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AN APPLICATION OF THE COALESCENCE THEORY TO BRANCHING RANDOM WALKS

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Abstract

In a discrete-time single-type Galton–Watson branching random walk $\{Z_n, \zeta_n\}_{n\geq 0}$, where Z_n is the population of the nth generation and ζ_n is a collection of the positions on $\mathbb{R}$ of the Z_n individuals in the nth generation, let Y_n be the position of a randomly chosen individual from the nth generation and $Z_n(x)$ be the number of points in ζ_n that are less than or equal to x for $x \in \mathbb{R}$. In this paper we show in the explosive case (i.e. $m = \mathbb{E}(Z_1 \mid Z_0 = 1) = \infty$) when the offspring distribution is in the domain of attraction of a stable law of order α, $0 < \alpha < 1$, that the sequence of random functions $\{Z_n(x)/Z_n \colon -\infty < x < \infty\}$ converges in the finite-dimensional sense to $\{\delta_x \colon -\infty < x < \infty\}$, where $\delta_x \equiv \mathbf{1}_{\{N\leq x\}}$ and N is an $N(0, 1)$ random variable.

Keywords: Branching process; branching random walk; coalescence; supercritical; infinite mean

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1. Introduction

A branching random walk is a branching tree such that with each line of descent a random walk is associated.

Let $\{Z_n\}_{n\geq 0}$ be a discrete-time single-type Galton–Watson branching process with offspring distribution $\{p_j\}_{j\geq 0}$. Let $Z_0 = 1$. Then there is a unique probability measure on the space of family trees initiated by this ancestor.

On this family tree, we impose the following movement structure. If an individual is located at x in the real line $\mathbb{R}$, and, upon death, produces k children, then these k children move to $x + X_{kj}$ for $1 \leq j \leq k$, where $(X_{k1}, X_{k2}, \dots, X_{kk})$ is a random vector with a joint distribution π_k on $\mathbb{R}^k$ for each k. The random vector $X_k \equiv (X_{k1}, X_{k2}, \dots, X_{kk})$ is stochastically independent of the history up to that generation as well as the movement of the offspring of other individuals.

Let $\zeta_n \equiv \{x_{ni} \colon 1 \leq i \leq Z_n\}$ be the positions of the Z_n individuals of the nth generation. For each $n \geq 0$, ζ_n is a collection of random numbers on $\mathbb{R}$ and, hence, is a point process. The sequence of pairs of $\{Z_n, \zeta_n\}_{n\geq 0}$ is called a *branching random walk*. The probability distribution of this process is completely specified by

- the offspring distribution $\{p_j\}_{j\geq 0}$;
- the family of probability measures $\{\pi_k\}_{k\geq 1}$;

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- the initial population size Z_0; and
- the locations $\zeta_0 \equiv \{x_{0i},\ 1 \le i \le Z_0\}$ of the initial ancestors.

It is clear that $\{\zeta_n\}_{n\ge 0}$ is a Markov chain whose state space is the set of all finite subsets of $\mathbb{R}$ and that the movement along any one line of descent is that of a classical random walk. Thus, if $\{X_{ki}\}_{k\ge 1, i\ge 1}$ are independent and identically distributed (i.i.d.) with mean μ and finite variance σ^2, then the location of an individual of the nth generation should be approximately Gaussian with mean $n\mu$ and variance $n\sigma^2$ by the central limit theorem. This suggests that if $Z_n \to \infty$ as $n \to \infty$ and if $x_n = \sigma\sqrt{n}x + n\mu$, then $Z_n(x_n)/Z_n$ could have $\Phi(x)$, the standard $N(0, 1)$ cumulative distribution function (CDF), as its limit. Or, if $X_{k,1}$ is in the domain of attraction of a stable law of order α, $0 < \alpha \le 2$, then there exist a_n and b_n such that $Z_n(a_n + b_n y)/Z_n$ converges to a standard stable law CDF as $n \to \infty$. This turns out to be true in the supercritical case ($1 < m = \sum_{j=1}^{\infty} jp_j < \infty$); see [2] for the details. More results related to the central limit theorem on branching random walks can been found in [1], [4], and [8].

2. Main results

In this paper we consider the explosive Galton–Watson branching process such that the offspring distribution $\{p_j\}_{j\ge 0}$ is in the domain of a stable law of order α with $0 < \alpha < 1$ and, hence, with $m \equiv \sum_{j=0}^{\infty} jp_j = \infty$. (See also [5], [6], and [7].)

Theorem 2.1. *Let $p_0 = 0$ and $\{p_j\}_{j\ge 0}$ satisfy $\sum_{j>x} p_j \sim x^{-\alpha}L(x)$ as $x \uparrow \infty$, where $0 < \alpha < 1$ and $L(\cdot)$ is slowly varying at ∞. Let $\{X_{k,i}\}_{k\ge 1,\, 1\le i\le k}$ be identically distributed. Let $\mathbb{E}X_{k,1} = 0$ and $\mathbb{E}X_{k,1}^2 = \sigma^2 < \infty$. Then, for any fixed $y \in \mathbb{R}$,*

(a) *$\mathbb{P}(Y_n \le \sqrt{n}\sigma y) \to \Phi(y)$ as $n \to \infty$;*

(b) *$Z_n(\sqrt{n}\sigma y)/Z_n \xrightarrow{\mathrm{D}} \delta_y$ as $n \to \infty$, where δ_y is Bernoulli$(\Phi(y))$, i.e. $\mathbb{P}(\delta_y = 1) = \Phi(y) = 1 - \mathbb{P}(\delta_y = 0)$.*

The result in Theorem 2.1(b) can be strengthened to the joint convergence of

$$\frac{Z_n(\sqrt{n}\sigma y_i)}{Z_n}, \qquad i = 1, 2, \ldots, k,$$

for $y_1, y_2, \ldots, y_k \in \mathbb{R}$.

We have the following theorem.

Theorem 2.2. *Under the hypothesis of Theorem 2.1,*

(a) *for any $-\infty < y_1 < y_2 < \infty$,*

$$\left(\frac{Z_n(\sqrt{n}\sigma y_1)}{Z_n}, \frac{Z_n(\sqrt{n}\sigma y_2)}{Z_n}\right) \xrightarrow{\mathrm{D}} (\delta_{y_1}, \delta_{y_2}),$$

which takes the values $(0, 0)$, $(0, 1)$, and $(1, 1)$ with probabilities $1 - \Phi(y_2)$, $\Phi(y_2) - \Phi(y_1)$, and $\Phi(y_1)$, respectively;

(b) *for any $-\infty < y_1 < y_2 < \cdots < y_k < \infty$,*

$$\left(\frac{Z_n(\sqrt{n}\sigma y_i)}{Z_n} : 1 \le i \le k\right) \xrightarrow{\mathrm{D}} (\delta_{y_1}, \ldots, \delta_{y_k}),$$

where each δ_{y_i} *is* 0 *or* 1*, and, furthermore,*

$$\delta_{y_i} = 1 \quad \Rightarrow \quad \delta_{y_j} = 1 \quad \text{for } j \geq i$$

and

$$\begin{aligned}\mathbb{P}(\delta_{y_1} = 0, \delta_{y_2} = 0, \ldots, \delta_{y_{j-1}} = 0, \delta_{y_j} = 1, \ldots, \delta_{y_k} = 1) &= \mathbb{P}(\delta_{y_{j-1}} = 0, \delta_{y_j} = 1) \\ &= \Phi(y_j) - \Phi(y_{j-1}).\end{aligned}$$

Remark 2.1. Theorem 2.2 suggests that

$$\left\{ Z_n(y) = \frac{Z_n(\sqrt{n}\sigma y)}{Z_n}, \ -\infty < y < \infty \right\}$$

converges in the Skorokhod space $D(-\infty, \infty)$ weakly to

$$\{X(y) \equiv \mathbf{1}_{\{N \leq y\}}, \ -\infty < y < \infty\},$$

where N is an $N(0, 1)$ random variable.

Since we have the finite-dimensional convergence (by Theorem 2.2), only tightness needs to be established.

3. Proofs of the main results

Let $\{Z_n\}_{n\geq 1}$ be a discrete-time single-type Galton–Watson branching process with offspring distribution $\{p_j\}_{j\geq 0}$ and initiated size Z_0. Pick two individuals from the population in the nth generation (assuming that $Z_n \geq 2$) by simple random sampling without replacement and trace their lines of descent backward in time until they meet for the first time. Call this common ancestor *the last common ancestor* or *the most recent common ancestor* of these two randomly chosen individuals. Let $\tau_{n,2}$ be the generation number of this common ancestor.

The following has been shown in [3].

Theorem 3.1. *Let* $p_0 = 0$*, and let* $m = \sum_{j=1}^{\infty} j p_j = \infty$*. Furthermore, for some* $0 < \alpha < 1$ *and a function* $L\colon (1, \infty) \to (0, \infty)$ *slowly varying at* ∞*, let*

$$\frac{\sum_{j>x} p_j}{x^{\alpha} L(x)} \to 1 \quad \text{as } x \to \infty.$$

Then, for almost all trees $\mathcal{T}$ *and* $k = 1, 2, \ldots$*, as* $n \to \infty$*,*

$$\mathbb{P}(\tau_{n,2} < k \mid \mathcal{T}) \to 0,$$
$$\mathbb{P}(n - \tau_{n,2} < k) \to \pi_2(k) \quad \text{exists,}$$

and $\pi_2(k) \uparrow 1$ *as* $k \uparrow \infty$*.*

To prove Theorem 2.1, we need the following result whose proof is straightforward and thus omitted.

Lemma 3.1. *If* $\{X_n\}_{n\geq 1}$ *is a sequence of random variables with values in* $[0, 1]$ *such that*

$$\lim_{n\to\infty} \mathbb{E}X_n^2 = \lim_{n\to\infty} (\mathbb{E}X_n)^2 = \lambda, \qquad 0 < \lambda < 1,$$

then X_n *converges in distribution to a Bernoulli random variable* X *with* $\mathbb{P}(X = 0) = 1 - \lambda$ *and* $\mathbb{P}(X = 1) = \lambda$*.*

3.1. Proof of Theorem 2.1

(a) Recall that the $\zeta_n \equiv \{x_{ni} : 1 \le i \le Z_n\}$ are the positions of the Z_n individuals of the nth generation. For any fixed $y \in \mathbb{R}$, let

$$\delta_{n,i} = \begin{cases} 1 & \text{if } x_{n,i} \le \sqrt{n}\sigma y, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$Z_n(\sqrt{n}\sigma y) = \sum_{i=1}^{Z_n} \delta_{n,i}.$$

So,

$$\begin{aligned}
\mathbb{E}\bigg(\frac{Z_n(\sqrt{n}\sigma y)}{Z_n}\bigg) &= \mathbb{E}\bigg(\frac{1}{Z_n}\sum_{i=1}^{Z_n}\delta_{n,i}\bigg) \\
&= \mathbb{E}\bigg(\frac{1}{Z_n}\sum_{i=1}^{Z_n}\mathbb{E}(\delta_{n,i} \mid Z_n)\bigg) \\
&= \mathbb{E}\bigg(\frac{1}{Z_n}\sum_{i=1}^{Z_n}\mathbb{E}(\delta_{n,1})\bigg) \\
&= \mathbb{E}(\delta_{n,1}) \\
&= \mathbb{P}(x_{n,1} \le \sqrt{n}\sigma y) \\
&= \mathbb{P}(x_{0,1} + S_n \le \sqrt{n}\sigma y) \\
&= \mathbb{P}(S_n \le \sqrt{n}\sigma y - x_{0,1}),
\end{aligned}$$

where $S_n = \sum_{i=1}^{n} \eta_i$, $\{\eta_i\}_{i\ge 1}$ are i.i.d. copies with distribution π_1, and $x_{0,1}$ is the location of the initial ancestor of the nth generation individual located at the position $x_{n,1}$. Since $\mathbb{E}X_{k,1} = 0$ and $\mathbb{E}X_{k,1}^2 = \sigma^2 < \infty$, by the central limit theorem we have

$$\mathbb{P}\bigg(\frac{S_n}{\sqrt{n}\sigma} \le y - \frac{x_{0,1}}{\sqrt{n}\sigma}\bigg) \to \Phi(y) \quad \text{as } n \to \infty.$$

Hence, as $n \to \infty$,

$$\mathbb{P}(Y_n \le \sqrt{n}\sigma y) = \mathbb{E}(\mathbb{P}(Y_n \le \sqrt{n}\sigma y \mid Z_n)) = \mathbb{E}\bigg(\frac{Z_n(\sqrt{n}\sigma y)}{Z_n}\bigg) \to \Phi(y).$$

(b) From (a), we already know that, for any fixed $y \in \mathbb{R}$,

$$\mathbb{E}\bigg(\frac{Z_n(\sqrt{n}\sigma y)}{Z_n}\bigg) \to \Phi(y) \quad \text{as } n \to \infty.$$

By Lemma 3.1, it suffices to show that, for any fixed $y \in \mathbb{R}$, we also have

$$\mathbb{E}\bigg(\frac{Z_n(\sqrt{n}\sigma y)}{Z_n}\bigg)^2 \to \Phi(y) \quad \text{as } n \to \infty.$$

Recall that, for any fixed $y \in \mathbb{R}$,

$$\delta_{n,i} = \begin{cases} 1 & \text{if } x_{n,i} \le \sqrt{n}\sigma y, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathbb{E}\bigg(\frac{Z_n(\sqrt{n}\sigma y)}{Z_n}\bigg)^2 = \mathbb{E}\bigg(\frac{1}{Z_n^2}\sum_{i=1}^{Z_n}\delta_{n,i}^2\bigg) + \mathbb{E}\bigg(\frac{1}{Z_n^2}\sum_{i\ne j=1}^{Z_n}\delta_{n,i}\delta_{n,j}\bigg).$$

Firstly, it is known that, in the explosive case under the assumption that $p_0 = 0$, $\mathbb{P}(Z_n \to \infty) = 1$. Also, we have

$$\mathbb{P}\bigg(0 < \frac{1}{Z_n^2}\sum_{i=1}^{Z_n}\delta_{n,i}^2 < \frac{1}{Z_n}\bigg) = 1.$$

Hence,

$$\mathbb{P}\bigg(\frac{1}{Z_n^2}\sum_{i=1}^{Z_n}\delta_{n,i}^2 \to 0\bigg) = 1,$$

so, by the bounded convergence theorem,

$$\mathbb{E}\bigg(\frac{1}{Z_n^2}\sum_{i=1}^{Z_n}\delta_{n,i}^2\bigg) \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

Secondly, by the symmetry consideration conditioned on the branching tree (but not the random walk), we have

$$\begin{aligned}\mathbb{E}\bigg(\frac{1}{Z_n^2}\sum_{i\ne j=1}^{Z_n}\delta_{n,i}\delta_{n,j}\bigg) &= \mathbb{E}\bigg(\frac{1}{Z_n^2}\sum_{i\ne j=1}^{Z_n}\mathbb{E}(\delta_{n,i}\delta_{n,j} \mid Z_n)\bigg) \\ &= \mathbb{E}\bigg(\frac{1}{Z_n^2}\sum_{i\ne j=1}^{Z_n}\mathbb{E}(\delta_{n,1}\delta_{n,2} \mid Z_n)\bigg) \\ &= \mathbb{E}\bigg(\frac{Z_n(Z_n-1)}{Z_n^2}\bigg)\mathbb{E}(\delta_{n,1}\delta_{n,2}).\end{aligned}$$

Note that, by the bounded convergence theorem,

$$\mathbb{E}\bigg(\frac{Z_n(Z_n-1)}{Z_n^2}\bigg) \to 1 \quad \text{as } n \to \infty. \tag{3.2}$$

Now, let $\tau_{n,2}$ be the generation number of the last common ancestor of any two randomly chosen individuals in the nth generation. Then, by Theorem 3.1 we have

$$n - \tau_{n,2} \xrightarrow{\text{D}} \tilde{\tau}_2 \quad \text{as } n \to \infty$$

for some random variable $\tilde{\tau}_2$. Let x_{τ_n} be the position of the last common ancestor of these two individuals corresponding to the positions $x_{n,1}$ and $x_{n,2}$. Then we can write

$$x_{n,i} = x_{\tau_n} + Y_{n,i}, \qquad i = 1, 2,$$

where $Y_{n,i}$ is the net displacement of the individual with position $x_{n,i}$ from generation τ_n to n.

Clearly, $Y_{n,1}$ and $Y_{n,2}$ are independent. Moreover, x_{τ_n}, $Y_{n,1}$, and $Y_{n,2}$ can be written as

$$x_{\tau_n} = x_{0,1} + \sum_{j=1}^{\tau_{n,2}} \eta_j \quad \text{and} \quad Y_{n,i} = \sum_{j=1}^{n-\tau_{n,2}} \eta_{i,j} \quad \text{for } i = 1, 2,$$

respectively, where $\{\eta_j\}_{j\geq 1}$, $\{\eta_{1,i}\}_{j\geq 1}$, and $\{\eta_{2,i}\}_{j\geq 1}$ are i.i.d. copies with distribution π_1 and are independent of each other. Therefore,

$$\begin{aligned}
&\mathbb{E}(\delta_{n,1}\delta_{n,2}) \\
&\quad = \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{x_{n,1}\leq\sqrt{n}\sigma y\}}\mathbf{1}_{\{x_{n,2}\leq\sqrt{n}\sigma y\}} \mid n - \tau_{n,2})) \\
&\quad = \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{\sum_{j=1}^{\tau_{n,2}} \eta_j \leq \sqrt{n}\sigma y - x_{0,1} - \sum_{j=1}^{n-\tau_{n,2}} \eta_{1,j}\}}\mathbf{1}_{\{\sum_{j=1}^{\tau_{n,2}} \eta_j \leq \sqrt{n}\sigma y - x_{0,1} - \sum_{j=1}^{n-\tau_{n,2}} \eta_{2,j}\}} \mid n - \tau_{n,2})) \\
&\quad = \mathbb{E}\biggl(\mathbb{P}\biggl(\sum_{j=1}^{\tau_{n,2}} \eta_j \leq \sqrt{n}\sigma y - x_{0,1} - \max\biggl\{\sum_{j=1}^{n-\tau_{n,2}} \eta_{1,j}, \sum_{j=1}^{n-\tau_{n,2}} \eta_{2,j}\biggr\} \biggm| n - \tau_{n,2}\biggr)\biggr).
\end{aligned}$$

Since $n - \tau_{n,2} \xrightarrow{\text{D}} \tilde{\tau}_2$ as $n \to \infty$ and $\mathbb{P}(\tilde{\tau}_2 < \infty) = 1$, we have, for $i = 1, 2$,

$$\sum_{j=1}^{n-\tau_{n,2}} \eta_{i,j} \xrightarrow{\text{D}} \sum_{j=1}^{\tilde{\tau}_2} \eta_{i,j} \quad \text{as } n \to \infty.$$

Also, $\tau_{n,2} \xrightarrow{\text{D}} \infty$ and $\tau_{n,2}/n \xrightarrow{\text{D}} 1$ as $n \to \infty$. Hence, as $n \to \infty$,

$$\begin{aligned}
&\mathbb{P}\biggl(\sum_{j=1}^{\tau_{n,2}} \eta_j \leq \sqrt{n}\sigma y - x_{0,1} - \max\biggl\{\sum_{j=1}^{n-\tau_{n,2}} \eta_{1,j}, \sum_{j=1}^{n-\tau_{n,2}} \eta_{2,j}\biggr\} \biggm| n - \tau_{n,2}\biggr) \\
&\qquad \to \Phi(y) \quad \text{with probability 1.}
\end{aligned}$$

Then, by the bounded convergence theorem,

$$\mathbb{E}(\delta_{n,1}\delta_{n,2}) \to \Phi(y) \quad \text{as } n \to \infty. \tag{3.3}$$

So, (3.1), (3.2), and (3.3) together imply that

$$\mathbb{E}\biggl(\frac{Z_n(\sqrt{n}\sigma y)}{Z_n}\biggr)^2 \to \Phi(y) \quad \text{as } n \to \infty,$$

completing the proof.

3.2. Proof of Theorem 2.2

(a) Let $-\infty < y_1 < y_2 < \infty$ be any two fixed real numbers. Then

$$\mathbb{P}\biggl(\frac{Z_n(\sqrt{n}\sigma y_1)}{Z_n} \leq \frac{Z_n(\sqrt{n}\sigma y_2)}{Z_n}\biggr) = 1.$$

So,

$$\mathbb{P}\biggl(\frac{Z_n(\sqrt{n}\sigma y_1)}{Z_n} = 1, \frac{Z_n(\sqrt{n}\sigma y_2)}{Z_n} = 0\biggr) = 0 \quad \text{for any } n = 1, 2, \ldots,$$

and, hence,

$$\mathbb{P}\bigg(\frac{Z_n(\sqrt{n}\sigma y_1)}{Z_n} = 1,\ \frac{Z_n(\sqrt{n}\sigma y_2)}{Z_n} = 0\bigg) \to 0 \quad \text{as } n \to \infty.$$

Also, by Theorem 2.1, we have

$$\frac{Z_n(\sqrt{n}\sigma y_i)}{Z_n} \xrightarrow{\text{D}} \delta_{y_i} \quad \text{as } n \to \infty,$$

where δ_i is a Bernoulli random variable with $\mathbb{P}(\delta_{y_i} = 1) = \Phi(y_i) = 1 - \mathbb{P}(\delta_{y_i}) = 0$ for $i = 1, 2$. Therefore, as $n \to \infty$,

$$\mathbb{P}\bigg(\frac{Z_n(\sqrt{n}\sigma y_1)}{Z_n} = 0,\ \frac{Z_n(\sqrt{n}\sigma y_2)}{Z_n} = 0\bigg) = \mathbb{P}\bigg(\frac{Z_n(\sqrt{n}\sigma y_2)}{Z_n} = 0\bigg) \to 1 - \Phi(y_2)$$

and

$$\mathbb{P}\bigg(\frac{Z_n(\sqrt{n}\sigma y_1)}{Z_n} = 1,\ \frac{Z_n(\sqrt{n}\sigma y_2)}{Z_n} = 1\bigg) = \mathbb{P}\bigg(\frac{Z_n(\sqrt{n}\sigma y_1)}{Z_n} = 1\bigg) \to \Phi(y_1).$$

Moreover, since $(\delta_{y_1}, \delta_{y_2})$ only take values on the set $\{(0, 0), (0, 1), (1, 1)\}$,

$$\mathbb{P}\bigg(\frac{Z_n(\sqrt{n}\sigma y_1)}{Z_n} = 0,\ \frac{Z_n(\sqrt{n}\sigma y_2)}{Z_n} = 1\bigg) \to \Phi(y_2) - \Phi(y_1),$$

completing the proof of part (a).

(b) The proof of part (b) is similar to the above and is hence omitted.

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