; the initial CPI-U rate, $I(t_0)$; and time-to-maturity, $T - t$. Let $cI(t)/I(t_0)$ denote the nominal coupon payments, which are the real coupon payments multiplied by an inflation adjustment ratio, $I(t)/I(t_0)$. To derive the analytical solution for the DPO value, we first follow Jarrow and Yildirim [2003] to decompose the cash flow structure of a TIPS bond into a series of zero-coupon bonds of different maturities plus, at maturity, a principal amount that is determined by the greater of the original principal at par or the inflation-adjusted principal; that is,

$$\max\left[F(T), F(t_0)\right] = F(T) + \frac{F(t_0)}{I(t_0)} \cdot \mathrm{DPO}(T) \qquad (9)$$

where $\mathrm{DPO}(T)$ denotes the deflation protection option with payoff $\max[I(t_0) - I(T), 0]$ at maturity T. Given that the principal amount now translates into an inflation-adjusted principal $F(T)$ plus $F(t_0)/I(t_0)$ units of the deflation protection option, we use the following lemma to depict the fair value of a TIPS bond.

Lemma 2. *The time t fair value of a TIPS coupon-bearing bond,* $B_{TIPS}(t,T)$*, adopted to filtration* $\mathbb{F}(t)$ *under the* $\mathbb{Q}$ *measure thus takes the following form:*

$$B_{TIPS}(t,T)=\beta_N(t)\left[\begin{array}{l}\displaystyle\sum_{s=t+1}^{T}\frac{c}{I(t_0)}\cdot E_t^{\mathbb{Q}}\left(\frac{I(s)P_R(s,s)}{\beta_N(s)}\right)\\ +E_t^{\mathbb{Q}}\left(\dfrac{max[F(T),F(t_0)]}{\beta_N(T)}\right)\end{array}\right]$$

$$=\sum_{s=t+1}^{T}[cI(t)P_R(t,s)+F(t_0)I(t)P_R(t,T)+F(t_0)DPO(t)]/I(t_0) \tag{10}$$

where $\beta_N(t)=exp\left[-\int_0^t r_N(u)du\right]$ *is the nominal money market account,* $P_R(t,T)$ *is the price of a T-maturity real zero-coupon bond,* $I(t)P_R(t,T)$ *denotes the price of a T-maturity inflation-linked zero-coupon bond, and* $\mathrm{E}_t^{\mathbb{Q}}(\cdot)$ *denotes the conditional expectation adapted to filtration* $\mathbb{F}(t)$ *under the risk-neutral* $\mathbb{Q}$ *measure. In particular, the payoff structure of the DPO at maturity T is given by*

$$\begin{aligned}\mathrm{DPO}(T)&=[I(t_0)-I(T)]1_{\{I(T)<I(t_0)\}}\\ &=[I(t_0)P_N(T,T)-I(T)P_R(T,T)]1_{\{I(T)P_R(T,T)<I(t_0)P_N(T,T)\}}\end{aligned} \tag{11}$$

where $P_N(t,T)$ *and* $P_R(t,T)$ *denote prices for the T-maturity nominal and real zero-coupon bonds, and* $1_{\{\cdot\}}$ *is an indicator function.*[4]

Equation 11 of Lemma 2 clearly depicts the deflation protection scheme offered by the DPO. This interpretation is not new but rather is drawn on the currency analogy for inflation as identified by Amin and Jarrow [1991] and Jarrow and Yildirim [2003]. That is, the real rates are regarded as the interest rates in the foreign currency, whereas the inflation index is reminiscent of the exchange rate between the U.S. dollar and the foreign currency. As such, the embedded DPO resembles a *T*-maturity European put option written on the current CPI-U, $I(T)$, with the initial CPI-U, $I(t_0)$, acting as the option's strike price. Specifically, TIPS holders are granted the right to exchange $I(t_0)$ units of the nominal zero-coupon bond for an inflation-linked zero-coupon bond. A falling inflation rate below the initial CPI-U reference level ($I(t) < I(t_0)$) would indicate the in-the-moneyness of the option and result in TIPS investors receiving the original principal at par. On the other hand, if the current CPI-U is above or equal to its initial reference level ($I(t) \geq I(t_0)$)—indicating the out-of-the-moneyness of the DPO value—TIPS investors will be granted the inflation-adjusted principal.

Theorem 1. *Adapted to filtration* $\mathbb{F}(t)$ *under the* $\mathbb{Q}$ *measure, the DPO value embedded in TIPS is given by*

$$\begin{aligned}\mathrm{DPO}(t)=\;&I(t_0)P_N(t,T)\sum_{m=0}^{\infty}p(m)e^{L_2(m)}\Phi[-d_2(m,t,T)]\\ &-I(t)P_R(t,T)\sum_{m=0}^{\infty}p(m)e^{L_1(m)}\Phi[-d_1(m,t,T)]\end{aligned} \tag{12}$$

where $p(m)=e^{-\lambda^{\mathbb{Q}}(T-t)}[\lambda^{\mathbb{Q}}(T-t)]^m/m!$ *is the probability mass function for Poisson distribution* $M(T-t)$ *with intensity* $\lambda^{\mathbb{Q}}(T-t)$ *conditional on m number of jumps occuring over a period* $[t, T]$. $\Phi(\cdot)$ *is the distribution function of a standard normal with*

$$d_1(m,t,T)=\frac{\ln\dfrac{I(t)P_R(t,T)}{I(t_0)P_N(t,T)}+\dfrac{1}{2}\delta^2(m,t,T)+L_1(m)-L_2(m)}{\delta(m,t,T)}$$

$$d_2(m,t,T)=\frac{\ln\dfrac{I(t)P_R(t,T)}{I(t_0)P_N(t,T)}-\dfrac{1}{2}\delta^2(m,t,T)+L_1(m)-L_2(m)}{\delta(m,t,T)}$$

where $\delta^2(m,t,T)=V_1^2(m,t,T)-2\xi(m,t,T)V_1(m,t,T)V_2(m,t,T)+V_2^2(m,t,T)$, and $L_1(m)$ and $L_2(m)$ are defined by

$$\begin{aligned}L_1(m)=\;&m\left(\theta_I^{\mathbb{Q}}+\theta_R^{\mathbb{Q}}+\frac{\nu_R^2(T-t)^2}{2}-\phi_{RI}\nu_R\nu_I(T-t)+\frac{\nu_I^2}{2}\right)\\ &-\lambda^{\mathbb{Q}}\left(e^{\theta_R^{\mathbb{Q}}+\theta_I^{\mathbb{Q}}+\frac{\nu_R^2(T-t)^2+2\phi_{RI}\nu_R(T-t)\nu_I+\nu_I^2}{2}}-1\right)(T-t)\end{aligned}$$

$$L_2(m)=m\left(\theta_N^{\mathbb{Q}}+\frac{\nu_N^2(T-t)^2}{2}\right)-\lambda^{\mathbb{Q}}\left(e^{\theta_N^{\mathbb{Q}}+\frac{\nu_N^2(T-t)^2}{2}}-1\right)(T-t)$$

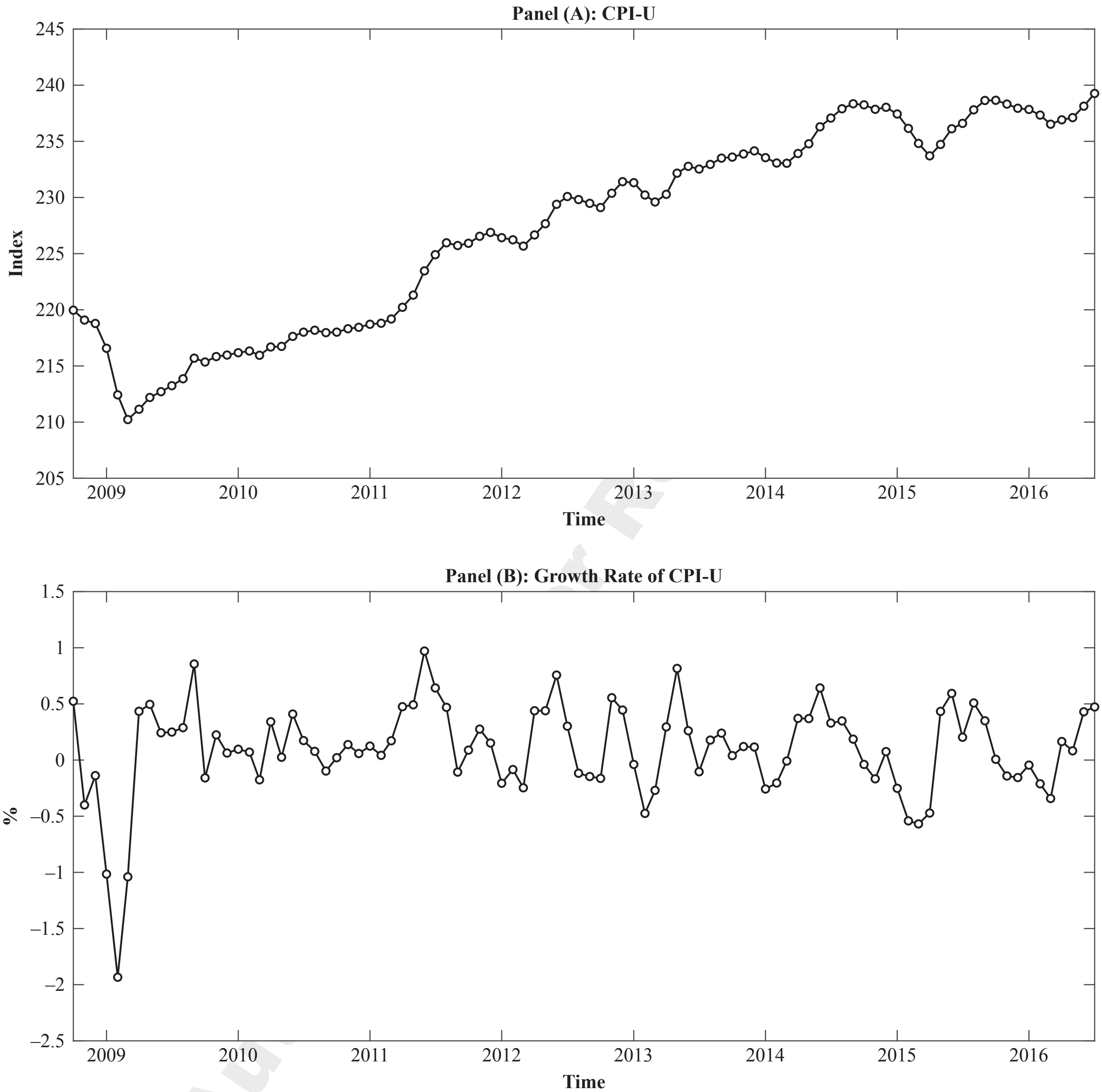
and other notations follow as previously defined.

Theorem 1 is referred to as the JY-CJ. Note that the analytic pricing formula for the DPO value depicted by Theorem 1 is in fact a generalized model incorporating correlated jump risks and thus is capable of adapting to different jump-risk specifications according to one's preferences. In this study, we demonstrate two such important cases. First,

[4] Given the martingale conditions given by Equations 7 and 8 of Proposition 1 and the cash flow decomposition of Equation 9, one immediately arrives at Lemma 2.

E X H I B I T   1

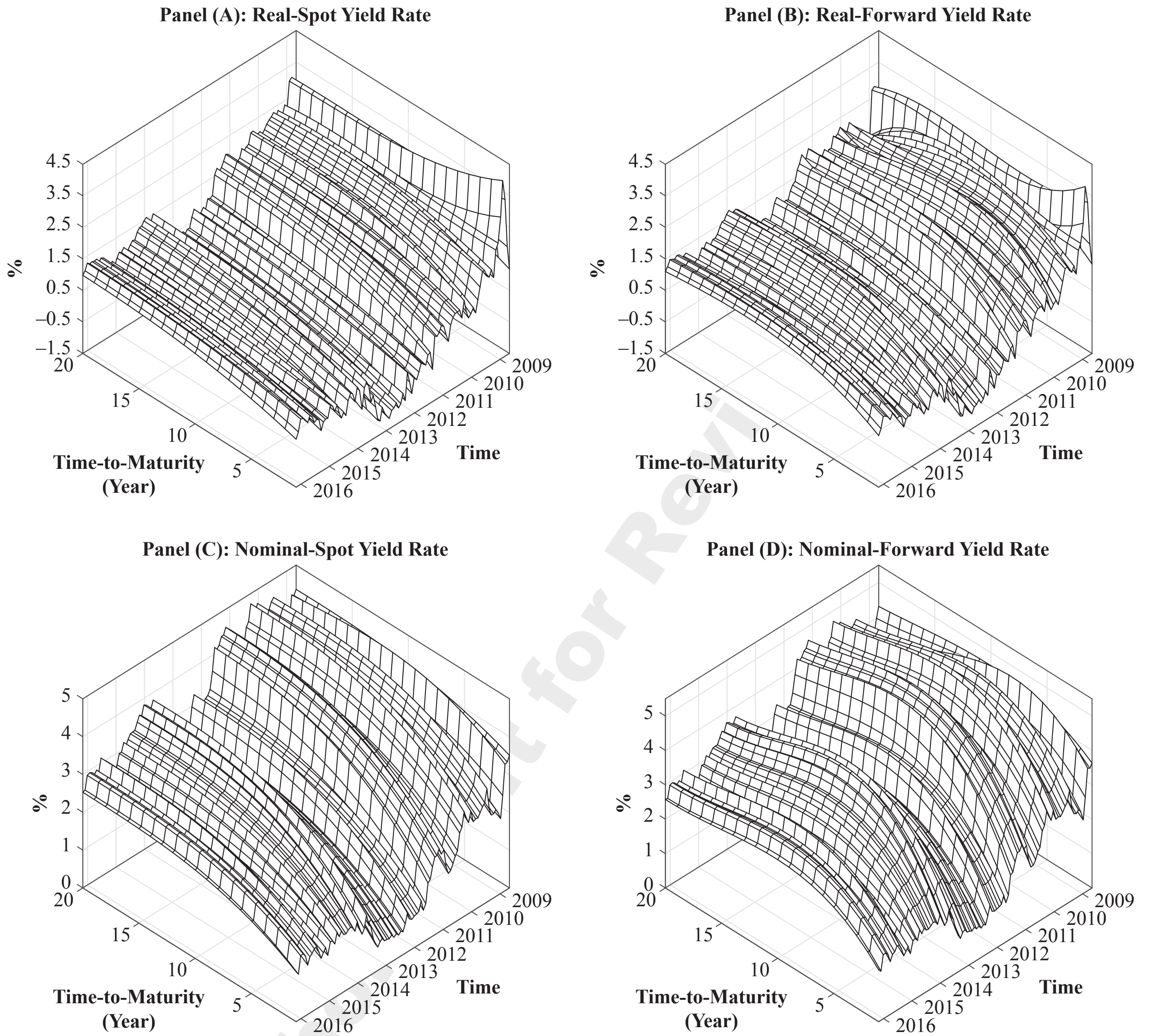
The U.S. Monthly CPI-U Data from September 2008 to May 2016 are Obtained from the U.S. Bureau of Labor Statistics



Notes: The descriptive statistics are as follows. Panel A reports 227.3884 for the average CPI-U, 8.7641 for the standard deviation, 210.2280 for the minimum level, 229.5395 for the median, 239.2610 for the maximum level, −0.3161 for skewness, and 1.6830 for kurtosis. Panel B reports 0.0950% for the average CPI-U growth rate, 0.4161% for the standard deviation, 1.9339% for the minimum level, 0.1182% for the median, 0.9704% for the maximum level, −1.4340 for skewness, and 8.2999 for kurtosis.

Exhibit 2

Data for 1,103 U.S. Treasury and 55 TIPS Bonds from September 2, 2008 to March 31, 2016



Note: Daily real and nominal yield rates are calibrated using piecewise cubic Hermite interpolating polynomials.

Source: Obtained from TreasuryDirect (http://www.treasurydirect.gov/).

with $h_{N,2} = h_{R,2} = h_{I,2} = 0$, the DPO pricing formula retrieves Merton's [1976] jump-diffusion option-pricing formula, where jump risk is assumed to be idiosyncratic and diversifiable. The associated measure is hereby referred to as the Merton measure. Second, by setting $h_{N,1}\hat{\sigma}_N(t,T) = h_{R,1}\hat{\sigma}_R(t,T) = h_{I,1}\sigma_I(t) = h_{N,2} = h_{R,2} = h_{I,2}$, one arrives at a modified Esscher measure (Gerber and Shiu [1994]) in the spirit of Ballotta [2005], which accounts for the presence of systematic/nondiversifiable jump risk. Hereafter, we may simply refer to this modified Esscher measure as the Esscher measure.

Given our interest in nailing down the impact of correlated jump risks on the DPO value and, in turn, their pricing influence on TIPS, subsequent discussions of the empirical results and pricing performance of the JY-CJ model need to rely on two degenerate cases of

Theorem 1 as the benchmarks for comparison: (1) when the correlation between any two jump amplitudes is set to zero ($\phi_{IR} = \phi_{IN} = \phi_{NR} = 0$), which results in the independent occurrences of jump events and is referred to as the JY model with independent jumps (the JY-IJ model); and (2) when there is no jump risk ($\lambda = \theta_I = \theta_R = \theta_N = \nu_I = \nu_R = \nu_N = 0$). In this case, the correlated geometric Brownian motions of Jarrow and Yildirim [2003] are retrieved. We refer to this case as the JY model.

EMPIRICAL AND NUMERICAL RESULTS

Data and Yield Curve Calibration

Our study sample consists of the daily prices of 1,013 Treasury bonds and 55 TIPS bonds from September 2, 2008 to March 31, 2016.[5] Exhibit 1 presents the time series data of monthly CPI-Us from September 2008 to July 2016, highlighting a deflationary episode begun by the deepening of the financial crisis and followed by recession. Over the period from September 2008 to March 2009, the CPI-U fell substantially to 4.0034% because of increasing deflationary concerns—documenting a negative year-over-year change in CPI-U for the first time since 1955. Panels A and B of Exhibit 2 show the daily real spot and forward yield curves constructed using a piecewise cubic Hermite interpolating polynomial based on U.S. Treasury bond data. Panels C and D in Exhibit 2 illustrate the daily nominal spot and forward yield curves over the same period.

Parameter Estimates and LRTs

Using the daily 1- to 20-year nominal and real yield curves calibrated from the bond data, we derive estimates of daily CPI-Us by linearly interpolating the monthly CPI-U data. Exhibit 3 reports parameter estimates for the JY, JY-IJ, and JY-CJ models, respectively. Without the presence of jumps, the JY model is estimated solely by maximum likelihood. From the first column of Exhibit 3, the mean-reverting forces of the nominal and real short rates (b_N and b_R) are 11.6100% and 10.2671%, respectively; the volatilities of both the nominal and the real zero-coupon bonds (i.e., $a_N e^{-b_N(T-t)}$ and $a_R e^{-b_R(T-t)}$) are positively related to the bond maturities.

[5] The sample data are collected from TreasuryDirect (http://www.treasurydirect.gov/).

In addition, Exhibit 3 exhibits positive correlations between (1) the nominal and the real zero-coupon bonds ($\rho_{NR} = 70.2337\%$) and (2) the real zero-coupon bonds and the inflation rate ($\rho_{RI} = 0.3162\%$). However, these correlation coefficients are not significantly different from zero. The correlation between the nominal zero-coupon bonds and the inflation rate is negative ($\rho_{IN} = -3.0195\%$).

The second and third columns of Exhibit 3 report the results of parameter estimation for the JY-IJ and JY-CJ models based on the EM algorithm. For JY-IJ, the means (θ_N, θ_R, and θ_I) and standard deviations (ν_N, ν_R, ν_I) for the jump amplitudes of the nominal forward rate, the real forward rate, and the CPI-U are (−0.0069%, −0.0002%, and −0.0050%) and (0.0678%, 0.0841%, and 0.0503%), respectively. Jump intensity (λ) is estimated as 14.5229%, suggesting a rate of occurrence of 0.145229 jump events per day. It takes approximately 6.8857 (=1/λ) days on average for a jump event to occur in our study sample. For JY-CJ, the mean-reverting forces of the short rates (b_N and b_R) are 11.8558% and 10.3665%, respectively; again, a positive volatility-maturity relationship can be identified between the nominal and real zero-coupon bonds. Average jump amplitudes (θ_N, θ_R, and θ_I) are (−0.0060%, 0.0008% [not significant], and −0.0045%), indicating the presence of downward jumps, and it takes approximately 6.3691 (=1/0.157008) days on average for a jump event to occur. In addition, the jump amplitudes of the nominal forward rate, the real forward rate, and the CPI-U are correlated. Finally, for all models, the standard errors of all parameters are obtained using cross-sectional nonlinear regressions over zero-coupon bonds with 1- to 20-year maturities.

Note that the jump frequency (of approximately once every six to seven days) estimated over our sample period is closely comparable to those of Das [2002], Johannes [2004], and Tauchen and Zhou [2011]. Das's [2002] Poisson–Gaussian model for interest rates, in particular, is equipped with an estimated jump frequency of once every five days. In general, the finance literature seems to document more intense occurrence of jumps in interest rates and Treasury bond prices than in stock prices/indexes. Eraker, Johannes, and Polson [2003], for example, found that the estimated jump frequency/intensity for S&P 500 Index returns is only 0.0060 to 0.0066 per day. Based on 11 U.S. individual stocks, Maheu and Mccurdy [2004] found that their averaged

EXHIBIT 3

Estimated Parameters for Inflation Rates, Real Forward Rates, and Nominal Forward Rates

	JY Model		JY-IJ Model		JY-CJ Model	
	Parameters (%)	S.E. (%)	Parameters (%)	S.E. (%)	Parameters (%)	S.E. (%)
μ_N	0.0012***	(0.0003)	0.00011***	(0.0003)	0.0011***	(0.0003)
μ_R	0.0009***	(0.0003)	0.0009***	(0.0002)	0.0009***	(0.0003)
μ_I	0.0042***	(0.0001)	0.0042***	(0.0001)	0.0042***	(0.0001)
σ_N	0.1496***	(0.0012)	0.1261***	(0.0012)	0.1210***	(0.0012)
σ_R	0.1706***	(0.0013)	0.1342***	(0.0012)	0.1294***	(0.0013)
σ_I	0.0234***	(0.0001)	0.0133***	(0.0001)	0.0131***	(0.0001)
γ_N	11.6100***	(0.0684)	11.8364***	(0.0778)	11.8558***	(0.0758)
γ_R	10.2671***	(0.0612)	10.3994***	(0.0728)	10.3665***	(0.0793)
ρ_{NR}	70.2273***	(0.2678)	79.5197***	(0.2386)	77.8530***	(0.2668)
ρ_{IN}	–3.0915***	(0.5215)	0.5974	(0.6147)	0.9370*	(0.6106)
ρ_{RI}	0.3162	(0.5084)	0.3174	(0.6289)	0.3641	(0.6002)
θ_N			–0.0069***	(0.0014)	–0.0060***	(0.0014)
θ_R			–0.0002	(0.0015)	0.0008	(0.0015)
θ_I			–0.0050***	(0.0007)	–0.0045***	(0.0007)
ν_N			0.0678***	(0.0011)	0.0777***	(0.0011)
ν_R			0.0841***	(0.0011)	0.0932***	(0.0012)
ν_I			0.0503***	(0.0006)	0.0487***	(0.0005)
λ			14.5229***	(0.2688)	15.7008***	(0.2837)
ϕ_{NR}					54.3579***	(1.3032)
ϕ_{IN}					–8.8433***	(1.5380)
ϕ_{RI}					0.5135	(1.9434)
Λογ–Λικελιηοοδ	570,400		584,717		585,279	
ΛΡΤ			28,635***		1,123***	

Notes: Model parameters for the JY model are estimated using maximum likelihood. For the JY-IJ and JY-CJ models, parameter estimates are obtained by using the EM algorithm. In the following, indexes i, j ∈ (N, R) denote the nominal or real interest rates. μ_i denotes the instantanous mean. a_i denotes the volatility coefficient. b_i is the mean-reverting force. σ_I denotes the volatility coefficent for inflation. ρ_{ij} denotes the correlation coefficient between the Brownian motions. θ_i and ν_i are the mean and standard deviation for the jump amplitudes, and ϕ_{ij} is the correlation coefficient between the jump amplitudes. Standard errors of the parameters are reported in the parentheses. The degrees of freedom for the LRT—the difference between the numbers of parameters in any two models—are reported in the parentheses. The second column is the null (versus alternative) hypothesis that the rates of return on bonds and CPI-U follow the JY model (versus the JY-IJ model). The third column is the null (versus alternative) hypothesis that the rates of return on bonds and CPI-U follow the JY-IJ model (versus the JY-CJ model).

**** and * indicate, respectively, statistical significance levels at the 1% and 10% levels.*

jump frequency is about 0.05 per day, or equivalently, once every month (approximately 20 business days).

The last row of Exhibit 3 reports the LRT statistics for the system of multivariate jump diffusions depicted by Equations 1–3. The goodness-of-fit of respective models is tested over the market CPI-U quotes and the bond data. The last number in Column 3 in Exhibit 3 is the LRT statistic for the JY-IJ jump-risk specification against the JY-CJ jump-risk specification. To be specific, it is the test statistic for the null hypothesis that all additional coefficients in the JY-CJ jump-risk specification with respect to the JY-IJ jump-risk specification are equal to zero. The result shows that we reject the null hypothesis at the 1% significance level, indicating that the JY-CJ jump-risk specification fits the data better than the JY-IJ jump-risk specification. Similarly, we find that JY-IJ fits the data better than JY at the 1% significance level. In sum, we conclude that the JY-CJ jump-risk specification is the best specification for the market data. These results support our rationale for hypothesizing the correlated nature of inflation- and interest-rate jump uncertainties.

EXHIBIT 4
Pricing Performance Analysis

Issue Date	Maturity Date	Initial CPI-U	Coupon Rate	Without DPO	Absolute Percentage Pricing Errors (%): With DPO: JY	JY-IJ: Merton	JY-IJ: Esscher	JY-CJ: Merton	JY-CJ: Esscher
Panel A: Full sample									
				2.93#	2.92	2.91	1.23	2.92	0.89##
Panel B: 5-Year									
10/15/2004	04/15/2010	189.45	0.875	0.40#	0.40	0.40	0.38##	0.40	0.39
04/15/2006	04/15/2011	198.49	2.375	0.98#	0.96	0.97	0.51##	0.97	0.71
04/15/2007	04/15/2012	202.92	2.000	1.78#	1.75	1.77	0.67##	1.77	0.86
04/15/2008	04/15/2013	211.37	0.625	2.73#	2.57	2.61	0.48##	2.64	0.73
04/15/2009	04/15/2014	211.63	1.250	2.78#	2.74	2.75	0.43##	2.76	0.53
04/15/2010	04/15/2015	216.71	0.500	2.78#	2.74	2.75	0.56##	2.76	0.68
04/15/2011	04/15/2016	220.73	0.125	2.88#	2.85	2.87	0.43##	2.88	0.62
04/15/2012	04/15/2017	227.13	0.125	2.70#	2.62	2.67	0.39	2.68	0.36##
04/15/2013	04/15/2018	231.16	0.125	2.78#	2.65	2.71	0.39	2.73	0.36##
04/15/2014	04/15/2019	234.32	0.125	2.13#	1.98	2.02	0.19##	2.06	0.31
04/15/2015	04/15/2020	234.18	0.125	2.03#	1.90	1.91	0.14##	1.95	0.26
Panel C: 10-Year									
01/15/2000	01/15/2010	168.25	4.250	0.39#	0.39	0.39	0.38##	0.39	0.39
01/15/2001	01/15/2011	174.05	3.500	1.03#	1.03	1.03	0.73##	1.03	0.92
01/15/2002	01/15/2012	177.56	3.375	1.69#	1.69	1.69	0.77##	1.69	1.11
07/15/2002	07/15/2012	179.80	3.000	1.99#	1.99	1.99	0.77##	1.99	1.03
07/15/2003	07/15/2013	183.66	1.875	2.46#	2.46	2.46	0.53##	2.46	0.82
01/15/2004	01/15/2014	184.77	2.000	2.41#	2.41	2.41	0.38##	2.41	0.59
07/15/2004	07/15/2014	188.50	2.000	2.61#	2.61	2.61	0.49##	2.61	0.66
01/15/2005	01/15/2015	190.95	1.625	2.57#	2.57	2.57	0.53##	2.57	0.66
07/15/2005	07/15/2015	194.51	1.875	2.99#	2.99	2.99	0.59##	2.99	0.82
01/15/2006	01/15/2016	198.48	2.000	2.97#	2.97	2.97	0.43##	2.97	0.66
07/15/2006	07/15/2016	201.95	2.500	3.27#	3.27	3.27	0.43##	3.27	0.67
01/15/2007	01/15/2017	201.66	2.375	3.17#	3.17	3.17	0.37##	3.17	0.44
07/15/2007	07/15/2017	207.26	2.625	3.69#	3.69	3.68	0.42##	3.69	0.48
01/15/2008	01/15/2018	209.50	1.625	3.62#	3.62	3.60	0.55	3.61	0.39##
07/15/2008	07/15/2018	215.64	1.375	4.17#	4.15	4.13	0.93	4.15	0.42##
01/15/2009	01/15/2019	214.70	2.125	3.93#	3.92	3.90	1.16	3.92	0.41##
07/15/2009	07/15/2019	213.52	1.875	4.36#	4.36	4.36	1.48	4.36	0.43##
01/15/2010	01/15/2020	216.25	1.375	3.99#	3.99	3.99	1.38	3.99	0.41##
07/15/2010	07/15/2020	218.09	1.250	4.37#	4.36	4.36	1.78	4.36	0.46##
01/15/2011	01/15/2021	218.75	1.125	3.84#	3.84	3.84	1.73	3.84	0.41##
07/15/2011	07/15/2021	225.38	0.625	4.16#	4.15	4.15	2.06	4.15	0.36##
01/15/2012	01/15/2022	226.33	0.125	3.65#	3.65	3.64	1.98	3.65	0.31##
07/15/2012	07/15/2022	229.96	0.125	3.93#	3.93	3.91	2.18	3.92	0.41##
01/15/2013	01/15/2023	230.82	0.125	3.23#	3.22	3.20	1.83	3.22	0.34##
07/15/2013	07/15/2023	232.72	0.375	3.33#	3.32	3.29	1.86	3.31	0.30##
01/15/2014	01/15/2024	233.33	0.625	2.70#	2.69	2.65	1.56	2.68	0.23##
07/15/2014	07/15/2024	237.45	0.125	2.99#	2.96	2.88	2.02	2.94	0.25##
01/15/2015	01/15/2025	236.85	0.250	2.84#	2.81	2.73	2.21	2.79	0.22##
07/15/2015	07/15/2025	237.14	0.375	3.29#	3.25	3.15	2.85	3.23	0.27##
01/15/2016	01/15/2026	237.61	0.625	3.70#	3.64	3.50	3.41	3.61	0.32##

(continued)

EXHIBIT 4 *(continued)*
Pricing Performance Analysis

Issue Date	Maturity Date	Initial CPI-U	Coupon Rate	Without DPO	Absolute Percentage Pricing Errors (%) — With DPO: JY	JY-IJ: Merton	JY-IJ: Esscher	JY-CJ: Merton	JY-CJ: Esscher
Panel D: 20-Year									
07/15/2004	01/15/2025	188.50	2.375	2.91#	2.91	2.91	2.79	2.91	2.05##
01/15/2006	01/15/2026	198.48	2.000	2.66#	2.66	2.66	2.63	2.66	2.11##
01/15/2007	01/15/2027	201.66	2.375	2.57#	2.57	2.57	2.56	2.57	2.34##
01/15/2008	01/15/2028	209.50	1.750	2.45#	2.44	2.43	2.45	2.44	2.36##
01/15/2009	01/15/2029	214.70	2.500	2.39#	2.39	2.38	2.39	2.39	2.36##
Panel E: 30-Year									
04/15/1998	04/15/2028	161.74	3.625	2.04#	2.04	2.04	2.04	2.04	1.98##
04/15/1999	04/15/2029	164.39	3.875	2.10#	2.10	2.10	2.10	2.10	2.08##
10/15/2001	04/15/2032	177.50	3.375	2.11#	2.11	2.11##	2.11	2.11	2.11
02/15/2010	02/15/2040	216.14	2.125	0.32#	0.25	0.24##	0.32	0.27	0.32
02/15/2011	02/15/2041	218.99	2.125	0.28#	0.20	0.19##	0.28	0.22	0.28
02/15/2012	02/15/2042	225.96	0.750	0.16#	0.12	0.10	0.16	0.01##	0.16
02/15/2013	02/15/2043	229.91	0.625	0.02##	0.32#	0.22	0.02	0.14	0.02
02/15/2014	02/15/2044	233.06	1.375	0.00##	0.37#	0.28	0.00	0.18	0.00
02/15/2015	02/15/2045	235.48	0.750	0.22##	0.66#	0.54	0.22	0.31	0.22

Notes: This table analyzes the pricing of TIPS using the JY-CJ model—under a Merton (idiosyncratic) or Esscher (systematic) measure—relative to the JY and JY-IJ models, with or without the embedded DPO. Pricing performance is measured by the APPE: $APPE_i = 1/T_i \Sigma_{t=1}^{T_i} 100 \times \left|\hat{V}_{i,t} - V_{i,t}\right| / V_{i,t}$, *where* $\hat{V}_{i,t}$ *denotes the TIPS prices derived from the JY-CJ model, and* $V_{i,t}$ *denotes their market quotes.* T_i *is the holding period for each TIPS bond i. The notations # and ## indicate, respectively, the maximum and minimum APPEs.*

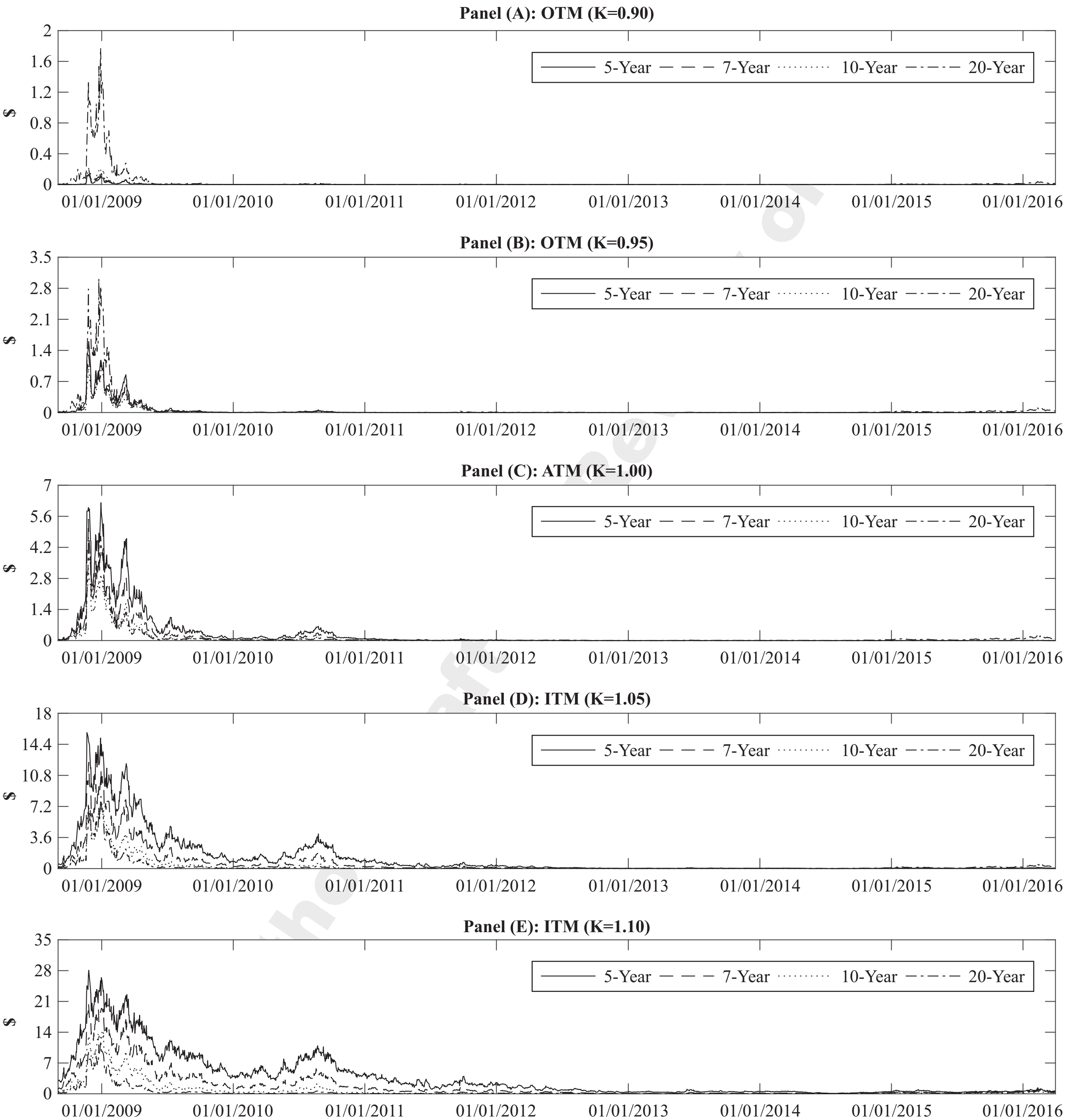
Pricing Performance Analysis

Exhibit 4 reports the in-sample pricing performance results for the overall fit of the JY, JY-IY, and JY-CJ models to the market data.[6] Pricing performance is measured by the absolute percentage pricing error (APPE)—defined as the ratio of the absolute pricing error (APE) to the market price—where APE is the absolute difference between the model price and market price. The fifth column of Exhibit 4 is the base case when the TIPS are priced without the DPO value, which gives rise to an average APPE of 2.93%.

The sixth to ninth columns of Exhibit 4 are the cases in which the DPO value is explicitly considered for the pricing of TIPS. The sixth column of Exhibit 4 reports the APPEs for the JY model; the seventh and ninth columns of Exhibit 4 report the APPEs for the JY-IJ and JY-CJ models under the Merton measure; and the eighth and tenth columns report the APPEs for the JY-IJ and JY-CJ models under the Esscher measure. The average APPE for the JY model is 2.92% whereas the JY-IJ and JY-CJ models under the Merton measure report average APPEs of 2.91% and 2.92%, respectively. On the other hand, both the JY-IJ and JY-CJ models under the Esscher measure show markedly incremental gains in pricing performance relative to the JY model, with their APPEs being reduced to 1.23% and 0.89%, respectively. This result strengthens the proposition that the accurate pricing of TIPS would entail a systematic jump-risk setting aided by an explicit presence of the DPO value. On the other hand, this result also suggests that the market prices of TIPS seem to already embed a certain level of risk premium associated with inflation- and interest-rate jump risks. A direct implication of this finding is that, for TIPS to properly function as an effective hedge against inflation risk, a higher level of unexpected inflation risk—more than what the market already participates—is required.

[6] The market data make up the sample consisting of TIPS bonds issued from September 2, 2008 to March 31, 2016.

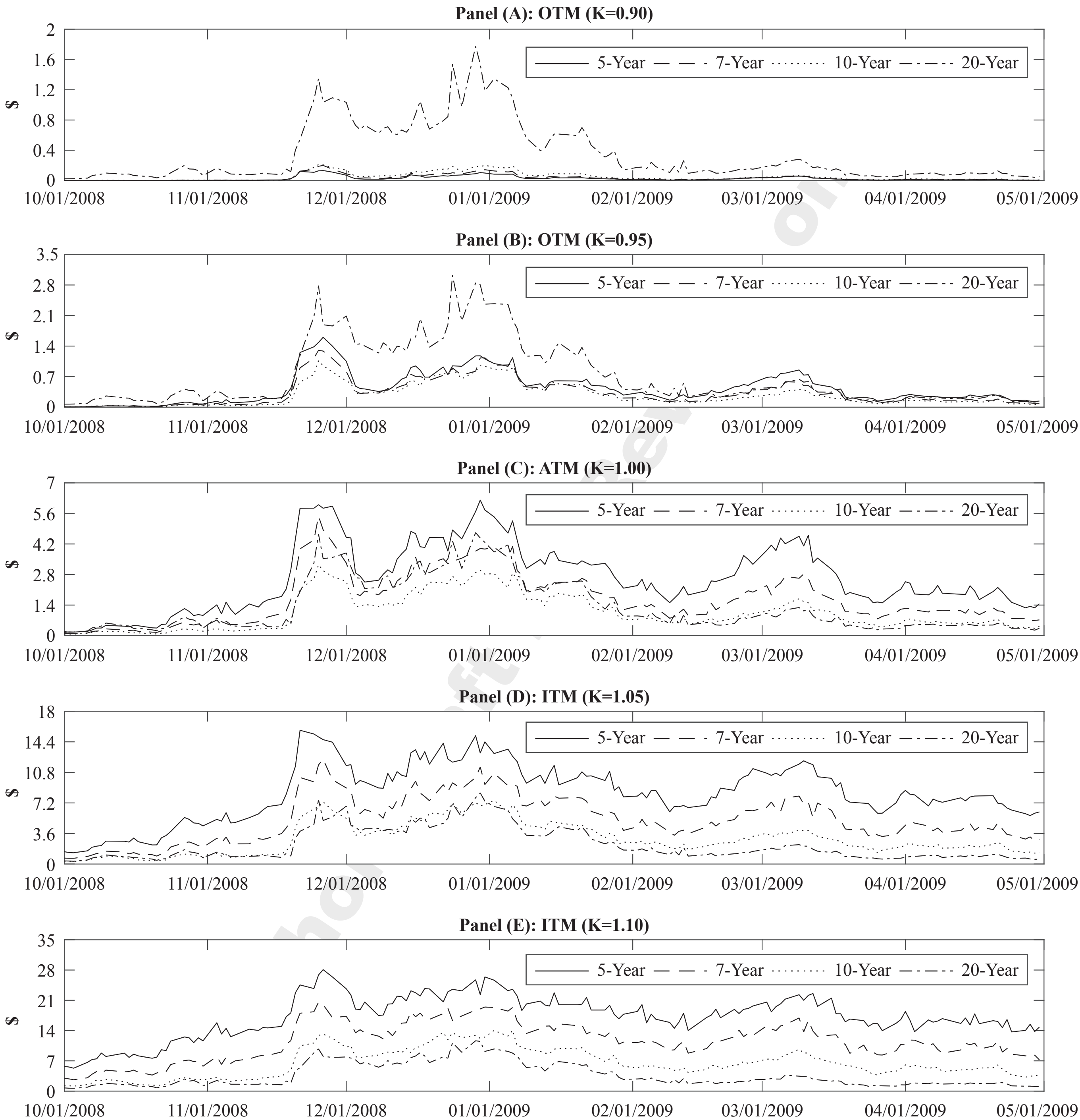
Exhibit 5
DPO Prices Computed with Theorem 1 Under the Esscher Measure from September 2, 2008 to March 31, 2016



ATM = at-the-money; ITM = in-the-money; OTM = out-of-the-money.

Notes: Using the parameter estimates of Exhibit 3, the figure illustrates the DPO prices computed with Theorem 1 (the JY-CJ model) under the Esscher measure (the systematic jump-risk setting) for the 5-, 7-, 10-, and 20-year TIPS bonds issued from September 2, 2008 to March 31, 2016. The initial reference CPI-U is 218.85 as of September 2, 2008; the strike prices of DPO are set at the 90%, 95%, 100%, 105%, and 110% levels of the initial CPI-U.

EXHIBIT 6
DPO Prices Computed with Theorem 1 Under the Esscher Measure from October 1, 2008 to April 30, 2009



ATM = at-the-money; ITM = in-the-money; OTM = out-of-the-money.

Notes: Using the parameter estimates of Exhibit 3, the figure illustrates the DPO prices computed with Theorem 1 (the JY-CJ model) under the Esscher measure (systematic jump-risk setting) for the 5-, 7-, 10-, and 20-year TIPS bonds issued during the subprime financial crisis (October 1, 2008 to April 30, 2009). The initial reference CPI-U is 218.85 as of September 2, 2008; the strike prices of DPO are set at the 90%, 95%, 100%, 105%, and 110% levels of the initial reference CPI-U.

Furthermore, the pricing performance gain of JY-CJ relative to JY-IJ under the Esscher measure suggests that the Merton measure, which assigns zero probabilities to the likelihoods of systematic/nondiversifiable jump events, is likely to underestimate the risk premiums required by TIPS investors: An extra return needs to be built in to compensate TIPS investors for the co-dependent jump risks that they bear. As for the bond markets, we are certainly not the first to make such assertions. Altman [1989], for example, showed that corporate bond investors in fact expect significantly higher returns than risk-free bond investors, even when accounting for the impact of defaults. Collin-Dufresne, Goldstein, and Hugonnier [2003] argued that correlated jump risks, in the form of systematic contagion, would require risk premiums that are several times higher than direct jump-risk premiums. Consistent with their findings, the TIPS investors in this article are assumed to require compensation not only for enduring the inflation- and interest-rate jump risks but also the correlated interplay among them. In our pricing framework, such co-dependent jump-risk premiums are encapsulated in the prices of the underlying bonds, under the risk-neutral measure, by the set of martingale conditions that ensures an arbitrage-free market. That, we argue, inevitably contributes to the incremental pricing performance of the JY-CJ model.

Finally, the TIPS prices from the JY-CJ model are found to provide a superior fit to the market prices of the 10-year TIPS issued after the 2008 crisis. Because the 10-year notes accounted for over 70% of total trading activity in TIPS in terms of daily trading volumes (Fleming and Krishnan [2012]), this superior fit may not be a mere coincidence but may in fact be representative of average investor perceptions regarding the inflation- and interest-rate environments. In this study, the representation of such information content, although aided by the correlated jump-risk specification, resolves into the DPO value. Further illustrated by Exhibit 5 are the DPO values associated with TIPS of 5-, 7-, 10-, and 20-year maturities, respectively, from 2008 to 2016. Consistent with Grishchenko, Vanden, and Zhang [2016], we find the DPO values to be time-varying and small in absolute terms. Most interestingly, as shown by Exhibit 6, the DOP values exhibit an interesting pattern especially for the late-2008 to early-2009 period. In stark contrast to the stagflation period of the 1970s, the late-2008 to early-2009 period was a time of turmoil induced by spurred fears of deflation and sudden upward jumps in real spot rates. Such a period defines a worst scenario for TIPS: Negative inflation is bad for TIPS, and rising real yields implies direct losses due to the extended duration profiles. Reflected by the peeked positive values in Exhibit 5 is the indispensable role of the DPO value in the valuation of TIPS in times of prevailing deflationary concerns confounded by rising real yields.

Although one can always benefit from an out-of-sample pricing performance analysis that employs data previously unused for parameter estimation, direct observations of Exhibit 1 (for the CPI-U) and Exhibit 2 (for the nominal/real interest rates) suggest that the patterns of their propagating paths tend to vary indistinguishably—which is most likely to induce time homogeneity in the estimated parameters. In this case, we expect the out-of-sample pricing performance results to remain quantitatively similar to those in Exhibit 4.

CONCLUSIONS

This pricing framework adds to the existing literature on TIPS valuation in the following respects. First, we theoretically quantify the DPO value embedded in TIPS—a motivation previously seen as inadequate for the inflationary environment prior to 1997 but that became a focus of attention when the 2008 financial crisis brought along deflationary concerns. Second, our pricing framework directly addresses the co-occurrence of inflation- and interest-rate uncertainties, which have always been considered vital to the pricing of TIPS (Christensen, Lopez, and Rudebusch [2012]; Grishchenko, Vanden, and Zhang [2016]). Third, this pricing framework is sufficiently general to incorporate various jump-diffusion settings in the literature. More importantly, we find that the pricing of TIPS with DPO explicitly identified under a systematic jump-risk setting (the Esscher measure) outperforms that under an idiosyncratic jump-risk setting (the Merton measure). Most interestingly, model prices from this pricing framework closely conform to the market prices of the 10-year TIPS issued following the 2008 crisis. Consistent with Grishchenko, Vanden, and Zhang [2016], we find the DPO values to be time varying, to be small in absolute terms, and to exhibit an interesting pattern, especially during the late-2008 to early-2009 crisis period.

Appendix A

PROOF OF PROPOSITION 1

By the Itô–Doeblin formula

$$\frac{P_N(T,T)}{\beta_N(T)} = \frac{P_N(t,T)}{\beta_N(t)} \exp[G_1(t,T)] \tag{A-1}$$

$$\frac{I(T)\beta_R(T)}{\beta_N(T)} = \frac{I(t)\beta_R(t)}{\beta_N(t)} \exp[G_2(t,T)] \tag{A-2}$$

$$\frac{I(T)P_R(T,T)}{\beta_N(T)} = \frac{I(t)P_R(t,T)}{\beta_N(t)} \exp[G_3(t,T)] \tag{A-3}$$

where $\beta_N(t) = \exp\left[-\int_0^t r_N(u)du\right]$ and $\beta_R(t) = \exp\left[-\int_0^t r_R(u)du\right]$ represent the nominal and real money market accounts, respectively; the exponential growth factors $G_1(t,T)$, $G_2(t,T)$, and $G_3(t,T)$ take the following forms:

$$G_1(t,T) = -\int_t^T \hat{\alpha}_N(u,T)du - \int_t^T \hat{\sigma}_N(u,T)dW_N(u) + \sum_{k=1}^{M(T-t)} \ln \tilde{Y}_{N,k},$$

$$G_2(t,T) = \int_t^T \left[\mu_I(u) + r_R(u) - r_N(u) - \frac{1}{2}\sigma_I^2(u) - \lambda\left(e^{\theta_I + \frac{\nu_I^2}{2}} - 1\right)\right]du + \int_t^T \sigma_I(u)dW_I(u) + \sum_{k=1}^{M(T-t)} \ln Y_{I,k},$$

$$G_3(t,T) = G_2(t,T) - \int_t^T \hat{\alpha}_R(u,T)du - \int_t^T \hat{\sigma}_R(u,T)dW_R(u) + \sum_{k=1}^{M(T-t)} \ln \tilde{Y}_{R,k}$$

Under the assumption that the compound Poisson process is independent of the standard Brownian motions, the conditional Radon–Nikodým derivative thus permits a decomposition into two independent martingale processes under the $\mathbb{P}$ measure; that is,

$$\frac{\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|\mathbb{F}(T)}{\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|\mathbb{F}(t)} = \exp[\eta_1(t,T) + \eta_2(t,T)] \tag{A-4}$$

with

$$\begin{aligned}\eta_1(t,T) = &-\frac{1}{2}\int_t^T [h_{N,1}^2\hat{\sigma}_N^2(u,T) + h_{R,1}^2\hat{\sigma}_R^2(u,T) + h_{I,1}^2\sigma_I^2 \\ &+ 2h_{N,1}h_{R,1}\rho_{NR}\hat{\sigma}_N(u,T)\hat{\sigma}_R(u,T) - 2h_{R,1}h_{I,1}\rho_{RI}\hat{\sigma}_R(u,T)\sigma_I(u) \\ &- 2h_{I,1}h_{N,1}\rho_{IN}\sigma_I(u)\hat{\sigma}_N(u,T)]du - \int_t^T h_{N,1}\hat{\sigma}_N(u,T)dW_N(u) \\ &- \int_t^T h_{R,1}\hat{\sigma}_R(u,T)dW_R(u) + \int_t^T h_{I,1}\sigma_I(u)dW_I(u)\end{aligned}$$

and

$$\begin{aligned}\eta_2(t,T) = &-\lambda[\psi(1;h_{N,2},h_{R,2},h_{I,2}) - 1](T-t) \\ &+ \sum_{k=1}^{M(T-t)} (h_{N,2}\ln\tilde{Y}_{N,k} + h_{R,2}\ln\tilde{Y}_{R,k} + h_{I,2}\ln Y_{I,k})\end{aligned}$$

where $\psi(c;h_{N,2},h_{R,2},h_{I,2})$ is a moment-generating function applied to identify the distributional characteristics of the random component $(h_{N,2}\ln\tilde{Y}_{N,k} + h_{R,2}\ln\tilde{Y}_{R,k} + h_{I,2}\ln Y_{I,k})$; that is,

$$\begin{aligned}&\psi(c;h_{N,2},h_{R,2},h_{I,2}) \\ &= \mathrm{E}\left[e^{c(h_{N,2}\ln\tilde{Y}_{N,k} + h_{R,2}\ln\tilde{Y}_{R,k} + h_{I,2}\ln Y_{I,k})}\right] \\ &= \exp\left[c\theta(h_{N,2},h_{R,2},h_{I,2}) + \frac{c^2\nu^2(h_{N,2},h_{R,2},h_{I,2})}{2}\right]\end{aligned} \tag{A-5}$$

with $\theta\ (h_{N,2}, h_{R,2}, h_{I,2})$ and $\nu^2(h_{N,2}, h_{R,2}, h_{I,2})$ representing the mean and variance, respectively, as follows

$$\begin{aligned}&\theta(h_{N,2},h_{R,2},h_{I,2}) = -(h_{N,2}\theta_N + h_{R,2}\theta_R)(T-t) + h_{I,2}\theta_I, \\ &\nu^2(h_{N,2},h_{R,2},h_{I,2}) \\ &\quad = (h_{N,2}^2\nu_N^2 + h_{R,2}^2\nu_R^2 + 2h_{N,2}h_{R,2}\phi_{NR}\nu_N\nu_R)(T-t)^2 \\ &\quad\quad + 2(h_{R,2}h_{I,2}\phi_{RI}\nu_R\nu_I + h_{I,2}h_{N,2}\phi_{IN}\nu_I\nu_N)(T-t) + h_{I,2}^2\nu_I^2\end{aligned}$$

For the standard Brownian motions $W_N(T\text{–}t)$, $W_R(T\text{–}t)$, and $W_I(T\text{–}t)$, we have

$$\begin{aligned}&E_t^{\mathbb{Q}}\left[e^{\int_t^T dW_N(u) + \int_t^T dW_R(u) + \int_t^T dW_I(u)}\right] = E_t\left[e^{\eta_1(t,T)} \cdot e^{\int_t^T dW_N(u) + \int_t^T dW_R(u) + \int_t^T dW_I(u)}\right] \\ &= \exp\left[\begin{aligned}&-\int_t^T h_{N,1}\hat{\sigma}_N(u,T)(1+\rho_{NR}+\rho_{IN})du \\ &-\int_t^T h_{R,1}\hat{\sigma}_R(u,T)(1+\rho_{RI}+\rho_{NR})du \\ &+\int_t^T h_{I,1}\sigma_I(u)(1+\rho_{RI}+\rho_{IN})du \\ &+\frac{1}{2}(3+2\rho_{NR}+2\rho_{RI}+2\rho_{IN})(T-t)\end{aligned}\right]\end{aligned} \tag{A-6}$$

which gives rise to a set of standard Brownian martingale representations for the nominal interest rate, the real interest rate, and the inflation index, under the $\mathbb{Q}$ measure

$$W_N^{\mathbb{Q}}(T-t) = W_N(T-t) + \int_t^T h_{N,1}\hat{\sigma}_N(u,T)\left(1+\rho_{NR}+\rho_{IN}\right)du \tag{A-7}$$

$$W_R^{\mathbb{Q}}(T-t) = W_R(T-t) + \int_t^T h_{R,1}\hat{\sigma}_R(u,T)(1+\rho_{RI}+\rho_{NR})du \tag{A-8}$$

$$W_I^{\mathbb{Q}}(T-t) = W_I(T-t) - \int_t^T h_{I,1}\sigma_I(u)(1+\rho_{RI}+\rho_{IN})du \tag{A-9}$$

For the compound Poisson process

$$\sum_{k=1}^{M(T-t)}\left(\ln\tilde{Y}_{N,k} + \ln\tilde{Y}_{R,k} + \ln Y_{I,k}\right)$$

$$\begin{aligned}
&\mathrm{E}_t^{\mathbb{Q}}\left(e^{\sum_{k=1}^{M(T-t)}(\ln\tilde{Y}_{N,k}+\ln\tilde{Y}_{R,k}+\ln Y_{I,k})}\right) = \mathrm{E}_t\left(e^{\eta_2(t,T)}\cdot e^{\sum_{k=1}^{M(T-t)}(\ln\tilde{Y}_{N,k}+\ln\tilde{Y}_{R,k}+\ln Y_{I,k})}\right)\\
&= \exp\left\{\begin{array}{l}\lambda\psi(1;h_{N,2},h_{R,2},h_{I,2})\\ \left[\dfrac{\psi(1;1+h_{N,2},1+h_{R,2},1+h_{I,2})}{\psi(1;h_{N,2},h_{R,2},h_{I,2})}-1\right](T-t)\end{array}\right\}
\end{aligned} \tag{A-10}$$

The jump frequency $M(T–t)$ under the $\mathbb{Q}$ measure obeys the Poisson distribution with the intensity $\lambda^{\mathbb{Q}}(T-t)$; that is,

$$M(T-t)\sim \text{Poisson}[\lambda^{\mathbb{Q}}(T-t)], \lambda^{\mathbb{Q}} := \lambda\cdot\psi(1;h_{N,2},h_{R,2},h_{I,2}) \tag{A-11}$$

and the correlated jump amplitudes for the nominal interest rate, the real interest rate, and the inflation index under the $\mathbb{Q}$ measure are as follows

$$\ln\tilde{Y}_{N,k} \overset{IID}{\sim} \text{Normal}[\theta_N^{\mathbb{Q}}, \nu_N^2(T-t)^2] \tag{A-12}$$

$$\ln\tilde{Y}_{R,k} \overset{IID}{\sim} \text{Normal}[\theta_R^{\mathbb{Q}}, \nu_R^2(T-t)^2] \tag{A-13}$$

$$\ln Y_{I,k} \overset{IID}{\sim} \text{Normal}(\theta_I^{\mathbb{Q}}, \nu_I^2). \tag{A-14}$$

where

$$\begin{aligned}\theta_N^{\mathbb{Q}} &= -\theta_N(T-t) + h_{N,2}\nu_N(T-t)\\ &\quad[\nu_N(T-t)+\phi_{NR}\nu_R(T-t)+\phi_{IN}\nu_I]\end{aligned}$$

$$\begin{aligned}\theta_R^{\mathbb{Q}} &= -\theta_R(T-t) + h_{R,2}\nu_R(T-t)\\ &\quad[\nu_R(T-t)+\phi_{NR}\nu_N(T-t)+\phi_{RI}\nu_I]\end{aligned}$$

$$\theta_I^{\mathbb{Q}} = \theta_I + h_{I,2}\nu_I\left[\nu_I + \phi_{RI}\nu_R(T-t) + \phi_{IN}\nu_N(T-t)\right]$$

By the fundamental theorem of asset pricing (Harrison and Kreps [1978]; Harrison and Pliska [1981]), the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure under which the discounted asset price processes are martingales. Hence, we require the discount price processes for the nominal zero-coupon bond $P_N(t,T)$, the inflation-adjusted money market account $I(t)\beta_R(t)$, and the inflation-linked zero-coupon bond $I(t)P_R(t,T)$ to be martingales under the $\mathbb{Q}$ measure

$$\frac{P_N(t,T)}{\beta_N(t)} = \mathrm{E}_t^{\mathbb{Q}}\left[\frac{P_N(T,T)}{\beta_N(T)}\right] \Leftrightarrow \mathrm{E}_t^{\mathbb{Q}}\{\exp[G_1(t,T)]\} = 1 \tag{A-15}$$

$$\frac{I(t)\beta_R(t)}{\beta_N(t)} = \mathrm{E}_t^{\mathbb{Q}}\left[\frac{I(T)\beta_R(T)}{\beta_N(T)}\right] \Leftrightarrow \mathrm{E}_t^{\mathbb{Q}}\{\exp[G_2(t,T)]\} = 1 \tag{A-16}$$

$$\frac{I(t)P_R(t,T)}{\beta_N(t)} = \mathrm{E}_t^{\mathbb{Q}}\left[\frac{I(T)P_R(T,T)}{\beta_N(T)}\right] \Leftrightarrow \mathrm{E}_t^{\mathbb{Q}}\{\exp[G_3(t,T)]\} = 1 \tag{A-17}$$

Using the functions $G_1(t,T)$, $G_2(t,T)$, $G_3(t,T)$ as derived in Equations A-1 to A-3 and the conditional Radon–Nikodým derivative of Equation A-4, the required martingale conditions of this pricing framework are as follows:

$$\begin{aligned}\hat{\alpha}_N(t,T) &= \left[h_{N,1}(1+\rho_{NR}+\rho_{IN})+\frac{1}{2}\right]\hat{\sigma}_N^2(t,T)\\ &\quad+\lambda^{\mathbb{Q}}\left(e^{\theta_N^{\mathbb{Q}}+\frac{\nu_N^2(T-t)^2}{2}}-1\right)\end{aligned} \tag{A-18}$$

$$\begin{aligned}\mu_I(t) &= r_N(t) - r_R(t) - h_{I,1}\sigma_I^2(t)(1+\rho_{RI}+\rho_{IN})\\ &\quad+\lambda\left(e^{\theta_I+\frac{\nu_I^2}{2}}-1\right)-\lambda^{\mathbb{Q}}\left(e^{\theta_I^{\mathbb{Q}}+\frac{\nu_I^2}{2}}-1\right)\end{aligned} \tag{A-19}$$

$$\begin{aligned}\hat{\alpha}_R(t,T) &= \left[h_{R,1}(1+\rho_{RI}+\rho_{NR})+\frac{1}{2}\right]\hat{\sigma}_R^2(t,T) - \rho_{RI}\hat{\sigma}_R(t,T)\sigma_I(t)\\ &\quad+\lambda^{\mathbb{Q}}\left(e^{\theta_R^{\mathbb{Q}}+\theta_I^{\mathbb{Q}}+\frac{\nu_R^2(T-t)^2+2\phi_{RI}\nu_R(T-t)\nu_I+\nu_I^2}{2}} - e^{\theta_I^{\mathbb{Q}}+\frac{\nu_I^2}{2}}\right)\end{aligned} \tag{A-20}$$

Thus, under the $\mathbb{Q}$ measure, the risk-neutral dynamic processes for the nominal zero-coupon bond $P_N(t, T)$, the inflation-adjusted money market account $I(t)\beta_R(t)$, and the inflation-linked zero-coupon bond $I(t)P_R(t, T)$ are

$$\frac{P_N(T,T)}{\beta_N(T)} = \frac{P_N(t,T)}{\beta_N(t)} \exp[G_1^{\mathbb{Q}}(t,T)] \tag{A-21}$$

$$\frac{I(T)\beta_R(T)}{\beta_N(T)} = \frac{I(t)\beta_R(t)}{\beta_N(t)} \exp\left[G_2^{\mathbb{Q}}(t,T)\right] \tag{A-22}$$

$$\frac{I(T)P_R(T,T)}{\beta_N(T)} = \frac{I(t)P_R(t,T)}{\beta_N(t)} \exp\left[G_3^{\mathbb{Q}}(t,T)\right] \tag{A-23}$$

where

$$\begin{aligned} G_1^{\mathbb{Q}}(t,T) = {} & \int_t^T \left[-\frac{1}{2}\hat{\sigma}_N^2(u,T) - \lambda^{\mathbb{Q}} \left(e^{\theta_N^{\mathbb{Q}} + \frac{\nu_N^2 (T-t)^2}{2}} - 1 \right) \right] du \\ & - \int_t^T \hat{\sigma}_N(u,T)\, dW_N^{\mathbb{Q}}(u) + \sum_{k=1}^{M(T-t)} \ln \tilde{Y}_{N,k} \end{aligned}$$

$$\begin{aligned} G_2^{\mathbb{Q}}(t,T) = {} & \int_t^T \left[-\frac{1}{2}\sigma_I^2(u) - \lambda^{\mathbb{Q}} \left(e^{\theta_I^{\mathbb{Q}} + \frac{\nu_I^2}{2}} - 1 \right) \right] du \\ & + \int_t^T \sigma_I(u)\, dW_I^{\mathbb{Q}}(u) + \sum_{k=1}^{M(T-t)} \ln Y_{I,k} \end{aligned}$$

$$\begin{aligned} G_3^{\mathbb{Q}}(t,T) = {} & \int_t^T \left[\begin{array}{l} -\frac{1}{2}\left[\sigma_I^2(u) - 2\rho_{RI}\hat{\sigma}_R(u,T)\sigma_I(u) + \hat{\sigma}_R^2(u,T)\right] \\ -\lambda^{\mathbb{Q}} \left(e^{\theta_R^{\mathbb{Q}} + \theta_I^{\mathbb{Q}} + \frac{\nu_R^2(T-t)^2 + 2\phi_{RI}\nu_R(T-t)\nu_I + \nu_I^2}{2}} - 1 \right) \end{array} \right] du \\ & + \int_t^T \sigma_I(u)\, dW_I^{\mathbb{Q}}(u) - \int_t^T \hat{\sigma}_R(u,T)\, dW_R(u) + \sum_{k=1}^{M(T-t)} \ln Y_{I,k} + \sum_{k=1}^{M(T-t)} \ln \tilde{Y}_{R,k} \end{aligned}$$

This completes our proof of Proposition 1.

APPENDIX B

PROOF OF THEOREM 1

At maturity, the TIPS investor will receive cash flows either from the original principal $F(t_0)$ or the adjusted principal $F(T) = F(t_0)I(T)/I(t_0)$. That is

$$\begin{aligned} \max[F(T), F(t_0)] &= F(T) + \frac{F(t_0)}{I(t_0)} \cdot \max[I(t_0) - I(T), 0] \\ &= F(T) + \frac{F(t_0)}{I(t_0)} \cdot \text{DPO}(T) \end{aligned} \tag{B-1}$$

which is analogous to a European put option written on the inflation-linked zero-coupon bond with $I(t_0)$ as its strike price. The payoff of DPO at maturity is thus defined by

$$\begin{aligned} \text{DPO}(T) &= [I(t_0) - I(T)]1_{\{I(T)<I(t_0)\}} \\ &= [I(t_0)P_N(T,T) - I(T)P_R(T,T)]1_{\{I(T)P_R(T,T)<I(t_0)P_N(T,T)\}} \end{aligned} \tag{B-2}$$

Under the $\mathbb{Q}$ measure and using the law of iterated expectations, the fair value of DPO at time t is

$$\begin{aligned} \text{DPO}(t) = {} & \beta_n(t)I(t_0)\text{E}_t^{\mathbb{Q}}\left[\text{E}_t^{\mathbb{Q}}\left(\frac{P_N(T,T)}{\beta_N(T)} 1_{\{I(T)P_R(T,T)<I(t_0)P_N(T,T)\}} \mid m \right) \right] \\ & - \beta_n(t)\text{E}_t^{\mathbb{Q}}\left[\text{E}_t^{\mathbb{Q}}\left(\frac{I(T)P_R(T,T)}{\beta_N(T)} 1_{\{I(T)P_R(T,T)<I(t_0)P_N(T,T)\}} \mid m \right) \right] \end{aligned} \tag{B-3}$$

where $M(T-t) = m$ is the expected number of jumps over the period $[t,T]$. Given m, the dynamic processes are

$$\begin{aligned} \frac{I(T)P_R(T,T)}{\beta_N(T)} = {} & \frac{I(t)P_R(t,T)}{\beta_N(t)} \\ & \times \exp\left[-\frac{1}{2}V_1^2(m,t,T) + L_1(m) + V_1(m,t,T)\varepsilon_1^{\mathbb{Q}} \right] \end{aligned} \tag{B-4}$$

$$\frac{P_N(T,T)}{\beta_N(T)} = \frac{P_N(t,T)}{\beta_N(t)} \exp\left[-\frac{1}{2}V_2^2(m,t,T) + L_2(m) + V_2(m,t,T)\varepsilon_2^{\mathbb{Q}} \right] \tag{B-5}$$

where

$$\begin{aligned} \varepsilon_1^{\mathbb{Q}} := {} & \frac{1}{V_1(m,t,T)} \left[\begin{array}{l} \int_t^T \sigma_I(u)\, dW_I^{\mathbb{Q}}(u) - \int_t^T \hat{\sigma}_R(u,T)\, dW_R(u) \\ + \sum_{k=1}^{m} \ln Y_{I,k} + \sum_{k=1}^{m} \ln \tilde{Y}_{R,k} - m(\theta_I^{\mathbb{Q}} + \theta_R^{\mathbb{Q}}) \end{array} \right] \\ & \sim \text{Normal}(0,1) \end{aligned}$$

$$\begin{aligned} \varepsilon_2^{\mathbb{Q}} := {} & \frac{1}{V_2(m,t,T)} \left[-\int_t^T \hat{\sigma}_N(u,T)\, dW_N^{\mathbb{Q}}(u) + \sum_{k=1}^{m} \ln \tilde{Y}_{N,k} - m\theta_N^{\mathbb{Q}} \right] \\ & \sim \text{Normal}(0,1) \end{aligned}$$

$$\begin{aligned} \text{Corr}(\varepsilon_1^{\mathbb{Q}}, \varepsilon_1^{\mathbb{Q}}) = {} & \frac{1}{V_1(m,t,T)V_2(m,t,T)} \\ & \times \left[\begin{array}{l} \int_t^T [\rho_{NR}\hat{\sigma}_N(u,T)\hat{\sigma}_R(u,T) - \rho_{IN}\sigma_I(u)\hat{\sigma}_N(u,T)]du \\ + m[\phi_{NR}\nu_N\nu_R(T-t)^2 + \phi_{IN}\nu_I\nu_N(T-t)] \end{array} \right] := \xi(m,t,T) \end{aligned}$$

$$V_1^2(m,t,T) = \int_t^T [\hat{\sigma}_R^2(u,T) - 2\rho_{RI}\hat{\sigma}_R(u,T)\sigma_I(u) + \sigma_I^2(u)]du + m[\nu_R^2(T-t)^2 - 2\phi_{RI}\nu_R\nu_I(T-t) + \nu_I^2]$$

$$V_2^2(m,t,T) = \int_t^T \hat{\sigma}_N^2(u,T)du + m\nu_N^2(T-t)^2$$

$$L_1(m) = m\left(\theta_I^{\mathbb{Q}} + \theta_R^{\mathbb{Q}} + \frac{\nu_R^2(T-t)^2}{2} - \phi_{RI}\nu_R\nu_I(T-t) + \frac{\nu_I^2}{2}\right) - \lambda^{\mathbb{Q}}\left(e^{\theta_R^{\mathbb{Q}}+\theta_I^{\mathbb{Q}}+\frac{\nu_R^2(T-t)^2+2\phi_{RI}\nu_R(T-t)\nu_I+\nu_I^2}{2}} - 1\right)(T-t)$$

$$L_2(m) = m\left(\theta_N^{\mathbb{Q}} + \frac{\nu_N^2(T-t)^2}{2}\right) - \lambda^{\mathbb{Q}}\left(e^{\theta_N^{\mathbb{Q}}+\frac{\nu_N^2(T-t)^2}{2}} - 1\right)(T-t)$$

For the inner expectation in the first term of Equation B-3, we have

$$\begin{aligned} &\mathrm{E}_t^{\mathbb{Q}}\left(\frac{P_N(T,T)}{\beta_N(T)} 1_{\{I(T)P_R(T,T)<I(t_0)P_N(T,T)\}} \mid m\right) \\ &= \frac{P_N(t,T)}{\beta_N(t)} e^{L_2(m)} \mathrm{E}_t^{\mathbb{Q}}\left(e^{-\frac{1}{2}V_2^2(m,t,T)+V_2(m,t,T)\varepsilon_2^{\mathbb{Q}}} 1_{\{I(T)P_R(T,T)<I(t_0)P_N(T,T)\}} \mid m\right) \\ &= \frac{P_N(t,T)}{\beta_N(t)} e^{L_2(m,t,T)} \Phi[-d_2(m,t,T)] \end{aligned} \tag{B-6}$$

The inner expectation in the second term of Equation B-3 leads to the follwing form:

$$\begin{aligned} &\mathrm{E}_t^{\mathbb{Q}}\left(\frac{I(T)P_R(T,T)}{\beta_N(T)} 1_{\{I(T)P_R(T,T)<I(t_0)P_N(T,T)\}} \mid m\right) \\ &= \frac{I(t)P_R(t,T)}{\beta_N(t)} e^{L_1(m)} \mathrm{E}_t^{\mathbb{Q}}\left(e^{-\frac{1}{2}V_1^2(t,T)+V_1(t,T)\varepsilon_1^{\mathbb{Q}}} 1_{\{I(T)P_R(T,T)<I(t_0)P_N(T,T)\}} \mid m\right) \\ &= \frac{I(t)P_R(t,T)}{\beta_N(t)} e^{L_1(m)} \Phi[-d_1(m,t,T)] \end{aligned} \tag{B-7}$$

where

$$d_1(m,t,T) = \frac{\ln\frac{I(t)P_R(t,T)}{I(t_0)P_N(t,T)} + \frac{1}{2}\delta^2(m,t,T) + L_1(m) - L_2(m)}{\delta(m,t,T)}$$

$$d_2(m,t,T) = \frac{\ln\frac{I(t)P_R(t,T)}{I(t_0)P_N(t,T)} - \frac{1}{2}\delta^2(m,t,T) + L_1(m) - L_2(m)}{\delta(m,t,T)}$$

$$\delta^2(m,t,T) = V_1^2(m,t,T) - 2\xi(m,t,T)V_1(m,t,T)V_2(m,t,T) + V_2^2(m,t,T)$$

Therefore, the DPO pricing formula is

$$\begin{aligned} \mathrm{DPO}(t) = {} & I(t_0)P_N(t,T)\sum_{m=0}^{\infty} p(m)e^{L_2(m)}\Phi[-d_2(m,t,T)] \\ & - I(t)P_R(t,T)\sum_{m=0}^{\infty} p(m)e^{L_1(m)}\Phi[-d_1(m,t,T)] \end{aligned} \tag{B-8}$$

where $p(m) = e^{-\lambda^{\mathbb{Q}}(T-t)}[\lambda^{\mathbb{Q}}(T-t)]^m / m!$ is the probability mass function of the Poisson distribution with the intensity $\lambda^{\mathbb{Q}}(T-t)$. This completes our proof of Theorem 1.

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