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Allocating unfunded liability in pension valuation under uncertainty

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Abstract

This paper studied the cost allocation for the unfunded liability in a defined benefit pension scheme incorporating the stochastic phenomenon of its returns. In the recent literature represented by Cairns and Parker [Insurance: Mathematics and Economics 21 (1997) 43], Haberman [Insurance: Mathematics and Economics 11 (1992) 179; Insurance: Mathematics and Economics 13 (1993) 45; Insurance: Mathematics and Economics 14 (1994) 219; Insurance: Mathematics and Economics 14 (1997) 127], Owadally and Haberman [North American Actuarial Journal 3 (1999) 105], the fund level is modeled based on the plan dynamics and the returns are generated through several stochastic processes to reflect the current realistic economic perspective to see how the contribution changed as the cost allocation period increased. In this study, we generalize the previous constant value assumption in cost amortization by modeling the returns and valuation rates simultaneously. Taylor series expansion is employed to approximate the unconditional and conditional moments of the plan contribution and fund level. Hence the stability of the plan contribution and the fund size under different allocation periods could be estimated, which provide valuable information adding to the previous works.
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1. Introduction

1.1. Preliminary

There are many indeterminate economic and demographic factors in pension funding such as the volatility of plan returns, the inflation rates, the employees' turnovers and the new entrants' participation. Owing to these uncertainties, it is inevitable for mismatches in plan valuation and sometimes wild margin of errors (i.e., gains or losses) in forecasting the plan financial status to occur. Hence the pension actuary has to properly plan a strategy to allocate these mismatches in advance and disclose such information in the financial balance sheet when it occurs.

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In particular, in a defined benefit pension plan, the benefits are promised by the plan sponsor and the financial soundness is especially vital for the plan participants.

This study investigates the stability of the cost allocation for unfunded liability and focuses on the uncertainty arisen from the assumptions on plan returns and valuation rates. How to properly disclose the cost of a pension plan under uncertainty has become a crucial issue in pension fund valuation. Since statistical fluctuations may appear and inevitably generate difference between the expected and actual results, it is necessary to allocate such discrepancy during specific time horizon. In order to reduce the impact of errors on the annual pension cost and stabilize the volatility of the plan contribution due to stochastic fluctuations, it is important to monitor these errors. The pension fund usually accumulates a large amount of assets in advance in order to match its promised obligations in the future. Since the gains and losses affect the pension fund performance directly, careful investigation of the effect using different amortization strategies is especially important.

Cairns and Parker (1997), Owadally and Haberman (1999) treated fund returns as independently identically distributed (i.i.d.) random variables to see how the contribution changed as the cost allocation period increased. While evaluating the proportion $1/\ddot{a}_{\overline{M|}}$ of the unfunded liability to allocate the deficit by the spread method of amortization; however, the discount factor is assumed to be constant. In this study, the force of interests used in discount factors are modeled through several stochastic processes to reflect the possible economic perspectives. The major improvements are summarized as follows:

1. The fund returns are modeled through several stochastic processes to derive $1/\ddot{a}_{\overline{M|}}, R_t$ (instead of $1/\ddot{a}_{\overline{M|}}$ since R_t are random for all t) in this paper, while constant valuation rate is used in previous work in computing $1/\ddot{a}_{\overline{M|}}$.
2. The fund and contribution have been assumed to be stationary, i.e., the means of the fund and contributions are viewed as constants not depending on time, t. However, we do not restrict these assumptions.

1.2. Literature review

Researches in pension valuation in recent decades can be found in Bowers et al. (1982), McKenna (1982), Dufresne (1988, 1989), Haberman (1992–1994, 1997), Mandl and Mazurova (1996), Gerrard and Haberman (1996), Haberman and Wong (1997), Cairns and Parker (1997), Owadally and Haberman (1999, 2000).

Dufresne (1988, 1989) discussed the contribution rate and fund level when the return rates of the plan's assets were modeled based on an i.i.d. sequence of random variables over a fixed time horizon. Haberman (1992, 1993) compared different funding methods of computing the expectations and variation in fund sizes and contribution levels with a time delay when real rates of return were assumed to be generated from i.i.d. and first-order autoregressive (AR) processes. Haberman and Wong (1997) derived the moment and variation in the contribution rate and fund level under different pension funding methods. The real investment rates of return were modeled through a moving average (MA) process considering the optimal allocation period. Haberman (1997) proposed the contribution rate risk and discussed which periods for spreading valuation surpluses and deficiencies could be chosen to minimize the risk.

Pension application of AR(1) models have been considered by Haberman (1994), Mandl and Mazurova (1996), Cairns and Parker (1997). Using MA(1) models can be found in Haberman and Wong (1997), Bedard (1999), Owadally and Haberman (2000) for both AR(1) and MA(1) cases (see, e.g., (Bedard, 1999), and references therein). We extend their research by modeling the returns and valuation rates simultaneously through the plausible term structure, AR and MA time series models. The outline of this article is as follows. Section 2 describes the general framework and the notations used in our model. In Sections 3 and 4, we formulate several potential stochastic models for the interest rates in amortizing the unfunded liability to investigate the mean and the variance of the contribution level and the fund size. Numerical illustrations obtained from Taylor series approximation based on an actual data are summarized in Section 5. Section 6 contains the conclusion and identifies the potential areas for the future research. In Appendix A, we explain in detail our approximation.

2. Allocating unfunded liability

The fund size F_t may not be equal to the accrued liability AL_t at time t when actuarial cost methods including the projected unit credit (PUC) and entry age normal (EAN) cost methods are used. As a result, the unfunded accrued liability at time t, UAL_t, occurs. The unfunded accrued liability is defined as the excess of the accrued liability over the fund size, i.e.,

$$\mathrm{UAL}_t = \mathrm{AL}_t - F_t. \tag{1}$$

Hence, a strategy must be set up to allocate this unfunded accrued liability and properly disclose such information in the financial balance sheet. This means that the total contribution at time t should be split into two parts: the normal cost NC_t and a fraction of the unfunded liability $k\,\mathrm{UAL}_t$, to compensate the mismatch under some allocation strategies. In this study, the amortization is recomputed each year on the basis of the current unfunded liability. Hence the annual contribution could be formulated as

$$\text{Total } C_t = \mathrm{NC}_t + k\,\mathrm{UAL}_t, \tag{2}$$

where k depends on the allocation period and the valuation rate we chose to amortize the unfunded accrued liability. In the literature, a number of ways of presenting k have been investigated:

(a) $k = 1/\ddot{a}_{\overline{M|},R_t}$ where the annuity is calculated using the (deterministic) valuation rate of interest, so that attention focuses on M.
(b) k itself is considered the parameter, most recently in a proportional control framework.

Also see Dufresne (1988), Cairns and Parker (1997), Owadally and Haberman (1999, 2000), Cairns (2000). We follow the approach (a) but extend it by allowing k to depend on t and be based on the estimated forward interest rate rather than on a constant value. Let M be the period of amortization, then k is formulated by $1/\ddot{a}_{\overline{M|},R_t}$ where $\ddot{a}_{\overline{M|},R_t}$ is the present value of the certain annuity payment from the beginning of the tth year to the end of the $(t+M-2)$th year calculated at assumed forward interest rate R_j where $j = t, t+1, \dots, t+M-2$. The widely used method is to amortize the unfunded liability with a series of constant dollar payments. It means that UAL_t is level amortized within the next M years. We should recognize in advance that the forward interest rate R_j during the period j up to $j+1$ might not be a constant. Instead, R_j behaves randomly, which can be described through a sequence of stochastic processes. If we have past data to which a parsimonious model has been fitted then we can employ the model to forecast the future value of R_j. As a result, k can be formulated as a function of these forecast values. Therefore, $1/\ddot{a}_{\overline{M|},R_t}$ has the following presentation:

$$\frac{1}{\ddot{a}_{\overline{M|},R_t}} = \left(1 + \sum_{j=2}^{M} \prod_{i=1}^{j-1} \frac{1}{1+R_{t+i-1}}\right)^{-1}. \tag{3}$$

Since M affects contribution size in pension funding, an optimal M becomes vital in cost allocation. To set up an optimal rule, a performance measure is required. In this study, the performance criterion originally proposed by Dufresne (1988) and several subsequent authors is adopted using the first and second moments of the contribution and fund level. The idea behind this approach is to investigate the relationship between the mean, the variance of C_t (or F_t) and the chosen period of amortization M simultaneously. This can lead to an optimal value of M^* at which $\mathrm{Var}(C_t)$ (or $\mathrm{Var}(F_t)$) reaches its minimum under the given $E(C_t)$ (or $E(F_t)$). The optimal spread period M^* obtained from this approach is beneficial to the decision maker in balancing his risk and expected goal.

3. Modeling the uncertainty

First, the constant force of interest δ_t between year t and $t+1$ is investigated. It satisfies $1+R_t=\exp(\delta_t)$. Then, i.i.d., AR(1), and MA(1) models are employed as the force of interest to derive $\mathrm{Var}(C_t)$ as a function of M. Thus, k can be rewritten as the form according to Eq. (3):

$$k_t=\left(1+\sum_{j=2}^{M}\exp\left(-\prod_{i=1}^{j-1}\delta_{t+i-1}\right)\right)^{-1}, \tag{4}$$

where δ_{t+i-1} is the force of interest during the period $t+i-1$ up to $t+i$, $i=1,2,\dots,M-1$.

Let B_t be the benefit outgo in year t. To simplify the calculations, the amount of benefit payments are assumed to be provided at the beginning of each year. In other words, B_t is simulated at the beginning of year t. The plan contribution is assumed to be made at the beginning of each year. The pension fund F_{t+1} in year $t+1$ is formulated as

$$F_{t+1}=\mathrm{e}^{\delta_t}(F_t+C_t-B_t). \tag{5}$$

As mentioned above, C_t depends on the normal cost NC_t and on the unfunded accrued liability UAL_t. Combining Eqs. (1) and (2) yields the overall contribution:

$$C_t=\mathrm{NC}_t+k_t(\mathrm{AL}_t-F_t). \tag{6}$$

The normal cost NC_t is previously computed by the given actuarial cost methods, such as EAN cost method, under given actuarial assumptions including decrement rates, future salary increase and the valuation rate of interest. AL_t and B_t depend on these assumptions also. To focus on the issue of contribution, NC_t, AL_t and B_t are obtained based on Chang (1999). From the expectation and variance of the contribution C_t and the fund F_{t+1} in the following year, the optimal solution $E(k_t)$ can be determined.

In brief, the approach to determine the optimal cost allocation could be summarized as follows:

1. The contribution cash inflow and benefit payments are assumed to occur at the beginning of the year.
2. Initial NC_t, AL_t, B_t and F_t are obtained from the plan balance sheet in year t.
3. The corresponding equations for the contribution and fund are

$$C_t=\mathrm{NC}_t+k_t(\mathrm{AL}_t-F_t),\qquad F_{t+1}=\mathrm{e}^{\delta_t}(F_t+C_t-B_t).$$

4. Approximation

It is difficult to derive the moments of C_t and F_{t+1} directly. In order to investigate the contribution and fund level, the variance of C_t is estimated through approximation. The multi-variable Taylor series expansion is adopted to perform the estimation. Notations used in this paper are as follows:

Notation 1. Let $f:\mathbf{R}^n\to\mathbf{R}$ be a continuous function. If $\vec{X}$ is a point in $\mathbf{R}^n$ where all second-order partial derivatives of f exist and if $\vec{Y}=(y_1,y_2,\dots,y_n)$ is an arbitrary point in $\mathbf{R}^n$, we can write

1. $D_jf(\vec{X})$: the partial derivative of f w.r.t. the jth coordinate;
2. $D_{i,j}f(\vec{X})$: the partial derivative of D_jf w.r.t. the ith coordinate;
3. $f'(\vec{X};\vec{Y})=\sum_{j=1}^{n}D_jf(\vec{X})y_j$, $f''(\vec{X};\vec{Y})=\sum_{i=1}^{n}\sum_{j=1}^{n}D_{i,j}f(\vec{X})y_iy_j$.

We hereby need to define some functions which will be used later.

Definition 2. Let the forward rates $\vec{X} = (\delta_t, \delta_{t+1}, \ldots, \delta_{t+M-2})$ be any point in $\mathbf{R}^{M-1}$ and f, g, and h be all real valued functions defined on $\mathbf{R}^{M-1}$:

$$f(\vec{X}) \equiv \left(1 + \sum_{j=2}^{M} \exp\left(-\sum_{i=1}^{j-1} \delta_{t+i-1}\right)\right)^{-1}, \qquad g(\vec{X}) \equiv \mathrm{e}^{\delta_t} \left(1 + \sum_{j=2}^{M} \exp\left(-\sum_{i=1}^{j-1} \delta_{t+i-1}\right)\right)^{-1},$$

$$h(\vec{X}) \equiv \mathrm{e}^{2\delta_t} \left(1 + \sum_{j=2}^{M} \exp\left(-\sum_{i=1}^{j-1} \delta_{t+i-1}\right)\right)^{-1}.$$

Note that $\delta_t, \delta_{t+1}, \ldots, \delta_{t+M-2}$ are random variables. Let $\bar{\delta}_{t+i-1}$ be the expected value of δ_{t+i-1}, for $i = 1, 2, \ldots, M-1$. Within the neighborhood of $\vec{\delta} = (\bar{\delta}_t, \bar{\delta}_{t+1}, \ldots, \bar{\delta}_{t+M-2})$, by Taylor series expansion, $f(\vec{X})$ can be estimated as

$$f(\vec{X}) \cong f(\vec{\delta}) + f'(\vec{\delta}; \vec{X} - \vec{\delta}).$$

Calculations of the error terms and numerical upper bounds from approximation are given in Appendix A. From now on, we will use $f(\vec{\delta}) + f'(\vec{\delta}; \vec{X} - \vec{\delta})$, $g(\vec{\delta}) + g'(\vec{\delta}; \vec{X} - \vec{\delta})$ and $h(\vec{\delta}) + h'(\vec{\delta}; \vec{X} - \vec{\delta})$ in replace of the above three functions in Definition 2.

First, we derive the general forms of the expectations and variances of contribution and fund level. Then, plausible stochastic models of interest rate are selected to investigate the relationship between interest rate assumption and spread period.

Several functions of $k_t = 1/\ddot{a}_{\overline{M|}, R_t}$ using the Taylor series expansion are summarized as

1. $E(k_t) \cong f(\vec{\delta})$;
2. $\mathrm{Var}(k_t) \cong \sum_{i,j} D_i f(\vec{\delta}) D_j f(\vec{\delta}) \times \mathrm{Cov}(\delta_{t+i-1}, \delta_{t+j-1})$;
3. $E(k_t \cdot \mathrm{e}^{\delta_t}) \cong g(\vec{\delta})$;
4. $\mathrm{Var}(k_t \cdot \mathrm{e}^{\delta_t}) \cong \sum_{i,j} D_i g(\vec{\delta}) D_j g(\vec{\delta}) \times \mathrm{Cov}(\delta_{t+i-1}, \delta_{t+j-1})$;
5. $\mathrm{Var}(\mathrm{e}^{\delta_t}) \cong \mathrm{Var}(\mathrm{e}^{\bar{\delta}_t} + \mathrm{e}^{\bar{\delta}_t}(\delta_t - \bar{\delta}_t)) = \mathrm{e}^{2\bar{\delta}_t} \times \mathrm{Var}(\delta_t)$;
6. $\mathrm{Cov}(\mathrm{e}^{\delta_t}, k_t \cdot \mathrm{e}^{\delta_t}) \cong h(\vec{\delta}) - \mathrm{e}^{\bar{\delta}_t} \times g(\vec{\delta}) = 0$.

Hence

$$E(C_t) \cong \mathrm{NC}_t + (\mathrm{AL}_t - F_t) f(\vec{\delta}), \tag{7}$$

$$\mathrm{Var}(C_t) \cong (\mathrm{AL}_t - F_t)^2 \times \sum_{i,j} D_i f(\vec{\delta}) D_j f(\vec{\delta}) \times \mathrm{Cov}(\delta_{t+i-1}, \delta_{t+j-1}), \tag{8}$$

$$E(F_{t+1}) \cong \mathrm{e}^{\bar{\delta}_t}(F_t + \mathrm{NC}_t - B_t) + g(\vec{\delta})(\mathrm{AL}_t - F_t), \tag{9}$$

$$\begin{aligned} \mathrm{Var}(F_{t+1}) \cong{}& \mathrm{e}^{2\bar{\delta}_t} \times \mathrm{Var}(\delta_t) \times (F_t + \mathrm{NC}_t - B_t)^2 + \sum_{i,j} D_i g(\vec{\delta}) D_j g(\vec{\delta}) \\ & \times \mathrm{Cov}(\delta_{t+i-1}, \delta_{t+j-1}) \times (\mathrm{AL}_t - F_t)^2. \end{aligned} \tag{10}$$

In the following sections, we investigate the stability of the contribution and the fund size through their unconditional and conditional means and standard deviations. In unconditional approach, we assume that there is no given prior information and i.i.d., AR(1) and MA(1) are employed to model the forward rate process. In conditional approach, we use Vasicek model to characterize the forward rate pattern and calculate the means and standard deviations based on the initial rate.

4.1. *Independent and identical distribution*

Assume that the force of forward rates δ_t forms an i.i.d. sequence of random variables with $E(\delta_t) = \delta$ and $\mathrm{Var}(\delta_t) = \sigma_\delta^2$. The force of interests are assumed to resemble white noises. Dufresne (1988), Cairns and Parker (1997), Owadally and Haberman (1999) treated returns as i.i.d. to see how $\mathrm{Var}(C_t)$ changed as M increases and gave the condition when $\mathrm{Var}(C_t)$ reached a minimum in i.i.d. case. The results allowing the plan returns and valuation rates to be i.i.d. are summarized as follows:

1. $E(C_t) \cong \mathrm{NC}_t + (\mathrm{AL}_t - F_t) f(\vec{\delta})$;
2. $\mathrm{Var}(C_t) \cong (\mathrm{AL}_t - F_t)^2 \times \sigma_\delta^2 \times \sum_{j=1}^{M-1} (D_j f(\vec{\delta}))^2$;
3. $E(F_{t+1}) \cong \mathrm{e}^{\delta}(F_t + \mathrm{NC}_t - B_t) + g(\vec{\delta})(\mathrm{AL}_t - F_t)$;
4. $\mathrm{Var}(F_{t+1}) \cong \mathrm{e}^{2\delta} \times \sigma_\delta^2 \times (F_t + \mathrm{NC}_t - B_t)^2 + (\mathrm{AL}_t - F_t)^2 \times \sigma_\delta^2 \times \sum_{j=1}^{M-1} (D_j g(\vec{\delta}))^2$.

Note that $\vec{\delta} = (\delta, \delta, \dots, \delta)$ is a point in R^{M-1} in this case.

4.2. *AR model of AR(1)*

Consider the AR(1) model for the force of interest as follows:

$$\delta_t - \delta = \phi(\delta_{t-1} - \delta) + \sigma\epsilon_t,$$

where δ is the expected value of δ_t, $|\phi| < 1$ and ϵ_t, $t = 1, 2, \dots$, an independent and identically distributed sequence of standard normal random variables. Then we have

$$E(\delta_t) = \delta, \qquad \mathrm{Var}(\delta_t) = \frac{\sigma^2}{1-\phi^2}, \qquad \mathrm{Cov}(\delta_t, \delta_s) = \frac{\phi^{|t-s|}}{1-\phi^2}\sigma^2.$$

Haberman (1994), Mandl and Mazurova (1996), Cairns and Parker (1997), Owadally and Haberman (1999) have considered this model and focused on variation in the unconditional moments of C_t and F_t with ϕ. The results allowing the plan returns and valuation rates to be random are summarized as follows:

1. $E(C_t) \cong \mathrm{NC}_t + (\mathrm{AL}_t - F_t) f(\vec{\delta})$;
2. $\mathrm{Var}(C_t) \cong (\mathrm{AL}_t - F_t)^2 \times \frac{\sigma^2}{1-\phi^2} \times \left\{ \sum_{j=1}^{M-1} (D_j f(\vec{\delta}))^2 + \sum_{i \neq j} D_i f(\vec{\delta}) D_j f(\vec{\delta}) \phi^{|j-i|} \right\}$;
3. $E(F_{t+1}) \cong \mathrm{e}^{\delta}(F_t + \mathrm{NC}_t - B_t) + g(\vec{\delta})(\mathrm{AL}_t - F_t)$;
4. $\mathrm{Var}(F_{t+1}) \cong \mathrm{e}^{2\delta} \times \frac{\sigma^2}{1-\phi^2} \times (F_t + \mathrm{NC}_t - B_t)^2 + (\mathrm{AL}_t - F_t)^2 \times \frac{\sigma^2}{1-\phi^2}$
$\qquad \times \left\{ \sum_{j=1}^{M-1} (D_j g(\vec{\delta}))^2 + \sum_{i \neq j} D_i g(\vec{\delta}) D_j g(\vec{\delta}) \phi^{|j-i|} \right\}$.

4.3. *Moving average model of MA(1)*

The force of interests δ_t are assumed to satisfy the following relation:

$$\delta_t = \delta + a_t - \phi a_{t-1}, \quad a_t \sim \mathrm{IIDN}(0, \sigma_a^2),$$

where a_t, $t = 1, 2, \dots$, is an independent identically distributed sequence of normal random variables with zero mean and variance σ_a^2.

We have

$$E(\delta_t) = \delta, \qquad \mathrm{Var}(\delta_t) = (1+\phi^2)\sigma_a^2, \qquad \mathrm{Cov}(\delta_t, \delta_s) = \begin{cases} -\phi\sigma_a^2, & \text{if } |t-s| = 1 \\ 0, & \text{otherwise} \end{cases},$$

and from the properties of the log-normal distribution

$$E(\mathrm{e}^{\delta_t}) = \exp(\delta + \tfrac{1}{2}(1+\phi^2)\sigma_a^2), \qquad \mathrm{Var}(\mathrm{e}^{\delta_t}) = \exp(2\delta + (1+\phi^2)\sigma_a^2)(\exp((1+\phi^2)\sigma_a^2) - 1).$$

Haberman and Wong (1997), Bedard (1999) discussed the variability of pension contributions and fund levels in the model of MA returns. Employing the Taylor approximation, we obtained the following results:

1. $E(C_t) \cong \mathrm{NC}_t + (\mathrm{AL}_t - F_t) f(\vec{\delta})$;
2. $\mathrm{Var}(C_t) \cong (\mathrm{AL}_t - F_t)^2 \times \left\{(1+\phi^2)\sigma_a^2 \sum_{j=1}^{M-1}(D_j f(\vec{\delta}))^2 - 2\phi\sigma_a^2 \sum_{j=1}^{M-2} D_j f(\vec{\delta}) D_{j+1} f(\vec{\delta})\right\}$;
3. $E(F_{t+1}) \cong \exp(\delta + \frac{1}{2}(1+\phi^2)\sigma_a^2)(F_t + \mathrm{NC}_t - B_t) + g(\vec{\delta})(\mathrm{AL}_t - F_t)$;
4. $\mathrm{Var}(F_{t+1}) \cong \exp(2\delta + (1+\phi^2)\sigma_a^2)(\exp((1+\phi^2)\sigma_a^2) - 1)(F_t + \mathrm{NC}_t - B_t)^2 + (\mathrm{AL}_t - F_t)^2 \times \left\{(1+\phi^2)\sigma_a^2 \sum_{j=1}^{M-1}(D_j g(\vec{\delta}))^2 - 2\phi\sigma_a^2 \sum_{j=1}^{M-2} D_j g(\vec{\delta}) D_{j+1} g(\vec{\delta})\right\}$.

4.4. *Conditional approach through Vasicek model*

The previous subsections contain the approximated unconditional results of the plan return and valuation rates under i.i.d., AR(1) and MA(1) assumptions. These time series models are however restricted in reflecting the economic perspectives in terms of time horizon. In this subsection, the concept of the term structure of interest rates is employed to investigate the first and second moment of the plan contribution and fund level incorporating current fund performance. Many stochastic interest rate models based on the term structure have been discussed. Among these models, the general structure of single-factor models proposed by Vasicek (1977), Cox et al. (1985), Hull and White (1987) are widely employed in financial literatures. Expression for the general model is as follows:

$$\mathrm{d}R = u(R,t)\,\mathrm{d}t + w(R,t)\,\mathrm{d}X, \quad \mathrm{d}X \sim N(0,\mathrm{d}t), \tag{11}$$

where $u(R,t)$ and $w(R,t)$ represent the drift coefficient and the diffusion coefficient, respectively, in the stochastic process, $R(t)$. $u(R,t)$ and $w(R,t)$ are functions of the random variables R and t. If these independent variables R and t are considered in Eq. (11), the complexity of this model would increase. Hence $w(R,t)$ and $u(R,t)$ are reduced to unknown constants u and w in our study. Under this assumption, Eq. (11) is expressed as

$$\mathrm{d}R = \alpha(\gamma - R)\,\mathrm{d}t + \rho\,\mathrm{d}X. \tag{12}$$

The mean reverting process was originally proposed by Vasicek (1977). The drift coefficient $\alpha(\gamma - R)$ shows that the long-term structure of the plan return approaches γ with velocity α. In real life, return rates are quoted at discrete time intervals. Therefore, a practical lower bound $\mathrm{d}t$ for the basic time-step exists. In order to investigate the explicit solution, the Vasicek model is rewritten into a discrete form as

$$\Delta R = \alpha(\gamma - R)\Delta t + \rho\Delta X, \quad \alpha > 0, \quad \Delta X \sim N(0,\Delta t). \tag{13}$$

Discrete format of Vasicek model can also be modeled as AR(1) model. The proposed model can easily be extended to reflect more realistic economic scenario. If we denote the instantaneous interest rate between year $t-1$ and t by r_{t-1}, then R_{t+j} can be expressed as:

$$R_{t+j} = \frac{1-(1-\alpha)^{j+1}}{\alpha}(\alpha\gamma + \rho\Delta X) + (1-\alpha)^{j+1} r_{t-1}. \tag{14}$$

The conditional expectation and variance of R_{t+j}, given $R_{t-1} = r_{t-1}$ is derived as follows:

$$\mu_{t+j|t-1} \equiv E(R_{t+j}|R_{t-1} = r_{t-1}) = [1-(1-\alpha)^{j+1}]\gamma + (1-\alpha)^{j+1} r_{t-1},$$

$$\sigma^2_{t+j|t-1} \equiv \mathrm{Var}(R_{t+j}|R_{t-1} = r_{t-1}) = \left(\frac{1-(1-\alpha)^{j+1}}{\alpha}\rho\right)^2$$

and the covariance of R_{t+i} and R_{t+j} at given $R_{t-1} = r_{t-1}$ is

$$\gamma_{t+i,t+j|t-1} \equiv \mathrm{Cov}(R_{t+i}, R_{t+j}|R_{t-1} = r_{t-1}) = \frac{[1-(1-\alpha)^{i+1}][1-(1-\alpha)^{j+1}]}{\alpha^2}\rho^2.$$

In the beginning of this section, we have defined three functions, $f(\vec{X})$, $g(\vec{X})$ and $h(\vec{X})$. To reduce the complexity, these three functions are used. Let $\vec{R} = (R_t, R_{t+1}, \dots, R_{t+M-2}) \in \mathbf{R}^{M-1}$, $l : \mathbf{R}^{M-1} \to \mathbf{R}^{M-1}$ be defined by

$$l(\vec{R}) = (\ln(1+R_t),\ \ln(1+R_{t+1}), \dots, \ln(1+R_{t+M-2}))$$

and the new expressions for $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ will be

$$\tilde{f}(\vec{R}) \equiv f(l(\vec{R})) = \left(1 + \sum_{j=2}^{M}\prod_{i=1}^{j-1}\frac{1}{1+R_{t+i-1}}\right)^{-1},$$

$$\tilde{g}(\vec{R}) \equiv g(l(\vec{R})) = (1+R_t)\left(1 + \sum_{j=2}^{M}\prod_{i=1}^{j-1}\frac{1}{1+R_{t+i-1}}\right)^{-1},$$

$$\tilde{h}(\vec{R}) \equiv h(l(\vec{R})) = (1+R_t)^2\left(1 + \sum_{j=2}^{M}\prod_{i=1}^{j-1}\frac{1}{1+R_{t+i-1}}\right)^{-1}.$$

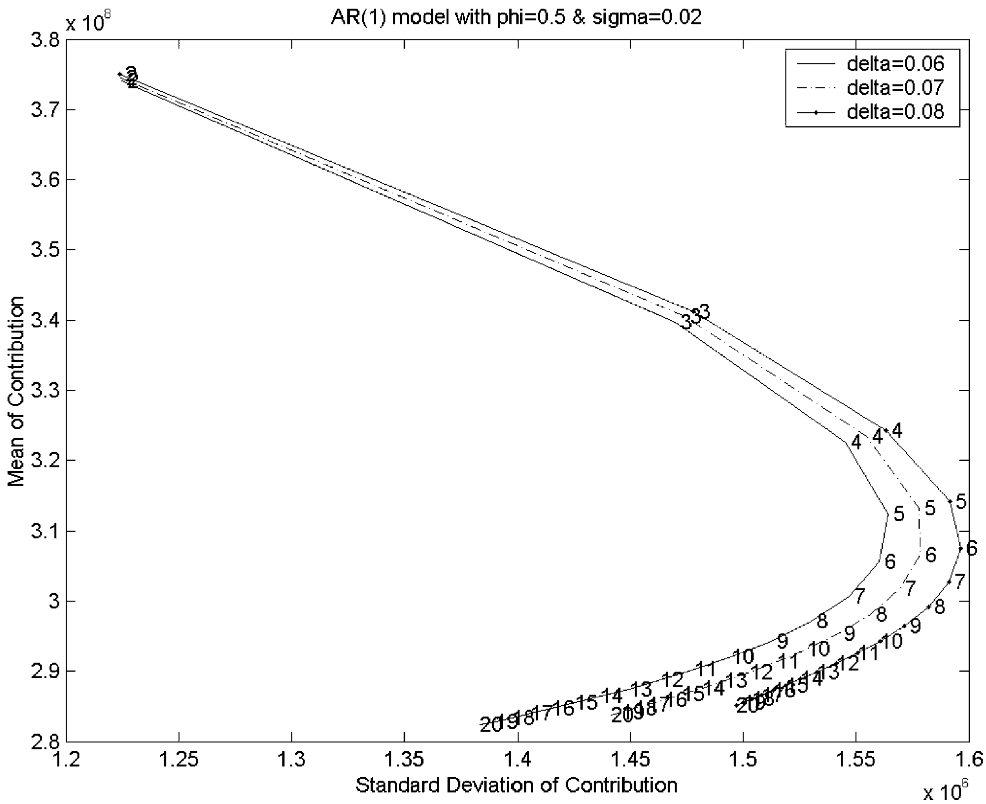

Fig. 1. Graph shows the pattern of mean versus standard deviation of contribution in AR(1) model with each point on the curves related to the identified values of M at time $t = 1997$.

Using this approach, the approximations of $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ can be obtained. By noting that

$$\tilde{f}(\vec{R}) \approx \tilde{f}(\vec{\mu}) + \tilde{f}'(\vec{\mu}; \vec{R} - \vec{\mu}),$$

where $\vec{\mu} = (\mu_{t|t-1}, \mu_{t+1|t-1}, \dots, \mu_{t+M-2|t-1})$ in R^{M-1} and $\vec{R}$ is in the neighborhood of $\vec{\mu}$ and we have

$$\tilde{f}'(\vec{\mu}; \vec{R} - \vec{\mu}) = \sum_{j=1}^{M-1} D_j \tilde{f}(\vec{\mu})(R_{t+j-1} - \mu_{t+j-1|t-1}).$$

Before investigating the optimal amortization period for the unfunded liability, the conditional expectation and variance of C_t and F_{t+1} using the Taylor series expansion are as follows:

$$\begin{aligned}
&E(C_t|R_{t-1} = r_{t-1}) \approx \mathrm{NC}_t + (\mathrm{AL}_t - F_t)\tilde{f}(\vec{\mu}),\\
&\mathrm{Var}(C_t|R_{t-1} = r_{t-1}) \approx (\mathrm{AL}_t - F_t)^2 \mathrm{Var}(\tilde{f}'(\vec{\mu}; \vec{R} - \vec{\mu}))\\
&\quad = (\mathrm{AL}_t - F_t)^2 \left[\sum_{j=1}^{M-1} (D_j \tilde{f}(\vec{\mu}))^2 \sigma^2_{t+j-1|t-1} + \sum_{i \neq j} D_i \tilde{f}(\vec{\mu}) D_j \tilde{f}(\vec{\mu}) \gamma_{t+i-1,t+j-1|t-1}\right],\\
&\quad E(F_{t+1}|R_{t-1} = r_{t-1}) \approx (1 + \mu_{t|t-1})(F_t + \mathrm{NC}_t - B_t) + (\mathrm{AL}_t - F_t)\tilde{g}(\vec{\mu}),
\end{aligned}$$

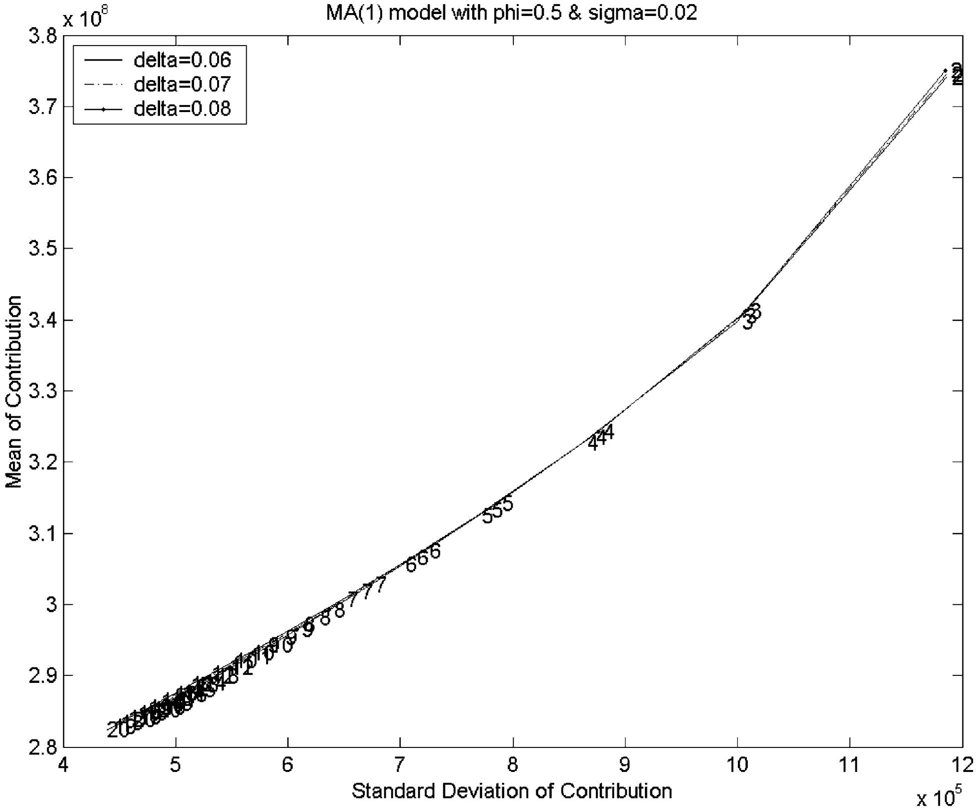

Fig. 2. Graph shows the pattern of mean versus standard deviation of contribution in MA(1) model with each point on the curves related to the identified values of M at time $t = 1997$.

$$
\begin{aligned}
\mathrm{Var}(F_{t+1}|R_{t-1}=r_{t-1}) &= \mathrm{Var}[(1+R_t)(F_t+C_t-B_t)] \\
&= \mathrm{Var}[(1+R_t)(F_t+\mathrm{NC}_t-B_t+k_t(\mathrm{AL}_t-F_t))] \\
&= (F_t+\mathrm{NC}_t-B_t)^2\mathrm{Var}(1+R_t)+(\mathrm{AL}_t-F_t)^2\mathrm{Var}(k_t(1+R_t))+2(F_t+\mathrm{NC}_t-B_t)(\mathrm{AL}_t-F_t) \\
&\quad \times\mathrm{Cov}(1+R_t,k_t(1+R_t)) \approx (F_t+\mathrm{NC}_t-B_t)^2\sigma_t^2+(\mathrm{AL}_t-F_t)^2 \\
&\quad \times\left[\sum_{j=1}^{M-1}(D_j\tilde{g}(\vec{\mu}))^2\mathrm{Var}(R_{t+j-1})+\sum_{i\neq j}D_i\tilde{g}(\vec{\mu})D_j\tilde{g}(\vec{\mu})\mathrm{Cov}(R_{t+i-1},R_{t+j-1})\right] \\
&\quad +2(F_t+\mathrm{NC}_t-B_t)(\mathrm{AL}_t-F_t)(\tilde{h}(\vec{\mu})-(1+\mu_t)\tilde{g}(\vec{\mu}))=(F_t+\mathrm{NC}_t-B_t)^2\sigma_t^2+(\mathrm{AL}_t-F_t)^2 \\
&\quad \times\left[\sum_{j=1}^{M-1}(D_j\tilde{g}(\vec{\mu}))^2\sigma^2_{t+j-1|t-1}+\sum_{i\neq j}D_i\tilde{g}(\vec{\mu})D_j\tilde{g}(\vec{\mu})\gamma_{t+i-1,t+j-1|t-1}\right].
\end{aligned}
$$

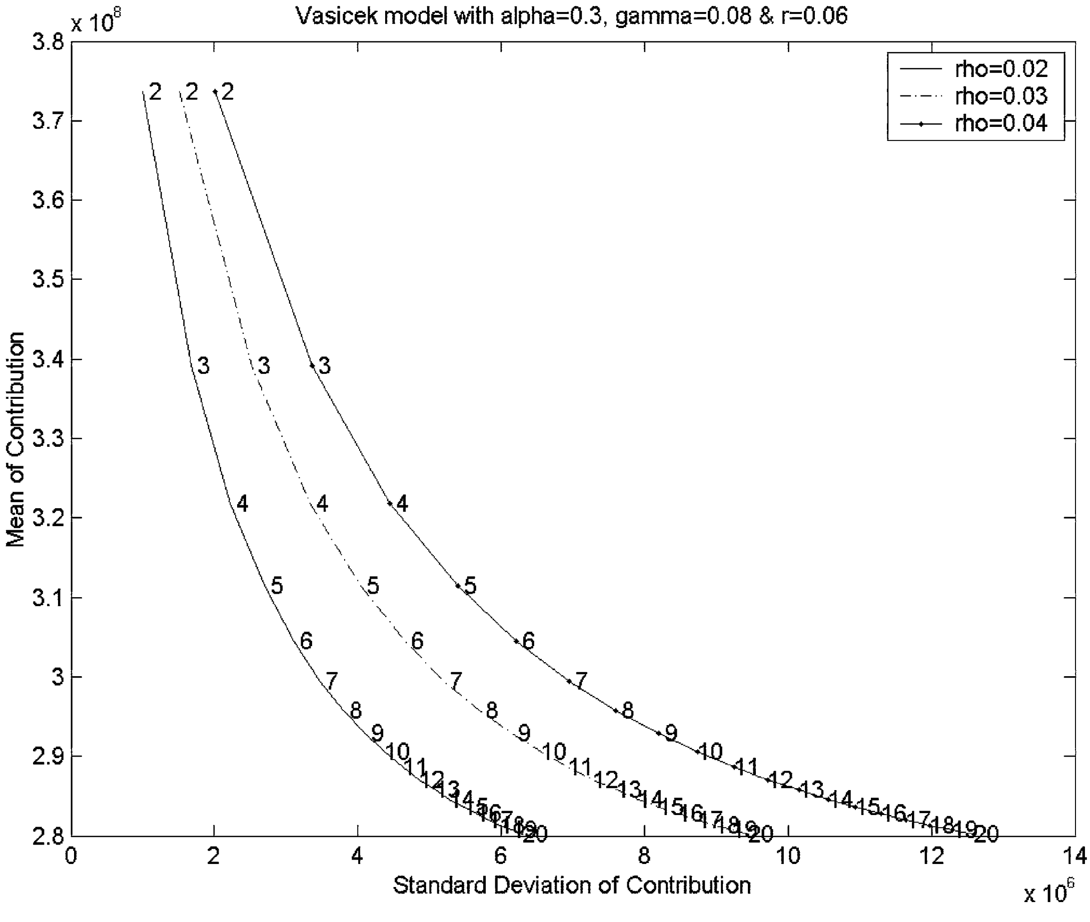

Fig. 3. Graph shows the pattern of mean versus standard deviation of contribution in Vasicek model with each point on the curves related to the identified values of M at time $t=1997$.

5. Numerical illustrations

In this section, we illustrate and evaluate the numerical approximation proposed in Section 4 in cost allocation for a realistic pension plan. Taiwan public employees retirement system (Tai-PERS) is used for illustration purpose. The cash flows of the benefit payment, accrued liability and normal cost in 20 years starting from 1997 are estimated based on 50 dynamic simulations using EAN cost method and open group assumption. Detailed benefit scheme and the procedure in performing the calculations can also be found in Chang (1999, 2000). The specific pension financial information of this plan at time $t = 1997$ is specified as follows: $B_t = 106, 636, 560$, $\mathrm{AL}_t = 585, 530, 240$, $\mathrm{NC}_t = 264, 658, 176$, $F_t = 373, 211, 585$ (measured in NT dollar). Since Tai-PERS provides a comprehensive compensation plan for its member and the funding policy is constrained by the current government regulation, this scheme starts with a significant deficit (i.e., $F_t < \mathrm{AL}_t$). The numerical results using stochastic models (i.e., unconditional approach using AR(1), MA(1) and conditional approach using Vasicek model) in generating the forward rates are investigated in detail.

Figs. 1–6 illustrate the estimated mean and standard deviation of contribution level and fund size based on these models. Fig. 1 shows how the standard deviation and expected contribution varies by M under AR(1). The expected interest rates are set at different values to monitor the impact on funding stability given various cost allocation

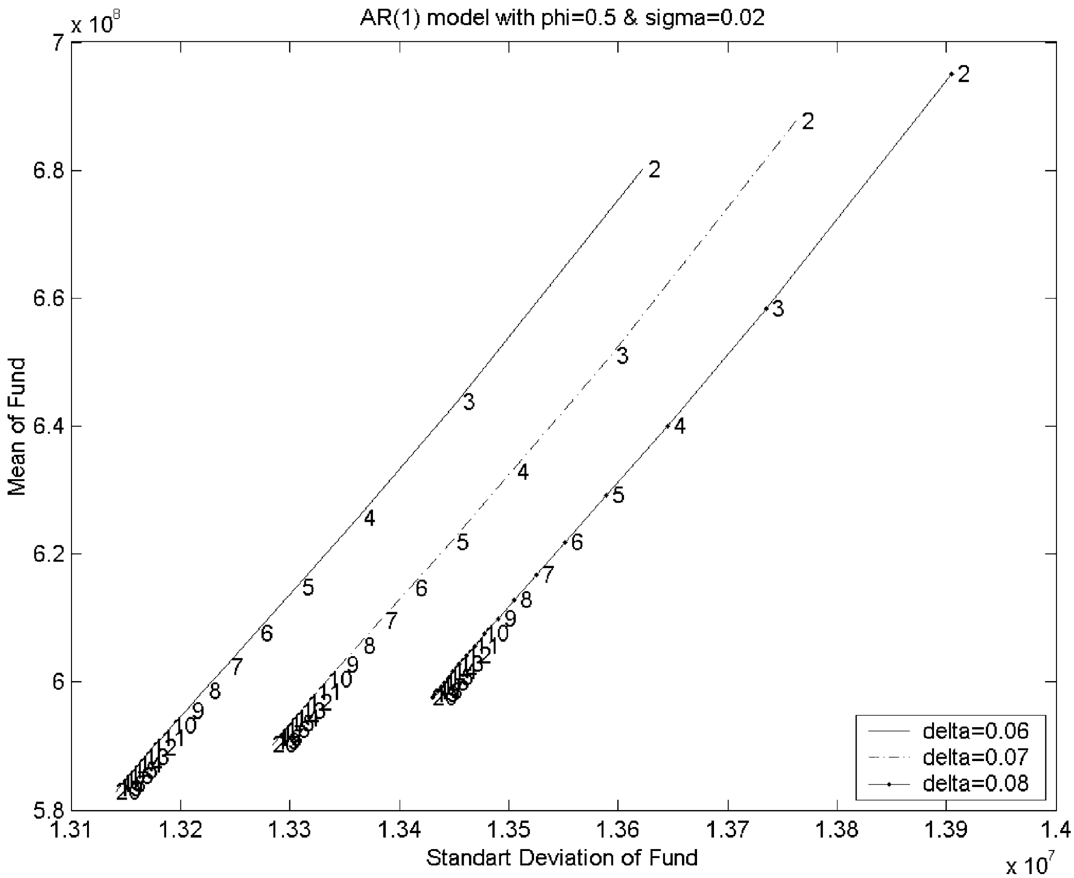

Fig. 4. Graph shows the pattern of mean versus standard deviation of fund in AR(1) model with each point on the curves related to the identified values of M at time $t = 1997$.

periods. It shows that the variances of contributions are higher when interest rates increase in each given scenario. Increasing the cost allocation period from $M = 2$ results in decreasing the expected contributions, while the variances of the contribution increase and then decrease gradually after $M = 6$. Based on outcomes from these scenarios, the plan manager can choose an optimal M according to his aimed financial status.

Fig. 2 shows how the standard deviation and expected contribution varies by M under MA(1). The expected contributions decrease and their variance increase when larger M is used in amortizing the unfunded. There are no significant difference on the patterns between different expected interest rates. In Fig. 3, Vasicek model is used to investigate the funding stability. The volatility of the interest rates are varied to analyze the mean and variance of the contribution at different M and investigate the optimal cost allocation. The volatility of the returns is set to be 4%, 3% and 2% to monitor the funding stability. It shows that variance of the contributions are larger when volatility of interest rates increases. Changing the cost allocation M results in decreasing the expected contribution and increasing in its variation. When M increases and more than 6, the variance increases dramatically. Hence, increasing M over a certain level fund manager can suffer large funding instability.

The numerical results for the stability of fund sizes are plotted in Figs. 4–6. Fig. 4 shows that the variance of fund size decreases when larger cost allocation M is used under AR(1). Then different interest rates are selected to analyze their impacts on the stability of fund. It shows that variation increases when interest rate increases. Using

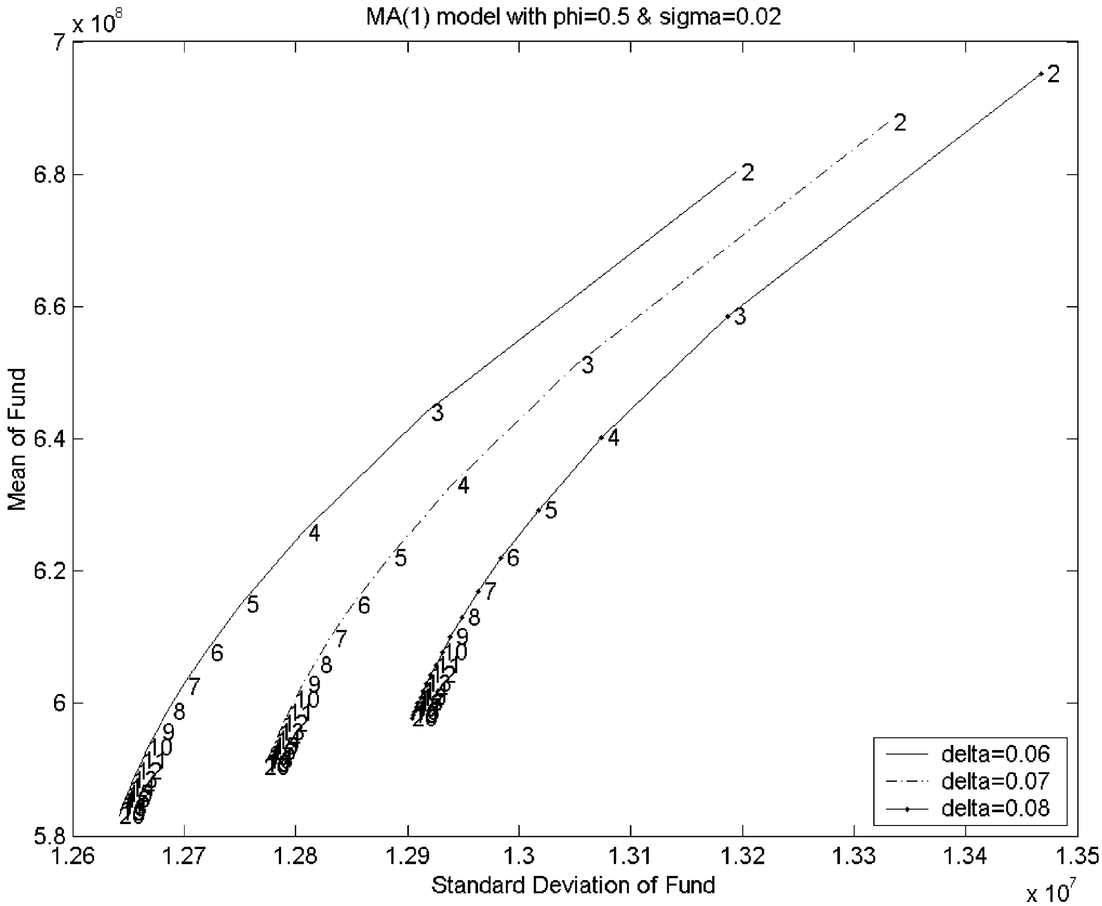

Fig. 5. Graph shows the pattern of mean versus standard deviation of fund in MA(1) model with each point on the curves related to the identified values of M at time $t = 1997$.

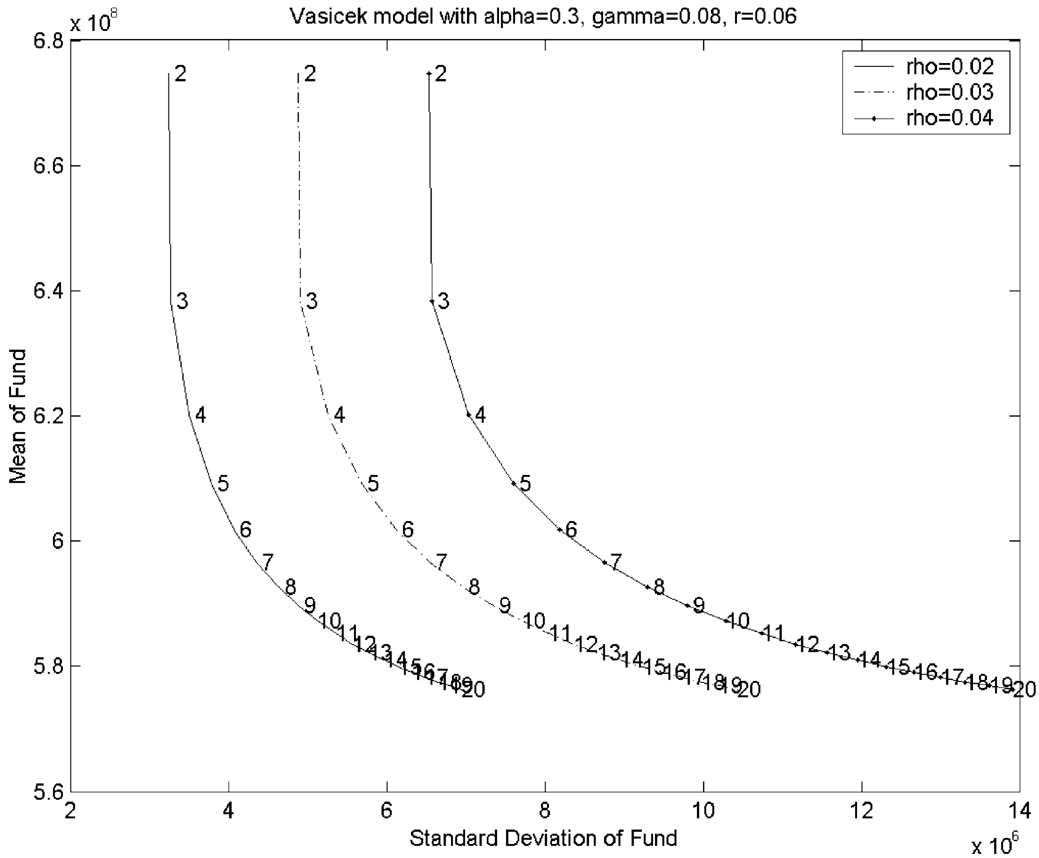

Fig. 6. Graph shows the pattern of mean versus standard deviation of fund in Vasicek model with each point on the curves related to the identified values of M at time $t = 1997$.

large M in allocating the unfunded liability results in smaller variation in fund size. Fig. 5 shows that the variance of fund size decreases when larger cost allocation M is used under MA(1). The pattern is similar with that in AR(1), except some small differences in shape.

Based on the numerical investigation of these results, the shapes of variation of fund level and contribution as function of M shown in Figs. 1 and 4 for AR results resemble those of Haberman (1994), Cairns and Parker (1997). While, the shapes presented for MA results have also shown the similar patterns with those of Haberman and Wong (1997), Bedard (1999) for Figs. 2 and 5.

Fig. 6 indicates that larger volatility of the interest rates in Vasicek model generates larger variation in fund sizes. As M increases in cost allocation, larger variation results in fund level. Hence select larger M may cause volatile fund levels, while employing smaller M in allocating the costs may be intervened by the political reasons and confront the plan short term insolvency. Hence the decision maker need to carefully measure the trade-off and reach a reasonable conclusion.

6. Concluding remarks

This paper studies the mean and standard deviation of the contribution and fund under several plausible stochastic models. The Taylor series expansion is used in approximating the mean and variance as functions of the allocation period M. The empirical results presented in this paper can provide valuable information in cost allocation. The optimal contribution can then be determined from the trade-off between the expected contribution and the associated variation.

In practice, the valuation actuary may need to select the proper stochastic model for the interest rates before he sets up the cost allocation period for the unfunded liability. In future research, these results will be extended to monitor the optimal cost allocation period in a more general framework.

Appendix A. The error terms of $f(\vec{X})$, $g(\vec{X})$ and $h(\vec{X})$

Consider the neighborhood of $\vec{\delta}$, by the Taylor series expansion, we have

$$f(\vec{X}) = f(\vec{\delta}) + f'(\vec{\delta}; \vec{X}-\vec{\delta}) + \frac{1}{2!} f''(\vec{Z}_x; \vec{X}-\vec{\delta}),$$
$$g(\vec{X}) = g(\vec{\delta}) + g'(\vec{\delta}; \vec{X}-\vec{\delta}) + \frac{1}{2!} g''(\vec{W}_x; \vec{X}-\vec{\delta}),$$
$$h(\vec{X}) = h(\vec{\delta}) + h'(\vec{\delta}; \vec{X}-\vec{\delta}) + \frac{1}{2!} h''(\vec{U}_x; \vec{X}-\vec{\delta}),$$

where $\vec{Z}_x$, $\vec{W}_x$ and $\vec{U}_x$ are on the line segment with two endpoints $\vec{\delta}$ and $\vec{X}$. If $f(\vec{X})$, $g(\vec{X})$ and $h(\vec{X})$ are replaced by $f(\vec{\delta}) + f'(\vec{\delta}; \vec{X}-\vec{\delta})$, $g(\vec{\delta}) + g'(\vec{\delta}; \vec{X}-\vec{\delta})$ and $h(\vec{\delta}) + h'(\vec{\delta}; \vec{X}-\vec{\delta})$, respectively, then the error terms will be $(1/2!)f''(\vec{Z}_x; \vec{X}-\vec{\delta})$, $(1/2!)g''(\vec{W}_x; \vec{X}-\vec{\delta})$ and $(1/2!)h''(\vec{U}_x; \vec{X}-\vec{\delta})$ accordingly. We could estimate the error terms and show that the error terms could be quite small when $\vec{X}$ is sufficiently close to $\vec{\delta}$.

Lemma 3. *Recall that* $\vec{X} = (\delta_t, \delta_{t+1}, \dots, \delta_{t+M-2})$ *and* $\vec{\delta} = (\bar{\delta}_t, \bar{\delta}_{t+1}, \dots, \bar{\delta}_{t+M-2})$. *Let* $(1/2!)f''(\vec{Z}_x; \vec{X}-\vec{\delta})$, $(1/2!)g''(\vec{W}_x; \vec{X}-\vec{\delta})$ *and* $(1/2!)h''(\vec{U}_x; \vec{X}-\vec{\delta})$ *be defined as above. Then*

1. $\left|\frac{1}{2!} f''(\vec{Z}_x; \vec{X}-\vec{\delta})\right| < \frac{1}{2!} r^2 \sum_{i=1}^{M-1} \left[\sum_{j=1}^{M-1} \left(\frac{r_2(i,M-1)}{(r_1(0,M-1))^2} + \frac{2r_2(0,j-1)r_2(i,M-1)}{(r_1(0,M-1))^3}\right) + \sum_{j=i+1}^{M-1} \frac{r_2(i,j-1)}{(r_1(0,M-1))^3}\right]$ *for* $|\delta_{t+j-1} - \bar{\delta}_{t+j-1}| < r, \forall j = 1, 2, \dots, M-1$.
2. $\left|\frac{1}{2!} g''(\vec{W}_x; \vec{X}-\vec{\delta})\right| < \frac{1}{2!} r^2 \sum_{i=2}^{M-1} \left[\sum_{j=1}^{M-1} \left(\frac{r_2(i,M-1)}{(r_1(0,M-1))^2} + \frac{2r_2(0,j-1)r_2(i-1,M-2)}{(r_1(0,M-1))^3}\right) + \sum_{j=i+1}^{M-1} \frac{r_2(i-1,j-2)}{(r_1(0,M-1))^3}\right] + \left(\sum_{j=1}^{M-1} \frac{2r_2(j-1,M-2)r_2(1,M-1)}{(r_1(0,M-1))^3}\right) + \left(\sum_{i=1}^{M-1} \frac{r_2(i-1,M-2)}{(r_1(0,M-1))^2}\right) + \frac{e^{\bar{\delta}_{\langle t\rangle}+r}}{r_1(0,M-1)}$ *for* $|\delta_{t+j-1} - \bar{\delta}_{t+j-1}| < r$, $\forall j = 1, 2, \dots, M-1$.
3. $\left|\frac{1}{2!} h''(\vec{U}_x; \vec{X}-\vec{\delta})\right| < \frac{1}{2!} r^2 \sum_{i=2}^{M-1} \left[\sum_{j=1}^{M-2} e^{\bar{\delta}_t+r} \left(\frac{r_2(i-1,M-2)}{(r_1(0,M-1))^2} + \frac{2r_2(0,j-1)r_2(i-1,M-2)}{(r_1(0,M-1))^3}\right) + \sum_{j=i+1}^{M-1} \frac{r_2(i-1,j-2)}{(r_1(0,M-1))^3}\right] + \sum_{j=1}^{M-1} e^{\bar{\delta}_{\langle t\rangle}+r} \left(\frac{r_2(j-1,M-2)}{(r_1(0,M-1))^2} + \frac{2r_2(j-1,M-2)r_2(1,M-1)}{(r_1(0,M-1))^3}\right) + \sum_{i=1}^{M-1} \left(\frac{2\,e^{\bar{\delta}_{\langle t\rangle}+r} r_2(i-1,M-2)}{(r_1(0,M-1))^2}\right) + \frac{4\,e^{2(\bar{\delta}_{\langle t\rangle}+r)}}{r_1(0,M-1)}$ *for* $|\delta_{t+j-1} - \bar{\delta}_{t+j-1}| < r, \forall j = 1, 2, \dots, M-1$, *where*

$$r_1(i,j) = \sum_{n=i}^{j} \exp\left(-\sum_{l=1}^{n} (\bar{\delta}_{t+l-1} + r)\right), \quad j \geq i, \qquad r_2(i,j) = \sum_{n=i}^{j} \exp\left(-\sum_{l=1}^{n} (\bar{\delta}_{t+l-1} - r)\right), \quad j \geq i.$$

Proof. We only prove the case of $f(\vec{X})$; the others can be obtained by the same technique.

Let $S(t, t+j-1) = \exp(-\sum_{n=1}^{j} \delta_{t+n-1})$, $j = 1, 2, \dots, M-1$, $S(t, t-1) = 1$,
$Z_t(i,j) = \sum_{n=i}^{j} S(t, t+n-1), \quad i, j = 1, 2, \dots, M-1$ and $j \geq i$.

Table 1
Values of $f(x)$, upper bounds of $(1/2)f''(x)$ and their ratios (in%) in various M and r

M	$f(x)$ at $x = 0.06$	$r = 0.001$		$r = 0.005$		$r = 0.01$		$r = 0.02$	
		Upper bound of $(1/2)f''(x)$	$(1/2)f''(x)/f(0.06)$ (%)	Upper bound of $(1/2)f''(x)$	$(1/2)f''(x)/f(0.06)$ (%)	Upper bound of $(1/2)f''(x)$	$(1/2)f''(x)/f(0.06)$ (%)	Upper bound of $(1/2)f''(x)$	$(1/2)f''(x)/f(0.06)$ (%)
2	0.51499550	0.00000025	0.000049	0.00000641	0.001244	0.00002592	0.005033	0.00010599	0.020581
3	0.35352116	0.00000072	0.000203	0.00001821	0.005151	0.00007438	0.021040	0.00031030	0.087773
4	0.27292910	0.00000124	0.000455	0.00003180	0.011650	0.00013118	0.048065	0.00055813	0.204498
5	0.22468966	0.00000181	0.000804	0.00004663	0.020752	0.00019424	0.086450	0.00084263	0.375018
6	0.19262622	0.00000240	0.001248	0.00006255	0.032473	0.00026308	0.136575	0.00116324	0.603883
7	0.16980588	0.00000303	0.001786	0.00007952	0.046830	0.00033760	0.198817	0.00152104	0.895754
8	0.15276214	0.00000369	0.002417	0.00009752	0.063837	0.00041786	0.273536	0.00191771	1.255354
9	0.13956914	0.00000438	0.003138	0.00011655	0.083504	0.00050395	0.361074	0.00235512	1.687422
10	0.12907129	0.00000510	0.003949	0.00013661	0.105838	0.00059599	0.461749	0.00283531	2.196698
11	0.12053322	0.00000584	0.004848	0.00015771	0.130842	0.00069410	0.575857	0.00336033	2.787884
12	0.11346463	0.00000662	0.005833	0.00017986	0.158514	0.00079841	0.703667	0.00393226	3.465629
13	0.10752606	0.00000742	0.006903	0.00020306	0.188847	0.00090905	0.845424	0.00455319	4.234502
14	0.10247500	0.00000825	0.008055	0.00022732	0.221833	0.00102612	1.001341	0.00522517	5.098968
15	0.09813362	0.00000911	0.009288	0.00025265	0.257455	0.00114974	1.171608	0.00595020	6.063365
16	0.09436849	0.00001000	0.010599	0.00027905	0.295698	0.00128000	1.356383	0.00673025	7.131883
17	0.09107758	0.00001092	0.011988	0.00030651	0.336537	0.00141698	1.555795	0.00756722	8.308545
18	0.08818151	0.00001186	0.013452	0.00033504	0.379949	0.00156076	1.769944	0.00846294	9.597183
19	0.08561761	0.00001283	0.014989	0.00036465	0.425903	0.00171141	1.998902	0.00941916	11.001426
20	0.08333570	0.00001383	0.016597	0.00039532	0.474366	0.00186898	2.242710	0.01043753	12.524677

Thus

$$f(\vec{X}) = \frac{1}{Z_t(0, M-1)}, \qquad D_j f(\vec{X}) = \frac{Z_t(j, M-1)}{(Z_t(0, M-1))^2}, \ j = 1, 2, \dots, M-1,$$

$$D_{i,j} f(\vec{X}) = \begin{cases} D_i f(\vec{X}) - \dfrac{2Z_t(0, j-1)Z_t(i, M-1)}{(Z_t(0, M-1))^3}, & \text{if } \ j \le i \\[2ex] D_i f(\vec{X}) + \dfrac{Z_t(i, j-1)}{(Z_t(0, M-1))^3} - \dfrac{2Z_t(0, j-1)Z_t(i, M-1)}{(Z_t(0, M-1))^3}, & \text{if } \ j > i. \end{cases}$$

For $|\delta_{t+j-1} - \bar{\delta}_{t+j-1}| < r, \forall j = 1, 2, \dots, M-1,$
we have,

$$\begin{aligned} &\bar{\delta}_{t+j-1} - r < \delta_{t+j-1} < \bar{\delta}_{t+j-1} + r \\ &\quad \Rightarrow -(\bar{\delta}_{t+j-1} + r) < -\delta_{t+j-1} < -(\bar{\delta}_{t+j-1} - r) \\ &\quad \Rightarrow \exp(-(\bar{\delta}_{t+j-1} + r)) < \exp(-\delta_{t+j-1}) < \exp(-(\bar{\delta}_{t+j-1} - r)) \\ &\quad \Rightarrow \exp\left(-\sum_{l=1}^{n}(\bar{\delta}_{t+j-1} + r)\right) < S(t, t+n-1) < \exp\left(-\sum_{l=1}^{n}(\bar{\delta}_{t+j-1} - r)\right) \\ &\quad \Rightarrow \sum_{n=i}^{j}\exp\left(-\sum_{l=1}^{n}(\bar{\delta}_{t+j-1} + r)\right) < Z_t(i, j) < \sum_{n=i}^{j}\exp\left(-\sum_{l=1}^{n}(\bar{\delta}_{t+j-1} - r)\right), \end{aligned}$$

i.e.,

$$r_1(i, j) < Z_t(i, j) < r_2(i, j).$$

By a straightforward process, we get an upper bound for $(1/2!) f''(\vec{Z}_x; \vec{X} - \vec{\delta})$:

$$\begin{aligned} &\left|\frac{1}{2!} f''(\vec{Z}_x; \vec{X} - \vec{\delta})\right| \\ &\quad < \frac{1}{2!} r^2 \sum_{i=1}^{M-1}\left[\sum_{j=1}^{M-1}\left(\frac{r_2(i, M-1)}{r_1(0, M-1)^2} + \frac{2r_2(0, j-1)r_2(i, M-1)}{(r_1(0, M-1))^3}\right) + \sum_{j=i+1}^{M-1}\frac{r_2(i, j-1)}{(r_1(0, M-1))^3}\right]. \qquad \square \end{aligned}$$

The upper bounds of $(1/2!) f''(\vec{Z}_x; \vec{X} - \vec{\delta})$ and the error ratios $(1/2!) f''(\vec{Z}_x; \vec{X} - \vec{\delta})/f(\vec{\delta})$ given M between 2 and 20 and $r = 0.001, 0.005, 0.01$ and 0.02 are evaluated. The results are listed in Table 1 for numerical illustrations. In our numerical calculations, we set $\vec{\delta} = 0.06 \times (1, \dots, 1)_{1\times(M-1)}$ for simplicity.

References

Bedard, D., 1999. Stochastic pension funding: proportional control and bilinear processes. ASTIN Bulletin 29, 271–293.

Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., Nesbitt, C.J., 1982. Notes on the dynamics of pension funding. Insurance: Mathematics and Economics 1, 261–270.

Cairns, A.J.G., 2000. Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time. ASTIN Bulletin 30, 19–55.

Cairns, A.J., Parker, G., 1997. Stochastic pension fund modelling. Insurance: Mathematics and Economics 21, 43–79.

Chang, S.C., 1999. Optimal pension funding through dynamic simulations: the case of Taiwan public employees retirement system. Insurance: Mathematics and Economics 24, 187–199.

Chang, S.C., 2000. Realistic pension funding: a stochastic approach. Journal of Actuarial Practice 8, 5–42.

Cox, J.C., Ingersoll, J., Ross, S., 1985. A theory of the term structure of interest rates. Econometrica 53, 385–407.
Dufresne, D., 1988. Moments of pension fund contributions and fund levels when rates of return are random. Journal of the Institute of Actuaries 115, 535–544.
Dufresne, D., 1989. Stability of pension systems when rates of return are random. Insurance: Mathematics and Economics 6, 129–134.
Gerrard, R.J., Haberman, S., 1996. Stability of pension systems when gains/losses are amortized and rates of return are autoregressive. Insurance: Mathematics and Economics 18, 59–71.
Haberman, S., 1992. Pension funding with time delays: a stochastic approach. Insurance: Mathematics and Economics 11, 179–189.
Haberman, S., 1993. Pension funding with time delays and autoregressive rates of investment return. Insurance: Mathematics and Economics 13, 45–56.
Haberman, S., 1994. Autoregressive rates of return and the variability of pension contributions and fund levels for a defined benefit pension scheme. Insurance: Mathematics and Economics 14, 219–240.
Haberman, S., 1997. Stochastic investment returns and contribution rate risk in a defined benefit pension scheme. Insurance: Mathematics and Economics 14, 127–139.
Haberman, S., Wong, P.L.Y., 1997. Moving average rates of return and the variability of pension contributions and fund levels for a defined benefit pension scheme. Insurance: Mathematics and Economics 20, 115–135.
Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatilities. Journal of Finance 42, 281–300.
Mandl, P., Mazurova, L., 1996. Harmonic analysis of pension funding methods. Insurance: Mathematics and Economics 17, 203–214.
McKenna, F.W., 1982. Pension plan cost risk. Journal of Risk and Insurance 49, 193–217.
Owadally, M.I., Haberman, S., 1999. Pension fund dynamics and gain/losses due to random rates of investment return. North American Actuarial Journal 3, 105–117.
Owadally, M.I., Haberman, S., 2000. Efficient amortization of actuarial gains/losses and optimal funding in pension plans. Actuarial Research Paper No. 133. Department of Actuarial Science and Statistics, City University, London.
Vasicek, O.A., 1977. An equilibrium characterization of the term structure. Journal of Financial Economics 5, 177–188.