國 立 政 治 大 學 應 用 數 學 系

碩 士 學 位 論 文

以階段型機率分佈表示
異質生成衝擊系統
A System Subject to Non-Homogeneous Pure Birth Shocks with Phase-Type Distributions

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中文摘要

考慮一個衝擊系統，它的衝擊依據異質生成過程而產生。這個系統有兩種類型的損壞。類型一的損壞可以被修理消除。類型二的損壞可以被不定期置換消除。假設兩個連續衝擊之間的時間間隔服從階段型分佈。例如，在一個特殊的階段型分佈—亞指數分佈—之下，我們發現穩定機率存在的條件。在這個模型下探討年齡置換策略，我們導出置換週期內的期望成本率。為了找到最小化期望成本率的最佳定期置換年齡，我們提供一個有效率的演算法並開發一個 MALAB 工具來實現。一系列數值範例促使我們發現新的定理，它比以前的定理更簡單，更實際，更直觀。該定理表明最佳定期置換年齡的存在性。

關鍵字：衝擊模型、階段型分佈、異質生成過程、再生過程、馬可夫過程、年齡置換策略、穩定機率

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Abstract

We consider a system subject to shocks which occur according to a non-homogeneous pure birth process. The system has two types of failures. Type-I failure can be removed by a repair. Type-II failure can be removed by an unplanned replacement. We assume that the inter-arrival time between consecutive shocks follows phase-type distributions. For example, under a special PH-distribution that is a hypo-exponential distribution, we find the conditions of the existence of stationary probability. Under this model we investigate the age replacement policy. We derive the expected cost rate of a replacement cycle. To find the optimal planned replacement age that minimizes the expected cost rate, we give an efficient algorithm and develop a MALAB tool for implementation. A series of numerical examples motivate us to write a new theorem. That is simpler, more practical, and more intuitive than a previous theorem. This theorem shows the existence of the optimal planned replacement age.

Keywords: Shock model, Phase-type distribution, Non-homogeneous pure birth process, Renewal process, Markov process, Age replacement policy, Stationary probability

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Chapter 1

Introduction

Sheu et al. [13] present a non-homogeneous pure birth shock model. Assume the probability that a machine undertakes repairable failure or deterioration will increase with age. Through preventive maintenance policy, we can minimize operational costs and catastrophic failure risks. One well-known preventive maintenance policy is age replacement which is widely used and easy to implement. As a shock occurs, the system gets into failure state. Hillier and Lieberman [8] introduce an example of a machine which have four states: (i) new, (ii) minor deterioration, (iii) major deterioration, and (iv) breakdown. With the random variable X_t denoting the state of the machine at week t, the stochastic process $\{X_t : t = 0, 1, 2, \cdots\}$ is a discrete time Markov chain. With the different one-step transition probabilities among states of the process, the machine reaches different operation modes. Without loss of generality, in this thesis, we only consider the system has two types of failures: (i) minor failure, and (ii) catastrophic failure.

There are many literatures which deal with the replacement of a system subject to shocks. According to the inter-arrival time between consecutive shocks, these model can be divided into (i) homogeneous Poisson process (PP), (ii) nonhomogeneous Poisson process (NHPP), (iii) nonhomogeneous pure birth process (NHPBP), and (iv) renewal process.

Cox [6] defines the fundamental shock model which is called the ordinary renewal process. Start a new system at zero time. The system fails at time X_1 and is immediately replaced by a new system with failure time X_2. Then the second failure will occur at time $X_1 + X_2$. Let this process continue. This system is called an ordinary renewal process if $\{X_1, X_2, \cdots\}$ are independent identically distributed random variables, all with probability density function

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$f(x)$. But this model does not consider the accumulation of shocks. Esary et al. [7] provide a more complicated model. They consider a system subject to shocks which occur according to a PP. Each shock causes a random damage. The damages on shocks are independent and identically distributed. The system will breakdown when the accumulated damage exceeds a specified threshold. A-Hameed and Proschan [1] extend the results obtained by Esary et al. [7] and consider a system subject to shocks which occur according to a NHPP. A-Hameed and Proschan [2] extend the above two results and consider a system subject to shocks which occur according to a nonstationary pure birth process: given k shocks have occurred in $[0, t]$, the probability of a shock occurring in $(t, t+\Delta]$ is $\lambda_k \lambda(t)\Delta + o(\Delta)$. Sheu et al. [13] investigate the maintenance or replacement policies under the NHPBP shock process.

For a shock model, the probability of an event occurring during an arbitrarily small interval is defined by $P_{k,k+1}(h) = \Pr\{X(t+h) - X(t) = 1 \,|\, X(t) = k\}$. We compare $P_{k,k+1}(h)$ of the four cases: (i) PP, $P_{k,k+1}(h) = \lambda h + o(h)$ as $h \to 0$, (ii) NHPP, $P_{k,k+1}(h) = \lambda(t)h + o(h)$ as $h \to 0$, (iii) homogeneous pure birth process (PBP), $P_{k,k+1}(h) = \lambda_k h + o(h)$ as $h \to 0$, (iv) NHPBP, $P_{k,k+1}(h) = \lambda_k(t)h + o(h)$ as $h \to 0$. The definition of Poisson process and pure birth process can be found in Taylor and Karlin [14]. Note that the hazard rate of PP is constant but that of NHPP is dependent on the age of the system, so is PBP which is dependent on the number of shocks. Moreover the hazard rate of NHPBP not only depends on the age of the system, but also depends on the number of shocks. Therefore the NHPBP is more appropriate for prescribing the system's deterioration process.

The NHPBP is more suitable for charactering the practical system's deterioration process than the PP and the NHPP. However, the cumulative probability function of the lifetime of the system is not easy to calculate. In order to make the calculation easier, a proper assumption about the distribution of the inter-arrival time between consecutive shocks should be considered. One suitable distribution is the phase-type distribution since it can be represented as matrix exponential forms with closure property, see Buchholz et al. [5]. Under this assumption, we investigate the age replacement policy and give an algorithm to compute the optimal planned replacement age.

Under the age replacement policy, the system is replaced at the planned replacement age or at failure, whichever occurs first. Barlow and Hunter [4] provide the standard model of the classical age replacement policy. Its objective is to minimize the expected cost rate of a

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replacement cycle which is the ratio of expected cost over a replacement cycle to expected length of a replacement cycle. The optimal planned replacement age corresponds to the minimum of the objective function.

Although Sheu et al. [13] provide an appropriate model for the system's deterioration process, the distribution of the inter-arrival time between any consecutive shocks is given by a general assumption without specific form which may cause computational difficulties when evaluating an optimal replacement policy. In this thesis, we give an analysis of how a phase-type distribution can be used to provide an efficient algorithm in order to evaluate the optimal policy, e.g., the optimal planned replacement age.

Both continuous (CPH) and discrete (DPH) phase-type distributions were first described in detail by Neuts [11]. They are widely used in distribution approximation due to their computational advantages and easy integration in complex stochastic models. It is known that the PH-distribution can approximate an arbitrary probability distribution with high accuracy by Asmussen et al. [3]. Weibull distribution is on of the functions for which it is easy to find the satisfactory PH-approximation. Maier and O'Cinneide [9] has proved that phase-type distribution is closed under convolutions and mixtures. Detailed calculation can refer to Buchholz et al. [5, pp. 24-25] and Nielsen [12, pp. 15-17]. Montoro-Cazorla et al. [10] consider a shock model whose inter-arrival times between any consecutive shocks follow phase-type (PH-) distributions. They apply the closure property of PH-distribution to express the lifetime of the system as PH-distribution. PH-distribution has matrix exponential form, and therethrough we can use numerical computation to solve the problems of shock models.

This thesis is organized as follows. In Chapter 2, a system subject to NHPBP shocks with phase-type distribution is considered. In Chapter 3, we find the conditions of the existence of stationary probability. In Chapter 4, the expected cost rate of a replacement cycle is formulated and the optimization of the age replacement policy has been developed. In Chapter 6, several numerical examples are given to the shock model to illustrate the algorithm and we find a new theorem which shows the existence of the optimal planned replacement age. Finally, Chapter 7 concludes this thesis.

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Chapter 2

Model Formulation

2.1 Definitions of NHPBP and Phase-Type Distributions

In this thesis, we consider a system subject to shocks which occur according to a non-homogeneous or non-stationary pure birth process defined below.

Definition. (Sheu et al. [13]) If a counting process $\{N(t) : t \geq 0\}$ is a non-homogeneous continuous time Markov process with following conditions:

(i) $N(0) = 0$,

(ii) $\Pr\{N(t+h) - N(t) = 1 | N(t) = k\} = \lambda_k(t)h + o(h)$,

(iii) $\Pr\{N(t+h) - N(t) \geq 2 | N(t) = k\} = o(h)$,

(iv) the process has independent increments,

then the process is called a non-homogeneous or non-stationary pure birth process (denoted by NHPBP or NSPBP) with the intensity function $\{\lambda_k(t), k = 0, 1, 2, \cdots\}$.

We only consider the case that the inter-arrival times between any consecutive shocks follow phase-type distributions.

Definition. The distribution $H(\cdot)$ on $[0, \infty)$ is a phase-type distribution with representation (β, A), if it is the distribution of the time until absorption in a Markov process on the states $\{1, \cdots, m, m+1\}$ with generator

$$\begin{bmatrix} A & \boldsymbol{a} \\ 0 & 0 \end{bmatrix},$$

and initial probability vector (β, β_{m+1}), where β is a row m-vector. We assume that the states $\{1, \cdots, m\}$ are all transient and the state $\{m+1\}$ is an absorbing state. Throughout this thesis

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$\mathbf{1}$ denotes a column vector with all components equal to one. The dimension of $\mathbf{1}$ is determined by the context. The matrix A of order m is non-singular with negative diagonal entries and non-negative off-diagonal entries and satisfies $-A\mathbf{1} = \boldsymbol{a} \geq 0$. The vector $\boldsymbol{a}$ is called the absorption vector. The distribution $H(\cdot)$ is given by

$$H(t) = 1 - \alpha \exp(At)\mathbf{1}, t \geq 0.$$

It will be denoted that $H(\cdot)$ follows a PH(α, A) distribution.

2.2 Assumptions of the System

Consider a system subject to shocks which occur according to a non-homogeneous pure birth process (denoted by NHPBP). As a shock occurs, the system enters one of two types of failure:

(i) type-I failure (minor failure), which is removed by a repair.

(ii) type-II failure (catastrophic failure), which is removed by an unplanned replacement.

Let $\{s_k\}_{k=1}^{\infty}$ be the sequence of the failure type at every shock since the last replacement, defined by $s_k \in \{1, 2\}, \forall k \in \mathbb{N}$, where 1 represents the type-I failure and 2 represents the type-II failure. The sample space is denoted by

$$\Omega = \{s \mid s \equiv \{s_1, s_2, \cdots\}, s_k \in \{1, 2\}, \forall k \in \mathbb{N}\}.$$

Let $M : \Omega \to \mathbb{N}$ be the number of shocks until the first type-II failure since the last replacement, which is a random variable defined by

$$M(s) = \min\{k \in \mathbb{N} \mid s_k = 2\}, \forall s \in \Omega.$$

Now, we define the probability of survival of the system. For all $k \geq 1$, the kth shock carries out either the type-I failure with probability q_k or the type-II failure with probability $\theta_k = 1 - q_k$. Note $q_k = \Pr\{s_k = 1\}$ and $\theta_k = \Pr\{s_k = 2\}, \forall k \geq 1$. Let p_k be the probability that the system breakdown when the kth cumulated shocks occurs for all $k \geq 1$. Note $p_k = \Pr\{M = k\}$,

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$\forall k \geq 1$. The p_k is defined by

$$p_k = \left(\prod_{i=1}^{k-1} q_i \right) \theta_k, \; k > 1, \tag{2.1}$$

and $p_1 = \theta_1$. Note $\{p_k\}_{k=1}^{\infty}$ is a discrete probability distribution and $\sum_{k=1}^{\infty} p_k = 1$. The survival function $\overline{P}_k$ of M is defined by

$$\overline{P}_k = \Pr\{M > k\} = \prod_{i=1}^{k} q_i, \; k \geq 1, \tag{2.2}$$

which is the probability that the first k cumulated shocks carry out type-I failures. Therefore we have $\overline{P}_{k+1} = q_{k+1}\overline{P}_k$.

The system update strategy is the age replacement policy. There are two types of replacements:

(i) unplanned replacement, which is caused by type-II failure.

(ii) planned replacement, which occurs when the system reaches age T.

Therefore, the system is replaced at any type-II failure or at age T. A replacement cycle is the time interval between two consecutive replacements.

The cost of unplanned (due to type-II failure) and planned (due to planned replacement time) replacement is given by R_1 and R_2. We denote by $c_k(t)$, the cost of the kth repair at time t, and denote by $r_k(t) = E[c_k(t)]$, the expected cost of the kth repair at time t for all $k \geq 1$. Let $m_k(t)$ be the cost per unit time of maintenance of the system at time t and the cumulated shocks is k for all $k \geq 0$.

The system satisfies the following conditions:

(1) The system is monitored continuously and failures are detected immediately.

(2) Repairs and replacements are completed instantaneously.

(3) The system becomes new after a replacement, i.e., $N(t) = 0$.

(4) We assume that M is independent of the shock process $\{N(t) : t \geq 0\}$.

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2.3 Lifetime of the System

Let $X^{(k)}$ be the inter-arrival time between the kth and the $(k+1)$th shocks, for all $k \geq 0$, where $X^{(0)}$ is the time until the arrival of the first shock. These inter-arrival times follow PH-distributions represented by PH$(\beta^{(k)}, A^{(k)})$, for all $k \geq 0$ of order n_k. We focus on a special PH-distribution that is a hypo-exponential distribution and the intensity matrix is

$$A^{(k)} = \begin{bmatrix} -\alpha_1^{(k)} & \alpha_1^{(k)} & & \\ & -\alpha_2^{(k)} & \alpha_2^{(k)} & \\ & & \ddots & \ddots \\ & & & -\alpha_{n_k}^{(k)} \end{bmatrix}.$$

Let $T^{(k)}$ be the time point of the occurrence of the kth shock, which is defined by

$$T^{(k)} = \sum_{i=0}^{k-1} X^{(i)},\ k \geq 1.$$

Let $T^{(0)}$ be the initial time of the system, clearly $T^{(0)} = 0$. These random variables follow the PH-distributions which are represented by PH$(g^{(k)}, G^{(k)})$, for all $k \geq 1$. The random variable $T^{(k)}$ is the convolution of $X^{(0)}, X^{(1)}, \cdots, X^{(k-1)}$, thus the matrix $G^{(k)}$ is

$$G^{(k)} = \begin{bmatrix} A^{(0)} & \boldsymbol{a}^{(0)}\beta^{(1)} & & \\ & A^{(1)} & \boldsymbol{a}^{(1)}\beta^{(2)} & \\ & & \ddots & \ddots \\ & & & A^{(k-1)} \end{bmatrix},\ k \geq 1, \tag{2.3}$$

and the initial vector is given by $g^{(k)} = (\beta^{(0)}, 0, \cdots, 0),\ k \geq 1$. The cumulative distributions of these $T^{(k)}$ are

$$H_k(t) = \Pr\{T^{(k)} \leq t\} = 1 - g^{(k)} \exp(G^{(k)}t)\mathbf{1},\ k \geq 1. \tag{2.4}$$

Denote by $h_k(t)$ the probability density function of $T^{(k)}$.

Let T_s be the lifetime (natural death) of the system. The distribution of T_s is a mixture of PH-distributions $H_k(t)$ for all $k \geq 1$, which is represented by PH(v_s, V_s) with

$$v_s = (p_1 g^{(1)}, p_2 g^{(2)}, \cdots), \quad V_s = \begin{bmatrix} G^{(1)} & & \\ & G^{(2)} & \\ & & \ddots \end{bmatrix},$$

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and the cumulative distribution of T_s is

$$H_s(t) = \Pr\{T_s \leq t\} = 1 - v_s \exp(V_s t)\mathbf{1} = 1 - \sum_{k=1}^{\infty} p_k g^{(k)} \exp(G^{(k)} t)\mathbf{1}. \tag{2.5}$$

Denote by $h_s(t)$ the probability density function of T_s.

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Chapter 3

The Stability of the System

In Chapter 2, we define a system with PH-distribution subject to NHPBP. The system has two types of failures and becomes new after a replacement. Now we consider the case that the system without planned replacement, i.e., $T = \infty$. Then the system is replaced only due to type-II failure. When we look at this system for a long time, we will find that the system breaks naturally and becomes new over and over again.

We consider a system subject to NHPBP which is a recurrent Markov process. We will give the transition rate matrix of the system and find the stationary probability of it.

3.1 The Stationary Probability

Under the assumptions in Chapter 2, we denote by $X^{(k)}$ the inter-arrival time between the kth and the $(k+1)$th shocks, for all $k \geq 0$. These inter-arrival time which follow special PH-distributions, given by hypo-exponential distributions, represented by PH$(\beta^{(k)}, A^{(k)})$, for all $k \geq 0$ of order n_k. We may assume that $\beta^{(k)} = (1, 0, \cdots, 0)$, for all $k \geq 0$, i.e., $X^{(k)}$ start at the first phase. We may assume that $n_k = m$, for all $k \geq 0$, for some $m \in \mathbb{N}$, i.e., $X^{(k)}$ have the same number of phases.

The system state is the cumulated shocks k and begin at $k = 0$. For all $k \geq 0$, if the next shock is type-I failure, then the state becomes to $k + 1$. If the next shock is type-II failure, then

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the state becomes to 0. Therefore the transition rate matrix Q is

$$Q = \begin{bmatrix} A^{(0)} + A_0^{(0)} & A_1^{(0)} & & & \\ A_0^{(1)} & A^{(1)} & A_1^{(1)} & & \\ A_0^{(2)} & & A^{(2)} & A_1^{(2)} & \\ \vdots & & & \ddots & \ddots \end{bmatrix}, \tag{3.1}$$

where

$$A_0^{(k)} = \theta_{k+1}\boldsymbol{a}^{(k)}\beta^{(0)} \text{ and } A_1^{(k)} = q_{k+1}\boldsymbol{a}^{(k)}\beta^{(k+1)}, \forall k \geq 0.$$

The matrix $A_0^{(k)}$ means that the system state is k and the next shock is type-II failure, so the system is replaced by a new one and the state becomes to 0. The matrix $A_1^{(k)}$ means that the system state is k and the next shock is type-I failure, so the state becomes to $k+1$.

Let $\pi = (\pi_0, \pi_1, \pi_2, \cdots)$ be the stationary probability of Q, where $\pi_k = (\pi_{k1}, \pi_{k2}, \cdots, \pi_{km})$ for all $k \geq 0$. Since $\pi Q = \mathbf{0}$, we have

$$\begin{cases} \pi_0 A^{(0)} + \sum_{k=0}^{\infty} \pi_k A_0^{(k)} = \mathbf{0}, & (3.2a) \\ \pi_k A_1^{(k)} + \pi_{k+1} A^{(k+1)} = \mathbf{0}, \forall k \geq 0. & (3.2b) \end{cases}$$

From equations (3.2a) and (3.2b), we get

$$\pi_{km} q_{k+1} \alpha_m^{(k)} - \pi_{k+1,1} \alpha_1^{(k+1)} = 0, \forall k \geq 0. \tag{3.3}$$

and

$$\begin{cases} \pi_{k,1}\alpha_1^{(k)} - \pi_{k,2}\alpha_2^{(k)} = 0, \forall k \geq 0, \\ \pi_{k,2}\alpha_2^{(k)} - \pi_{k,3}\alpha_3^{(k)} = 0, \forall k \geq 0, \\ \vdots \\ \pi_{k,m-1}\alpha_{m-1}^{(k)} - \pi_{k,m}\alpha_m^{(k)} = 0, \forall k \geq 0. \end{cases} \tag{3.4}$$

From equation (3.4), we get

$$\pi_{kj} = \frac{\alpha_1^{(k)}}{\alpha_j^{(k)}}\pi_{k1}, \forall k \geq 0 \quad \text{and} \quad \pi_k = \pi_{k1}\left(1, \frac{\alpha_1^{(k)}}{\alpha_2^{(k)}}, \frac{\alpha_1^{(k)}}{\alpha_3^{(k)}}, \cdots, \frac{\alpha_1^{(k)}}{\alpha_m^{(k)}}\right), \forall k \geq 0. \tag{3.5}$$

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From equations (3.3) and (3.5), we have

$$\pi_{k1} = \pi_{k-1,m} q_k \frac{\alpha_m^{(k-1)}}{\alpha_1^{(k)}} = \pi_{k-1,1} q_k \frac{\alpha_1^{(k-1)}}{\alpha_1^{(k)}}, \forall k \geq 1.$$

Thus, by Mathematical Induction, we get

$$\pi_{k1} = \pi_{01} \frac{\alpha_1^{(0)}}{\alpha_1^{(k)}} \left(\prod_{i=1}^{k} q_i \right), \forall k \geq 1. \tag{3.6}$$

From equations (3.5) and (3.6), we have

$$\begin{aligned}
1 &= \sum_{k=0}^{\infty} \sum_{j=1}^{m} \pi_{kj} \\
&= \pi_{01} \left(1 + \frac{\alpha_1^{(0)}}{\alpha_2^{(0)}} + \frac{\alpha_1^{(0)}}{\alpha_3^{(0)}} + \cdots + \frac{\alpha_1^{(0)}}{\alpha_m^{(0)}} \right) + \sum_{k=1}^{\infty} \pi_{k1} \left(1 + \frac{\alpha_1^{(k)}}{\alpha_2^{(k)}} + \frac{\alpha_1^{(k)}}{\alpha_3^{(k)}} + \cdots + \frac{\alpha_1^{(k)}}{\alpha_m^{(k)}} \right) \\
&= \pi_{01} \alpha_1^{(0)} b_0 + \pi_{01} \alpha_1^{(0)} \sum_{k=1}^{\infty} b_k \left(\prod_{i=1}^{k} q_i \right) \\
&= \pi_{01} \left[\alpha_1^{(0)} b_0 + \alpha_1^{(0)} \sum_{k=1}^{\infty} b_k \left(\prod_{i=1}^{k} q_i \right) \right],
\end{aligned}$$

where

$$b_k = \frac{1}{\alpha_1^{(k)}} + \frac{1}{\alpha_2^{(k)}} + \cdots + \frac{1}{\alpha_m^{(k)}}, \forall k \geq 0. \tag{3.7}$$

Hence

$$\pi_{01} = \left[\alpha_1^{(0)} b_0 + \alpha_1^{(0)} \sum_{k=1}^{\infty} b_k \left(\prod_{i=1}^{k} q_i \right) \right]^{-1}. \tag{3.8}$$

3.2 The Conditions of the Existence of Stationary Probability

In this chapter, we will find the condition such that the stationary probability exists. In Chapter 3.1, we find the solution of π_{01}. From equation (3.8), we only need to find the condition such that the following series exists:

$$\sum_{k=1}^{\infty} \left(\frac{1}{\alpha_1^{(k)}} + \frac{1}{\alpha_2^{(k)}} + \cdots + \frac{1}{\alpha_m^{(k)}} \right) \left(\prod_{i=1}^{k} q_i \right) \tag{3.9}$$

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The following theorem gives the sufficient conditions for the existence of the series (3.9).

Theorem 1. The series (3.9) is convergent under the the assumption that either
(i) the limit $\lim_{k\to\infty} \frac{\alpha_j^{(k)}}{\alpha_j^{(k+1)}}$ exists and less than one, for all $j = 1, 2, \cdots, m$, or
(ii) the sequence $\{q_i\}$ is decreasing with $q_{n_0} < 1$ for some $n_0 \geq 1$, and the following sets $\{\alpha_1^{(k)} | k \geq 0\}, \{\alpha_2^{(k)} | k \geq 0\}, \cdots, \{\alpha_m^{(k)} | k \geq 0\}$ are bounded below by a positive real number b.

Proof. (i) Since q_i is probability, so $q_i \leq 1, \forall i \geq 1$. Thus $\prod_{i=1}^{k} q_i \leq 1$. Therefore

$$\sum_{k=1}^{\infty} \left(\frac{1}{\alpha_1^{(k)}} + \frac{1}{\alpha_2^{(k)}} + \cdots + \frac{1}{\alpha_m^{(k)}} \right) \left(\prod_{i=1}^{k} q_i \right) \leq \sum_{k=1}^{\infty} \left(\frac{1}{\alpha_1^{(k)}} + \frac{1}{\alpha_2^{(k)}} + \cdots + \frac{1}{\alpha_m^{(k)}} \right),$$

and we only need to show the right hand side is convergent.

For each $1 \leq j \leq m$, $\{1/\alpha_j^{(k)}\}_{k=1}^{\infty}$ is a sequence and $\alpha_j^k > 0, \forall k \geq 1$. By hypothesis, we have

$$\lim_{k\to\infty} \left| \frac{\alpha_j^{(k)}}{\alpha_j^{(k+1)}} \right| = \lim_{k\to\infty} \frac{\alpha_j^{(k)}}{\alpha_j^{(k+1)}} < 1, \forall 1 \leq j \leq m.$$

Therefore by Ratio Test, we have $\sum_{k=1}^{\infty} \frac{1}{\alpha_j^{(k)}}$ converges absolutely, $\forall j = 1, 2, \cdots, m$. Then

$$\sum_{k=1}^{\infty} \left| \frac{1}{\alpha_1^{(k)}} + \frac{1}{\alpha_2^{(k)}} + \cdots + \frac{1}{\alpha_m^{(k)}} \right| \leq \sum_{k=1}^{\infty} \left| \frac{1}{\alpha_1^{(k)}} \right| + \sum_{k=1}^{\infty} \left| \frac{1}{\alpha_2^{(k)}} \right| + \cdots + \sum_{k=1}^{\infty} \left| \frac{1}{\alpha_m^{(k)}} \right|$$

is convergent. Therefore the series

$$\sum_{k=1}^{\infty} \left(\frac{1}{\alpha_1^{(k)}} + \frac{1}{\alpha_2^{(k)}} + \cdots + \frac{1}{\alpha_m^{(k)}} \right)$$

is absolutely convergent and also is convergent. Hence the series (3.9) is converge.

(ii) By hypothesis, we have $\{\alpha_j^{(k)} | k \geq 0\}$ is bounded below, $\forall j = 1, 2, \cdots, m$, by a positive number b. So for each $j = 1, 2, \cdots, m$, we have $b \leq \alpha_j^{(k)}, k \geq 0$. Then for each $j = 1, 2, \cdots, m$, we have $\frac{1}{\alpha_j^{(k)}} \leq \frac{1}{b}, \forall k \geq 0$. Thus

$$\frac{1}{\alpha_1^{(k)}} + \frac{1}{\alpha_2^{(k)}} + \cdots + \frac{1}{\alpha_m^{(k)}} \leq \frac{m}{b}, \forall k \geq 0.$$

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Therefore the set $\{\frac{1}{\alpha_1^{(k)}} + \frac{1}{\alpha_2^{(k)}} + \cdots + \frac{1}{\alpha_m^{(k)}} | k \geq 0\}$ is bounded above by $\dfrac{m}{b}$.

Now, we have

$$\sum_{k=1}^{\infty} \left(\frac{1}{\alpha_1^{(k)}} + \frac{1}{\alpha_2^{(k)}} + \cdots + \frac{1}{\alpha_m^{(k)}} \right) \left(\prod_{i=1}^{k} q_i \right) \leq \frac{m}{b} \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k} q_i \right),$$

and we only need to show that the series $\sum\limits_{k=1}^{\infty} \left(\prod\limits_{i=1}^{k} q_i \right)$ is convergent.

By hypothesis, there is an $n_0 \geq 1$ such that $q_{n_0} < 1$, and we have

$$\sum_{k=1}^{\infty} \left(\prod_{i=1}^{k} q_i \right) = \sum_{k=1}^{n_0-1} \left(\prod_{i=1}^{k} q_i \right) + \sum_{k=n_0}^{\infty} \left(\prod_{i=1}^{k} q_i \right).$$

It is sufficient to show the series $\sum\limits_{k=n_0}^{\infty} \left(\prod\limits_{i=1}^{k} q_i \right)$ is convergent.

Since the sequence $\{q_i\}_{i=1}^{\infty}$ is decreasing and $q_i \leq 1, \forall i \geq 1$, we have

$$\sum_{k=n_0}^{\infty} \left(\prod_{i=1}^{k} q_i \right) = \left(\prod_{i=1}^{n_0-1} q_i \right) \sum_{k=n_0}^{\infty} \left(\prod_{i=n_0}^{k} q_i \right) \leq \left(\prod_{i=1}^{n_0-1} q_i \right) \sum_{k=1}^{\infty} q_{n_0}^k \leq \sum_{k=1}^{\infty} q_{n_0}^k.$$

The right hand side is a geometric series with common ratio less than 1, thus it is convergent. Therefore the series $\sum\limits_{k=n_0}^{\infty} \left(\prod\limits_{i=1}^{k} q_i \right)$ is convergent. Hence the series (3.9) is converge. $\square$

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Chapter 4

Age Replacement Policy

The $P_k(t)$ be the transition probability of the system at time t given $N(0) = 0$, write

$$P_k(t) \equiv \Pr\{N(t) = k \mid N(0) = 0\},$$

which can be defined by

$$P_k(t) = \begin{cases} 1 - H_1(t), & \text{if } k = 0, \\ H_k(t) - H_{k+1}(t), & \text{if } k \geq 1. \end{cases} \tag{4.1}$$

Since $H_k(t)$ is the cumulative distribution of $T^{(k)}$, we have

$$H_k(t) = \Pr\{T^{(k)} \leq t\} = \Pr\{N(t) \geq k\} = \sum_{i=k}^{\infty} P_i(t).$$

Hence $P_k(t) = H_k(t) - H_{k+1}(t)$ for all $k \geq 1$.

Now, we will find the relationship between $P_k(t)$ and the probability density function of $T^{(k+1)}$. One of the conditions of a NHPBP is

$$\Pr\{N(t+h) - N(t) = 1 | N(t) = k\} = \lambda_k(t)h + o(h),$$

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where $\lambda_k(t)$ is called the intensity function. Then we have

$$\begin{aligned}\Pr\{N(t+h)-N(t)=1|N(t)=k\} &= \frac{\Pr\{N(t+h)-N(t)=1 \text{ and } N(t)=k\}}{\Pr\{N(t)=k\}} \\ &= \frac{\Pr\{t<T^{(k+1)}\leq t+h\}}{\Pr\{N(t)=k\}}\end{aligned}$$

We know that

$$\Pr\{t<T^{(k+1)}\leq t+h\} = \int_t^{t+h} h_{k+1}(s)ds = h_{k+1}(t)h + o(h)$$

where $h_{k+1}(t)$ is the probability density function of $T^{(k+1)}$. Therefore it gives

$$\lambda_k(t) + \frac{o(h)}{h} = \frac{h_{k+1}(t)}{\Pr\{N(t)=k\}}$$

Taking $h \to \infty$ we have

$$\lambda_k(t) = \frac{h_{k+1}(t)}{P_k(t)}, \tag{4.2}$$

which can be rewritten by

$$h_{k+1}(t) = \lambda_k(t)P_k(t).$$

4.1 Expected Cost Functions

Let $H_s(t)$ be the cumulative distribution of T_s we have defined before, $\overline{H}_s(t) = 1 - H_s(t)$ be its survival function, and $h_s(t)$ be its density function.

Let T be the planned replacement age. Consider the lifetime of system T_s and the planned replacement age T together. Let $T_s^* = \min\{T_s, T\}$ be the length of a replacement cycle. Then the expected length of a replacement cycle is given by

$$L(T) = E[T_s^*] = E[\min\{T_s, T\}] = \int_0^T t \cdot h_s(t)dt + T \cdot \overline{H}_s(T) = \int_0^T \overline{H}_s(t)dt.$$

In Section 2.2, we have defined the parameters of cost. The cost of unplanned and planned replacement is given by R_1 and R_2. Denote by $r_k(t)$, the expected cost of the kth repair at time t for $k \geq 1$. Denote by $m_k(t)$, the cost per unit time of maintenance of the system at time $t \in [T^{(k)}, T^{(k+1)})$ for all $k \geq 0$ (note $T^{(0)} = 0$).

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Now, we can defined the cost function. Let W be the all costs over a replacement cycle T_s^* which is defined by

$$W = R_2\mathbf{I}_{[T_s>T]} + R_1\mathbf{I}_{[T_s\leq T]} + \sum_{k=1}^{\infty} c_k(T^{(k)})\mathbf{I}_{[M>k]}\mathbf{I}_{[T^{(k)}\leq T]} + \int_0^T m_{N(t)}(t)\mathbf{I}_{[M>N(t)]}dt.$$

Therefore, the expected cost over a replacement cycle is given by

$$C(T) = E[W] = R_2\overline{H}_s(T) + R_1H_s(T) + \sum_{k=1}^{\infty}\int_0^T r_k(t)\overline{P}_k h_k(t)dt + \int_0^T \sum_{k=0}^{\infty} m_k(t)\overline{P}_k P_k(t)dt$$

From (4.2), we have shown that $h_{k+1}(t) = \lambda_k(t)P_k(t)$. Thus, we have

$$\begin{aligned} C(T) &= R_2\overline{H}_s(T) + R_1H_s(T) + \sum_{k=1}^{\infty}\int_0^T r_k(t)\overline{P}_k\lambda_{k-1}(t)P_{k-1}(t)dt + \int_0^T \sum_{k=0}^{\infty} m_k(t)\overline{P}_k P_k(t)dt \\ &= R_2 + (R_1 - R_2)H_s(T) + \int_0^T \sum_{k=0}^{\infty} r_{k+1}(t)\overline{P}_{k+1}\lambda_k(t)P_k(t)dt + \int_0^T \sum_{k=0}^{\infty} m_k(t)\overline{P}_k P_k(t)dt \end{aligned}$$

In Chapter 2.2, we have shown that $\overline{P}_{k+1} = q_{k+1}\overline{P}_k$. Thus, it yields

$$C(T) = R_2 + (R_1 - R_2)H_s(T) + \int_0^T \sum_{k=0}^{\infty} \left[r_{k+1}(t)q_{k+1}\lambda_k(t) + m_k(t)\right] P_k(t)\overline{P}_k dt$$

The expected cost rate of a replacement cycle is given by $J_C(T) = \dfrac{C(T)}{L(T)}$, that is

$$J_C(T) = \frac{R_2 + (R_1 - R_2)H_s(T) + \int_0^T \sum_{k=0}^{\infty} \left[r_{k+1}(t)q_{k+1}\lambda_k(t) + m_k(t)\right] P_k(t)\overline{P}_k dt}{\int_0^T \overline{H}_s(t)dt}. \quad (4.3)$$

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4.2 The Optimal Planned Replacement Age

We want to determine the optimal planned replacement time T^*, by using the first derivative test. Taking the first-order derivative of $J_C(T)$, we get

$$\begin{aligned} J_C'(T) = &\left\{ \left[(R_1 - R_2)h_s(T) + \sum_{k=0}^{\infty} \left[r_{k+1}(T) q_{k+1} \lambda_k(T) + m_k(T) \right] P_k(T) \overline{P}_k \right] \int_0^T \overline{H}_s(t) dt \right. \\ &\left. - \left[R_2 + (R_1 - R_2) H_s(T) + \int_0^T \sum_{k=0}^{\infty} \left[r_{k+1}(t) q_{k+1} \lambda_k(t) + m_k(t) \right] P_k(t) \overline{P}_k dt \right] \overline{H}_s(T) \right\} \\ &\times \left(\int_0^T \overline{H}_s(t) dt \right)^{-2}. \end{aligned}$$

Setting $J_C'(T) = 0$, we find the optimal condition for planned replacement time

$$\begin{aligned} &\varphi_C(T) \int_0^T \overline{H}_s(t) dt \\ &- \left[(R_1 - R_2) H_s(T) + \int_0^T \sum_{k=0}^{\infty} \left[r_{k+1}(t) q_{k+1} \lambda_k(t) + m_k(t) \right] P_k(t) \overline{P}_k dt \right] = R_2, \end{aligned} \tag{4.4}$$

where denoting

$$\varphi_C(T) = \frac{1}{\overline{H}_s(T)} \left[(R_1 - R_2) h_s(T) + \sum_{k=0}^{\infty} \left[r_{k+1}(T) q_{k+1} \lambda_k(T) + m_k(T) \right] P_k(T) \overline{P}_k \right]. \tag{4.5}$$

In order to find the optimal T^*, we consider the relationship between $\varphi_C(T)$ and $J_C(T)$. Taking $T \to \infty$, we have

$$\lim_{T \to \infty} J_C(T) = \frac{1}{\int_0^{\infty} \overline{H}_s(t) dt} \left[R_1 + \int_0^{\infty} \sum_{k=0}^{\infty} \left[r_{k+1}(t) q_{k+1} \lambda_k(t) + m_k(t) \right] P_k(t) \overline{P}_k dt \right].$$

Now, we take a theorem to check weather a problem have a finite and unique T^*.

Theorem 2. (Sheu et al. [13]) Assume that

(1) $r_{k+1}(t) q_{k+1} \lambda_k(t) + m_k(t)$ is non-decreasing in (k, t) and $r_{k+1}(t) q_{k+1} \lambda_k(t) + m_k(t) \to c_R$ uniformly as $t \to \infty$.

(2) $\lambda_k(t)$ is increasing in (k, t).

(3) $\overline{P}_k$ is a discrete increasing failure rate (IFR) function.

If $\lim_{T \to \infty} \varphi_C(T) > \lim_{T \to \infty} J_C(T)$, then there exists a finite and unique T^* which minimizes $J_C(T)$

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and such that $\varphi_C(T^*) = J_C(T^*)$. Otherwise, the optimal age replacement policy is $T^* = \infty$, i.e., there is no planned replacement.

Proof. In order to find the optimal planned replacement time T^*, we take the first-order derivative of (4.3) and set it equal to zero. Then we obtain the equation (4.4). Let $Q(T)$ be the left-hand side of (4.4), that is

$$\begin{aligned} Q(T) =& \varphi_C(T) \int_0^T \overline{H}_s(t)dt \\ & - \left[(R_1 - R_2)H_s(T) + \int_0^T \sum_{k=0}^{\infty} \left[r_{k+1}(t) q_{k+1} \lambda_k(t) + m_k(t) \right] P_k(t) \overline{P}_k dt \right]. \end{aligned} \tag{4.6}$$

If $\overline{P}_k$ is a discrete IFR and $\lambda_k(t)$ is increasing in (k, t), then $\overline{H}_s(T)$ is IFR, which is proved by Theorem 2.4 in A-Hameed and Proschan [2]. Thus, under assumptions (1)-(3), $\varphi_C(T)$ is increasing in T and $\varphi_C'(T) \geq 0$. Now, we prove that $Q(T)$ is also increasing in T. Taking the first-order derivative of $Q(T)$, we have

$$\begin{aligned} Q'(T) =& \varphi_C'(T) \int_0^T \overline{H}_s(t)dt + \varphi_C(T) \overline{H}_s(T) \\ & - \left[(R_1 - R_2)h_s(T) + \sum_{k=0}^{\infty} \left[r_{k+1}(T) q_{k+1} \lambda_k(T) + m_k(t) \right] P_k(T) \overline{P}_k \right] \\ =& \varphi_C'(T) \int_0^T \overline{H}_s(t)dt. \end{aligned}$$

Since $\int_0^T \overline{H}_s(t)dt \geq 0$ and $\varphi_C'(T) \geq 0$, then we can deduce that $Q'(T) \geq 0$. Hence $Q(T)$ is increasing.

Assume $\lim_{T\to\infty} \varphi_C(T) > \lim_{T\to\infty} J_C(T)$, then we have

$$\begin{aligned} \lim_{T\to\infty} Q(T) =& \lim_{T\to\infty} \varphi_C(T) \int_0^\infty \overline{H}_s(t)dt \\ & - \left[(R_1 - R_2) + \int_0^\infty \sum_{k=0}^{\infty} \left[r_{k+1}(t) q_{k+1} \lambda_k(t) + m_k(t) \right] P_k(t) \overline{P}_k dt \right] \\ >& \lim_{T\to\infty} J_C(T) \int_0^\infty \overline{H}_s(t)dt \\ & - \left[(R_1 - R_2) + \int_0^\infty \sum_{k=0}^{\infty} \left[r_{k+1}(t) q_{k+1} \lambda_k(t) + m_k(t) \right] P_k(t) \overline{P}_k dt \right] \\ =& R_2. \end{aligned}$$

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Therefore $\lim_{T\to\infty} Q(T) > R_2$.

Since $Q(0) = 0 < R_2 < \lim_{T\to\infty} Q(T)$, there exists a finite and unique T^* (i.e., $0 < T^* < \infty$) such that $Q(T^*) = R_2$. The optimal planned replacement age T^* minimizes $J_C(T)$.

Finally, we prove that $\varphi_C(T^*) = J_C(T^*)$. Since $Q(T^*) = R_2$, we have

$$\varphi_C(T^*) \int_0^{T^*} \overline{H}_s(t)dt$$
$$- \left[(R_1 - R_2)H_s(T^*) + \int_0^{T^*} \sum_{k=0}^{\infty} \left[r_{k+1}(t) q_{k+1} \lambda_k(t) + m_k(t) \right] P_k(t) \overline{P}_k dt \right] = R_2.$$

Then add the second term to the right-hand side and divide it by $\int_0^{T^*} \overline{H}_s(t)dt$, we get $J_C(T^*)$. Hence $\varphi_C(T^*) = J_C(T^*)$. □

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Chapter 5

Algorithmic Computation

Before starting this chapter, we list the notations defined in previous chapters as follows.

Notation.

M	the number of shocks until the first type-II failure since the last replacement;
q_k	the probability that the kth carries out the type-I failure, $\forall k \geq 1$;
θ_k	the probability that the kth carries out the type-II failure, $\forall k \geq 1$, $\theta_k = 1 - q_k$;
p_k	the probability that the system breakdown when the kth cumulated shocks occurs, $p_k = \Pr\{M = k\}$, $k \geq 1$;
$\overline{P}_k$	the survival function of M, $\overline{P}_k = \Pr\{M > k\}$, $\forall k \geq 0$;
$X^{(k)}$	the inter-arrival time between any consecutive shocks, $X^{(k)} \sim \mathrm{PH}(\beta^{(k)}, A^{(k)})$, $\forall k \geq 0$;
$T^{(k)}$	the time point of the occurrence of the kth shock, $T^{(k)} \sim \mathrm{PH}(g^{(k)}, G^{(k)})$, $\forall k \geq 1$;
T_s	the lifetime (natural death) of the system, $T_s \sim \mathrm{PH}(v_s, V_s)$;
$H_k(t)$	the cumulated distribution function of $T^{(k)}$, $\forall k \geq 1$;
$h_k(t)$	the probability density function of $T^{(k)}$, $\forall k \geq 1$;
$H_s(t)$	the cumulated distribution function of T_s;
$h_s(t)$	the probability density function of T_s;
$\overline{H}_s(t)$	the survival function of T_s;
$P_k(t)$	the transition probability of the system at time t given $N(0) = 0$, $\forall k \geq 0$;
$\lambda_k(t)$	the intensity function of the system at time t when the cumulated shocks is k, $\forall k \geq 0$;

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Notation. (Continue)

R_1	the cost of unplanned replacement;
R_2	the cost of planned replacement, $R_2 \leq R_1$;
$r_k(t)$	the expected cost of the kth repair at time t, $\forall k \geq 1$;
$m_k(t)$	the cost per unit time of maintenance at time t when the cumulated shocks is k, $\forall k \geq 0$;
$C(T)$	the expected cost over a replacement cycle;
$L(T)$	the expected length of a replacement cycle;
$J_C(T)$	the expected cost rate of a replacement cycle, $J_C(T) = \frac{C(T)}{L(T)}$;
$\varphi_C(T)$	defined in equation (4.5);
$Q(T)$	defined in equation (4.6);
T^*	the optimal planned replacement time;

In this chapter, we give an efficient algorithm to calculate the optimal planned replacement age. We define a shock model whose inter-arrival times between any consecutive shocks $X^{(k)}$ follow PH-distributions. Then the sequence $\{T^{(k)}\}_{k=1}^{\infty}$ and T_s all follow PH-distributions. Now, Compute $H_s(t)$ by $\sum_{k=1}^{\infty} p_k g^{(k)} \exp(G^{(k)}t)\mathbf{1}$ is more convenient than by $v_s \exp(V_s t)\mathbf{1}$. Hence our algorithm is efficient. By definition, the computation of CDF of a PH-distribution only involves in matrix computation, especially is of the special type matrix exponential, refer to equations (2.4) and (2.5). Let X be an $n \times n$ real matrix. The exponential of X is denote by $\exp(X)$ which is given by the following series

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

Therefore the computation of CDF of a PH-distribution essentially is matrix multiplication.

In Sheu et al. [13], it starts from general intensity functions $\lambda_k(t)$ to express the shock model. To determine the optimal age T^*, we must perform through complicated computation to get survival function of the lifetime of system T_s. Also the PDFs $\{h_k(t)\}_{k=1}^{\infty}$ of $\{T^{(k)}\}_{k=1}^{\infty}$ are not easy to be expressed under this definition.

We abandon the general definition of the shock model and define the distribution of inter-arrival times between any consecutive shocks directly. Therefore, we can easily and efficiently compute the CDFs and PDFs of $T^{(k)}$ and T_s.

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5.1 The Structure of the Algorithm

The input of our algorithm only needs $\{q_k\}_{k=1}^{\infty}$ and $\{X^{(k)}\}_{k=0}^{\infty}$. The parameters are R_1, R_2, $\{r_k(t)\}_{k=1}^{\infty}$, and $\{m_k(t)\}_{k=0}^{\infty}$, which are defined as individual cost. We assume the sequence $\{q_k\}_{k=1}^{\infty}$ is decreasing with $q_{n_0} < 1$ for some $n_0 \geq 1$. We assume $\beta^{(k)} = (1, 0, \cdots, 0)$ and $A^{(k)}$ is of order m for all $k \geq 0$. The goal is to calculate the optimal planned replacement age T^* for the system with PH-distribution under upper triangle intensity matrix.

We divide the algorithm into two parts: the first part is used to compute the basic elements of the system; the second part is used to compute the the optimal planned replacement age T^* and the optimal expected cost rate $J_C(T^*)$. The Basic elements include: $\{q_k\}_{k=1}^{\infty}$, $\{\theta_k\}_{k=1}^{\infty}$, $\{p_k\}_{k=1}^{\infty}$, $\{\overline{P}_k\}_{k=0}^{\infty}$, $\{H_k(t)\}_{k=1}^{\infty}$, $\{h_k(t)\}_{k=1}^{\infty}$, $H_s(t)$, $h_s(t)$, $\overline{H}_s(t)$, $\{P_k(t)\}_{k=0}^{\infty}$, and $\{\lambda_k\}_{k=0}^{\infty}$.

The steps of the first part are listed in the following:

Input: $\{q_k\}_{k=1}^{\infty}$ and $\{X^{(k)}\}_{k=0}^{\infty}$ in terms of $\{\beta^{(k)}\}_{k=0}^{\infty}$ and $\{A^{(k)}\}_{k=0}^{\infty}$

1. From $\{q_k\}_{k=1}^{\infty}$, compute $\{\theta_k\}_{k=1}^{\infty}$, $\{p_k\}_{k=1}^{\infty}$ by equation (2.1) and $\{\overline{P}_k\}_{k=0}^{\infty}$ by equation (2.2).
2. From $\{\beta^{(k)}\}_{k=0}^{\infty}$ and $\{A^{(k)}\}_{k=0}^{\infty}$, compute $\{g^{(k)}\}_{k=1}^{\infty}$ and $\{G^{(k)}\}_{k=1}^{\infty}$ by equation (2.3).
3. Compute $\{H_k(t)\}_{k=1}^{\infty}$ and $\{h_k(t)\}_{k=1}^{\infty}$ by equation (2.4).
4. Compute $H_s(t)$ and $h_s(t)$ by equation (2.5). Then compute $\overline{H}_s(t) = 1 - H_s(t)$.
5. From $\{H_k(t)\}_{k=1}^{\infty}$, compute $\{P_k(t)\}_{k=0}^{\infty}$ by equation (4.1).

Output: $\{\overline{P}_k\}_{k=0}^{\infty}$, $\{h_k(t)\}_{k=1}^{\infty}$, $H_s(t)$, $h_s(t)$, $\overline{H}_s(t)$, and $\{P_k(t)\}_{k=0}^{\infty}$.

Since the computation of the second part involves $\lambda_k(t)$ (see equation 4.2), it has division form. However, when we compute $\lambda_k(t)$ as t is increasing and the error becomes larger and larger (see Figure 5.1). It is due to the numerical accuracy.

From equation (4.5), we rewrite it as

$$\varphi_C(T) = \frac{1}{\overline{H}_s(T)} \left[(R_1 - R_2) h_s(T) + c_r(T) + c_m(T) \right], \tag{5.1}$$

where

$$c_r(T) = \sum_{k=1}^{\infty} r_k(T) h_k(T) \overline{P}_k \quad \text{and} \quad c_m(T) = \sum_{k=0}^{\infty} m_k(T) P_k(T) \overline{P}_k.$$

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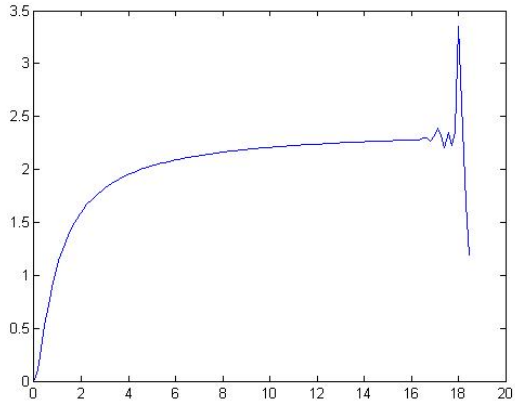

Figure 5.1: An example of $\lambda_k(t)$.

From equation (4.6), we have

$$Q(T) = \varphi_C(T)\int_0^T \overline{H}_s(t)dt - \left[(R_1 - R_2)H_s(T) + \int_0^T [c_r(t) + c_m(t)]dt\right]. \tag{5.2}$$

The steps of the second part are listed in the following:

Input: $\{q_k\}_{k=1}^{\infty}$, $\{\overline{P}_k\}_{k=0}^{\infty}$, $\{h_k(t)\}_{k=1}^{\infty}$, $H_s(t)$, $h_s(t)$, $\overline{H}_s(t)$, and $\{P_k(t)\}_{k=0}^{\infty}$.

Given: R_1, R_2, $\{r_k(t)\}_{k=1}^{\infty}$, and $\{m_k(t)\}_{k=0}^{\infty}$

1. Compute $\varphi_C(T)$ by equation (5.1).
2. Compute $Q(T)$ by equation (5.2).
3. Find the root of $Q(T) - R_2 = 0$ and denote it by T^*.
4. We have $J_C(T^*) = \varphi_C(T^*)$.

Output: T^* and $J_C(T^*)$.

5.2 Summary of the Algorithm

To implement our algorithm, we develop a Matlab tool (see Appendix A). We will use the codes which are described in Appendix A to execute the following two algorithms.

Given the minor failure probability sequence $\{q_k\}_{k=1}^{N}$, the initial vector sequence $\{\beta^{(k)}\}_{k=0}^{N-1}$, and the intensity matrix sequence $\{A^{(k)}\}_{k=0}^{N-1}$, for convenience, we ignore the indices of $\beta^{(k)}$ and $A^{(k)}$ to $k = 1, 2, \cdots, N$. Denote $q \equiv \{q_k\}_{k=1}^{N}$.

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First, we compute the basic elements of the system. Denote $\theta \equiv \{\theta_k\}_{k=1}^N$, $p \equiv \{p_k\}_{k=1}^N$, $\overline{P} \equiv \{\overline{P}_k\}_{k=1}^N$.

Algorithm 1: Compute the Basic Elements of the System

Input : q, $\{\beta^{(k)}\}_{k=1}^N$, $\{A^{(k)}\}_{k=1}^N$

$\theta = \text{MajorFailureProbSeq}(q)$

$p = \text{FailureDistribution}(q, \theta)$

$[\overline{P}_0, \overline{P}] = \text{SurvivalOfSystem}(q)$

$[\{g^{(k)}\}_{k=1}^N, \{G^{(k)}\}_{k=1}^N] = \text{Convolution}(\{\beta^{(k)}\}_{k=1}^N, \{A^{(k)}\}_{k=1}^N)$

$t = \text{TimeAxis}()$

$\{H_k(t)\}_{k=1}^N = \text{CdfSequencePH}(\{g^{(k)}\}_{k=1}^N, \{G^{(k)}\}_{k=1}^N, t)$

$\{h_k(t)\}_{k=1}^N = \text{PdfSequencePH}(\{g^{(k)}\}_{k=1}^N, \{G^{(k)}\}_{k=1}^N, t)$

$H_s(t) = \text{CdfMixturePH}(p, \{H_k(t)\}_{k=1}^N)$

$h_s(t) = \text{PdfMixturePH}(p, \{h_k(t)\}_{k=1}^N)$

$[P_0(t), \{P_k(t)\}_{k=1}^{N-1}] = \text{TransitionProbOfSystem}(\{H_k(t)\}_{k=1}^N)$

Output: $\overline{P}_0$, $\overline{P}$, $\{h_k(t)\}_{k=1}^N$, $H_s(t)$, $h_s(t)$, $P_0(t)$, $\{P_k(t)\}_{k=1}^{N-1}$

Finally, we run the optimal algorithm to get $Q(T)$ and $J_C(T)$.

Algorithm 2: Compute the Optimal Planned Replacement Age

Input : q, $\overline{P}_0$, $\overline{P}$, $\{h_k(t)\}_{k=1}^N$, $H_s(t)$, $h_s(t)$, $P_0(t)$, $\{P_k(t)\}_{k=1}^{N-1}$

Given : R_1, R_2, $\{r_k(T)\}_{k=1}^N$, $m_0(T)$, $\{m_k(T)\}_{k=1}^{N-1}$

$[C_r(T), c_r(T)] = \text{ExpectedRepairCost}(\{r_k(T)\}_{k=1}^N, \{h_k(T)\}_{k=1}^N, \overline{P})$

$[C_m(T), c_m(T)] = \text{EMCost}(m_0(T), \{m_k(T)\}_{k=1}^{N-1}, P_0(T), \{P_k(T)\}_{k=1}^{N-1}, \overline{P}_0, \overline{P})$

$\varphi_C(T) = \text{PhiC}(R_1, R_2, H_s(T), h_s(T), c_r(T), c_m(T))$

$Q(T) = \text{Qfun}(R_1, R_2, \varphi_C(T), H_s(T), C_r(T), C_m(T))$

$J_C(T) = \text{ExpectedCostRate}(R_1, R_2, H_s(T), C_r(T), C_m(T))$

Output: $Q(T)$, $J_C(T)$

Since the set $S = \{x \in \mathbb{R} \,|\, Q(x) = R_2\}$ may contain more than one element, we can not give an algorithm to compute all T^*. But one can find a T^* in the set S, since

$$J_C(T^*) \leq J_C(x), \text{ for all } x \in S.$$

Remark. As the symbol used in MATLAB for Algorithms 1 and 2, we need to define legal symbol in Matlab for these notation. For example, we can use `Jc` to define the symbol $J_C(T)$.

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Chapter 6

Numerical Examples

In this chapter, we give several examples of shock models with Erlang distribution, hypo-exponential distribution, Coxian distribution, hyper-Erlang distribution, and intensity matrices are upper-triangle matrices.

Example 1. Consider shocks with Erlang distributions which are represented by PH$(\beta^{(k)}, A^{(k)})$, for all $k \geq 0$ of order 3. Let $q_k = 0.8$ for all $k \geq 1$. Define $\beta^{(k)} = (1, 0, 0)$ and

$$A^{(k)} = \begin{bmatrix} -2.4 & 2.4 & 0 \\ 0 & -2.4 & 2.4 \\ 0 & 0 & -2.4 \end{bmatrix}, \text{ for all } k \geq 0.$$

Let $R_1 = 1500$ and $R_2 = 1000$. Define constant c_k by a randomly generated sequence, i.e.,

$$\begin{aligned} \{c_k\}_{k=1}^{\infty} = \{&1629.4, 1811.6, 254, 1826.8, 1264.7, 195.1, 557, 1093.8, 1915, 1929.8, \\ &315.2, 1941.2, 1914.3, 970.8, 1600.6, 283.8, 843.5, 1831.5, 1584.4, 1919, \\ &2000, 2000, \cdots\}, \end{aligned}$$

let $r_k(t) = c_k$ for all $k \geq 1$. Let $m_k(t) = 0.5k + 0.2$ for all $k \geq 0$.

We truncate the sequence at $N = 40$ and compute $T^* = 23.4234$ by our algorithm. The graphs of $Q(T)$ and $J_C(T)$ are shown at the Figure 6.1 and 6.2.

The proof of theorem 2 states that $Q(T)$ is increasing. However, in the above case, observe that $Q(T)$ is not increasing, but $J_C(T)$ still has minimum. This motivates us to find another theorem.

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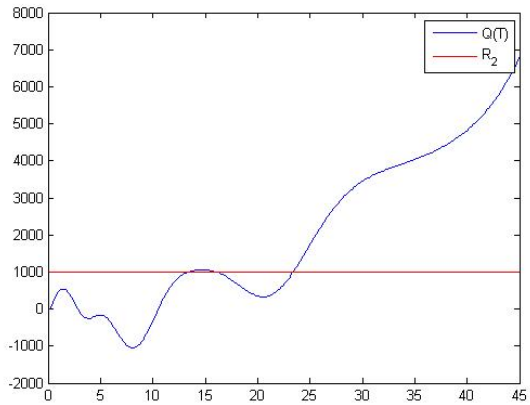

Figure 6.1: $Q(T)$ of an Erlang distribution

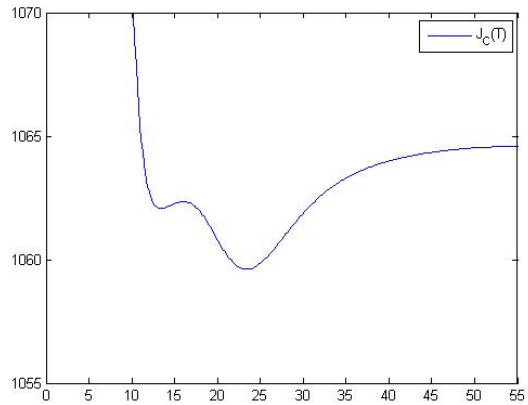

Figure 6.2: $J_C(T)$ of an Erlang distribution

Theorem 3. Assume that

(1) $r_{k+1}(T)$ and $m_k(T)$ are continuous and bounded above for all $k \geq 0$.

(2) $\{q_k\}_{k=1}^{\infty}$ is decreasing and there is some natural number n_0 such that $q_{n_0} < 1$.

(3) For all $\epsilon > 0$, there is a $T_\epsilon > 0$ such that for all $x \in [0, T_\epsilon)$, $\overline{H}_s(x) > \epsilon$.

Given an $\epsilon > 0$, if there is a number $u \in (0, T_\epsilon)$ such that $Q(u) > R_2$, then there is a $T^* \in (0, u)$ which minimizes $J_C(T)$ and $\varphi_C(T^*) = Q(T^*)$. Otherwise, the optimal age replacement policy is $T^* = \infty$, i.e., there is no planned replacement.

Proof. Given an $\epsilon > 0$, we will show that $Q(T)$ is continuous on $[0, T_\epsilon)$. Then by Intermediate Value Theorem, there is a real number $T^* \in (0, u)$ such that $Q(T^*) = R_2$ and minimizes $J_C(T)$, since we have $Q(0) = 0 < R_2 < Q(u)$. The proof of $\varphi_C(T^*) = J_C(T^*)$ is the same as in theorem 2.

Now, we prove that $Q(T)$ is continuous on $[0, T_\epsilon)$. First, we have

$$r_{k+1}(T)h_{k+1}(T)\overline{P}_{k+1} + m_k(T)P_k(T)\overline{P}_k$$

is continuous on $[0, \infty)$ and positive for all $k \geq 0$.

By assumption (1), we have $r_{k+1}(T)$ and $m_k(T)$ are continuous and bounded above for all $k \geq 0$. Suppose $r_{k+1}(T)$ and $m_k(T)$ are bounded above by B for all $k \geq 0$.

By assumption (2), we have $\sum_{k=n_0}^{\infty} \overline{P}_k = \sum_{k=n_0}^{\infty} \prod_{i=1}^{k} q_i \leq \sum_{k=n_0}^{\infty} q_{n_0}^k$. The right hand side is a geometric series with common ratio less than 1, thus $\sum_{k=1}^{\infty} \overline{P}_k$ converges. Therefore there is an

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$N_\epsilon \in \mathbb{N}$ such that

$$\sum_{k=N_\epsilon}^{\infty} \overline{P}_k < \frac{\epsilon}{2B}.$$

Since $h_{k+1}(T) \leq 1$ and $P_k(T) \leq 1$, for all $T \in [0, \infty)$ we have

$$r_{k+1}(T)h_{k+1}(T)\overline{P}_{k+1} + m_k(T)P_k(T)\overline{P}_k \leq r_{k+1}(T)\overline{P}_{k+1} + m_k(T)\overline{P}_k, \forall k \geq 0.$$

Since $\overline{P}_{k+1} = q_{k+1}\overline{P}_k$, $\{\overline{P}_k\}_{k=0}^{\infty}$ is decreasing. Therefore for all $T \in [0, \infty)$ we have

$$r_{k+1}(T)\overline{P}_{k+1} + m_k(T)\overline{P}_k \leq [r_{k+1}(T) + m_k(T)]\overline{P}_k, \forall k \geq 0.$$

Let

$$s_n(T) = \sum_{k=0}^{n} \left[r_{k+1}(T)h_{k+1}(T)\overline{P}_{k+1} + m_k(T)P_k(T)\overline{P}_k\right],$$

and

$$f(T) = \sum_{k=0}^{\infty} \left[r_{k+1}(T)h_{k+1}(T)\overline{P}_{k+1} + m_k(T)P_k(T)\overline{P}_k\right].$$

From the above results, for all $n \geq N_\epsilon$ and for all $T \in [0, \infty)$ we have

$$\begin{aligned}
|s_n(T) - f(T)| &= \sum_{k=n+1}^{\infty} \left[r_{k+1}(T)h_{k+1}(T)\overline{P}_{k+1} + m_k(T)P_k(T)\overline{P}_k\right] \\
&\leq \sum_{k=N_\epsilon}^{\infty} \left[r_{k+1}(T)h_{k+1}(T)\overline{P}_{k+1} + m_k(T)P_k(T)\overline{P}_k\right] \\
&\leq \sum_{k=N_\epsilon}^{\infty} [r_{k+1}(T) + m_k(T)]\overline{P}_k \\
&\leq 2B \sum_{k=N_\epsilon}^{\infty} \overline{P}_k < \epsilon.
\end{aligned}$$

Therefore $s_n \to f$ uniformly on $[0, \infty)$, and hence $f(T)$ is continuous on $[0, \infty)$.

By assumption (3), there is a $T_\epsilon > 0$ such that for all $x \in [0, T_\epsilon)$, $\overline{H}_s(x) > \epsilon$. Since $H_s(T)$ is continuous on $[0, \infty)$, $\dfrac{1}{\overline{H}_s(T)}$ is continuous on $[0, T_\epsilon)$.

Form above, we have $\varphi_C(T)$ is continuous on $[0, T_\epsilon)$. Hence $Q(T)$ is continuous on $[0, T_\epsilon)$.

□

Remark. It is easy to see that example 1 satisfies the assumptions of theorem 3.

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Example 2. Consider shocks with phase-type distributions which are represented by PH$(\beta^{(k)}, A^{(k)})$, for all $k \geq 0$ of order 3. Assume the intensity matrix $A^{(k)}$ are upper-triangle matrix defined by

$$A^{(0)} = \begin{bmatrix} -0.5000 & 0.1383 & 0.1203 \\ 0 & -1.3000 & 1.0526 \\ 0 & 0 & -1.6429 \end{bmatrix}, A^{(1)} = \begin{bmatrix} -1.8333 & 0.5677 & 0.5165 \\ 0 & -1.9545 & 0.9470 \\ 0 & 0 & -2.0385 \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} -2.1000 & 0.3598 & 0.8861 \\ 0 & -2.1471 & 1.2398 \\ 0 & 0 & -2.1842 \end{bmatrix}, A^{(3)} = \begin{bmatrix} -2.2143 & 0.6136 & 1.1816 \\ 0 & -2.2391 & 1.6197 \\ 0 & 0 & -2.2600 \end{bmatrix},$$

$$A^{(4)} = \begin{bmatrix} -2.2778 & 1.1315 & 0.3842 \\ 0 & -2.2931 & 1.0082 \\ 0 & 0 & -2.3065 \end{bmatrix}, A^{(5)} = \begin{bmatrix} -2.3182 & 1.3518 & 0.7711 \\ 0 & -2.3286 & 0.8546 \\ 0 & 0 & -2.3378 \end{bmatrix},$$

$$A^{(6)} = \begin{bmatrix} -2.3462 & 1.0331 & 0.3125 \\ 0 & -2.3537 & 0.4887 \\ 0 & 0 & -2.3605 \end{bmatrix}, A^{(7)} = \begin{bmatrix} -2.3667 & 1.0384 & 0.5883 \\ 0 & -2.3723 & 1.3416 \\ 0 & 0 & -2.3776 \end{bmatrix},$$

$$A^{(8)} = \begin{bmatrix} -2.3824 & 0.4739 & 1.1197 \\ 0 & -2.3868 & 0.8944 \\ 0 & 0 & -2.3909 \end{bmatrix}, A^{(9)} = \begin{bmatrix} -2.3947 & 0.3810 & 1.0092 \\ 0 & -2.3983 & 0.9622 \\ 0 & 0 & -2.4016 \end{bmatrix}.$$

For $k \geq 10$, we set

$$A^{(k)} = \begin{bmatrix} -2.4048 & 0.2761 & 0.1964 \\ 0 & -2.4077 & 1.0950 \\ 0 & 0 & -2.4104 \end{bmatrix}.$$

The other parameters are set the same as that in example 1.

Consider $N = 40$ and compute $T^* = 5.4054$ by our algorithm. The graphs of $Q(T)$ and $J_C(T)$ are illustrated at the following Figures 6.3 and 6.4.

Example 3. Consider shocks with hypo-exponential distributions which are represented by PH$(\beta^{(k)}, A^{(k)})$, for all $k \geq 0$ of order 3. For $k \geq 0$, the intensity matrix is defined by

$$A^{(k)} = \begin{bmatrix} -\alpha_1^{(k)} & \alpha_1^{(k)} & 0 \\ 0 & -\alpha_2^{(k)} & \alpha_2^{(k)} \\ 0 & 0 & -\alpha_3^{(k)} \end{bmatrix}.$$

For $k = 0, 1, 2, \cdots, 9$ and for $i = 1, 2, 3$, define $\alpha_i^{(k)} = 2.5 - \dfrac{1}{x_i + k + 0.5}$, where $x_1 = 0$, $x_2 = \frac{1}{3}$, and $x_3 = \frac{2}{3}$. For $k \geq 10$, $\alpha_1^{(k)} = 2.4048$, $\alpha_2^{(k)} = 2.4077$, and $\alpha_3^{(k)} = 2.4104$. Actually, $\alpha_i^{(k)}$ are the same as the diagonal elements of example 2. The other parameters are set the same as that in example 1.

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The graphs of $Q(T)$ and $J_C(T)$ are shown at Figures 6.5 and 6.6. We find $T^* = 4.2042$.

Example 4. Consider shocks with Coxian distributions which are represented by PH$(\beta^{(k)}, A^{(k)})$, for all $k \geq 0$ of order 3. For $k \geq 0$, the intensity matrix is defined by

$$A^{(k)} = \begin{bmatrix} -\alpha_1^{(k)} & 0.3\alpha_1^{(k)} & 0 \\ 0 & -\alpha_2^{(k)} & 0.3\alpha_2^{(k)} \\ 0 & 0 & -\alpha_3^{(k)} \end{bmatrix}.$$

For $k \geq 0$ and for $i = 1, 2, 3$, $\alpha_i^{(k)}$ are the same as that in example 2. The other parameters are set the same as that in example 1. The graphs of $Q(T)$ and $J_C(T)$ are shown at Figures 6.7 and 6.8. We find $T^* = 4.3544$.

Example 5. Consider shocks with hyper-Erlang distributions which are represented by PH$(\beta^{(k)}, A^{(k)})$, for all $k \geq 0$ of order 3. The intensity matrices are given by

$$A^{(k)} = \begin{bmatrix} -1.2 & 1.2 & 0 & 0 \\ 0 & -1.2 & 0 & 0 \\ 0 & 0 & -2.4 & 2.4 \\ 0 & 0 & 0 & -2.4 \end{bmatrix}, \text{ for all } k \geq 0.$$

The other parameters are set the same as that in example 1. The graphs of $Q(T)$ and $J_C(T)$ are shown at Figures 6.9 and 6.10. We find $T^* = 31.6817$.

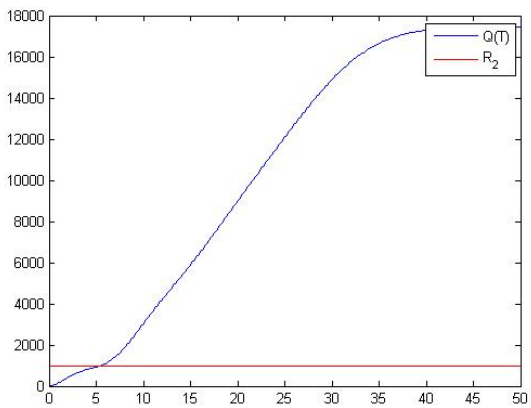

Figure 6.3: $Q(T)$ of $A^{(k)}$ being an upper-triangle matrix

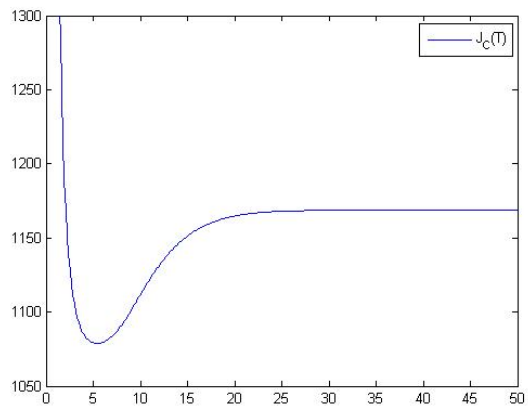

Figure 6.4: $J_C(T)$ of $A^{(k)}$ being an upper-triangle matrix

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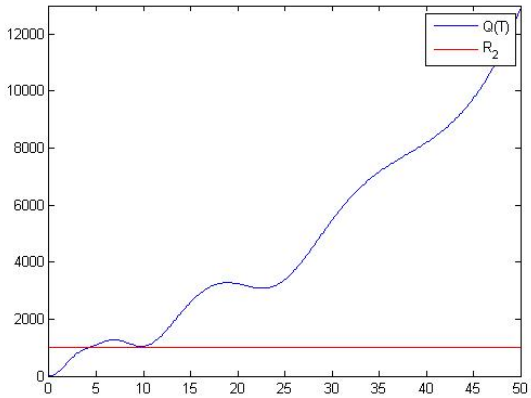

Figure 6.5: $Q(T)$ of a hypo-exponential distribution

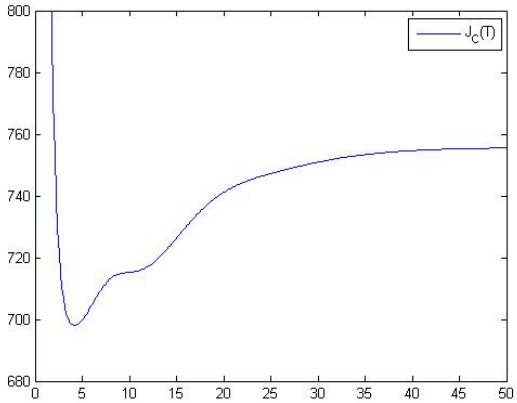

Figure 6.6: $J_C(T)$ of a hypo-exponential distribution

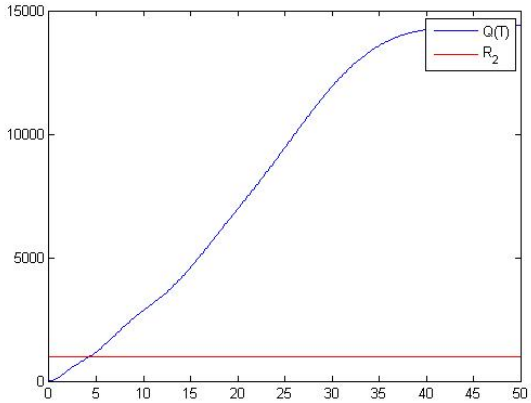

Figure 6.7: $Q(T)$ of a Coxian distribution

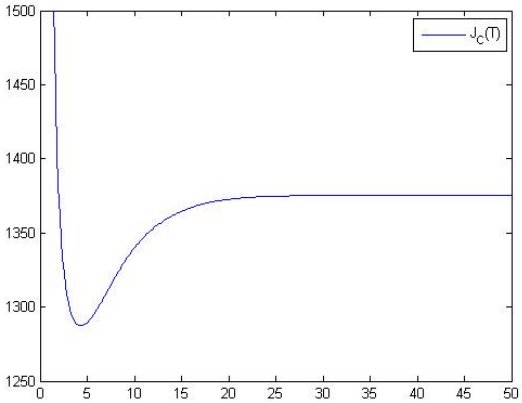

Figure 6.8: $J_C(T)$ of a Coxian distribution

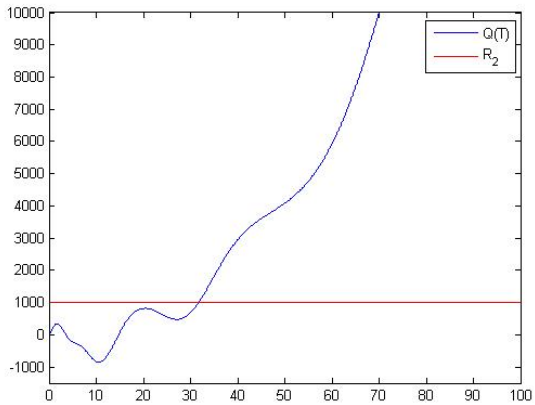

Figure 6.9: $Q(T)$ of a hyper-Erlang distribution

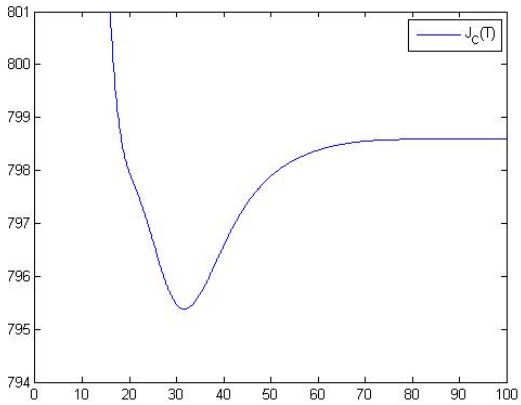

Figure 6.10: $J_C(T)$ of a hyper-Erlang distribution

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Chapter 7

Conclusion

We study the non-homogeneous pure birth shock model under the methodology of the matrix-analytic methods. We suppose the inter-arrival time between consecutive shocks follows a PH-distribution. Then the cumulative distribution function of the lifetime of the system is easy to express, see equation (2.5). The equation (2.5) is also one of the reasons why our algorithm is efficient. For the case of the intensity matrix that is hypo-exponential, we find the sufficient conditions of the existence of stationary probability of the shock model.

Under this model, we investigate the age replacement policy. The expected cost rate of a replacement cycle is developed. We apply the Theorem of Sheu et al. [13] (theorem 2) to show that the existence of the optimal planned replacement age which minimizes the expected cost rate. However, in numerical example 1, we find a case that $Q(T)$ does not satisfy the property in proof of theorem 2, see Figure 6.1. Therefore we develop a new theorem which gives more simple and practical conditions of the existence of the optimal planned replacement age.

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Appendix A

MATLAB Phase-Type Distribution Tool

We develop a tool which can implement our algorithm to compute the optimal planned replacement age T^* and the optimal expected cost rate $J_C(T^*)$.

A.1 Basic Program

A.1.1 Operators

1. $C = \text{AddMatrix}(A, B)$:

   Let A, B be matrices. Output the matrix $C = A \oplus B = \text{diag}(A, B)$.

```
function C = AddMatrix(A,B)
[m, n] = size(A);
[s, t] = size(B);
C(1:m, 1:n) = A;
C(m+1:m+s, n+1:n+t) = B;
```

2. $[g, G] = \text{ConvoluteMatrix}(\alpha, A, \beta, B)$:

   Let α, β be initial vectors and A, B be intensity matrices. Output the initial vector g and the matrix G, where $\text{PH}(g, G)$ is the convolution of $\text{PH}(\alpha, A)$ and $\text{PH}(\beta, B)$. The matrix G is defined by

$$G = \begin{bmatrix} A & \boldsymbol{a}\beta \\ & B \end{bmatrix},$$

   where $\boldsymbol{a} = -A\mathbf{1}$ is the absorption vector. Note $\boldsymbol{a}$ is a column vector and β is a row vector.

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```
function [g, G] = ConvoluteMatrix(alpha, A, beta, B)
m = length(A);
n = length(B);
G = AddMatrix(A, B); % self-defined function
a = -A*ones(m,1);
G(1:m, m+1:m+n) = a*beta;
g = [alpha, zeros(1,n)];
```

A.1.2 Functions

1. $F(t) = \text{MixtureDistribution}(w, \{P_i(t)\}_{i=1}^{N})$:

   Let $w = \{w_1, w_2, \cdots, w_N\}$ be a probability mass function and $P_i(t)$ be a CDF for all $i = 1, 2, \cdots, N$. Output the mixture distribution $F(t) = \sum_{i=1}^{N} w_i P_i(t)$.

```
function F = MixtureDistribution(w, P)
F = 0;
for i = 1:length(w)
    F = F + w(i)*P{i};
end
```

2. $H(t) = \text{CdfPH}(g, G, t)$:

   Let g be an initial vector and G be an intensity matrix. Output the CDF of PH(g, G) which is defined by $H(t) = 1 - g\exp(Gt)\mathbf{1}$. Note g is a row vector.

```
function H = CdfPH(g, G, t)
for i = 1:length(t)
    H(i) = 1 - g*expm(G*t(i))*ones(length(G),1);
end
```

3. $h(t) = \text{PdfPH}(g, G, t)$:

   Let g be an initial vector and G be a intensity matrix. Output the the PDF of PH(g, G) which is defined by $h(t) = -g\exp(Gt)G\mathbf{1}$. Note g is a row vector.

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```
function h = PdfPH(g,G,t)
for i = 1:length(t)
    h(i) = - g*expm(G*t(i))*G*ones(length(G),1);
end
```

A.1.3 Support Program

1. $t = \text{TimeAxis}()$:

   Output a time axis from a to b, i.e., $t = \{a = t_1 < t_2 < \cdots < t_M = b\}$. For all $i = 1, 2, \cdots, M-1$, we have $t_{i+1} - t_i = \frac{b-a}{M-1}$. Let $a = 0$, $b = 150$, and $M = 1000$.

```
function t = TimeAxis()
a = 0;
b = 150;
M = 1000;
t = linspace(a,b,M);
```

A.2 Program for Basic the Elements of the System

1. $\theta = \text{MajorFailureProbSeq}(q)$:

   Output a sequence $\theta = \{\theta_k\}_{k=1}^N$ of major failure probability θ_k. Note $\theta_k = 1 - q_k$ for all $k = 1, 2, \cdots, N$. Where $q = \{q_k\}_{k=1}^N$ is the sequence of minor failure probability.

```
function theta = MajorFailureProbSeq(q)
for k = 1:length(q)
    theta(k) = 1 - q(k);
end
```

2. $p = \text{FailureDistribution}(q, \theta)$:

   Output the sequence $\{p_k\}_{k=1}^N$ of failure probability $p_k = \left(\prod_{i=1}^{k-1} q_i\right)\theta_k$.

```
function p = FailureDistribution(q,theta)
p(1) = theta(1);
```


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```
for k = 2:length(q)
    p(k) = 1;
    for i = 1:k-1
        p(k) = p(k)*q(i);
    end
    p(k) = p(k)*theta(k);
end
```

3. $[\overline{P}_0, \overline{P}] = \text{SurvivalOfSystem}(q)$:
   Output the survival function of the system $\overline{P}_k = \Pr\{M > k\} = \prod_{i=1}^{k} q_i, \forall k = 1, 2, \cdots, N$ and $\overline{P}_0 = 1$. Note $\overline{P} = \{\overline{P}_k\}_{k=1}^{N}$. Where M is the number of shocks until the first type-II failure since the last replacement.

```
function [P0bar,Pkbar] = SurvivalOfSystem(q)
P0bar = 1;
Pkbar = zeros(size(q));
for k=1:length(q)
    Pkbar(k) = 1;
    for i=1:k
        Pkbar(k) = Pkbar(k)*q(i);
    end
end
```

4. $[\{g^{(k)}\}_{k=1}^{N}, \{G^{(k)}\}_{k=1}^{N}] = \text{Convolution}(\{\beta^{(k)}\}_{k=1}^{N}, \{A^{(k)}\}_{k=1}^{N})$:
   Output the initial vector $g^{(k)}$ and the intensity matrix $G^{(k)}$ of the PH-distribution $\text{PH}(g^{(k)}, G^{(k)})$ for all $k = 1, 2, \cdots, N$. Where $\text{PH}(g^{(k)}, G^{(k)})$ is the convolution of $\text{PH}(\beta_i, A_i)$ for all $i = 1, 2, \cdots, k$.

```
function [g,G] = Convolution(beta,A)
g{1} = beta{1};
G{1} = A{1};
for k = 2:length(beta)
```


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```
    [g{k},G{k}] = ConvoluteMatrix(g{k-1},G{k-1},beta{k},A{k});
end
```

5. $\{H_k(t)\}_{k=1}^N = \text{CdfSequencePH}(\{g^{(k)}\}_{k=1}^N, \{G^{(k)}\}_{k=1}^N, t)$:
   Output the CDF sequence of the PH-distributions $\text{PH}(g^{(k)}, G^{(k)})$ for all $k = 1, 2, \cdots, N$.

```
function H = CdfSequencePH(g,G,t)
for k = 1:length(g)
    H{k} = CdfPH(g{k},G{k},t);
end
```

6. $\{h_k(t)\}_{k=1}^N = \text{PdfSequencePH}(\{g^{(k)}\}_{k=1}^N, \{G^{(k)}\}_{k=1}^N, t)$:
   Output the PDF sequence of the PH-distributions $\text{PH}(g^{(k)}, G^{(k)})$ for all $k = 1, 2, \cdots, N$.

```
function h = PdfSequencePH(g,G,t)
for k = 1:length(g)
    h{k} = PdfPH(g{k},G{k},t);
end
```

7. $H_s(t) = \text{CdfMixturePH}(p, \{H_k(t)\}_{k=1}^N)$:
   Let $H_k(t)$ be the CDF of $\text{PH}(g^{(k)}, G^{(k)})$ for all $k = 1, 2, \cdots, N$. Output the CDF of T_s which defined by $H_s(t) = 1 - \sum_{k=1}^{N} p_k g^{(k)} \exp(G^{(k)}t)\mathbf{1}$. Note $p = \{p_k\}_{k=1}^N$.

```
function Hs = CdfMixturePH(p,H)
Hs = MixtureDistribution(p,H) + 1 - sum(p);
end
```

8. $h_s(t) = \text{PdfMixturePH}(p, \{h_k(t)\}_{k=1}^N)$:
   Let $h_k(t)$ be the PDF of $\text{PH}(g^{(k)}, G^{(k)})$ for all $k = 1, 2, \cdots, N$. Output the PDF of T_s which defined by $h_s(t) = -\sum_{k=1}^{N} p_k g^{(k)} \exp(G^{(k)}t)G^{(k)}\mathbf{1}$. Note $p = \{p_k\}_{k=1}^N$.

```
function hs = PdfMixturePH(p,h)
hs = MixtureDistribution(p,h);
end
```


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9. $[P_0(t), \{P_k(t)\}_{k=1}^{N-1}] = \text{TransitionProbOfSystem}(\{H_k(t)\}_{k=1}^{N})$:
Output the sequence of $P_k(t)$, the transition probability of the system at time t given $N(0) = 0$, which is defined by

$$P_k(t) = \begin{cases} 1 - H_1(t), & \text{if } k = 0, \\ H_k(t) - H_{k+1}(t), & \text{if } k \geq 1. \end{cases}$$

```
function [P0t,Pkt] = TransitionProbOfSystem(H)
P0t = 1-H{1};
for k = 1:length(H)-1
    Pkt{k} = H{k} - H{k+1};
end
```

A.3 Programs for the Optimal Algorithm

In this section, we use the programs defined above to compute the function $Q(T)$ and $J_C(T)$.

1. $[C_r(T), c_r(T)] = \text{ExpectedRepairCost}(\{r_k(T)\}_{k=1}^{N}, \{h_k(T)\}_{k=1}^{N}, \overline{P})$:
Output $c_r(T) = \sum_{k=1}^{N} r_k(T) h_k(T) \overline{P}_k$ and $C_r(T) = \int_0^T c_r(t)dt$. Note $\overline{P} = \{\overline{P}_k\}_{k=1}^{N}$.

```
function [Cr,cr] = ExpectedRepairCost(r,h,Pkbar)
t = TimeAxis();
cr = zeros(size(t));
for i = 1:length(r)
    cr = cr + r(i).*h{i}*Pkbar(i);
end
Cr = cumtrapz(t,cr);
```

2. $[C_m(T), c_m(T)] =$
$\text{EMCost}(m_0(T), \{m_k(T)\}_{k=1}^{N-1}, P_0(T), \{P_k(T)\}_{k=1}^{N-1}, \overline{P}_0, \overline{P})$:
Output $c_m(T) = \sum_{k=0}^{N-1} m_k(T) P_k(T) \overline{P}_k$ and $C_m(T) = \int_0^T c_m(t)dt$. Note $\overline{P} = \{\overline{P}_k\}_{k=1}^{N}$.

```
function [Cm,cm] = EMCost(m0,mk,P0t,Pkt,P0bar,Pkbar)
```


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```
t = TimeAxis();
cm = zeros(size(t));
cm = m0.*P0t*P0bar;
for i = 1:length(mk)
    cm = cm + mk(i).*Pkt{i}*Pkbar(i);
end
Cm = cumtrapz(t,cm);
```

3. $\varphi_C(T) = \text{PhiC}(R_1, R_2, H_s(T), h_s(T), c_r(T), c_m(T))$:
   Output $\varphi_C(T)$ which is defined by

$$\varphi_C(T) = \frac{1}{\overline{H}_s(T)} \left[(R_1 - R_2)h_s(T) + c_r(T) + c_m(T)\right].$$

```
function phiC = PhiC(R1,R2,Hs,hs,cr,cm)
phiC = ((R1-R2)*hs + cr + cm)./ (1-Hs);
```

4. $Q(T) = \text{Qfun}(R_1, R_2, \varphi_C(T), H_s(T), C_r(T), C_m(T))$:
   Output $Q(T)$ which is defined by

$$Q(T) = \varphi_C(T) \int_0^T \overline{H}_s(t)dt - \left[(R_1 - R_2)H_s(T) + C_r(T) + C_m(T)\right].$$

```
function Q = Qfun(R1,R2,phiC,Hs,Cr,Cm)
t = TimeAxis();
Q = phiC.*cumtrapz(t,1-Hs) - ((R1-R2)*Hs + Cr + Cm);
```

5. $J_C(T) = \text{ExpectedCostRate}(R_1, R_2, H_s(T), C_r(T), C_m(T))$:
   Output $J_C(T)$ which is defined by

$$J_C(T) = \frac{R_2 + (R_1 - R_2)H_s(T) + C_r(T) + C_m(T)}{\int_0^T \overline{H}_s(t)dt}.$$

```
function Jc = ExpectedCostRate(R1,R2,Hs,Cr,Cm)
t = TimeAxis();
F = cumtrapz(t,1-Hs);
Jc = (R2 + (R1-R2)*Hs + Cr + Cm)./ F;
```


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Appendix B

Special Phase-Type Distributions

Definition. (Hypo-exponential Distribution) Let PH(β, Θ) be a PH-distribution. It is called a hypo-exponential distribution if its intensity matrix has the following form

$$\Theta = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -\lambda_{n-2} & \lambda_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & -\lambda_{n-1} & \lambda_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & -\lambda_n \end{bmatrix}.$$

Definition. (Erlang Distribution) Let PH(β, E) be a PH-distribution. It is also an Erlang distribution if its intensity matrix has the following form

$$E = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & \lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 & 0 & -\lambda \end{bmatrix}.$$

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Definition. (Hyper-Erlang Distribution) Let PH(β, E_{hyper}) be a PH-distribution. It is called a hyper-Erlang distribution if its intensity matrix has the following form

$$E_{hyper} = \begin{bmatrix} E_1 & O & O & \cdots & O & O \\ O & E_2 & O & \ddots & O & O \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ O & O & \ddots & E_{n-2} & O & O \\ O & O & \cdots & O & E_{n-1} & O \\ O & O & \cdots & O & O & E_n \end{bmatrix},$$

where E_i is an intensity matrix of an Erlang distribution, for all $i = 1, 2, \cdots, n$. Note O is a zero matrix.

Definition. (Coxian Distribution) Let PH(β, C) be a PH-distribution. It is called a Coxian distribution if its intensity matrix has the following form

$$C = \begin{bmatrix} -\lambda_1 & p_1\lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & p_2\lambda_2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -\lambda_{n-2} & p_{n-2}\lambda_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & -\lambda_{n-1} & p_{n-1}\lambda_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & -\lambda_n \end{bmatrix},$$

where $p_i \in (0, 1]$ for all $i = 1, 2, \cdots, n$.

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