COALESCENCE ON CRITICAL AND SUBCRITICAL MULTITYPE BRANCHING PROCESSES

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Abstract

Consider a d-type ($d < \infty$) Galton–Watson branching process, conditioned on the event that there are at least $k \geq 2$ individuals in the nth generation, pick k individuals at random from the nth generation and trace their lines of descent backward in time till they meet. In this paper, the limit behaviors of the distributions of the generation number of the most recent common ancestor of any k chosen individuals and of the whole population are studied for both critical and subcritical cases. Also, we investigate the limit distribution of the joint distribution of the generation number and their types.

Keywords: Branching process; coalescence; critical; subcritical; multitype; line of descent

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1. Introduction

1.1. Branching processes

Let $\mathbf{Z}_n = (Z_{n,1}, Z_{n,2}, \ldots, Z_{n,d})$ be the population vector in the nth generation, $n = 0, 1, 2, \ldots$, where $Z_{n,i}$ is the number of individuals of type-i in the nth generation and let $|\mathbf{Z}_0| = 1$. We assume that each individual of type-i lives a unit of time and, upon death, produces children of all types according to the offspring distribution $\{p^{(i)}(\mathbf{j}) := p^{(i)}(j_1, j_2, \ldots, j_d)\}_{\mathbf{j} \in \mathbb{N}_0^d}$ and independently of other individuals, where $\mathbb{N}_0^d := \{\mathbf{j} := (j_1, j_2, \ldots, j_d) : j_i \in \mathbb{N}_0, 1 \leq i \leq d\}$, $\mathbb{N}_0$ is the set of nonnegative integers, and $p^{(i)}(j_1, j_2, \ldots, j_d)$ is the probability that a type-i parent produces j_1 children of type 1, $\ldots$, j_d children of type-d.

Let m_{ij} be the expected number of type-j offspring of a type-i individual in one generation for any $1 \leq i, j \leq d$. Then

$$\mathbf{M} := \{m_{ij} : 1 \leq i, j \leq d\}$$

is called the mean matrix. For a nonsingular and positive regular process, by the Perron–Frobenius theorem (see Athreya and Ney [3]), the matrix $\mathbf{M}$ has a maximal eigenvalue ρ which is positive, simple and has associated strictly positive right and left eigenvectors $\mathbf{u}$ and $\mathbf{v}$ which can be normalized so that

$$\mathbf{u} \cdot \mathbf{v} = 1 \quad \text{and} \quad \mathbf{u} \cdot \mathbf{1} = 1, \tag{1.1}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ in $\mathbb{N}_0^d$. The process is said to be supercritical, critical, or subcritical according to $1 < \rho < \infty$, $\rho = 1$, or $\rho < 1$, respectively.

1.2. The coalescence problem

For any integer $k \geq 2$, conditioned on the event $\{|\boldsymbol{Z}_n| \geq k\}$, pick k individuals at random from the nth generation by simple random sampling without replacement and trace their lines of descent backward in time till they meet. Let $X_{n,k}$ be that generation number called *the coalescence time* of these k individuals of the nth generation. We call the common ancestor of these chosen individuals in the $X_{n,k}$th generation their *most recent common ancestor*. Also, let T_n be the coalescence time of the whole population of the nth generation (T_n is also called *the total coalescence time*). The coalescence problem is to study the properties related to the most recent common ancestor such as the limit behaviors of the distributions of $X_{n,k}$ and T_n as $n \to \infty$. The coalescence problem has been studied for different branching processes. Athreya [1], [2] stated the results for the single-type Galton–Watson processes. Hong [4], [5] extended them to multitype Galton–Watson processes and also to supercritical and subcritical cases for Bellman–Harris processes.

This paper is organized as follows: two classical limit theorems for multitype Galton–Watson processes are stated in Section 2 and notations established will be used for the proofs. The main results for the critical cases are presented in Section 3 and are proved in Section 5. For the subcritical case, the theorems are in Section 4 and proofs are provided in Section 6.

2. Preliminaries and notation

Note that, when we need to consider the type of the initial ancestor, i.e. the process is initiated with an individual of type-i, we denote the process $\{\boldsymbol{Z}_n\}_{n\geq0}$ by

$$\boldsymbol{Z}_n^{(i)} = (Z_{n,1}^{(i)}, Z_{n,2}^{(i)}, \ldots, Z_{n,d}^{(i)}),$$

where $Z_{n,j}^{(i)}$ is the number of type-j individuals in the nth generation, $1 \leq j \leq d$.

To describe the growth rates of the populations in the critical and subcritical cases, we need their probability generating functions and some settings about their second moments.

Let $\boldsymbol{f}_n := (f_n^{(1)}, f_n^{(2)}, \ldots, f_n^{(d)})$ be the probability generating function of $\boldsymbol{Z}_n$. Also, when the second moments exists, we let

$$q_n^{(r)}(i, j) = \mathbb{E}(Z_{n,i}^{(r)} Z_{n,j}^{(r)} - \delta_{i,j} Z_{n,i}^{(r)}), \qquad 1 \leq i, j, r \leq d,$$

the quadratic forms

$$\boldsymbol{Q}_n^{(r)}[s] = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i q_n^{(r)}(i, j) s_j \quad \text{for } 1 \leq r \leq d,$$

the vectors of quadratic forms

$$\mathbb{Q}_n[s] = (\boldsymbol{Q}_n^{(1)}[s], \boldsymbol{Q}_n^{(2)}[s], \ldots, \boldsymbol{Q}_n^{(d)}[s]), \tag{2.1}$$

and let $\mathbb{Q}[s] = \mathbb{Q}_1[s]$, which plays an important role in the limit theorems.

In addition, throughout this paper, we adopt the following notation:

(i) the absolute value of the vector $\boldsymbol{x}$ is $|\boldsymbol{x}| = |x_1| + |x_2| + \cdots + |x_d|$;

(ii) the uniform norm of the vector $\boldsymbol{x}$ is $\|\boldsymbol{x}\| = \max\{|x_1|, |x_2|, \ldots, |x_d|\}$;

(iii) the vector $\boldsymbol{e}_i$ means the vector with 1 in the ith component and 0 elsewhere.

The following two well-known theorems describe the growth rates for the critical and subcritical branching processes.

Theorem 2.1. *(Critical case [6].) Let $\rho = 1$ and $\mathbb{E}\|\mathbf{Z}_1\|^2 < \infty$. Then,*

  (i) *we have*

$$\lim_{n\to\infty} n\mathbb{P}(\mathbf{Z}_n \neq \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{i}) = \frac{\mathbf{i} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]}.$$

  (ii) *If $\mathbf{w} \cdot \mathbf{v} > 0$ then $\mathbf{Z}_n \cdot \mathbf{w}/n$, conditioned on $\mathbf{Z}_n \neq \mathbf{0}$, converges in distribution to the random variable Y with density $f(x) = (1/\gamma_1)e^{-x/\gamma_1}$, $x \geq 0$, where $\gamma_1 = \mathbf{v} \cdot \mathbf{w}/\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]$.*

Theorem 2.2. *(Subcritical case [6].) Let $\rho < 1$. Then*

  (i) *there exists a random vector $\mathbf{Y}$ such that $(\mathbf{Z}_n \mid |\mathbf{Z}_n| > 0) \xrightarrow{\mathrm{D}} \mathbf{Y}$. Furthermore,*

$$\mathbb{E}\|\mathbf{Y}\| < \infty \quad \Longleftrightarrow \quad \mathbb{E}\|\mathbf{Z}_1\| \log \|\mathbf{Z}_1\| < \infty.$$

  (ii) *There exists a nonincreasing and positive $\mathbf{Q}(\cdot)$ such that $\mathbf{v} \cdot [\mathbf{1} - \mathbf{f}_n(\mathbf{s})]/\rho^n \downarrow \mathbf{Q}(\mathbf{s})$ as $n \to \infty$, $\mathbf{0} \leq \mathbf{s} \leq \mathbf{1}$, if and only if $\mathbb{E}\|\mathbf{Z}_1\| \log \|\mathbf{Z}_1\| < \infty$;*

  (iii) *$\lim_{n\to\infty}((\mathbf{1} - \mathbf{f}_n(\mathbf{s}))/\rho^n) = \mathbf{Q}(\mathbf{s})\mathbf{u}$;*

  (iv) *$\lim_{n\to\infty} \rho^{-n}\mathbb{P}(\mathbf{Z}_n \neq \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{i}) = \mathbf{Q}(\mathbf{0})(\mathbf{i} \cdot \mathbf{u})$, where $\mathbf{Q}(\mathbf{0}) = 1/\mathbf{u} \cdot \mathbb{E}\mathbf{Y}$.*

3. Main results for the critical case ($\rho = 1$)

For any $t < n$, let $\{\mathbf{Z}_{t,i,n-t}^{(l)} = (Z_{t,i,n-t}^{(l)1}, Z_{t,i,n-t}^{(l)2}, \ldots, Z_{t,i,n-t}^{(l)d})\}_{n\geq t}$ be the branching process initiated by the ith individual of type-l in the tth generation and let $J_{t,n}^{(l)}$ be the set of all $i \in \{1, 2, \ldots, Z_t^{(l)}\}$ such that $|\mathbf{Z}_{t,i,n-t}^{(l)}| > 0$, $1 \leq l \leq d$.

Theorem 3.1. *Let $\rho = 1$ and $\mathbb{E}\|\mathbf{Z}_1\|^2 < \infty$. On the event $A_n := \{|\mathbf{Z}_n| > 0\}$, for $t < n$, consider the random point process*

$$V_n := \left\{ \frac{\mathbf{Z}_{t,i,n-t}^{(l)}}{n - t} \;\middle|\; i \in J_{t,n}^{(l)}, 1 \leq l \leq d \right\}.$$

Let $n \to \infty$, $t \to \infty$, and $t/n \to \alpha$ for $\alpha \in (0, 1)$. Then, conditioned on A_n, the random point process V_n converges in distribution to a random point process $V := \{Y_i \mid 1 \leq i \leq N_\alpha\}$, where $\{Y_i = (v_1 Y_i, v_2 Y_i, \ldots, v_d Y_i)\}_{i\geq 1}$ are independent and identically distributed (i.i.d.) random vectors with $Y_i \sim \exp(1/\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}])$, N_α is a random variable independent of $\{Y_i\}_{i\geq 1}$ with distribution $\mathbb{P}(N_\alpha = j) = (1 - \alpha)\alpha^{j-1}$ for $j \geq 1$, $\mathbb{Q}$ is the quadratic form as defined in (2.1), and u and v are as in (1.1).

In Theorem 3.1 we showed the convergence of a point process constructed from the original branching process and, by this theorem, we are able to prove the results on the coalescence problems in Theorems 3.2 and 3.3.

Theorem 3.2. *Let $\rho = 1$ and $\mathbb{E}\|\mathbf{Z}_1\|^2 < \infty$. Then, for $k = 2, 3, \ldots$, there exists a random variable $\tilde{X}_k$ such that $(X_{n,k}/n \mid |\mathbf{Z}_n| \geq k) \xrightarrow{\mathrm{D}} \tilde{X}_k$ as $n \to \infty$ and, for any $\alpha \in (0, 1)$,*

$$\mathbb{P}(\tilde{X}_k < \alpha) = 1 - (1 - \alpha)F(1, 2; k + 1; \alpha) := H_k(\alpha),$$

where F is a hypergeometric function. Furthermore, $\lim_{\alpha\uparrow 1} H_k(\alpha) = 1$.

From Theorem 3.2 we see that the generation number of the most recent common ancestor grows like n. That is, the coalescence time X_n is not close either to the beginning of the tree or to the present time when n gets large.

Theorem 3.3. *Let $\rho = 1$ and $\mathbb{E}\|\mathbf{Z}_1\|^2 < \infty$. Then*

$$\left(\frac{T_n}{n} \,\middle|\, |\mathbf{Z}_n| > 0 \right) \xrightarrow{\mathrm{D}} \tilde{T} \quad as\ n \to \infty,$$

where $\tilde{T}$ has a uniform distribution in $(0, 1)$.

4. Main results for the subcritical case

For the subcritical case, Theorem 4.1 shows that the difference $n - X_{n,2}$ between the coalescence time $X_{n,2}$ and the current generation number n converges in distribution as $n \to \infty$ and it tells us that the coalescence time does not go right back to the beginning of the tree. Instead, it is close to the present time. Theorem 4.1 can be extended to the case for any $k = 2, 3, \ldots$.

Theorem 4.1. *Let $0 < \rho < 1$ and $\mathbb{E}\|\mathbf{Z}_1\| \log \|\mathbf{Z}_1\| < \infty$. Then there exists a random variable $\tilde{X}_2$ such that $(n - X_{n,2} \mid |\mathbf{Z}_n| \geq 2) \xrightarrow{\mathrm{D}} \tilde{X}_2$ as $n \to \infty$, and, for any $r = 0, 1, 2, \ldots$,*

$$\mathbb{P}(\tilde{X}_2 \leq r) = 1 - \frac{1}{\rho^r \mathbb{P}(|\mathbf{Y}| \geq 2)} \mathbb{E}(\phi(Y^{(1)}, Y^{(2)}, \ldots, Y^{(d)}, r)) := H_2(r),$$

where

$$\phi(t_1, t_2, \ldots, t_d, r) = \mathbb{E}\left(\frac{\sum_{l=1}^d \sum_{i \neq j=1}^{t_l} |\tilde{\mathbf{Z}}_{r,i}^{(l)}||\tilde{\mathbf{Z}}_{r,j}^{(l)}| + \sum_{l \neq p=1}^d \sum_{i=1}^{t_l} \sum_{j=1}^{t_p} |\tilde{\mathbf{Z}}_{r,i}^{(l)}||\tilde{\mathbf{Z}}_{r,j}^{(p)}|}{(\sum_{l=1}^d \sum_{i \neq j=1}^{t_l} |\tilde{\mathbf{Z}}_{r,i}^{(l)}|)(\sum_{l=1}^d \sum_{i \neq j=1}^{t_l} |\tilde{\mathbf{Z}}_{r,i}^{(l)}| - 1)} \right.$$
$$\left. \times 1_{\{\sum_{l=1}^d \sum_{i \neq j=1}^{t_l} |\tilde{\mathbf{Z}}_{r,i}^{(l)}| \geq 2\}} \right),$$

where I is the indicator function. Furthermore, $\{\tilde{\mathbf{Z}}_{r,i}^{(l)} : i \geq 1\}_{r \geq 0}$ are i.i.d. copies of $\{\mathbf{Z}_r^{(l)}\}_{r \geq 0}$, and $\lim_{r \to \infty} H_2(r) = 1$.

Theorem 4.2. *Let $0 < \rho < 1$ and $\mathbb{E}\|\mathbf{Z}_1\| \log \|\mathbf{Z}_1\| < \infty$. Then there exists a random variable $\tilde{T}$ such that $(n - T_n \mid |\mathbf{Z}_n| > 0) \xrightarrow{\mathrm{D}} \tilde{T}$ as $n \to \infty$, and, for any $r = 0, 1, 2, \ldots$,*

$$\mathbb{P}(\tilde{T} \leq r) = \rho^{-r} \mathbb{E}\left(\sum_{l=1}^d Y^{(l)} (1 - f_r^{(l)}(\mathbf{0}))(f_r^{(l)}(\mathbf{0}))^{Y^{(l)}-1} \prod_{p \neq l} (f_r^{(l)}(\mathbf{0}))^{Y^{(p)}} \right) := \pi(r),$$

where $\mathbf{Y}$ is the random vector with distribution $\{b(\mathbf{j})\}_{\mathbf{j} \in \mathbb{R}_+^d}$ defined as in Theorem 2.2(iv). Also, $\lim_{r \uparrow \infty} \pi(r) = 1$, i.e. $\tilde{T}$ is a proper random variable.

Next, we look at the limit of the joint distribution of the generation number $X_{n,2}$, the type of the most recent common ancestor, and the types of two randomly chosen individuals. In addition to $X_{n,2}$, we let η_n be the type of this common ancestor and $\zeta_{n,i}$ the type of the ith chosen individual.

Theorem 4.3. *Let $0 < \rho < 1$ and $\mathbb{E}\|\mathbf{Z}_1\| \log \|\mathbf{Z}_1\| < \infty$. Then*

$$\lim_{n \to \infty} \mathbb{P}(X_{n,2} = r, \eta_n = j, \zeta_{n,1} = i_1, \zeta_{n,2} = i_2 \mid |\mathbf{Z}_n| \geq 2) := \psi_2(r, j, i_1, i_2)$$

exists and $\sum_{(r,j,i_1,i_2)} \psi_2(r, j, i_1, i_2) = 1$.

The next result is an extension of the above theorem for $k \geq 2$.

Theorem 4.4. *Let $0 < \rho < 1$ and $\mathbb{E}\|\mathbf{Z}_1\| \log \|\mathbf{Z}_1\| < \infty$. Then*

$$\lim_{n \to \infty} \mathbb{P}(X_{n,k} = r, \, \eta_n = j, \, \zeta_{n,1} = i_1, \ldots, \zeta_{n,k} = i_k \mid |\mathbf{Z}_n| \geq k) := \psi_k(r, j, i_1, \ldots, i_k)$$

exists and $\sum_{(r,j,i_1,\ldots,i_k)} \psi_k(r, j, i_1, \ldots, i_k) = 1$.

5. Proofs of the main results for the critical case

Proof of Theorem 3.1. To prove the convergence of the random point process $\{V_n\}$, we consider the Laplace functional of V_n, i.e.

$$\varphi_n(\theta_1, \theta_2, \ldots, \theta_d, f_1, f_2, \ldots, f_d) := \mathbb{E}\left(\exp\left(-\sum_{l=1}^{d} \sum_{i \in J_{t,n}^{(l)}} \sum_{p=1}^{d} \theta_p f_p\left(\frac{Z_{t,i,n-t}^{(l)p}}{n-t}\right)\right) \,\middle|\, |\mathbf{Z}_n| > 0\right),$$

where $\theta_1, \theta_2, \ldots, \theta_d > 0$ and $f_1, f_2, \ldots, f_d : \mathbb{R}^+ \to \mathbb{R}^+$ are bounded and continuous functions. Let

$$Y_{n,t} = \exp\left(-\sum_{l=1}^{d} \sum_{i \in J_{t,n}^{(l)}} \sum_{p=1}^{d} \theta_p f_p\left(\frac{Z_{t,i,n-t}^{(l)p}}{n-t}\right)\right),$$

then, we have

$$\begin{aligned}
\varphi_n&(\theta_1, \theta_2, \ldots, \theta_d, f_1, f_2, \ldots, f_d) \\
&= \mathbb{E}(Y_{n,t} \mid |\mathbf{Z}_n| > 0) \\
&= \frac{\mathbb{E}(Y_{n,t} \mathbf{1}_{\{|\mathbf{Z}_n|>0\}})}{\mathbb{P}(|\mathbf{Z}_n| > 0)} \\
&= \frac{\mathbb{E}(\mathbb{E}(Y_{n,t} \mathbf{1}_{\{|\mathbf{Z}_n|>0\}} \mid \mathbf{Z}_j, \, j \leq t))}{\mathbb{P}(|\mathbf{Z}_n| > 0)} \\
&= \frac{\mathbb{E}(\mathbb{E}(Y_{n,t} \mathbf{1}_{\{|\mathbf{Z}_n|>0\}} \mid \mathbf{Z}_t))}{\mathbb{P}(|\mathbf{Z}_n| > 0)} \quad \text{(by the Markov property)} \\
&= \frac{\mathbb{E}(\mathbb{E}(Y_{n,t} \mathbf{1}_{\{|\mathbf{Z}_n|>0\}} \mathbf{1}_{\{|\mathbf{Z}_t|>0\}} \mid \mathbf{Z}_t))}{\mathbb{P}(|\mathbf{Z}_n| > 0)} \quad \text{(since } \mathbf{1}_{\{|\mathbf{Z}_n|>0\}} \subseteq \mathbf{1}_{\{|\mathbf{Z}_t|>0\}}) \\
&= \frac{\mathbb{P}(|\mathbf{Z}_t| > 0)\mathbb{E}(\mathbb{E}(Y_{n,t} \mid \mathbf{Z}_t) - \mathbb{E}(Y_{n,t} \mathbf{1}_{\{|\mathbf{Z}_n|=0\}} \mid \mathbf{Z}_t) \mid |\mathbf{Z}_t| > 0)}{\mathbb{P}(|\mathbf{Z}_n| > 0)}.
\end{aligned}$$

If we let $g_j^{(l)}(\theta) = \mathbb{E}(\exp(-\sum_{p=1}^{d} \theta_p f_p(Z_j^{(p)}/j)\mathbf{1}_{\{|\mathbf{Z}_j|>0\}}) \mid \mathbf{Z}_0 = \mathbf{e}_l)$, $\theta = (\theta_1, \theta_2, \ldots, \theta_d)$, and let $q_j^{(l)} = \mathbb{P}(|\mathbf{Z}_j| = 0 \mid \mathbf{Z}_0 = \mathbf{e}_l)$ for $j \geq 1$, then the above quantity is equal to

$$\frac{\mathbb{P}(|\mathbf{Z}_t| > 0)}{\mathbb{P}(|\mathbf{Z}_n| > 0)} \mathbb{E}\left(\prod_{l=1}^{d} (g_{n-t}^{(l)}(\theta))^{Z_t^{(l)}} - \prod_{l=1}^{d} (q_{n-t}^{(l)}(\theta))^{Z_t^{(l)}} \,\middle|\, |\mathbf{Z}_t| > 0\right). \tag{5.1}$$

For the convergence of the fractional coefficient of (5.1), by Theorem 2.1(i), we know that, as $t, n \to \infty$,

$$\frac{\mathbb{P}(|\mathbf{Z}_t| > 0)}{\mathbb{P}(|\mathbf{Z}_n| > 0)} = \frac{t\mathbb{P}(|\mathbf{Z}_t| > 0)}{n\mathbb{P}(|\mathbf{Z}_n| > 0)} \frac{n}{t} \to \left[\left(\frac{u_{i_0}}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]}\right)\left(\frac{u_{i_0}}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]}\right)^{-1}\right] \frac{1}{\alpha} = \frac{1}{\alpha}. \tag{5.2}$$

Next, we prove the convergence of the minuend of (5.1), i.e.

$$
g_j^{(l)}(\theta) = \mathbb{P}(|\mathbf{Z}_j| = 0 \mid \mathbf{Z}_0 = \mathbf{e}_l)
$$
$$
\times \mathbb{E}\left(\exp\left(-\sum_{p=1}^d \theta_p f_p\left(\frac{Z_j^{(p)}}{j} \right) 1_{\{|\mathbf{Z}_j|>0\}} \right) \,\Big|\, |\mathbf{Z}_j| = 0,\ \mathbf{Z}_0 = \mathbf{e}_l \right)
$$
$$
+ \mathbb{P}(|\mathbf{Z}_j| > 0 \mid \mathbf{Z}_0 = \mathbf{e}_l)
$$
$$
\times \mathbb{E}\left(\exp\left(-\sum_{p=1}^d \theta_p f_p\left(\frac{Z_j^{(p)}}{j} \right) 1_{\{|\mathbf{Z}_j|>0\}} \right) \,\Big|\, |\mathbf{Z}_j| > 0,\ \mathbf{Z}_0 = \mathbf{e}_l \right)
$$
$$
= q_j^{(l)} + (1 - q_j^{(l)})\mathbb{E}\left(\exp\left(-\sum_{p=1}^d \theta_p f_p\left(\frac{Z_j^{(p)}}{j} \right) 1_{\{|\mathbf{Z}_j|>0\}} \right) \,\Big|\, |\mathbf{Z}_j| > 0,\ \mathbf{Z}_0 = \mathbf{e}_l \right)
$$
$$
= 1 + (1 - q_j^{(l)})\left[\mathbb{E}\left(\exp\left(-\sum_{p=1}^d \theta_p f_p\left(\frac{Z_j^{(p)}}{j} \right) 1_{\{|\mathbf{Z}_j|>0\}} \right) \,\Big|\, |\mathbf{Z}_j| > 0,\ \mathbf{Z}_0 = \mathbf{e}_l \right) - 1 \right]
$$

and, hence, as a result of Theorems 2.1(i) and 2.1(ii), and the definition of the constant e, when $j \to \infty$,

$$
(g_j^{(l)}(\theta))^j = \left(1 + j(1 - q_j^{(l)}) \right.
$$
$$
\left. \times \frac{[\mathbb{E}(\exp(-\sum_{p=1}^d \theta_p f_p(Z_j^{(p)}/j)1_{\{|\mathbf{Z}_j|>0\}}) \mid |\mathbf{Z}_j| > 0,\ \mathbf{Z}_0 = \mathbf{e}_l) - 1]}{j} \right)^j
$$
$$
\to \exp\left(\frac{u_l}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]}(g(\theta) - 1) \right),
$$

where $g(\theta) = \mathbb{E}(\exp(-\sum_{p=1}^d \theta_p f_p(v_p Y)))$. The same idea can be applied to the subtrahend of (5.1). Therefore,

$$
\mathbb{E}\left(\prod_{l=1}^d (g_{n-t}^{(l)}(\theta))^{Z_t^{(l)}} - \prod_{l=1}^d (q_{n-t}^{(l)}(\theta))^{Z_t^{(l)}} \,\Big|\, |\mathbf{Z}_t| > 0 \right)
$$
$$
= \mathbb{E}\left(\prod_{l=1}^d (g_{n-t}^{(l)}(\theta))^{(n-t)[(t/(n-t))(Z_t^{(l)}/t)]} \right.
$$
$$
\left. - \prod_{l=1}^d ((1 - (1 - q_{n-t}^{(l)}))^{n-t})^{[(t/(n-t))(Z_t^{(l)}/t)]} \,\Big|\, |\mathbf{Z}_t| > 0 \right)
$$
$$
\to \mathbb{E}\left(\exp\left(\sum_{l=1}^d \frac{u_l}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]}(g(\theta) - 1)\frac{\alpha}{1-\alpha}v_l Y \right) - \exp\left(-\sum_{l=1}^d \frac{u_l}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]}\frac{\alpha}{1-\alpha}v_l Y \right) \right)
$$
$$
= \mathbb{E}\left(\exp\left(\frac{1}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]}(g(\theta) - 1)\frac{\alpha}{1-\alpha}Y \right) - \exp\left(-\frac{1}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]}\frac{\alpha}{1-\alpha}Y \right) \right)
$$
$$
= \frac{1-\alpha}{1-\alpha g(\theta)} - (1-\alpha) \tag{5.3}
$$

since

$$Y \sim \exp\left(\frac{1}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]}\right) \quad \text{as } n \to \infty, \, t \to \infty, \, t/n \to \alpha,$$

for $0 < \alpha < 1$. Hence, by (5.2) and (5.3),

$$\begin{aligned}
\varphi_n(\theta_1, \theta_2, \ldots, \theta_d, f_1, f_2, \ldots, f_d) &\to \frac{1}{\alpha}\left(\frac{1-\alpha}{1-\alpha g(\theta)} - (1-\alpha)\right) \\
&= \frac{(1-\alpha)g(\theta)}{1-\alpha g(\theta)} \\
&= \sum_{j=1}^{\infty}(1-\alpha)\alpha^{j-1}(g(\theta))^j \quad \text{as } t, n \to \infty.
\end{aligned}$$

Finally, let $\boldsymbol{V} := \{Y_i \mid 1 \leq i \leq N_\alpha\}$, where $\{Y_i = (v_1 Y_i, v_2 Y_i, \ldots, v_d Y_i)\}_{i \geq 1}$ are i.i.d. random vectors with $Y_i \sim \exp(1/\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}])$ and N_α is a random variable independent of $\{Y_i\}_{i \geq 1}$ with distribution $\mathbb{P}(N_\alpha = j) = (1-\alpha)\alpha^{j-1}$ for $j \geq 1$. Then, for any $\theta_1, \theta_2, \ldots, \theta_d > 0$ and any bounded, nonnegative and continuous functions $f_1, f_2, \ldots, f_d$, the Laplace functional of V is

$$\mathbb{E}\left(\exp\left(-\sum_{i=1}^{N_\alpha}\sum_{p=1}^{d}\theta_p f_p(Y_i^{(p)})\right)\right) = \sum_{j=1}^{\infty}(1-\alpha)\alpha^{j-1}(g(\theta))^j.$$

Therefore, by the continuous mapping theorem for random measures (see [7]), the sequence of random point processes $\{\boldsymbol{V}_n\}_{n \geq 1}$, conditioned on $\{|\boldsymbol{Z}_n| > 0\}$, converges in distribution to the random point process $\boldsymbol{V} := \{Y_i \mid 1 \leq i \leq N_\alpha\}$ as $n, t \to \infty$, $t/n \to \alpha$.   □

Proof of Theorem 3.2. For almost all trees $\mathcal{T}$ and for any $\alpha \in (0, 1)$, let $r = [n\alpha] + 1$. Conditioned on the set $\{|\boldsymbol{Z}_n| \geq k\}$, the event $\{X_{n,k} \geq r\}$ occurs if and only if all k individuals in the nth generation are chosen from the decedents of a single ancestor in the rth generation. So, we have

$$\mathbb{P}\left(\frac{X_{n,k}}{n} < \alpha \,\Big|\, |\boldsymbol{Z}_n| \geq k\right)$$
$$= 1 - \mathbb{E}(\mathbb{P}(X_{n,k} \geq r \mid \boldsymbol{Z}_n) \mid |\boldsymbol{Z}_n| \geq k)$$
$$= 1 - \mathbb{E}\left(\frac{\sum_{l=1}^{d}\sum_{i=1}^{Z_r^{(l)}}|\boldsymbol{Z}_{r,i,n-r}^{(l)}|(|\boldsymbol{Z}_{r,i,n-r}^{(l)}|-1)\cdots(|\boldsymbol{Z}_{r,i,n-r}^{(l)}|-k+1)}{|\boldsymbol{Z}_n|(|\boldsymbol{Z}_n|-1)\cdots(|\boldsymbol{Z}_n|-k+1)} \,\Big|\, |\boldsymbol{Z}_n| \geq k\right).$$

If we expand the numerator in the expectation, we have

$$1 - \frac{1}{\mathbb{P}(|\boldsymbol{Z}_n| \geq k \mid |\boldsymbol{Z}_n| > 0)}$$
$$\times \mathbb{E}\Bigg(\Bigg(\frac{\sum_{l=1}^{d}\sum_{i=1}^{Z_r^{(l)}}|\boldsymbol{Z}_{r,i,n-r}^{(l)}|^k}{|\boldsymbol{Z}_n|(|\boldsymbol{Z}_n|-1)\cdots(|\boldsymbol{Z}_n|-k+1)}$$
$$+ \sum_{s=1}^{k-1}\Bigg((-1)^s\Bigg(\sum_{\substack{1 \leq q_1 < q_2 < \cdots < q_s \leq k-1 \\ q_1, q_2, \ldots, q_s \in \mathbb{Z}}} q_1 q_2 \cdots q_s\Bigg)\sum_{l=1}^{d}\sum_{i=1}^{Z_r^{(l)}}|\boldsymbol{Z}_{r,i,n-r}^{(l)}|^{k-s}\Bigg)$$
$$\times [|\boldsymbol{Z}_n|(|\boldsymbol{Z}_n|-1)\cdots(|\boldsymbol{Z}_n|-k+1)]^{-1}\Bigg)\mathbb{1}_{\{|\boldsymbol{Z}_n|\geq k\}} \,\Big|\, |\boldsymbol{Z}_n| > 0\Bigg)$$

$$= 1 - \frac{1}{\mathbb{P}(|\mathbf{Z}_n| \geq k \mid |\mathbf{Z}_n| > 0)}$$

$$\times \mathbb{E}\Bigg(\Bigg(\Bigg[\sum_{l=1}^{d} \sum_{i \in J_r^{(l)}} \Bigg(\frac{|\mathbf{Z}_{r,i,n-r}^{(l)}|}{n-r}\Bigg)^k \Bigg(\sum_{l=1}^{d} \sum_{i \in J_r^{(l)}} \frac{|\mathbf{Z}_{r,i,n-r}^{(l)}|}{n-r}\Bigg)^{-k}\Bigg] 1_{\{|\mathbf{Z}_n| \geq k\}}$$

$$+ \frac{\sum_{l=1}^{d} \sum_{i=1}^{Z_r^{(l)}} |\mathbf{Z}_{r,i,n-r}^{(l)}|^k}{|\mathbf{Z}_n|(|\mathbf{Z}_n|-1)\cdots(|\mathbf{Z}_n|-k+1)} - \frac{\sum_{l=1}^{d} \sum_{i=1}^{Z_r^{(l)}} |\mathbf{Z}_{r,i,n-r}^{(l)}|^k}{|\mathbf{Z}_n|^k}$$

$$+ \sum_{s=1}^{k-1}\Bigg((-1)^s \Bigg(\sum_{\substack{1 \leq q_1 < q_2 < \cdots < q_s \leq k-1 \\ q_1, q_2, \ldots, q_s \in \mathbb{Z}}} q_1 q_2 \cdots q_s\Bigg) \sum_{l=1}^{d} \sum_{i=1}^{Z_r^{(l)}} |\mathbf{Z}_{r,i,n-r}^{(l)}|^{k-s}\Bigg)$$

$$\times [|\mathbf{Z}_n|(|\mathbf{Z}_n|-1)\cdots(|\mathbf{Z}_n|-k+1)]^{-1}\Bigg) 1_{\{|\mathbf{Z}_n| \geq k\}} \Bigg| |\mathbf{Z}_n| > 0\Bigg).$$

Now, we prove the convergence of the above equation. First, we know that

$$\mathbb{P}(|\mathbf{Z}_n| \geq k \mid |\mathbf{Z}_n| > 0) \to 1 \quad \text{as } n \to \infty.$$

Next, we show that the second and the third parts of the equation converge to 0. Thus,

$$\mathbb{E}\Bigg(\Bigg|\frac{\sum_{l=1}^{d} \sum_{i=1}^{Z_r^{(l)}} |\mathbf{Z}_{r,i,n-r}^{(l)}|^k}{|\mathbf{Z}_n|(|\mathbf{Z}_n|-1)\cdots(|\mathbf{Z}_n|-k+1)} - \frac{\sum_{l=1}^{d} \sum_{i=1}^{Z_r^{(l)}} |\mathbf{Z}_{r,i,n-r}^{(l)}|^k}{|\mathbf{Z}_n|^k}\Bigg| 1_{\{|\mathbf{Z}_n| \geq k\}} \Bigg| |\mathbf{Z}_n| > 0\Bigg)$$

$$= \mathbb{E}\Bigg(\frac{\sum_{l=1}^{d} \sum_{i=1}^{Z_r^{(l)}} |\mathbf{Z}_{r,i,n-r}^{(l)}|^k}{|\mathbf{Z}_n|^k}$$

$$\times 1_{\{|\mathbf{Z}_n| > k\}}\Bigg|\frac{1}{(1-1/|\mathbf{Z}_n|)\cdots(1-(k-1)/|\mathbf{Z}_n|)} - 1\Bigg| \Bigg| |\mathbf{Z}_n| > 0\Bigg)$$

$$\leq \mathbb{E}\Bigg(\Bigg|\frac{1}{(1-1/|\mathbf{Z}_n|)\cdots(1-(k-1)/|\mathbf{Z}_n|)} - 1\Bigg| \Bigg| |\mathbf{Z}_n| > 0\Bigg)$$

$$= \mathbb{P}\Bigg(\frac{|\mathbf{Z}_n|}{n} < \varepsilon \Bigg| |\mathbf{Z}_n| > 0\Bigg)$$

$$+ \mathbb{E}\Bigg(\Bigg|\frac{1}{(1-1/|\mathbf{Z}_n|)\cdots(1-(k-1)/|\mathbf{Z}_n|)} - 1\Bigg| 1_{\{\|\mathbf{Z}_n|/n \geq \varepsilon\}} \Bigg| |\mathbf{Z}_n| > 0\Bigg)$$

$$\leq \mathbb{P}\Bigg(\frac{|\mathbf{Z}_n|}{n} < \varepsilon \Bigg| |\mathbf{Z}_n| > 0\Bigg) + \Bigg|\frac{1}{(1-1/n\varepsilon)\cdots(1-(k-1)/n\varepsilon)} - 1\Bigg|$$

$$\to 1 - \exp\Bigg(-\frac{\mathbf{v} \cdot \mathbf{1}}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]}\varepsilon\Bigg) \quad \text{as} \to \infty$$

for any arbitrarily small $\varepsilon > 0$.

Hence, we have, as $n \to \infty$,

$$\mathbb{E}\Bigg(\Bigg|\frac{\sum_{l=1}^{d} \sum_{i=1}^{Z_r^{(l)}} |\mathbf{Z}_{r,i,n-r}^{(l)}|^k}{|\mathbf{Z}_n|(|\mathbf{Z}_n|-1)\cdots(|\mathbf{Z}_n|-k+1)} - \frac{\sum_{l=1}^{d} \sum_{i=1}^{Z_r^{(l)}} |\mathbf{Z}_{r,i,n-r}^{(l)}|^k}{|\mathbf{Z}_n|^k}\Bigg| 1_{\{|\mathbf{Z}_n| \geq k\}} \Bigg| |\mathbf{Z}_n| > 0\Bigg)$$

$$\to 0.$$

Similarly, for $s = 1, 2, \ldots, k$, and for any arbitrarily small $\varepsilon > 0$,

$$
\mathbb{E}\left(\frac{\sum_{l=1}^{d} \sum_{i=1}^{Z_r^{(l)}} |Z_{r,i,n-r}^{(l)}|^{k-s}}{|\mathbf{Z}_n|(|\mathbf{Z}_n| - 1) \cdots (|\mathbf{Z}_n| - k + 1)} 1_{\{|\mathbf{Z}_n| \geq k\}} \,\bigg|\, |\mathbf{Z}_n| > 0 \right)
$$

$$
\leq \mathbb{E}\left(\frac{|\mathbf{Z}_n|^{k-s}}{|\mathbf{Z}_n|(|\mathbf{Z}_n| - 1) \cdots (|\mathbf{Z}_n| - k + 1)} 1_{\{|\mathbf{Z}_n| \geq k\}} \,\bigg|\, |\mathbf{Z}_n| > 0 \right)
$$

$$
\leq \frac{k^k}{k!} \mathbb{E}(|\mathbf{Z}_n|^{-s} 1_{\{|\mathbf{Z}_n| \geq k\}} \mid |\mathbf{Z}_n| > 0)
$$

$$
\leq \frac{k^k}{k!}\left((n\varepsilon)^{-s} + \mathbb{P}\left(\frac{|\mathbf{Z}_n|}{n} \leq \varepsilon \,\bigg|\, |\mathbf{Z}_n| > 0 \right) \right)
$$

$$
\to \frac{k^k}{k!}\left(1 - \exp\left(-\frac{\mathbf{v} \cdot \mathbf{1}}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]} \varepsilon \right) \right) \quad \text{as } n \to \infty.
$$

Since $\varepsilon > 0$ is arbitrarily small, we have

$$
\mathbb{E}\left(\frac{\sum_{l=1}^{d} \sum_{i=1}^{Z_r^{(l)}} |Z_{r,i,n-r}^{(l)}|^{k-s}}{|\mathbf{Z}_n|(|\mathbf{Z}_n| - 1) \cdots (|\mathbf{Z}_n| - k + 1)} 1_{\{|\mathbf{Z}_n| \geq k\}} \,\bigg|\, |\mathbf{Z}_n| > 0 \right) \to 0 \quad \text{as } n \to \infty.
$$

Therefore, by the continuous mapping theorem and Theorem 3.1,

$$
\mathbb{P}\left(\frac{X_{n,k}}{n} < \alpha \,\bigg|\, |\mathbf{Z}_n| > 0 \right) \to 1 - \mathbb{E}\left(\frac{\sum_{i=1}^{N_\alpha} Y_i^k}{(\sum_{i=1}^{N_\alpha} Y_i)^k} \right) := H_k(\alpha) \quad \text{as } n \to \infty.
$$

Let $G_i(x, k) = Y_i^k / (\sum_{i=1}^{x} Y_i)^k$, so $H_k(\alpha) = 1 - \mathbb{E}(\sum_{i=1}^{x} G_i(N_\alpha, k))$. Next we determine the distribution of $G_i(x, k)$. Since $\{Y_i\}_{1 \leq i \leq x}$ are i.i.d. exponential random variables, $Z_i := \sum_{j \neq i} Y_j$ is independent of Y_i and has gamma$(x - 1, 1/\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}])$ distribution for $i = 1, 2, \ldots, x$. So, the joint probability density function (PDF) of Y_i and Z_i is

$$
f_{Y_i, Z_i}(y, z) = \frac{1}{(\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}])^x \Gamma(x - 1)} z^{x-2} \exp\left(-\frac{y + z}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]} \right).
$$

Let $U = (\sum_{i=1}^{x} Y_i)^k = (Y_i + Z_i)^k$ and $V = G_i(x, k)$, then the Jacobian of the transformation of U and V is $|J| = (1/k^2) u^{2/k-1} v^{1/k-1}$. So, the joint PDF of U and V is

$$
f_{U,V}(u, v) = f_{Y_i, Z_i}(y, z)|J|
$$

$$
= \frac{1}{k^2 (\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}])^x \Gamma(x - 1)} u^{x/k-1} v^{1/k-1} (1 - v^{1/k})^{x-2} \exp\left(\frac{u^{1/k}}{\mathbf{v} \cdot \mathbb{Q}[\mathbf{u}]} \right)
$$

and, hence, the PDF of $V = G_i(x, k)$ is

$$
f_{G_i(x,k)}(v) = \int_0^1 f_{U,V}(u, v) \, \mathrm{d}u = \frac{x - 1}{k} v^{1/k-1} (1 - v^{1/k})^{x-2}, \qquad 0 < v < 1.
$$

Therefore, by letting $t = v^{1/k}$ and from the definition of the beta function, we have

$$
\mathbb{E}(G_i(x, k)) = \int_0^1 v f_{G_i(x,k)}(v) \, \mathrm{d}v = (x - 1) \int_0^1 t^k (1 - t)^{x-2} \, \mathrm{d}t = \frac{k!\,(x - 1)!}{(k + x - 1)!}
$$

and then, since N_α has the geometric distribution with parameter $1 - \alpha$,

$$
\begin{aligned}
H_k(\alpha) &= 1 - \mathbb{E}\left(\sum_{i=1}^{N_\alpha} \mathbb{E}(G_i(N_\alpha, k) \mid N_\alpha) \right) \\
&= 1 - \mathbb{E}\left(N_\alpha \frac{k!\,(N_\alpha - 1)!}{(k + N_\alpha - 1)!} \right) \\
&= 1 - \mathbb{E}\left(\frac{k!\,N_\alpha!}{(k + N_\alpha - 1)!} \right) \\
&= 1 - \sum_{n=1}^{\infty} \frac{k!\,n!}{(k + n - 1)!}(1 - \alpha)\alpha^{n-1} \\
&= 1 - \sum_{n=0}^{\infty} \frac{k!\,(n + 1)!}{(k + n)!}(1 - \alpha)\alpha^{n} \\
&= 1 - (1 - \alpha)\sum_{n=0}^{\infty} \frac{n!\,(n + 1)!}{(k + 1)(k + 1)\cdots(k + n)}\frac{\alpha^n}{n!} \\
&= 1 - (1 - \alpha)F(1, 2; k + 1; \alpha),
\end{aligned}
$$

where F is the hypergeometric function.

Note that $N_\alpha \to \infty$ with probability 1 as $\alpha \to 1$. So, by the bounded convergence theorem, $\lim_{\alpha \to 1} H_k(\alpha) = 1 - 0 = 1$. Moreover, $H_k(0) = 0$. Therefore, H_k is a proper probability distribution and the proof is complete. $\qquad\square$

Proof of Theorem 3.3. For any $\alpha \in (0, 1)$ and any $n \in \mathbb{N}$, let $r = [n\alpha] + 1$. The event $\{T_n \geq r\}$ conditioned on $\{|\mathbf{Z}_n| > 0\}$ occurs if and only if all the individuals in the nth generation come from the $(n - r)$th generation of the tree initiated by exactly one individual in the rth generation. That is, $|\mathbf{Z}_{r,i,n-r}^{(l)}| = 0$ for all but one $l = 1, 2, \ldots, d$ and one $i = 1, 2, \ldots, Z_r^{(l)}$ and $|\mathbf{Z}_r| > 0$. Hence,

$$
\begin{aligned}
\mathbb{P}&\left(\frac{T_n}{n} > \alpha \,\middle|\, |\mathbf{Z}_n| > 0 \right) \\
&= \mathbb{P}(T_n \geq r \mid |\mathbf{Z}_n| > 0) \\
&= \frac{1}{\mathbb{P}(|\mathbf{Z}_n| > 0)}\mathbb{P}(|\mathbf{Z}_{r,i,n-r}^{(l)}| = 0 \text{ for all but one } l = 1, 2, \ldots, d \\
&\qquad\qquad\qquad\qquad\qquad \text{and one } i = 1, 2, \ldots, Z_r^{(l)} \text{ and } |\mathbf{Z}_r| > 0) \\
&= \frac{1}{\mathbb{P}(|\mathbf{Z}_n| > 0)}\mathbb{E}\left(\sum_{l=1}^{d}\sum_{i=1}^{Z_r^{(l)}} \mathbb{P}(|\mathbf{Z}_{r,i,n-r}^{(l)}| > 0)\prod_{j\neq i}\mathbb{P}(|\mathbf{Z}_{r,j,n-r}^{(l)}| = 0) \right. \\
&\qquad\qquad\qquad\qquad \left. \times \prod_{p\neq l}\prod_{j=1}^{Z_r^{(p)}}\mathbb{P}(|\mathbf{Z}_{r,j,n-r}^{(p)}| = 0)\mathbf{1}_{\{|\mathbf{Z}_r|>0\}} \right) \\
&= \frac{\mathbb{P}(|\mathbf{Z}_r| > 0)}{\mathbb{P}(|\mathbf{Z}_n| > 0)}\mathbb{E}\left(\sum_{l=1}^{d} Z_r^{(l)}g_{n-r}^{(l)}(1 - g_{n-r}^{(l)})^{Z_r^{(l)}-1}\prod_{p\neq l}(1 - g_{n-r}^{(p)})^{Z_r^{(p)}} \,\middle|\, |\mathbf{Z}_r| > 0 \right)
\end{aligned}
$$

$$= \frac{\mathbb{P}(|\boldsymbol{Z}_r| > 0)}{\mathbb{P}(|\boldsymbol{Z}_n| > 0)} \mathbb{E}\Bigg(\sum_{l=1}^{d} \frac{Z_r^{(l)}}{r}(n-r)g_{n-r}^{(l)}$$

$$\times \frac{r}{n-r}\left(1 - \frac{(n-r)g_{n-r}^{(l)}}{n-r}\right)^{(n-r)((Z_r^{(l)}-1)/r)(r/(n-r))}$$

$$\times \prod_{p \neq l}\left(1 - \frac{(n-r)g_{n-r}^{(p)}}{n-r}\right)^{(n-r)((Z_r^{(p)}/r)(r/(n-r)))} \Bigg| |\boldsymbol{Z}_r| > 0\Bigg),$$

where $g_n^{(l)} = \mathbb{P}(|\boldsymbol{Z}_n| > 0 \mid \boldsymbol{Z}_0 = \boldsymbol{e}_l)$. First, following similar lines as in the proof of Theorem 3.1, we have $\mathbb{P}(|\boldsymbol{Z}_r| > 0)/\mathbb{P}(|\boldsymbol{Z}_n| > 0)$ converges to $1/\alpha$ as $n \to \infty$. Secondly, let h_n be the function defined by

$$h_n(x_1, x_2, \ldots, x_d) := \sum_{l=1}^{d} x_l(n-r)g_{n-r}^{(l)} \frac{r}{n-r}\left(1 - \frac{(n-r)g_{n-r}^{(l)}}{n-r}\right)^{(n-r)(x_l-1/r)(r/(n-r))}$$

$$\times \prod_{p \neq l}\left(1 - \frac{(n-r)g_{n-r}^{(p)}}{n-r}\right)^{(n-r)x_p(r/(n-r))}$$

then, by Theorem 2.1(i), as $n \to \infty$, h_n converges to

$$h(x_1, x_2, \ldots, x_d) := \sum_{l=1}^{d} x_l \frac{u_l}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]} \frac{\alpha}{1-\alpha} \exp\left(-\frac{u_l}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]}x_l \frac{\alpha}{1-\alpha}\right)$$

$$\times \prod_{p \neq l} \exp\left(-\frac{u_p}{\boldsymbol{v}\mathbb{Q}[\boldsymbol{u}]}x_p \frac{\alpha}{1-\alpha}\right).$$

Finally, since $h_n \to h$ uniformly on any compact set since h_n and h are continuous and bounded, and, hence, as $n \to \infty$,

$$\mathbb{E}\left(h_n\left(\frac{Z_r^{(1)}}{r}, \frac{Z_r^{(2)}}{r}, \ldots, \frac{Z_r^{(d)}}{r}\right) \Bigg| |\boldsymbol{Z}_n| > 0\right) \to \mathbb{E}(h(v_1 Y, v_2 Y, \ldots, v_d Y))$$

then the limit on the right-hand side is equal to

$$\mathbb{E}\Bigg(\sum_{l=1}^{d} v_l Y \frac{u_l}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]} \frac{\alpha}{1-\alpha} \exp\left(-\frac{u_l}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]}v_l Y \frac{\alpha}{1-\alpha}\right) \prod_{p \neq l} \exp\left(-\frac{u_p}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]}v_p Y \frac{\alpha}{1-\alpha}\right)\Bigg)$$

$$= \mathbb{E}\left(\frac{Y}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]} \frac{\alpha}{1-\alpha} \exp\left(-\frac{Y}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]} \frac{\alpha}{1-\alpha}\right)\right).$$

So, for $\alpha \in (0, 1)$,

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{T_n}{n} > \alpha \Bigg| |\boldsymbol{Z}_n| > 0\right)$$

$$= \frac{1}{\alpha} \mathbb{E}\left(\frac{Y}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]} \frac{\alpha}{1-\alpha} \exp\left(-\frac{Y}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]} \frac{\alpha}{1-\alpha}\right)\right)$$

$$= \frac{1}{\alpha} \frac{1}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]} \frac{\alpha}{1-\alpha} \int_0^\infty y \exp\left(-\frac{1}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]} \frac{\alpha}{1-\alpha} y\right) \frac{1}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]} \exp\left(-\frac{1}{\boldsymbol{v} \cdot \mathbb{Q}[\boldsymbol{u}]} y\right) \mathrm{d}y$$

$$= 1 - \alpha.$$

Hence, $(T_n/n \mid |\mathbf{Z}_n| > 0) \overset{\mathrm{D}}{\to} \tilde{T}$ as $n \to \infty$, where $\tilde{T}$ is a uniform$(0, 1)$ random variable, and we prove Theorem 3.3. $\qquad\square$

6. Proofs of the main results for the subcritical case

Proof of Theorem 4.1. For any $r \geq 0$, conditioned on $\{|\mathbf{Z}_n| \geq 2\}$, the event $\{X_{n,2} < n - r\}$ occurs if and only if these two individuals are chosen from two trees initiated by two different ancestors who are either of the same type or of two different types in the $(n - r)$th generation. So,

$$
\begin{aligned}
&\mathbb{P}(n - X_{n,2} > r \mid |\mathbf{Z}_n| \geq 2) \\
&\quad = \mathbb{P}(X_{n,2} < n - r \mid |\mathbf{Z}_n| \geq 2) \\
&\quad = \mathbb{E}\left(\frac{\sum_{l=1}^{d} \sum_{i \neq j=1}^{Z_{n-r}^{(l)}} |\mathbf{Z}_{n-r,i,r}^{(l)}||\mathbf{Z}_{n-r,j,r}^{(l)}| + \sum_{l \neq p=1}^{d} \sum_{i=1}^{Z_{n-r}^{(l)}} \sum_{j=1}^{Z_{n-r}^{(p)}} |\mathbf{Z}_{n-r,i,r}^{(l)}||\mathbf{Z}_{n-r,j,r}^{(p)}|}{|\mathbf{Z}_n|(|\mathbf{Z}_n|-1)} \,\middle|\, |\mathbf{Z}_n| \geq 2 \right) \\
&\quad = \frac{1}{\mathbb{P}(|\mathbf{Z}_n| \geq 2, |\mathbf{Z}_n| > 0)} \\
&\qquad \times \mathbb{E}\left(\frac{\sum_{l=1}^{d} \sum_{i \neq j=1}^{Z_{n-r}^{(l)}} |\mathbf{Z}_{n-r,i,r}^{(l)}||\mathbf{Z}_{n-r,j,r}^{(l)}| + \sum_{l \neq p=1}^{d} \sum_{i=1}^{Z_{n-r}^{(l)}} \sum_{j=1}^{Z_{n-r}^{(p)}} |\mathbf{Z}_{n-r,i,r}^{(l)}||\mathbf{Z}_{n-r,j,r}^{(p)}|}{|\mathbf{Z}_n|(|\mathbf{Z}_n| - 1)} \right. \\
&\qquad\qquad \left. \times \mathbf{1}_{\{|\mathbf{Z}_n| \geq 2\}} \mathbf{1}_{\{|\mathbf{Z}_{n-r}| > 0\}} \right)
\end{aligned}
$$

since $\{|\mathbf{Z}_n| \geq 2\} \subseteq \{|\mathbf{Z}_{n-r}| > 0\}$. Also, the random variable $\mathbf{Z}_{n-r,i,r}^{(l)}$ is the branching process that is initiated by the ith individual of type-l in the $(n-r)$th generations of the original process and has been evolving for r generations, and, therefore, it has the same distribution as $\mathbf{Z}_r^{(l)}$; hence, the quantity can be written as

$$
\begin{aligned}
&\frac{\mathbb{P}(|\mathbf{Z}_{n-r}| > 0)}{\mathbb{P}(|\mathbf{Z}_n| \geq 2 \mid |\mathbf{Z}_n| > 0)\mathbb{P}(|\mathbf{Z}_n| > 0)} \\
&\quad \times \mathbb{E}\left(\frac{\sum_{l=1}^{d} \sum_{i \neq j=1}^{Z_{n-r}^{(l)}} |\tilde{\mathbf{Z}}_{r,i}^{(l)}||\tilde{\mathbf{Z}}_{r,j}^{(l)}| + \sum_{l \neq p=1}^{d} \sum_{i=1}^{Z_{n-r}^{(l)}} \sum_{j=1}^{Z_{n-r}^{(p)}} |\tilde{\mathbf{Z}}_{r,i}^{(l)}||\tilde{\mathbf{Z}}_{r,j}^{(p)}|}{(\sum_{l=1}^{d} \sum_{i \neq j=1}^{Z_{n-r}^{(l)}} |\tilde{\mathbf{Z}}_{r,i}^{(l)}|)(\sum_{l=1}^{d} \sum_{i \neq j=1}^{Z_{n-r}^{(l)}} |\tilde{\mathbf{Z}}_{r,i}^{(l)}| - 1)} \right. \\
&\qquad\qquad \left. \times \mathbf{1}_{\{\sum_{l=1}^{d} \sum_{i=1}^{Z_{n-r}^{(l)}} |\tilde{\mathbf{Z}}_{r,i}^{(l)}| \geq 2\}} \,\middle|\, |\mathbf{Z}_{n-r}| > 0 \right) \\
&\quad = \frac{\mathbb{P}(|\mathbf{Z}_{n-r}| > 0)}{\mathbb{P}(|\mathbf{Z}_n| \geq 2 \mid |\mathbf{Z}_n| > 0)\mathbb{P}(|\mathbf{Z}_n| > 0)} \mathbb{E}(\phi(Z_{n-r}^{(1)}, Z_{n-r}^{(2)}, \ldots, Z_{n-r}^{(d)}, r) \mid |\mathbf{Z}_{n-r}| > 0),
\end{aligned}
$$

where $\tilde{\mathbf{Z}}_{r,i}^{(l)} \sim \mathbf{Z}_r^{(l)}$ for all i and $l = 1, 2, \ldots, d$, and where

$$
\begin{aligned}
\phi(t_1, t_2, \ldots, t_d, r) = \mathbb{E}&\left(\frac{\sum_{l=1}^{d} \sum_{i \neq j=1}^{t_l} |\tilde{\mathbf{Z}}_{r,i}^{(l)}||\tilde{\mathbf{Z}}_{r,j}^{(l)}| + \sum_{l \neq p=1}^{d} \sum_{i=1}^{t_l} \sum_{j=1}^{t_p} |\tilde{\mathbf{Z}}_{r,i}^{(l)}||\tilde{\mathbf{Z}}_{r,j}^{(p)}|}{(\sum_{l=1}^{d} \sum_{i \neq j=1}^{t_l} |\tilde{\mathbf{Z}}_{r,i}^{(l)}|)(\sum_{l=1}^{d} \sum_{i \neq j=1}^{t_l} |\tilde{\mathbf{Z}}_{r,i}^{(l)}| - 1)} \right. \\
&\qquad \left. \times \mathbf{1}_{\{\sum_{l=1}^{d} \sum_{i=1}^{t_l} |\tilde{\mathbf{Z}}_{r,i}^{(l)}| \geq 2\}} \right).
\end{aligned}
$$

Since $\phi(\cdot, r)$ is continuous and bounded, by Theorem 2.2(i), there exists a random vector $\boldsymbol{Y} := (Y^{(1)}, Y^{(2)}, \ldots, Y^{(d)})$ such that, as $n \to \infty$, for any fixed $r \geq 0$,

$$\mathbb{E}(\phi(Z_{n-r}^{(1)}, Z_{n-r}^{(2)}, \ldots, Z_{n-r}^{(d)}, r) \mid |\boldsymbol{Z}_{n-r}| > 0) \to \mathbb{E}(\phi(Y^{(1)}, Y^{(2)}, \ldots, Y^{(d)}, r)).$$

Also, by Theorems 2.2(i) and 2.2(iv), as $n \to \infty$, it follows that $\mathbb{P}(|\boldsymbol{Z}_n| \geq 2 \mid |\boldsymbol{Z}_n| > 0)$ converges to $\mathbb{P}(|\boldsymbol{Y}| \geq 2)$ and $\mathbb{P}(|\boldsymbol{Z}_{n-r}| > 0)/\mathbb{P}(|\boldsymbol{Z}_n| > 0)$ converges to ρ^{-r}. Hence, for any $r \geq 0$, the probability $\mathbb{P}(n - X_{n,2} > r \mid |\boldsymbol{Z}_n| \geq 2)$ converges to

$$\frac{1}{\rho^r \mathbb{P}(|\boldsymbol{Y}| \geq 2)} \mathbb{E}(\phi(Y^{(1)}, Y^{(2)}, \ldots, Y^{(d)}, r)) := 1 - H_2(r).$$

Now, it remains to show that $1 - H_2(r) \to 0$ as $r \to \infty$. Recall that $f_r^{(l)}(\boldsymbol{0})$ is the probability of extinction for the individuals of type-l in the rth generation. Since the event that two individuals can be chosen from the rth generation implies that there are at least two individuals in the rth generation, we have

$$\begin{aligned}
0 \leq {} & \frac{1}{\rho^r} \mathbb{E}(\phi(Y^{(1)}, Y^{(2)}, \ldots, Y^{(d)}, r)) \\
\leq {} & \frac{1}{\rho^r} \mathbb{E}\Bigg(1 - \prod_{l=1}^{d}(f_r^{(l)}(\boldsymbol{0}))^{Y^{(l)}} \\
& - \sum_{l=1}^{d} Y^{(l)}(1 - f_r^{(l)}(\boldsymbol{0}))(f_r^{(l)}(\boldsymbol{0}))^{(Y^{(l)}-1)} \prod_{p \neq l}(f_r^{(p)}(\boldsymbol{0}))^{Y^{(p)}}\Bigg) \\
= {} & \mathbb{E}\Bigg(\frac{1 - \prod_{l=1}^{d}(f_r^{(l)}(\boldsymbol{0}))^{Y^{(l)}}}{\rho^r}\Bigg) \\
& - \sum_{l=1}^{d} \frac{1 - f_r^{(l)}(\boldsymbol{0})}{\rho^r} \mathbb{E}\Bigg(Y^{(l)}(f_r^{(l)}(\boldsymbol{0}))^{(Y^{(l)}-1)} \prod_{p \neq l}(f_r^{(p)}(\boldsymbol{0}))^{Y^{(p)}}\Bigg). \quad (6.1)
\end{aligned}$$

First, by Theorem 2.2 and the monotone convergence theorem, as $r \to \infty$,

$$\mathbb{E}\Bigg(\frac{1 - \prod_{l=1}^{d}(f_r^{(l)}(\boldsymbol{0}))^{Y^{(l)}}}{\rho^r}\Bigg) = \mathbb{E}(\rho^{-r}\mathbb{P}(\boldsymbol{Z}_r \neq \boldsymbol{0} \mid \boldsymbol{Z}_0 = \boldsymbol{Y})) \to \mathbb{E}\Bigg(\frac{\boldsymbol{u} \cdot \boldsymbol{Y}}{\boldsymbol{u} \cdot \mathbb{E}\boldsymbol{Y}}\Bigg) = 1.$$

Secondly, by Theorem 2.2(iii), we have

$$\mathbb{E}Y^{(l)} < \infty \quad \text{and} \quad \frac{1 - f_r^{(l)}(\boldsymbol{0})}{\rho^r} \to \frac{u_l}{\boldsymbol{u} \cdot \mathbb{E}\boldsymbol{Y}} \quad \text{as } r \to \infty.$$

Also, $f_r^{(l)}(\boldsymbol{0}) \to 1$ as $r \to \infty$. So, by the bounded convergence theorem, as $r \to \infty$, the subtrahend of (6.1) converges to $\sum_{l=1}^{d}(u_l/\boldsymbol{u} \cdot \mathbb{E}\boldsymbol{Y})\mathbb{E}Y^{(l)} = 1$. Therefore, we obtain

$$1 - H_2(r) \to 1 - 1 = 0 \quad \text{as } r \to \infty. \qquad \square$$

Proof of Theorem 4.2. Since, conditioned on $\{|\boldsymbol{Z}_n| > 0\}$, the event $\{T_n \geq n - r\}$ occurs if and only if the whole population comes from exactly one individual in the $(n - r)$th generation and

the trees initiated by other individuals in the same generation die out before the nth generation except this one, for any $r \geq 0$,

$$\mathbb{P}(n - T_n \leq r \mid |\mathbf{Z}_n| > 0) = \mathbb{P}(T_n \geq n - r \mid |\mathbf{Z}_n| > 0)$$

$$= \mathbb{E}\left(\sum_{l=1}^{d} \sum_{i=1}^{Z_{n-r}^{(l)}} \mathbb{P}(|\mathbf{Z}_{n-r,i,r}^{(l)}| > 0) \prod_{j \neq i} \mathbb{P}(|\mathbf{Z}_{n-r,j,r}^{(l)}| = 0)\right.$$

$$\left. \times \prod_{p \neq l} \prod_{j=1}^{Z_{n-r}^{(p)}} \mathbb{P}(|\mathbf{Z}_{n-r,j,r}^{(p)}| = 0) \,\middle|\, |\mathbf{Z}_n| > 0\right)$$

$$= \frac{\mathbb{P}(|\mathbf{Z}_{n-r}| > 0)}{\mathbb{P}(|\mathbf{Z}_n| > 0)} \mathbb{E}\left(\sum_{l=1}^{d} Z_{n-r}^{(l)}(1 - f_r^{(l)}(\mathbf{0}))(f_r^{(l)}(\mathbf{0}))^{Z_{n-r}^{(l)}-1}\right.$$

$$\left. \times \prod_{p \neq l} (f_r^{(l)}(\mathbf{0}))^{Z_{n-r}^{(p)}} \,\middle|\, |\mathbf{Z}_{n-r}| > 0\right).$$

Let

$$h(x_1, x_2, \ldots, x_d) = \mathbb{E}\left(\sum_{l=1}^{d} x_l(1 - f_r^{(l)}(\mathbf{0}))(f_r^{(l)}(\mathbf{0}))^{x_l-1} \prod_{p \neq l} (f_r^{(l)}(\mathbf{0}))^{x_p}\right),$$

then h is continuous at $(x_1, x_2, \ldots, x_d)$. So, by the continuous mapping theorem, as $n \to \infty$,

$$\mathbb{E}(h(Z_{n-r}^{(1)}, Z_{n-r}^{(2)}, \ldots, Z_{n-r}^{(d)}) \mid |\mathbf{Z}_{n-r}| > 0) \to \mathbb{E}(h(Y^{(1)}, Y^{(2)}, \ldots, Y^{(d)}))$$

and, hence, $\mathbb{P}(n - X_n \leq r \mid |\mathbf{Z}_n| > 0)$ converges to

$$\rho^{-r} \mathbb{E}\left(\sum_{l=1}^{d} Y^{(l)}(1 - f_r^{(l)}(\mathbf{0}))(f_r^{(l)}(\mathbf{0}))^{Y^{(l)}-1} \prod_{p \neq l} (f_r^{(l)}(\mathbf{0}))^{Y^{(p)}}\right) := \pi(r).$$

Also, along the same lines as in the proof of Theorem 4.1, we can prove that $\lim_{r \to \infty} \pi(r) = 1$ and which that $\pi(r)$ is a proper probability distribution. $\qquad \square$

Proof of Theorem 4.3. Let $\boldsymbol{\xi}_{n,j}^i = (\xi_{n,j}^{i(1)}, \xi_{n,j}^{i(2)}, \ldots, \xi_{n,j}^{i(d)})$ be the vector of offspring of the jth individual of type-i in the nth generation. Let

$$\mathbf{Z}_{p,r,s,n}^{j,l} = (Z_{p,r,s,n}^{j,l,(1)}, Z_{p,r,s,n}^{j,l,(2)}, \ldots, Z_{p,r,s,n}^{j,l,(d)})$$

be the branching process initiated by the sth child of type-l of the pth individual of type-j in the rth generation. Also, let $A_{n,i}$ be the type of the ancestor in the next generation of the most recent common ancestor of the ith chosen individual, $i = 1, 2$. Then the event $\{n - X_{n,2} = r, \eta_n = j, \zeta_{n,1} = i_1, \zeta_{n,2} = i_i\}$ is a disjoint union

$$E_1 := \{n - X_{n,2} = r, \eta_n = j, \zeta_{n,1} = \zeta_{n,2} = i, A_{n,1} = A_{n,2}\} \cup E_2$$
$$:= \{n - X_{n,2} = r, \eta_n = j, \zeta_{n,1} = \zeta_{n,2} = i, A_{n,1} \neq A_{n,2}\} \cup E_3$$
$$:= \{n - X_{n,2} = r, \eta_n = j, \zeta_{n,1} = i_1 \neq \zeta_{n,2} = i_2, A_{n,1} = A_{n,2}\} \cup E_4$$
$$:= \{n - X_{n,2} = r, \eta_n = j, \zeta_{n,1} = i_1 \neq \zeta_{n,2} = i_2, A_{n,1} \neq A_{n,2}\}.$$

First, conditioned on $\{|\mathbf{Z}_n| \geq 2\}$, the event E_1 occurs if and only if these two individuals are chosen from those of type-i in the nth generation and from two different trees initiated by two children, who are of the same type, of an individual of type-j in the $(n-r)$th generation. So,

$$
\mathbb{P}(E_1 \mid |\mathbf{Z}_n| \geq 2)
$$

$$
= \mathbb{E}\left(\frac{\sum_{p=1}^{Z_{n-r}^{(j)}} \sum_{l=1}^{d} \sum_{s \neq t=1}^{\xi_{n-r,p}^{j(l)}} Z_{p,n-r,s,r-1}^{j,l,(i)} Z_{p,n-r,t,r-1}^{j,l,(i)}}{|\mathbf{Z}_n|(|\mathbf{Z}_n| - 1)} \,\middle|\, |\mathbf{Z}_n| \geq 2 \right)
$$

$$
= \frac{\mathbb{P}(|\mathbf{Z}_{n-r}| > 0)}{\mathbb{P}(|\mathbf{Z}_n| \geq 2 \mid |\mathbf{Z}_n| > 0)\mathbb{P}(|\mathbf{Z}_n| > 0)}
$$

$$
\times \mathbb{E}\left(\frac{\sum_{p=1}^{Z_{n-r}^{(j)}} \sum_{l=1}^{d} \sum_{s \neq t=1}^{\xi_{n-r,p}^{j(l)}} Z_{p,n-r,s,r-1}^{j,l,(i)} Z_{p,n-r,t,r-1}^{j,l,(i)}}{|\mathbf{Z}_n|(|\mathbf{Z}_n| - 1)} 1_{\{|\mathbf{Z}_n| \geq 2\}} \,\middle|\, |\mathbf{Z}_{n-r}| > 0 \right). \quad (6.2)
$$

Let $\boldsymbol{\xi}^j = (\xi^{j(1)}, \xi^{j(2)}, \ldots, \xi^{j(d)})$ be i.i.d. copies of the vector of offspring of an individual of type-j, then $\boldsymbol{\xi}^j$ has the same distribution as $\boldsymbol{\xi}_{n-r,p}^j$. Also, let $\tilde{\mathbf{Z}}_{r-1,s}^l = (\tilde{Z}_{r-1,s}^{l(1)}, \tilde{Z}_{r-1,s}^{l(2)}, \ldots, \tilde{Z}_{r-1,s}^{l(\alpha)})$ be the i.i.d. copies of $\mathbf{Z}_{r-1}$ with $\mathbf{Z}_0 = e_l$, then $\tilde{\mathbf{Z}}_{r-1,s}^l$ has the same distribution as $\mathbf{Z}_{p,n-r,s,r-1}^{jl}$. So, the expectation in (6.2) is equal to

$$
\mathbb{E}\left(\frac{\sum_{p=1}^{Z_{n-r}^{(j)}} \sum_{l=1}^{d} \sum_{s \neq t=1}^{\xi^{j(l)}} \tilde{Z}_{r-1,s}^{l(i)} \tilde{Z}_{r-1,t}^{l(i)}}{|\mathbf{Z}_n|(|\mathbf{Z}_n| - 1)} 1_{\{\sum_{j=1}^{d} \sum_{p=1}^{Z_{n-r}^{(j)}} \sum_{l=1}^{d} \sum_{s=1}^{\xi^{j(l)}} |\tilde{\mathbf{Z}}_{r-1,s}^l| \geq 2\}} \,\middle|\, |\mathbf{Z}_{n-r}| > 0 \right).
$$

Let

$$
\varphi_1(x_1, x_2, \ldots, x_d, r)
$$

$$
= \mathbb{E}\left(\frac{\sum_{p=1}^{x_j} \sum_{l=1}^{d} \sum_{s \neq t=1}^{\xi^{j(l)}} \tilde{Z}_{r-1,s}^{l(i)} \tilde{Z}_{r-1,t}^{l(i)} 1_{\{\sum_{j=1}^{d} \sum_{p=1}^{x_j} \sum_{l=1}^{d} \sum_{s=1}^{\xi^{j(l)}} |\tilde{\mathbf{Z}}_{r-1,s}^l| \geq 2\}}}{(\sum_{j=1}^{d} \sum_{p=1}^{x_j} \sum_{l=1}^{d} \sum_{s=1}^{\xi^{j(l)}} |\tilde{\mathbf{Z}}_{r-1,s}^l|)(\sum_{j=1}^{d} \sum_{p=1}^{x_j} \sum_{l=1}^{d} \sum_{s=1}^{\xi^{j(l)}} |\tilde{\mathbf{Z}}_{r-1,s}^l| - 1)} \right),
$$

then, since $\varphi_1(\cdot, r)$ is continuous and $(\mathbf{Z}_{n-r} \mid |\mathbf{Z}_{n-r}| > 0) \xrightarrow{\mathrm{D}} \mathbf{Y}$ as $n \to \infty$,

$$
\mathbb{P}(E_1 \mid |\mathbf{Z}_n| \geq 2) \to \frac{1}{\rho^r \mathbb{P}(|\mathbf{Y}| \geq 2)} \mathbb{E}(\varphi_1(Y^{(1)}, Y^{(2)}, \ldots, Y^{(d)})).
$$

Similarly, there exist φ_l, $l = 2, 3, 4$, such that, as $n \to \infty$,

$$
\mathbb{P}(E_l \mid |\mathbf{Z}_n| \geq 2) \to \frac{1}{\rho^r \mathbb{P}(|\mathbf{Y}| \geq 2)} \mathbb{E}(\varphi_l(Y^{(1)}, Y^{(2)}, \ldots, Y^{(d)})).
$$

Let $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$, then, as $n \to \infty$,

$$
\mathbb{P}(n - X_{n,2} = r, \eta_n = j, \zeta_{n,1} = i_1, \zeta_{n,2} = i_2 \mid |\mathbf{Z}_n| \geq 2)
$$

$$
\to \frac{1}{\rho^r \mathbb{P}(|\mathbf{Y}| \geq 2)} \mathbb{E}(\varphi(Y^{(1)}, Y^{(2)}, \ldots, Y^{(d)}))
$$

$$
:= \psi_2(r, j, i_1, i_2).
$$

Moreover, since $(n - X_{n,2} \mid |Z_n| \geq 2) \overset{\text{D}}{\to} \tilde{X}_2$ as $n \to \infty$, it follows that $\{n - X_{n,2}\}_{n \geq 0}$ is tight and, also, $\{\eta_n\}_{n \geq 0}$, $\{\zeta_{n,1}\}_{n \geq 0}$, and $\{\zeta_{n,2}\}_{n \geq 0}$ only take values on the finite set $\{1, 2, \ldots, d\}$, so $\{(n - X_{n,2}, \eta_n, \zeta_{n,1}, \zeta_{n,2})\}_{n \geq 0}$ is tight. Therefore, the limit $\psi_2(r, j, i_1, i_2)$ of

$$\mathbb{P}(n - X_{n,2} = r, \; \eta_n = j, \; \zeta_{n,1} = i_1, \; \zeta_{n,2} = i_2)$$

is a probability mass function on $\mathbb{N}_0 \times \{1, 2, \ldots, d\} \times \{1, 2, \ldots, d\} \times \{1, 2, \ldots, d\}$. That is, $\sum_{(r,j,i_1,i_2)} \psi_2(r, j, i_1, i_2) = 1$. $\qquad \square$

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