國立政治大學應用數學系

碩士學位論文

具非線性連接之 Hindmarsh-Rose 神經元耦合系統的同步化研究

Synchronization of nonlinearly coupled systems of Hindmarsh-Rose neurons with time delays

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中文摘要

在此論文，我們研究 Hindmarsh-Rose 神經元耦合系統的同步化，我們所考慮的模型之耦合結構可以相等的一般性。模型所具備的耦合函數可以是非線性的，耦合矩陣可容許非零的非對角元素能有不同的正負號，並且我們也考慮耦合時間延遲。藉由 [33] 的同步化理論，我們推導出與時間延遲相關的同步化條件。我們提供兩個數值例子來表現本論文同步化理論之效用。

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Abstract

In this thesis, we investigate the synchronization of coupled systems of Hindmarsh-Rose neurons. The coupling scheme under consideration is general. The coupling functions could be non-linear. The connection matrix could have non-zero and non-diagonal entries with different signs. We also consider the transmission delays in the coupling terms of the coupled systems. We derive a delay-dependent criterion that leads to the synchronization of coupled neurons. Two examples with numerical simulations are illustrated to show the effectiveness of theoretical result.

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Chapter 1

Introduction

Synchronization is an important phenomenon in several biological, physiological complex networks systems [14, 31]. For instance, there has been an observation that neurons synchronize each other by coupled dynamic neural network [2, 8, 22, 23, 27, 32, 34, 37, 38]. The process of conveying neural information in brain is conducted through mutual interaction of neural populations [6]. In addition, the information delivering process in the neural systems, synchronization plays a key role in association and memory [26]. As the basic behavior of neuron, synchronization is an expression of neuron discharge, widely existing in the neural system like the visual cortex [25, 35]. However, it is shown that too much synchronization will do harm to the organism and cause such brain diseases as Alzheimer's disease, epilepsy, Parkinson's disease, and schizophrenia based on several physiological experiments [15, 24, 39]. Since those illnesses have a lot to do with the abnormal synchronization of neuron systems, it is worth to study the synchronization in coupled dynamic networks.

To obtain a deep understanding of neural network dynamics, lots of neural models such as the Hodgkin-Huxley model [21, 28], the FitzHugh-Nagumo model [22, 33], the Morris-Lecar model [1, 3] and the Hindmarsh-Rose model [17, 36] have been used by numerous researches for biological application. Among these models, Hindmarsh-Rose neurons were discovered from neuron cells in a pond snail which burst after it is depolarized by a pulse of short current [4, 36]. What's more, it has been shown that the Hindmarsh-Rose neuron model, a system of three ordinary differential equations, is able to produce abundant neural behavior like spiking, bursting, and chaotic behavior [7, 11, 13]. In this paper, we will investigate the synchronization of the coupled systems of Hindmarsh-Rose neurons.

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The dynamics of an isolated single Hindmarsh-Rose neuron could be depicted by the following system of ordinary differential equations [10]:

$$\begin{cases} \dot{x}_1(t) = x_2(t) - a(x_1(t))^3 + b(x_1(t))^2 - x_3(t) + I, \\ \dot{x}_2(t) = 1 - dx_1(t)^2 - x_2(t), \\ \dot{x}_3(t) = r(s(x_1(t) + x_0) - x_3(t)), \end{cases} \tag{1.1}$$

where x_1 represents the membrane potential of the neuron, x_2 the recovery variable, and x_3 the adaptation current. The external input current is represented by I to determine the output mode of the neuron. The parameters a, b, d, r, s, x_0 are all positive constants.

Time delay, which occurs in the propagation of action potentials along the axon, transferring signals across the synapse or other artificial units, is an important factor in the coupled neural systems [5,12]. It is essential to tackle nonlinear systems under delayed coupling. Therefore, in this paper, we shall consider the following coupled network consisting of N identical Hindmarsh-Rose neurons with coupling time delay:

$$\begin{cases} \dot{x}_{i,1}(t) = x_{i,2}(t) - ax_{i,1}(t)^3 + bx_{i,1}(t)^2 - x_{i,3}(t) + I \\ \qquad + c \sum_{j \in \mathcal{N}, j \neq i} a_{ij}(g(x_{j,1}(t-\tau)) - g(x_{i,1}(t-\tau))), \\ \dot{x}_{i,2}(t) = 1 - dx_{i,1}(t)^2 - x_{i,2}(t), \\ \dot{x}_{i,3}(t) = r(s(x_{i,1}(t) + x_0) - x_{i,3}(t)), \end{cases} \tag{1.2}$$

for $i \in \mathcal{N} := \{1, 2, ..., N\}$, where $c \geq 0$ is the coupling strength ;

$$A = [a_{ij}]_{N \times N}, \text{ with } a_{ii} := -\sum_{j=1, j \neq i}^{\mathcal{N}} a_{ij}, \tag{1.3}$$

is the connection matrix representing the topological structure of the network; $g : \mathbb{R} \to \mathbb{R}$ is the coupling function; $\tau \geq 0$ is the transmission delay. We note that system (1.2) is linearly coupled if g is linear, otherwise it is non-linearly coupled. Among the existing investigations on coupled Hindmarsh-Rose neurons, some of the investigations considered with linearly coupling

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function: $g(x) = x$, cf. [36]; while some others considered non-linearly coupling function: $g(x) = tanh(x)$, cf. [17]. Moreover, the work in [36] considered linearly coupled systems of two Hindmarsh-Rose neurons with time delays (i.e. $\tau \neq 0$); the work in [17] considered non-linearly coupled Hindmarsh-Rose neurons without delay (i.e. $\tau = 0$), cf. [17]. It is worth noting that most of the previous investigations on the synchronization of coupled systems required that the non-zero and off-diagonal entries of connection matrix have the same signs, cf. [9, 16–20, 30]; moreover, most of the previous investigations on the synchronization of coupled systems considered linear couplings, cf. [2, 27].

In the following, $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))$ denotes an arbitrary solution of system (1.2) and $(\mathbf{x}_1^t, \ldots, \mathbf{x}_N^t)$ is the corresponding evolution of system (1.2), where $\mathbf{x}_i^t \in \mathcal{C}([-\tau, 0]; \mathbb{R}^K)$, $i \in \mathcal{N}$, written as $\mathbf{x}_i^t(\theta) = \mathbf{x}_i(t + \theta)$ for $\theta \in [-\tau, 0]$. It is said that the system (1.2) achieves global synchronization if

$$x_{i,k}(t) - x_{j,k}(t) \to 0, \text{ as } t \to \infty, \text{ for all } i, j \in \mathcal{N}, \ k \in \{1, 2, 3\},$$

for every solution $(\mathbf{x}_1(t), \ldots, \mathbf{x}_3(t))$, where $\mathbf{x}_i(t) = (x_{i,1}(t), x_{i,2}(t), x_{i,3}(t))$. In this paper, we shall utilize the approach developed in [33] to investigate the global synchronization of coupled system (1.2) with the coupling function g in the following class:

$$\{g \in C^1 : \delta := g'(0) > g'(x) > 0, \ x \neq 0\}. \tag{1.4}$$

We emphasize that the connection matrix $A = [a_{ij}]_{N \times N}$, considered in this thesis, is allowed to have off-diagonal entries with different signs.

The remainder of this thesis is organized as follows. In chapter 2, we introduce the synchronization theory developed in [33]. In chapter 3, we establish the synchronization of non-linearly coupled systems of Hindmarsh-Rose neurons based on the theory in [33]. In chapter 4, we demonstrate two numerical examples to support our theory. In chapter 5, we give some discussions and conclusions.

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Chapter 2

Preliminaries

In this chapter, we shall introduce the synchronization theory developed in [33], the model considered in [33] is as follows:

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t), t) + c\sum_{j\in\mathcal{N}} a_{ij}(t)\mathbf{G}(\mathbf{x}_j(t-\tau(t))),\ i \in \mathcal{N},\ t \geq t_0, \tag{2.1}$$

where $\mathcal{N} = \{1, \ldots, N\}$, $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t)) \in \mathbb{R}^K$, $\mathbf{F} = (F_1, \ldots, F_K)$ is a smooth function describing the intrinsic dynamics of each subsystem, $c \geq 0$ is the coupling strength, and $a_{ij}(t)$, $i, j \in \mathcal{N}$, are bounded functions of t. Matrix $A(t) := [a_{ij}(t)]_{1\leq i,j\leq N}$ is referred to as the connection matrix and is assumed to satisfy the condition:

$$\sum_{j\in\mathcal{N}} a_{ij}(t) = \kappa(t), \quad \text{for all } i \in \mathcal{N} \text{ and } t \geq t_0. \tag{2.2}$$

The function $\mathbf{G} = (G_1, \ldots, G_K)$ is assumed to satisfy

$$G_k(\mathbf{x}_j(t-\tau(t))) = g_k(x_{j,k}(t-\tau(t))), \quad \text{for all } i, j \in \mathcal{K} \text{ and } t \geq t_0, \tag{2.3}$$

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where $\mathcal{K} := \{1, \ldots, K\}$, g_k is a non-decreasing and differentiable function, and $\tau(t) \in [0, \tau_M]$ stands for the time-dependent transmission delay. For later use, set

$$\bar{\kappa} = \sup\{|\kappa(t)| : t \geq t_0\}, \tag{2.4}$$

$$\check{\kappa} = \inf\{\kappa(t) : t \geq t_0\}, \tag{2.5}$$

$$\hat{\kappa} = \sup\{\kappa(t) : t \geq t_0\}, \tag{2.6}$$

$$\bar{a}_{ij} = \sup\{|a_{ij}(t)| : t \geq t_0\}, \tag{2.7}$$

$$\bar{\tau} = \sup\{\tau(t) : t \geq t_0\}. \tag{2.8}$$

In this section, $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))$ denotes an arbitrary solution of system (2.1), and $(\mathbf{x}_1^t, \ldots, \mathbf{x}_N^t)$ is the corresponding evolution of system (2.1), where $\mathbf{x}_i^t \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^K)$, $i \in \mathcal{N}$, are defined as $\mathbf{x}_i^t(\theta) = \mathbf{x}_i(t+\theta)$ for $\theta \in [-\tau_M, 0]$. System (2.1) is said to attain global (identical) synchronization, if

$$x_{i,k}(t) - x_{j,k}(t) \to 0, \text{ as } t \to \infty, \text{ for all } i, j \in \mathcal{N},\ k \in \mathcal{K},$$

for every solution $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))$, where $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t))$.

Let us introduce two assumptions as follows:
Assumption (D): All solutions of system (2.1) eventually enter and then remain in some compact set $\mathcal{Q}^N := \mathcal{Q} \times \cdots \times \mathcal{Q} \subset \mathbb{R}^{NK}$, where $\mathcal{Q} := [\check{q}_1, \hat{q}_1] \times \cdots \times [\check{q}_K, \hat{q}_K] \subset \mathbb{R}^K$.

Define

$$\begin{aligned} \mathcal{C}_{\mathcal{Q}} := \{(\Phi_1, \ldots, \Phi_N) : \Phi_i = (\phi_{i,1}, \ldots, \phi_{i,K}) \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^K), \\ \phi_{i,k}(\theta) \in [\check{q}_k, \hat{q}_k], \theta \in [-\tau_M, 0], i \in \mathcal{N},\ k \in \mathcal{K}\}. \end{aligned} \tag{2.9}$$

Decompose $F_k(E, t) - F_k(\tilde{E}, t)$ as

$$F_k(E, t) - F_k(\tilde{E}, t) = h_k(x_k, \tilde{x}_k, t) + w_k(E, \tilde{E}, t),$$

where $t \geq t_0$, $E = (x_1, \ldots, x_K)$, and $\tilde{E} = (\tilde{E}_1, \ldots, \tilde{E}_K)$.

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Assumption (F): For each $k \in \mathcal{K}$, there exist $\check{\mu}_k, \hat{\mu}_k \in \mathbb{R}$, $\rho_k^w \geq 0$, and $\bar{\mu}_{kl} \geq 0$, for $l \in \mathcal{K} - \{k\}$, such that for any $E, \tilde{E} \in \mathcal{Q}$, the following two properties hold for all $t \geq t_0$:

(F-i): $\begin{cases} \check{\mu}_k \leq h_k(x_k, \tilde{x}_k, t)/(x_k - \tilde{x}_k) \leq \hat{\mu}_k & \text{if } x_k - \tilde{x}_k \neq 0, \\ h_k(x_k, \tilde{x}_k, t) = 0 & \text{if } x_k - \tilde{x}_k = 0, \end{cases}$

(F-ii): $|w_k(E, \tilde{E}, t)| \leq \rho_k^w$, and $|w_k(E, \tilde{E}, t)| \leq \sum_{l \in \mathcal{K} - \{k\}} \bar{\mu}_{kl} |x_l - \tilde{x}_l|$.

For later use, define the following sets of indices:

$$\mathcal{A} := (\mathcal{N} - \{N\}) \times \mathcal{K} \text{ and } \mathcal{A}_{i,k} := \mathcal{A} - \{i\} \times \{k\}, \text{ where } (i,k) \in \mathcal{A}. \tag{2.10}$$

Assume that $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))$, where $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t))$, is an arbitrary solution of system (2.1). Setting

$$\mathbf{z}_i(t) := \mathbf{x}_i(t) - \mathbf{x}_{i+1}(t), i \in \mathcal{N} - \{N\}, \tag{2.11}$$

where $\mathbf{z}_i(t) = (z_{i,1}(t), \ldots, z_{i,K}(t))$, cf. (2.1) and (2.3), then $(\mathbf{z}_1(t), \ldots, \mathbf{z}_{N-1}(t))$ satisfies the following difference-differential system corresponding to system (2.1):

$$\dot{z}_{i,k}(t) = H_{i,k}(\mathbf{x}_1^t, \ldots, \mathbf{x}_N^t, t),\ (i,k) \in \mathcal{A},\ t \geq t_0, \tag{2.12}$$

where

$$\begin{aligned} H_{i,k}(\Phi_1, \ldots, \Phi_N, t) &:= F_k(\Phi_i(0), t) - F_k(\Phi_{i+1}(0), t) \\ &\quad + c \sum_{j \in \mathcal{N}} [a_{ij}(t) - a_{(i+1)j}(t)] g_k(\phi_{j,k}(-\tau(t))), \end{aligned} \tag{2.13}$$

for $\Phi_j = (\phi_{j,1}, \ldots, \phi_{j,K}) \in \mathcal{C}([-\tau_M, 0]; \mathbb{R}^K)$, $j \in \mathcal{N}$. Clearly, system (2.1) attains global synchronization if $z_{i,k}(t) \to 0$, as $t \to \infty$, for every $(i,k) \in \mathcal{A}$.

Via $A(t) = [a_{ij}(t)]_{1 \leq i,j \leq N}$, define matrix $\tilde{A}(t) = [\tilde{a}_{ij}(t)]_{1 \leq i,j \leq N}$, where

$$\tilde{a}_{ij}(t) = \begin{cases} a_{ii}(t) - \kappa(t), & \text{if } i = j \in \mathcal{N}, \\ a_{ij}(t), & \text{if } i, j \in \mathcal{N} \text{ and } i \neq j. \end{cases} \tag{2.14}$$

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We introduce matrix $\bar{A}(t)$:

$$\bar{A}(t) = [\alpha_{ij}(t)]_{1\le i,j\le N-1} := \mathbf{C}\tilde{A}(t)\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1} \in \mathbb{R}^{(N-1)\times(N-1)}, \tag{2.15}$$

where

$$\mathbf{C} := \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times N}.$$

Then, $\bar{A}(t)$ in (2.15) is well-defined, and satisfies

$$\mathbf{C}\tilde{A}(t) = \bar{A}(t)\mathbf{C}, \tag{2.16}$$

for all $t \ge t_0$. For later use, set

$$\bar{\alpha}_{ij} = \sup\{|\alpha_{ij}(t)| : t \ge t_0\}, \tag{2.17}$$

$$\breve{\alpha}_{ij} = \inf\{\alpha_{ij}(t) : t \ge t_0\}, \tag{2.18}$$

$$\hat{\alpha}_{ij} = \sup\{\alpha_{ij}(t) : t \ge t_0\}, \tag{2.19}$$

where $\alpha_{ij}(t)$, $1 \le i, j \le N-1$, are entries of $\bar{A}(t)$ defined in (2.15).

Proposition 2.1. *([33]) Consider system* (2.1) *which satisfies Assumptions (D) and (F). Then, functions $H_{i,k}$, $(i,k) \in \mathcal{A}$, defined in* (2.13)*, can be decomposed as*

$$\begin{aligned} H_{i,k}(\Phi_1,\ldots,\Phi_N,t) =\ & h_{i,k}(\phi_{i,k}(0),\phi_{i+1,k}(0),t) \\ & + \tilde{h}_{i,k}(\phi_{i,k},\phi_{i+1,k},t) + w_{i,k}(\Phi_1,\ldots,\Phi_N,t), \\ h_{i,k}(\phi_{i,k}(0),\phi_{i+1,k}(0),t) =\ & h_k(\phi_{i,k}(0),\phi_{i+1,k}(0),t), \\ \tilde{h}_{i,k}(\phi_{i,k},\phi_{i+1,k},t) =\ & c[\kappa(t)+\alpha_{ii}(t)][g_k(\phi_{i,k}(-\tau(t))) - g_k(\phi_{i+1,k}(-\tau(t)))], \\ w_{i,k}(\Phi_1,\ldots,\Phi_N,t) =\ & w_k(\Phi_i(0),\Phi_{i+1}(0),t) \\ & + c\sum_{j\in\mathcal{N}-\{i,N\}} \alpha_{ij}(t)[g_k(\phi_{j,k}(-\tau(t))) - g_k(\phi_{j+1,k}(-\tau(t)))]. \end{aligned}$$

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Moreover, for all $(i,k) \in \mathcal{A}$ *and all* $(\Phi_1, \ldots, \Phi_N) \in \mathcal{C}_{\mathcal{Q}}$, *where* $\Phi_i = (\phi_{i,1}, \ldots, \phi_{i,K})$, $i \in \mathcal{N}$, *the following three properties hold for all* $t \geq t_0$:

(H-i):
$$\begin{cases} \check{\mu}_k \leq \frac{h_{i,k}(\phi_{i,k}(0), \phi_{i+1,k}(0), t)}{[\phi_{i,k}(0) - \phi_{i+1,k}(0)]} \leq \hat{\mu}_k & \text{if } \phi_{i,k}(0) - \phi_{i+1,k}(0) \neq 0, \\ h_{i,k}(\phi_{i,k}(0), \phi_{i+1,k}(0), t) = 0 & \text{if } \phi_{i,k}(0) - \phi_{i+1,k}(0) = 0, \end{cases}$$

(H-ii): $|\tilde{h}_{i,k}(\phi_{i,k}, \phi_{i+1,k}, t)| \leq \rho^h_{ik}$, *and*

$$\begin{cases} \check{\beta}_{ik} \leq \frac{\tilde{h}_{i,k}(\phi_{i,k}, \phi_{i+1,k}, t)}{[\phi_{i,k}(-\tau(t)) - \phi_{i+1,k}(-\tau(t))]} \leq \hat{\beta}_{ik} & \text{if } \phi_{i,k}(-\tau(t)) - \phi_{i+1,k}(-\tau(t)) \neq 0, \\ \tilde{h}_{i,k}(\phi_{i,k}, \phi_{i+1,k}, t) = 0 & \text{if } \phi_{i,k}(-\tau(t)) - \phi_{i+1,k}(-\tau(t)) = 0, \end{cases}$$

(H-iii): $|w_{i,k}(\Phi_1, \ldots, \Phi_N, t)| \leq \rho^w_{ik}$, *and*

$$\begin{aligned} |w_{i,k}(\Phi_1, \ldots, \Phi_N, t)| \leq \sum_{(j,l) \in \mathcal{A}_{i,k}} & \{\bar{\mu}^{(jl)}_{ik} |\phi_{j,l}(0) - \phi_{j+1,l}(0)| \\ & + \bar{\beta}^{(jl)}_{ik} |\phi_{j,l}(-\tau(t)) - \phi_{j+1,l}(-\tau(t))|\}, \end{aligned}$$

for ρ^h_{ik} *and* ρ^w_{ik} *satisfying*

$$\rho^h_{ik} \geq 2c[\kappa(t) + \alpha_{ii}(t)]\rho^g_k \quad \text{and} \quad \rho^w_{ik} \geq \rho^w_k + 2c\rho^g_k \sum_{j \in \mathcal{N} - \{i,N\}} \bar{\alpha}_{ij},$$

respectively, and

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$$\check{\beta}_{ik} = \begin{cases} c[\kappa(t)+\alpha_{ii}(t)]\check{L}_k & \textit{if } \kappa(t)+\alpha_{ii}(t) \geq 0, \\ c[\kappa(t)+\alpha_{ii}(t)]\hat{L}_k & \textit{if } \kappa(t)+\alpha_{ii}(t) < 0, \end{cases}$$

$$\hat{\beta}_{ik} = \begin{cases} c[\kappa(t)+\alpha_{ii}(t)]\hat{L}_k & \textit{if } \kappa(t)+\alpha_{ii}(t) \geq 0, \\ c[\kappa(t)+\alpha_{ii}(t)]\check{L}_k & \textit{if } \kappa(t)+\alpha_{ii}(t) < 0, \end{cases}$$

$$\bar{\mu}_{ik}^{(jl)} = \begin{cases} \bar{\mu}_{kl} & \textit{if } i=j,\ k\neq l, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\bar{\beta}_{ik}^{(jl)} = \begin{cases} c\bar{\alpha}_{ij}\hat{L}_k & \textit{if } j\neq i,\ k=l, \\ 0 & \textit{otherwise}, \end{cases}$$

where

$$\rho_k^g := \max\{|g_k(x_i)| : x_i \in [\check{q}_k, \hat{q}_k]\} \geq 0, \tag{2.20}$$

$$\check{L}_k := \min\{g_k'(x_i) : x_i \in [\check{q}_k, \hat{q}_k]\} \geq 0, \tag{2.21}$$

$$\hat{L}_k := \max\{g_k'(x_i) : x_i \in [\check{q}_k, \hat{q}_k]\} \geq 0. \tag{2.22}$$

Herein, $\bar{\kappa}$, $\check{\kappa}$, and $\hat{\kappa}$ are defined in (2.6)*, $\mathcal{C}_{\mathcal{Q}}$ in* (2.9)*, $\mathcal{A}_{i,k}$ in* (2.10)*, $\bar{\alpha}_{ij}$, $\check{\alpha}_{ij}$, and $\hat{\alpha}_{ij}$ in* (2.17)*, $\check{\mu}_k$, $\hat{\mu}_k$, ρ_k^w, $\bar{\mu}_{kl}$, and functions h_k, w_k in Assumption (F).*

With $\check{\mu}_k$, $\hat{\mu}_k$, $\check{\beta}_{ik}$, $\hat{\beta}_{ik}$, $\bar{\mu}_{ik}^{(jl)}$, and $\bar{\beta}_{ik}^{(jl)}$, introduced in Proposition 2.1, and $\bar{\tau}$ defined in (2.8), an associated matrix can be defined as follows:

$$\mathbf{M} = [M^{(kl)}]_{1\leq k,l\leq K}, \tag{2.23}$$

for each $k, l \in \mathcal{K}$, $M^{(kl)} = [m_{ij}^{(kl)}]_{1\leq i,j\leq N-1}$ is an $(N-1)\times(N-1)$ matrix, and its entries are defined by

$$m_{ij}^{(kl)} = \begin{cases} \eta_{ik}, & \text{if } i=j\in\mathcal{N}-\{N\} \text{ and } k=l\in\mathcal{K}, \\ -\bar{L}_{ik}^{(jl)}, & \text{otherwise.} \end{cases} \tag{2.24}$$

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where

$$\bar{\beta}_{ik} := \max\{|\check{\beta}_{ik}|, |\hat{\beta}_{ik}|\}, \tag{2.25}$$

$$\eta_{ik} := -\hat{\mu}_k - \hat{\beta}_{ik} + \bar{\beta}_{ik}\bar{\tau}(\check{\mu}_k + \hat{\mu}_k + \check{\beta}_{ik} + \hat{\beta}_{ik}), \tag{2.26}$$

$$\bar{L}_{ik}^{(jl)} := \bar{\mu}_{ik}^{(jl)} + \bar{\beta}_{ik}^{(jl)}. \tag{2.27}$$

Let us introduce the following condition:

Condition (S): For all $(i,k) \in \mathcal{A}$, $\hat{\mu}_k + \hat{\beta}_{ik} < 0$ and

$$\bar{\beta}_{ik}\bar{\tau} < 3\rho_{ik}^h(\hat{\mu}_k + \hat{\beta}_{ik})/[(\check{\mu}_k + \hat{\mu}_k + \check{\beta}_{ik} + \hat{\beta}_{ik})(3\rho_{ik}^h + \rho_{ik}^w)].$$

We note that Condition (S) involves $\bar{\tau}$, and is thus delay-dependent.

Theorem 2.2. *([33]) Consider system* (2.1) *which satisfies Assumptions (D) and (F). Then, the system globally synchronizes if Condition (S) holds, and the Gauss-Seidel iterations for the linear system:*

$$\mathbf{Mv} = \mathbf{0}, \tag{2.28}$$

converge to zero, the unique solution of (2.28)*; or equivalently,*

$$\lambda_{\text{syn}} := \max_{1\le\sigma\le K\times(N-1)} \{|\lambda_\sigma| : \ \lambda_\sigma : \ \text{eigenvalue of } (D_{\mathbf{M}} - L_{\mathbf{M}})^{-1}U_{\mathbf{M}}\} < 1. \tag{2.29}$$

where $\mathbf{M}$ *is defined in* (2.23) *and* $D_{\mathbf{M}}$*,* $-L_{\mathbf{M}}$*, and* $-U_{\mathbf{M}}$ *represent the diagonal, strictly lower-triangular, and strictly upper-triangular parts of* $\mathbf{M}$*, respectively.*

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Chapter 3

Synchronization of Hindmarsh-Rose neurons

To apply the synchronization theory in [33], we first rewrite system (1.2) into the form of (2.1). Recalling (1.3), we rewrite the coupling part of (1.2) as follows:

$$\begin{aligned}
& c \sum_{j \in \mathcal{N}-\{i\}} a_{ij}(g(x_{j,1}(t-\tau)) - g(x_{i,1}(t-\tau))) \\
&= c\,[\sum_{j \in \mathcal{N}-\{i\}} a_{ij} g(x_{j,1}(t-\tau)) - \sum_{j \in \mathcal{N}-\{i\}} a_{ij} g(x_{i,1}(t-\tau))] \\
&= c\,[\sum_{j \in \mathcal{N}-\{i\}} a_{ij} g(x_{j,1}(t-\tau)) - g(x_{i,1}(t-\tau)) \sum_{j \in \mathcal{N}-\{i\}} a_{ij}] \\
&= c\,[\sum_{j \in \mathcal{N}-\{i\}} a_{ij} g(x_{j,1}(t-\tau)) + a_{ii} g(x_{i,1}(t-\tau))] \\
&= c \sum_{j \in \mathcal{N}} a_{ij} g(x_{j,1}(t-\tau)).
\end{aligned} \tag{3.1}$$

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By (3.1), system (1.2) can be written as follows :

$$\begin{cases} \dot{x}_{i,1}(t) = x_{i,2}(t) - a x_{i,1}(t)^3 + b x_{i,1}(t)^2 - x_{i,3}(t) + I \\ \qquad\qquad + c \sum_{j \in \mathcal{N}} a_{ij} g(x_{j,1}(t-\tau)), \\ \dot{x}_{i,2}(t) = 1 - d x_{i,1}(t)^2 - x_{i,2}(t), \\ \dot{x}_{i,3}(t) = r(s(x_{i,1}(t) + x_0) - x_{i,3}(t)), \end{cases} \tag{3.2}$$

for all $i \in \mathcal{N}$. Notably, the terms κ, defined in (2.2), now satisfies the diffusive coupling condition:

$$\kappa = \check{\kappa} = \hat{\kappa} = 0.$$

Accordingly,

$$\tilde{a}_{ij} = \begin{cases} a_{ii} - \kappa = a_{ii}, & \text{if } i = j \in \mathcal{N}, \\ a_{ij}, & \text{if } i, j \in \mathcal{N} \text{ and } i \neq j. \end{cases} \tag{3.3}$$

Thus, $\tilde{A}(t)$ in (2.14) is now $\tilde{A}(t) = A$. Applying (2.15) yields that $\bar{A}(t)$ satisfies $\bar{A}(t) = \bar{A} = [\alpha_{ij}]_{1 \le i,j \le N-1} := \mathbf{C} A \mathbf{C}^T (\mathbf{C}\mathbf{C}^T)^{-1} \in \mathbb{R}^{(N-1)\times(N-1)}$. Hence, $\check{\alpha}_{ij}$, $\hat{\alpha}_{ij}$, $\bar{\alpha}_{ij}$ in (2.17) are now

$$\check{\alpha}_{ij} = \hat{\alpha}_{ij} = \alpha_{ij}, \tag{3.4}$$

$$\bar{\alpha}_{ij} = |\alpha_{ij}|, \tag{3.5}$$

for all $i, j \in \mathcal{N} - \{N\}$. In addition, for system (3.2), the time delay $\tau(t)$ is now independent of t (i.e. $\tau(t) \equiv \tau$) and the functions $F_k(\mathbf{x}_i(t), t)$ and $\mathbf{G_k}(\mathbf{x}_j(t - \tau(t)))$ defined in (2.1) and (2.3)

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are now

$$
\begin{cases}
F_1(\mathbf{x}_i(t),t) = x_{i,2}(t) - ax_{i,1}(t)^3 + bx_{i,1}(t)^2 - x_{i,3}(t) + I, \\
F_2(\mathbf{x}_i(t),t) = 1 - dx_{i,1}(t)^2 - x_{i,2}(t), \\
F_3(\mathbf{x}_i(t),t) = r(s(x_{i,1}(t) + x_0) - x_{i,3}(t)),
\end{cases} \tag{3.6}
$$

$$
\begin{cases}
\mathbf{G}_1(\mathbf{x}_j(t-\tau(t))) = g(x_{j,1}(t-\tau)), \\
\mathbf{G}_2(\mathbf{x}_j(t-\tau(t))) = 0, \\
\mathbf{G}_3(\mathbf{x}_j(t-\tau(t))) = 0,
\end{cases} \tag{3.7}
$$

respectively.

Let us introduce a condition for system (3.2), which plays the role as Condition (D) for system (2.1):

Condition (D)*: All solutions of system (3.2) eventually enter and then remain in some compact set $\mathcal{Q}^N := \mathcal{Q}^* \times \cdots \times \mathcal{Q}^* \subset \mathbb{R}^{3N}$, where $\mathcal{Q}^* := [-\rho_1^*, \rho_1^*] \times [-\rho_2^*, \rho_2^*] \times [-\rho_3^*, \rho_3^*] \subset \mathbb{R}^3$ with $\rho_k^* \geq 0, k = 1, 2, 3$.

Next, let us establish Assumption (F) for system (3.2) under Condition (D)*.

Proposition 3.1. *Assume that Condition (D)* holds, then system* (3.2) *satisfies Assumption (F) with* $\check{\mu}_1 = -3a(\rho_1^*)^2 - 2b\rho_1^*$, $\hat{\mu}_1 = b^2/3a$, $\check{\mu}_2 = \hat{\mu}_2 = -1$, $\check{\mu}_3 = \hat{\mu}_3 = -r$, $\bar{\mu}_{12} = \bar{\mu}_{13} = 1$, $\bar{\mu}_{23} = \bar{\mu}_{32} = 0$, $\bar{\mu}_{21} = 2\rho_1^* d$, $\bar{\mu}_{31} = rs$, $\rho_1^w = 2(\rho_2^* + \rho_3^*)$, $\rho_2^w = 4d(\rho_1^*)^2$, $\rho_3^w = 2rs\rho_1^*$, *where* $\rho_k^*, k = 1, 2, 3$, *are defined in Condition (D)*.*

Proof. We first compute quantities $\check{\mu}_k$, $\hat{\mu}_k$, $\bar{\mu}_{kl}$ ρ_k^w, where $k, l \in \{1, 2, 3\}$ and $k \neq l$. By (3.6), applying the mean value theorem yields that

$$
\begin{cases}
F_1(E,t) - F_1(\tilde{E},t) = h_1(x_1, \tilde{x_1}, t) + w_1(E, \tilde{E}, t), \\
F_2(E,t) - F_2(\tilde{E},t) = h_2(x_2, \tilde{x_2}, t) + w_2(E, \tilde{E}, t), \\
F_3(E,t) - F_3(\tilde{E},t) = h_3(x_3, \tilde{x_3}, t) + w_3(E, \tilde{E}, t),
\end{cases} \tag{3.8}
$$

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for $E = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\tilde{E} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathbb{R}^3$, and $t \geq t_0$, where

$$h_1(x_1, \tilde{x_1}, t) = [-3as^2 + 2bs](x_1 - \tilde{x}_1), \tag{3.9}$$

$$w_1(E, \tilde{E}, t) = x_2 - \tilde{x}_2 - (x_3 - \tilde{x}_3), \tag{3.10}$$

$$h_2(x_2, \tilde{x_2}, t) = -(x_2 - \tilde{x}_2), \tag{3.11}$$

$$w_2(E, \tilde{E}, t) = -d(x_1^2 - \tilde{x}_1^2), \tag{3.12}$$

$$h_3(x_3, \tilde{x_3}, t) = -(x_3 - \tilde{x}_3), \tag{3.13}$$

$$w_3(E, \tilde{E}, t) = rs(x_1 - \tilde{x}_1), \tag{3.14}$$

and s is some number between x_1 and $\tilde{x}_1$. Recall Assumption (F), where $E = (x_1, x_2, x_3)$, $\tilde{E} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathcal{Q}^* = [-\rho_1^*, \rho_1^*] \times [-\rho_2^*, \rho_2^*] \times [-\rho_3^*, \rho_3^*]$ which leads to $s \in [-\rho_1^*, \rho_1^*]$; moreover,

$$-3a(\rho_1^*)^2 - 2b\rho_1^* \leq -3as^2 + 2bs \leq b^2/3a, \tag{3.15}$$

$$|w_1(E, \tilde{E}, t)| \leq 2(\rho_2^* + \rho_3^*), \tag{3.16}$$

$$|w_2(E, \tilde{E}, t)| \leq 4d(\rho_1^*)^2, \tag{3.17}$$

$$|w_3(E, \tilde{E}, t)| \leq 2rs\rho_1^*. \tag{3.18}$$

Based on (3.9)-(3.18), we shall show that system (3.2) satisfies Assumption(F) for which quantities are chosen as follows:

By (3.9) and (3.15), we can choose $\check{\mu}_1 = -3a(\rho_1^*)^2 - 2b\rho_1^*$ and $\hat{\mu}_1 = b^2/3a$.

By (3.11), we can choose $\check{\mu}_2 = \hat{\mu}_2 = -1$.

By (3.13), we can choose $\check{\mu}_3 = \hat{\mu}_3 = -r$.

By (3.10) and (3.16), we can choose $\bar{\mu}_{12} = \bar{\mu}_{13} = 1$ and $\rho_1^w = 2(\rho_2^* + \rho_3^*)$.

By (3.12) and (3.17), we can choose $\bar{\mu}_{21} = 2\rho_1^* d$, $\bar{\mu}_{23} = 0$ and $\rho_2^w = 4d(\rho_1^*)^2$.

By (3.14) and (3.18), we can choose $\bar{\mu}_{31} = rs$, $\bar{\mu}_{32} = 0$ and $\rho_3^w = 2rs\rho_1^*$. □

From Proposition 3.1, system (3.2) satisfies Assumption (D) and (F) under Condition (D)*. Accordingly, the assumption for the assertion in Proposition 2.1 holds for system (3.2) under Condition (D)*. In addition, the quantities in the assertion of Proposition 2.1 can be chosen as those in the following proposition.

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Proposition 3.2. *Assume that Condition (D)* holds. The assertion in Proposition 2.1 holds with*

$$\rho^h_{ik} = \begin{cases} 2c\bar{\alpha}_{ii}\rho^g, & \text{if } k = 1, \\ v, & \text{if } k = 2, 3, \end{cases} \tag{3.19}$$

$$\check{\beta}_{ik} = \begin{cases} c\alpha_{ii}\check{\delta}, & \text{if } k = 1,\ \alpha_{ii} \geq 0, \\ c\alpha_{ii}\delta, & \text{if } k = 1,\ \alpha_{ii} < 0, \\ 0, & \text{if } k = 2, 3, \end{cases} \tag{3.20}$$

$$\hat{\beta}_{ik} = \begin{cases} c\alpha_{ii}\delta & \text{if } k = 1,\ \alpha_{ii} \geq 0, \\ c\alpha_{ii}\check{\delta} & \text{if } k = 1,\ \alpha_{ii} < 0, \\ 0 & \text{if } k = 2, 3, \end{cases} \tag{3.21}$$

$$\rho^w_{ik} = \begin{cases} 2(\rho^*_2 + \rho^*_3) + 2c\rho^g \sum_{j \in \mathcal{N} - \{i,N\}} \bar{\alpha}_{ij}, & \text{if } k = 1, \\ 4d(\rho^*_1)^2, & \text{if } k = 2, \\ 2rs\rho^*_1, & \text{if } k = 3, \end{cases} \tag{3.22}$$

$$\bar{\mu}^{(jl)}_{ik} = \begin{cases} 1, & \text{if } i = j,\ (k,l) = (1,2) \text{ or } (1,3), \\ 2\rho^*_1 d, & \text{if } i = j,\ (k,l) = (2,1), \\ rs, & \text{if } i = j,\ (k,l) = (3,1), \\ 0, & \text{otherwise}, \end{cases} \tag{3.23}$$

$$\bar{\beta}^{(jl)}_{ik} = \begin{cases} c\bar{\alpha}_{ij}\delta, & \text{if } j \neq i,\ k = l = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.24}$$

where $\delta = \max\{g'(x_i) : x_i \in [-\rho^*_1, \rho^*_1]\}$ *is defined in* (1.4)*,* $\bar{\alpha}_{ii}$ *and* $\bar{\alpha}_{ij}$ *in* (3.4)*,* ρ^*_k *in Condition (D)*,* $\bar{\mu}_{kl}$ *in Proposition 3.1;* v *is an arbitrary positive number and* $\check{\delta} := \min\{g'(x) : x \in [-\rho^*_1, \rho^*_1]\}$.

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Proof. By (3.7), for $E = (x_1, x_2, x_3) \in \mathbb{R}^3$, we have

$$\begin{cases} g_1(x_1) = \mathbf{G}_1(E) = g(x), \\ g_1(x_2) = \mathbf{G}_2(E) = 0, \\ g_1(x_3) = \mathbf{G}_3(E) = 0. \end{cases} \tag{3.25}$$

By Proposition 2.1 and (3.25), $\check{L}_k$ and $\hat{L}_k$, $k = 1, 2, 3$, are now $\check{L}_1 = \check{\delta}$, $\hat{L}_1 = \delta = \max\{g'(x) : x \in [-\rho_1^*, \rho_1^*]\}$ and $\check{L}_2 = \hat{L}_2 = \check{L}_3 = \hat{L}_3 = 0$, where δ is defined in (1.4). By (3.25), ρ_k^g defined in (2.20), $k = 1, 2, 3$, can be chosen as

$$\rho_1^g = \rho^g,$$

$$\rho_2^g = \rho_3^g = v,$$

for an arbitrary $v > 0$. Combining those quantities of $\check{L}_k$, $\hat{L}_k$ and ρ_k^g, $k = 1, 2, 3$, chosen above as well as $\check{\mu}_k, \hat{\mu}_k, \bar{\mu}_{kl}$ and ρ_k^w, for $k, l \in \{1, 2, 3\}$ and $k \neq l$, chosen in Proposition 3.1, the quantities in the assertion of Proposition 2.1 can be determined as those in (3.19)-(3.24). □

Let us now introduce the following condition for system (3.2) which plays the role as Condition (S) for system (2.1):

Condition (S)*: $b^2/3a + c\alpha_{ii}\check{\delta} < 0$ and $\tau < \tilde{\tau}_i^*$, for all $i \in \{1, 2, \ldots, N-1\}$,

where

$$\tilde{\tau}_i^* := -3\bar{\rho}_i^h(b^2/3a + c\alpha_{ii}\check{\delta})/[\bar{\beta}_i(3\bar{\rho}_i^h + \bar{\rho}_i^w)],$$

with

$$\begin{aligned} \bar{\rho}_i^h &:= 2c\bar{\alpha}_{ii}\rho^g, \\ \bar{\beta}_i &:= c\alpha_{ii}\delta[b^2/3a - 3a(\rho_1^*)^2 - 2b\rho_1^* + c\alpha_{ii}(\delta + \check{\delta})], \\ \bar{\rho}_i^w &:= 2(\rho_2^* + \rho_3^*) + 2c\sum_{j\in\mathcal{N}-\{i,N\}} \bar{\alpha}_{ij}\rho^g. \end{aligned}$$

Therein, the quantities $\check{\delta}$ and ρ^g are defined in Propositions 3.2; ρ_k^*, $k = 1, 2, 3$, is defined in Condition (D)*; $\bar{\alpha}_{ij}$ and $\bar{\alpha}_{ii}$ are defined in (3.4).

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In the following lemma, we shall show the matrix $\mathbf{M}$ in (2.23) in terms of the quantities shown in Proposition 3.2. Moreover, the matrix M in (2.23) is determined by quantities in Proposition 2.1. Basically, the following lemma comes from Proposition 3.1 and 3.2.

Lemma 3.3. *Assume that Condition (D)* holds. Then, the matrix* $\mathbf{M} = [M^{(kl)}]_{1\le k,l\le K}$*, where* $M^{(kl)} = [m_{ij}^{(kl)}]_{1\le i,j\le N-1}$*, in* (2.23) *is now denoted by* $\tilde{\mathbf{M}} = [\tilde{M}^{(kl)}]_{1\le k,l\le K}$*, where* $\tilde{M}^{(kl)} = [\tilde{m}_{ij}^{(kl)}]_{1\le i,j\le N-1}$*, with*

$$\tilde{m}_{ij}^{(kl)} := \begin{cases} -b^2/3a - c\alpha_{ii}\check{\delta} - \tau\bar{\beta}_i, & \textit{if } i = j \textit{ and } (k,l) = (1,1), \\ 1, & \textit{if } i = j \textit{ and } (k,l) = (2,2), \\ r, & \textit{if } i = j \textit{ and } (k,l) = (3,3), \\ -\mu_{ik}^{(jl)} - \beta_{ik}^{(jl)}, & \textit{otherwise}, \end{cases} \tag{3.26}$$

where

$$\mu_{ik}^{(jl)} := \begin{cases} 1, & \textit{if } i = j, (k,l) = (1,2) \textit{ or } (1,3), \\ 2\rho_1^* d, & \textit{if } i = j, (k,l) = (2,1), \\ rs, & \textit{if } i = j, (k,l) = (3,1), \\ 0, & \textit{otherwise}, \end{cases} \tag{3.27}$$

$$\beta_{ik}^{(jl)} := \begin{cases} \beta_{ij}^*, & \textit{if } j \ne i, (k,l) = (1,1), \\ 0, & \textit{otherwise}. \end{cases} \tag{3.28}$$

Herein, $\beta_{ij}^* := c\bar{\alpha}_{ij}\delta$*,* $\check{\delta}$ *is defined in Proposition 3.2;* $\bar{\beta}_i$ *is defined in Condition (S1)*;* $\bar{\alpha}_{ij}$ *is defined in* (3.4)*;* ρ_1^* *is defined in Condition (D)*.*

Proof. From Propositions 3.1 and 3.2, system (3.2) satisfies Condition (D)* with $\mathcal{Q}^* = [-\rho_1^*, \rho_1^*] \times [-\rho_2^*, \rho_2^*] \times [-\rho_3^*, \rho_3^*]$, $\check{\mu}_k$, $\hat{\mu}_k$, ρ_k^w, for $k = 1,2,3$, and $\bar{\mu}_{kl}$, $k,l \in \{1,2,3\}$ and $k \ne l$, determined in Proposition 3.1. As seen from (3.7), the terms $\check{L}_k$, and $\hat{L}_k$, $k = 1,2,3$, defined in Proposition 2.1, are now chosen as those in Proposition 3.2. Notably, Condition (S)* implies that $\alpha_{ii} < 0$ for all $i = 1,\ldots,N-1$, because $b^2/3a > 0$, $c > 0$, and $\check{\delta} > 0$. Based on Propositions 3.1 and 3.2, system (3.2) satisfies the assertion of Proposition 2.1, with $\check{\mu}_1 = -3a(\rho_1^*)^2 - 2b\rho_1^*$, $\hat{\mu}_1 = b^2/3a$, $\check{\mu}_2 = \hat{\mu}_2 = -1$, $\check{\beta}_{i1} = c\alpha_{ii}\delta$, $\hat{\beta}_{i1} = c\alpha_{ii}\check{\delta}$, $\check{\beta}_{i2} = \hat{\beta}_{i2} = \check{\beta}_{i3} = \hat{\beta}_{i3} = 0$, $\bar{\mu}_{ik}^{(jl)} = \mu_{ik}^{(jl)}$,

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$\bar{\beta}_{ik}^{(jl)} = \beta_{ik}^{(jl)}$. In particular, $\bar{\tau} = \tau$ for system (3.2), cf. (2.8). As seen from the definition of η_{ik} in (2.27), η_{ik} is now $\eta_{ik} = \tilde{\eta}_{ik}$. Consider $\tilde{\eta}_{ik}$ satisfying

$$\tilde{\eta}_{ik} := -\hat{\mu}_k - \hat{\beta}_{ik} + \beta_{ik}^{*}\bar{\tau}(\check{\mu}_k + \hat{\mu}_k + \check{\beta}_{ik} + \hat{\beta}_{ik}), \tag{3.29}$$

where $\beta_{i1}^{*} = c\alpha_{ii}\delta$ and $\beta_{i2}^{*} = \beta_{i3}^{*} = 0$ by (2.25), (3.20), and (3.21). Thus, by (3.29),

$$\tilde{\eta}_{ik} = \begin{cases} -b^2/3a - c\alpha_{ii}\check{\delta} - \tau\bar{\beta}_i, & \text{if } k = 1, \\ 1, & \text{if } k = 2, \\ r, & \text{if } k = 3. \end{cases} \tag{3.30}$$

Moreover, by (3.23) and (3.24), $\bar{L}_{ik}^{(jl)}$, defined in (2.27), is now

$$\bar{L}_{ik}^{(jl)} = \mu_{ik}^{(jl)} + \beta_{ik}^{(jl)} \tag{3.31}$$

with $\mu_{ik}^{(jl)}$ and $\beta_{ik}^{(jl)}$ defined in (3.27) and (3.28). By (3.29) and (3.31), the matrix $\tilde{M}^{(kl)} = [\tilde{m}_{ij}^{(kl)}]_{1 \le i,j \le N-1}$ defined in (2.24), now satisfies

$$\tilde{m}_{ij}^{(kl)} = \begin{cases} \tilde{\eta}_{ik}, & \text{if } i = j \in \mathcal{N} - \{N\} \text{ and } k = l \in \mathcal{K}, \\ -\bar{L}_{ik}^{(jl)}, & \text{otherwise.} \end{cases} \tag{3.32}$$

Therefore, the entries of $\tilde{\mathbf{M}}$ defined in (3.26) come from (3.27), (3.28), (3.30), and (3.32). □

Theorem 3.4. *Assume that Conditions (D)* and (S)* hold. Then, the system* (3.2) *globally synchronizes if the Gauss-Seidel iterations for the linear system:*

$$\tilde{\mathbf{M}}\mathbf{v} = \mathbf{0} \tag{3.33}$$

converges to zero, the unique solution of (3.33)*; or equivalently,*

$$\tilde{\lambda}_{\text{syn}} := \max_{1 \le \sigma \le K \times (N-1)} \{|\tilde{\lambda}_\sigma| : \ \tilde{\lambda}_\sigma : \ \text{eigenvalue of } (D_{\tilde{\mathbf{M}}} - L_{\tilde{\mathbf{M}}})^{-1} U_{\tilde{\mathbf{M}}}\} < 1, \tag{3.34}$$

where $\tilde{\mathbf{M}}$ *is defined in Lemma 3.3, and* $D_{\tilde{\mathbf{M}}}$, $-L_{\tilde{\mathbf{M}}}$, $-U_{\tilde{\mathbf{M}}}$ *represent the diagonal, strictly lower-triangular and strictly upper-triangular parts of* $\tilde{\mathbf{M}}$*, respectively.*

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Proof. By Proposition 3.1, system (3.2) satisfies Assumption (D) and (F) under Condition (D)*. In addition, system (3.2) satisfies Condition (S) under Condition (S)*, and the matrix $\mathbf{M}$ in (2.28) is now $\tilde{\mathbf{M}}$ in Lemma 3.3. By Theorem 2.2, system (3.2) achieves global synchronization, if the Gauss-Seidel iterations for the linear system (3.33), converge to zero. Hence, we complete the proof. □

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Chapter 4

Numerical examples

In this chapter, we will illustrate two examples with numerical simulations to demonstrate the effectiveness of the theoretical result derived in this thesis.

Example 1. Consider three coupled Hindmarsh-Rose neurons (3.2) with $a = 1$, $b = 3$, $I = 3.0$, $d = 5$, $r = 0.005$, $s = 4$, $x_0 = 1.6$, $c = 50$, $g(x) = 10\, tanh(x/10)$, $\tau = 0.00001$ and

$$A = [a_{ij}]_{1\leq i,j\leq 3} = \begin{pmatrix} -1.0 & 0.6 & 0.4 \\ 0.4 & -0.8 & 0.4 \\ -0.1 & 0.6 & -0.5 \end{pmatrix}. \tag{4.1}$$

From (1.4), (2.6), and (2.8), we have $\bar{\kappa} = \check{\kappa} = \hat{\kappa} = 0$, $\bar{a} = 2.0$ and $\bar{\tau} = \tau = 0.00001$. By (2.15) and (4.1), we obtain

$$\bar{A} = [\alpha_{ij}]_{1\leq i,j\leq 2} = \begin{pmatrix} -1.4 & 0 \\ 0.5 & -0.9 \end{pmatrix}. \tag{4.2}$$

By (4.2), the quantities defined in (2.17) are now $\check{\alpha}_{11} = \hat{\alpha}_{11} = -1.4$, $\bar{\alpha}_{11} = 1.4$, $\check{\alpha}_{22} = \hat{\alpha}_{22} = -0.9$, $\bar{\alpha}_{22} = 0.9$, $\bar{\alpha}_{12} = 0$, $\bar{\alpha}_{21} = 0.5$. By numerical simulation cf. Figure 4.1, we can observe that the system satisfies Condition (D)* with $\mathcal{Q}^* = [-\rho_1^*, \rho_1^*] \times [-\rho_2^*, \rho_2^*] \times [-\rho_3^*, \rho_3^*]$, where

$$\rho_1^* = 2, \quad \rho_2^* = 9, \quad \rho_3^* = 3.5. \tag{4.3}$$

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We note that Figure 4.1 demonstrates the evolution for the solution of the considered system, evolved from $(3.5, 0.3, -2.1, 3.6, 0.4, -2.2, 3.7, 0.5, -2.3)$ at $t_0 = 0$. It appears that the solution eventually enters, and then remains in $\mathcal{Q}^* \times \mathcal{Q}^* \times \mathcal{Q}^*$, where $\mathcal{Q}^*$ is defined in (4.3). From Propositions 3.1 and 3.2, Lemma 3.3, and (4.3), we can obtain $b^2/3a = 3$, $\check{\delta} \approx 0.97104$, $\bar{\rho}_1^h \approx 276.32545$, $\bar{\rho}_2^h \approx 177.63779$, $\bar{\rho}_1^w \approx 25$, $\bar{\rho}_2^w \approx 123.68766$, $\bar{\beta}_1 \approx 11079.11062$, $\bar{\beta}_2 \approx 4916.11204$, $\tilde{\tau}_1^* \approx 0.00563$, $\tilde{\tau}_2^* \approx 0.00664$, $\beta_{12}^* = \beta_{13}^* = \beta_{23}^* = \beta_{31}^* = \beta_{32}^* = 0$, and $\beta_{21}^* = 25$. By the quantities above and Lemma 3.3, we can further verify that Condition (S)* holds and matrix $\tilde{\mathbf{M}}$ in (3.33) is approximately

$$\begin{pmatrix} 64.16221 & 0 & -1.0 & 0 & -1.0 & 0 \\ -25.0 & 40.19777 & 0 & -1.0 & 0 & -1.0 \\ -20.0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & -20.0 & 0 & 1.0 & 0 & 0 \\ -0.02 & 0 & 0 & 0 & 0.005 & 0 \\ 0 & -0.02 & 0 & 0 & 0 & 0.005 \end{pmatrix}. \tag{4.4}$$

By the matrix in (4.4), we can compute the corresponding value $\tilde{\lambda}_{\text{syn}} \approx 0.60$, cf. (2.29). Hence, the system attains global synchronization by Theorem 2.2. Figure 4.2 and 4.3 demonstrate that the evolution for the solution of the considered system, evolved from $(3.5, 0.3, -2.1, 3.6, 0.4, -2.2, 3.7, 0.5, -2.3)$ at $t_0 = 0$. It appears that the solution remains oscillatory. Figures 4.3(a), 4.3(b) and 4.3(c) show that the solution $(\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t))$, $\mathbf{x}_i(t) = (x_{i,1}(t), x_{i,2}(t), x_{i,3}(t))$, with $z_{i,k} = \mathbf{x}_{i,k}(t) - \mathbf{x}_{i+1,k}(t)$ converging to zero for $i = 1, 2$ and $k = 1, 2, 3$. This demonstrates that the solution synchronizes.

If we consider large coupling delay $\tau = 0.05$ instead of $\tau = 0.00001$, then Condition (S)* does not hold. Figures 4.4(a), 4.4(b) and 4.4(c) show that each of the solution does not synchronize, and exhibits asynchronous oscillatory behavior. This shows that large delay may destroy synchronization.

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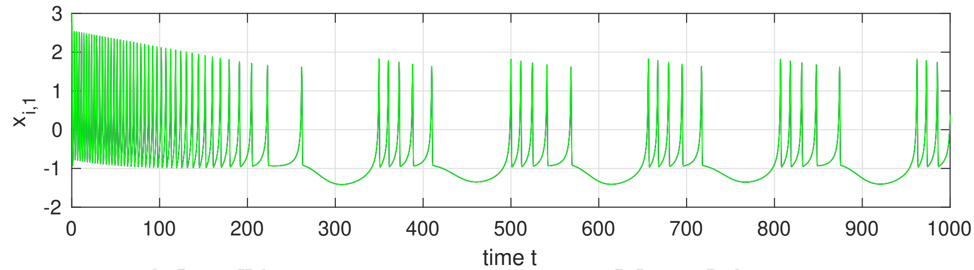
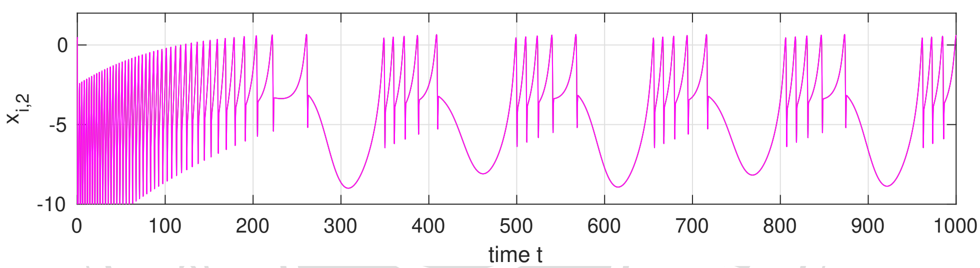
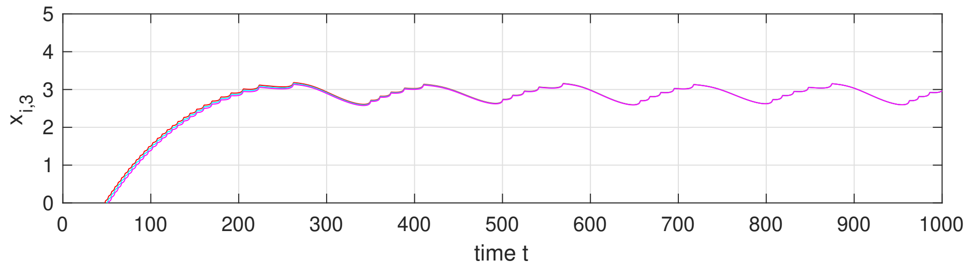


(a) $x_{i,1}$



(b) $x_{i,2}$



(c) $x_{i,3}$

Figure 4.1: Simulation for the solution of the system considered in Example 1, with $\tau = 0.00001$: evolution of components $x_{i,k}(t)$, $i = 1, 2, 3$, and $k = 1, 2, 3$.

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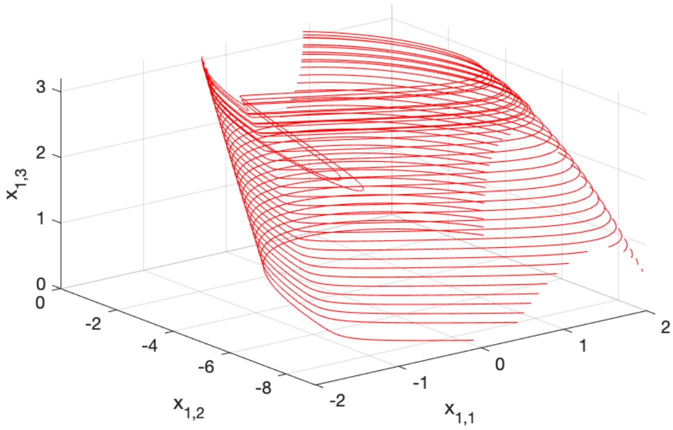
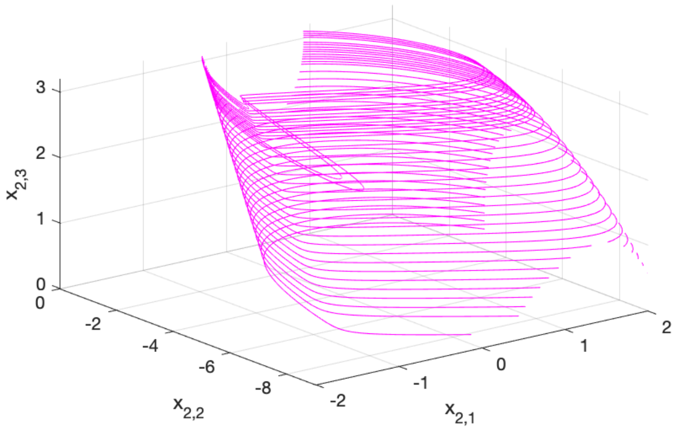
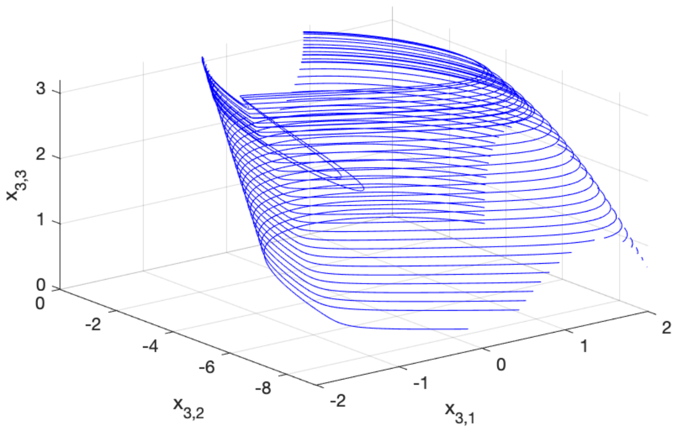


(a) $(x_{1,1}, x_{1,2}, x_{1,3})$



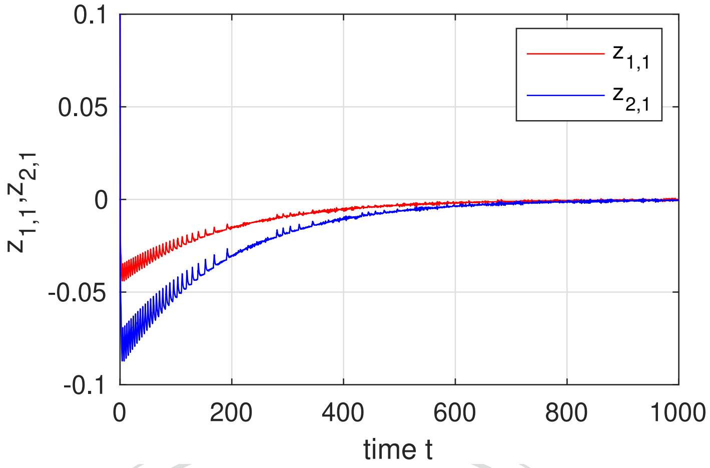
(b) $(x_{2,1}, x_{2,2}, x_{2,3})$



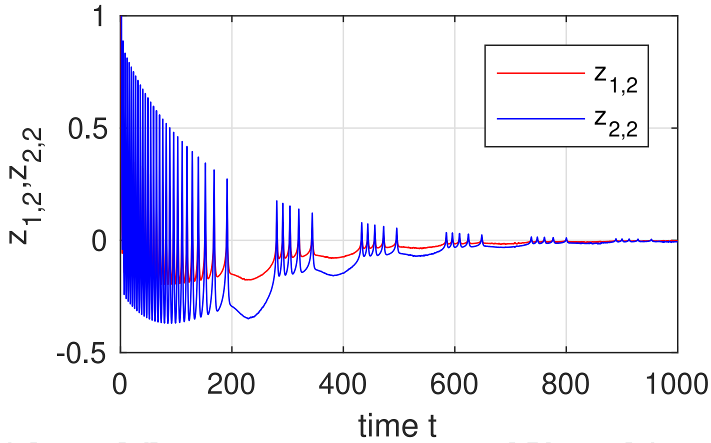
(c) $(x_{3,1}, x_{3,2}, x_{3,3})$

Figure 4.2: Simulation for the solution of the system considered in Example 1, with $\tau = 0.00001$: evolution of components $(x_{i,1}(t), x_{i,2}(t), x_{i,3}(t))$, $i = 1, 2, 3$.

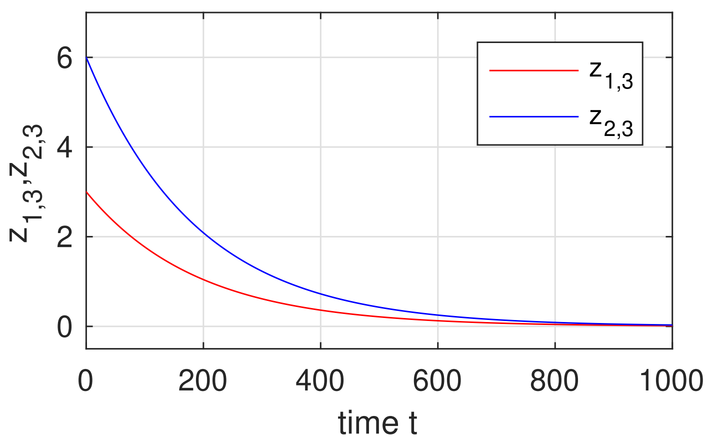
DOI:10.6814/NCCU202000086



(a) $z_{i,1}$



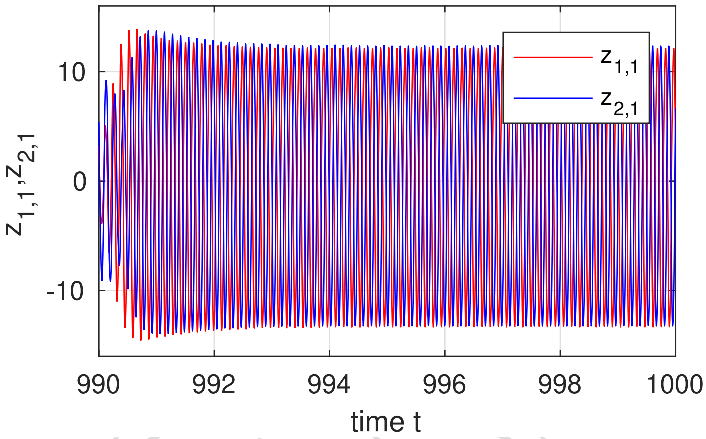
(b) $z_{i,2}$



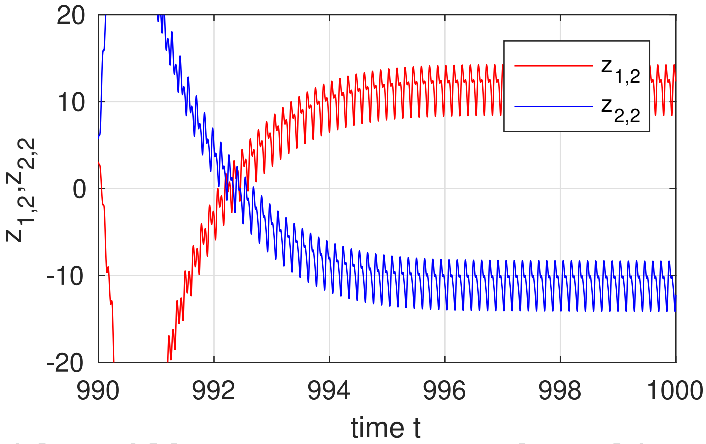
(c) $z_{i,3}$

Figure 4.3: Simulation for the solution of the system considered in Example 1, with $\tau = 0.00001$: evolution of components $z_{i,k}(t), i = 1, 2$, and $k = 1, 2, 3$.

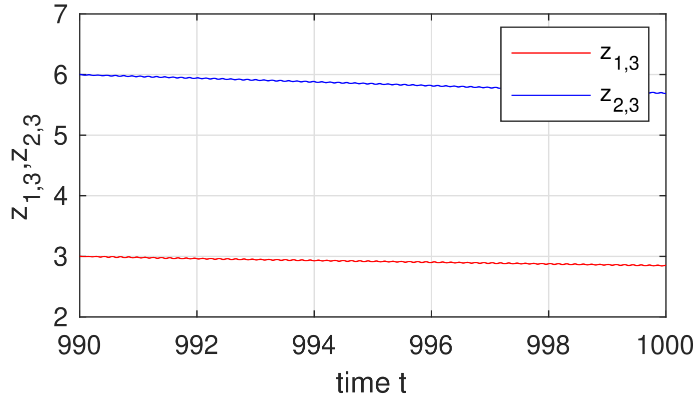
DOI:10.6814/NCCU202000086



(a) $z_{i,1}$



(b) $z_{i,2}$



(c) $z_{i,3}$

Figure 4.4: Simulation for the solution of the system considered in Example 1, with $\tau = 0.05$: evolution of components $z_{i,k}(t), i = 1, 2$, and $k = 1, 2, 3$.

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Example 2. Consider three coupled Hindmarsh-Rose neurons (3.2) with $a = 1$, $b = 3$, $I = 3.0$, $d = 5$, $r = 0.005$, $s = 4$, $x_0 = 1.6$, $c = 200$, $g(x) = [tanh(x) + x]/2$, $\tau = 0.00002$ and

$$A = [a_{ij}]_{1 \le i,j \le 3} = \begin{pmatrix} -0.4 & 0.3 & 0.1 \\ 0.2 & -0.7 & 0.5 \\ 0.1 & 0.8 & -0.9 \end{pmatrix}. \quad (4.5)$$

From (1.4), (2.6), and (2.8), we have $\bar{\kappa} = \check{\kappa} = \hat{\kappa} = 0$, $\bar{a} = 3.2$ and $\bar{\tau} = \tau = 0.00002$. By (2.15) and (4.5), we obtain

$$\bar{A} = [\alpha_{ij}]_{1 \le i,j \le 2} = \begin{pmatrix} -0.6 & 0.4 \\ 0.1 & -1.4 \end{pmatrix}. \quad (4.6)$$

By (4.6), the quantities defined in (2.17) are now $\check{\alpha}_{11} = \hat{\alpha}_{11} = -0.6$, $\bar{\alpha}_{11} = 0.6$, $\check{\alpha}_{22} = \hat{\alpha}_{22} = -1.4$, $\bar{\alpha}_{22} = 1.4$, $\bar{\alpha}_{12} = 0.4$, $\bar{\alpha}_{21} = 0.1$. By numerical simulation cf. Figure 4.5, we can observe that the system satisfies Condition (D)* with $\mathcal{Q}^* = [-\rho_1^*, \rho_1^*] \times [-\rho_2^*, \rho_2^*] \times [-\rho_3^*, \rho_3^*]$, where

$$\rho_1^* = 2, \ \ \rho_2^* = 9, \ \ \rho_3^* = 3.5. \quad (4.7)$$

We note that Figure 4.5 demonstrates the evolution for the solution of the considered system, evolved from $(0.7; 2.5; -2.8; 1; 2.7; -2.5; 0.5; 2.9; -2.2)$ at $t_0 = 0$. It appears that the solution eventually enters and then remains in $\mathcal{Q}^* \times \mathcal{Q}^* \times \mathcal{Q}^*$, where $\mathcal{Q}^*$ is defined in (4.7). From Propositions 3.1 and 3.2, Lemma 3.3, and (4.7), we can obtain $b^2/3a = 3$, $\check{\delta} \approx 0.53533$, $\bar{\rho}_1^h \approx 444.60414$, $\bar{\rho}_2^h \approx 1037.40965$, $\bar{\rho}_1^w \approx 321.40276$, $\bar{\rho}_2^w \approx 99.10069$, $\bar{\beta}_1 \approx 37694.82178$, $\bar{\beta}_2 \approx 195427.36302$, $\tilde{\tau}_1^* \approx 0.00165$, $\tilde{\tau}_2^* \approx 91427.56468$, $\beta_{12}^* = 99.99999$, $\beta_{21}^* = 25.0$, and $\beta_{13}^* = \beta_{23}^* = \beta_{31}^* = \beta_{32}^* = 0$. By the quantities above and Lemma 3.3, we can further verify that

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Condition (S)* holds and matrix $\tilde{\mathbf{M}}$ in (3.33) is approximately

$$\begin{pmatrix} 60.74648 & -80.0 & -1.0 & 0 & -1.0 & 0 \\ -20.0 & 144.36613 & 0 & -1.0 & 0 & -1.0 \\ -20.0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & -20.0 & 0 & 1.0 & 0 & 0 \\ -0.02 & 0 & 0 & 0 & 0.005 & 0 \\ 0 & -0.02 & 0 & 0 & 0 & 0.005 \end{pmatrix}. \quad (4.8)$$

By the matrix in (4.8), we can compute the corresponding value $\tilde{\lambda}_{\text{syn}} \approx 0.64$, cf. (2.29). Hence, the system attains global synchronization by Theorem 2.2. Figures 4.5 and 4.6 demonstrate that the evaluation for the solution of the considered system, evolved from $(0.7, 2.5, -2.8, 1, 2.7, -2.5, 0.5, 2.9, -2.2)$ at $t_0 = 0$. It shows that the solution remains oscillatory. Figures 4.7(a), 4.7(b) and 4.7(c) illustrate that the solution $(\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t))$, $\mathbf{x}_i(t) = (x_{i,1}(t), x_{i,2}(t), x_{i,3}(t))$, with $z_{i,k} = \mathbf{x}_{i,k}(t) - \mathbf{x}_{i+1,k}(t)$ converging to zero for $i = 1, 2$ and $k = 1, 2, 3$. This demonstrates that the solution synchronizes.

If we consider large coupling delay $\tau = 0.1$ instead of $\tau = 0.00002$, then Condition (S)* does not hold. Figures 4.8(a), 4.8(b) and 4.8(c) show that each of the solution does not synchronize, and exhibits asynchronous oscillatory behavior. This shows that large coupling delay may lead to asynchrony.

Remark 4.1. *Among the existing studies on synchronization of coupled systems, the synchronization theories in [9, 16–20, 30] required that all non-zero and off-diagonal entries of the connection matrix have the same sign. The connection matrices considered in Examples 1 and 2 have off-diagonal entries with the mixed signs, and do not satisfy the circulant condition required in [29]. Therefore, the synchronization of systems considered in Examples 1 and 2 can not be treated by previous approaches in [9, 16–20, 29, 30].*

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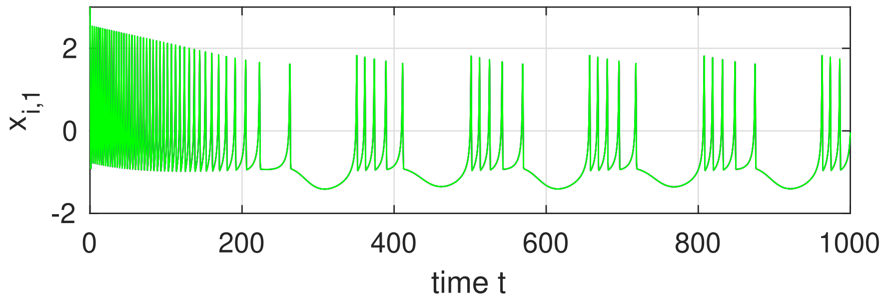
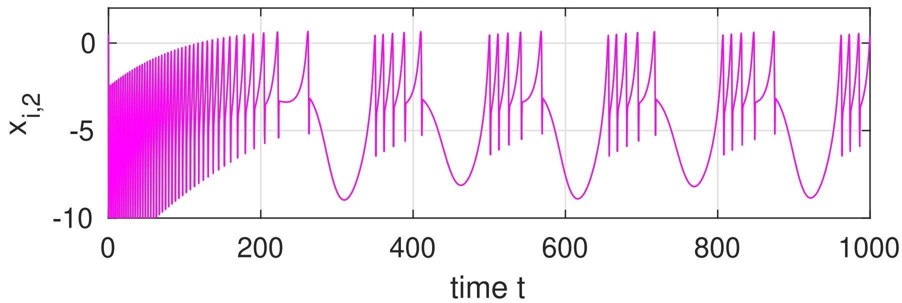
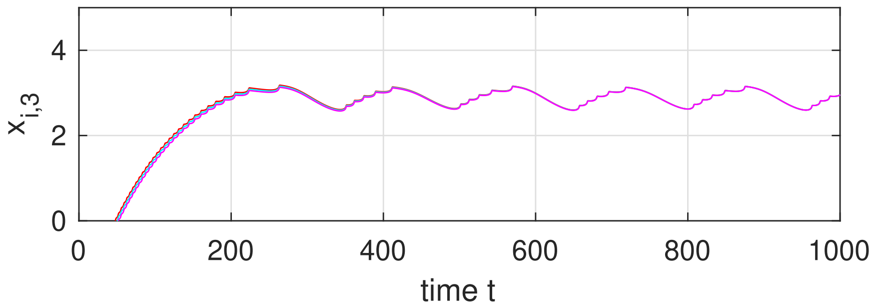


(a) $x_{i,1}$



(b) $x_{i,2}$



(c) $x_{i,3}$

Figure 4.5: Simulation for the solution of the system considered in Example 2, with $\tau = 0.00002$: evolution of components $x_{i,k}(t)$, $i = 1, 2, 3$, and $k = 1, 2, 3$.

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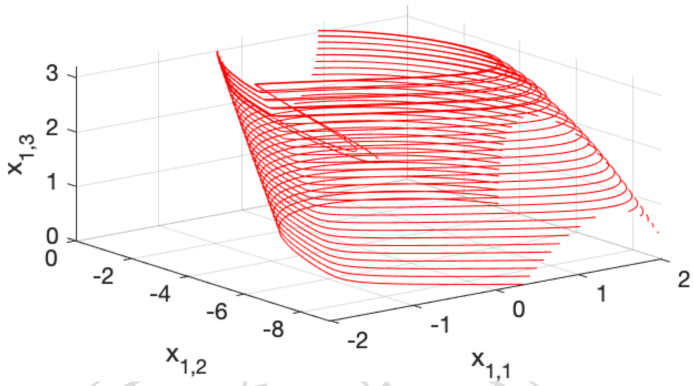
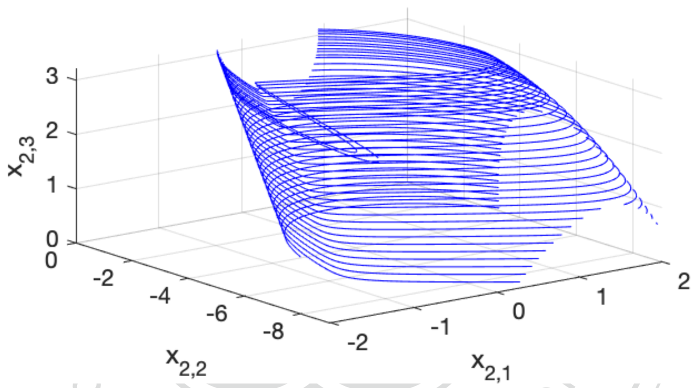
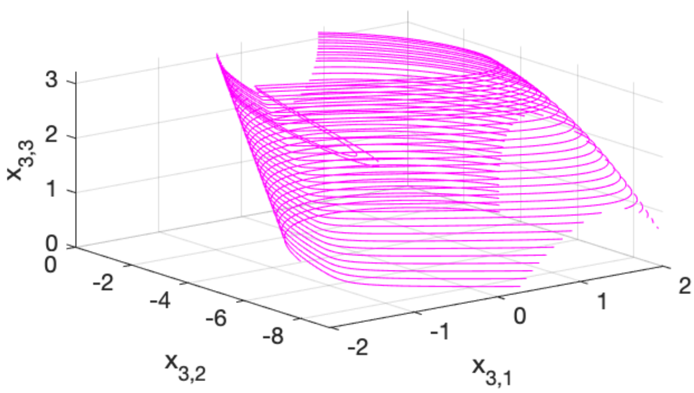


(a) $(x_{1,1}, x_{1,2}, x_{1,3})$



(b) $(x_{2,1}, x_{2,2}, x_{2,3})$



(c) $(x_{3,1}, x_{3,2}, x_{3,3})$

Figure 4.6: Simulation for the solution of the system considered in Example 2, with $\tau = 0.00002$: evolution of components $(x_{i,1}(t), x_{i,2}(t), x_{i,3}(t))$, $i = 1, 2, 3$.

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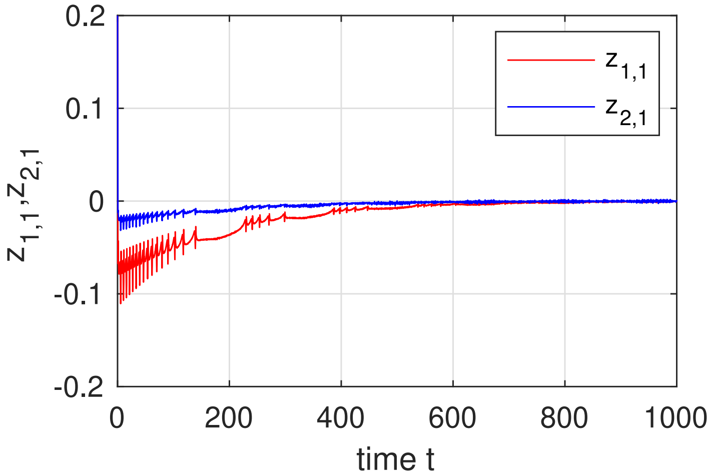
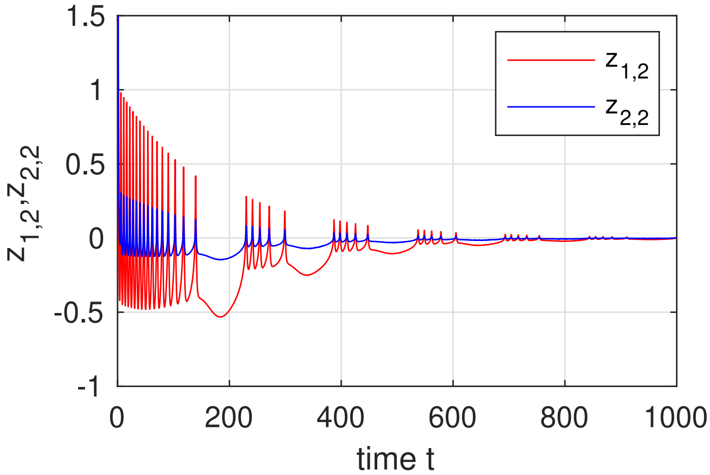
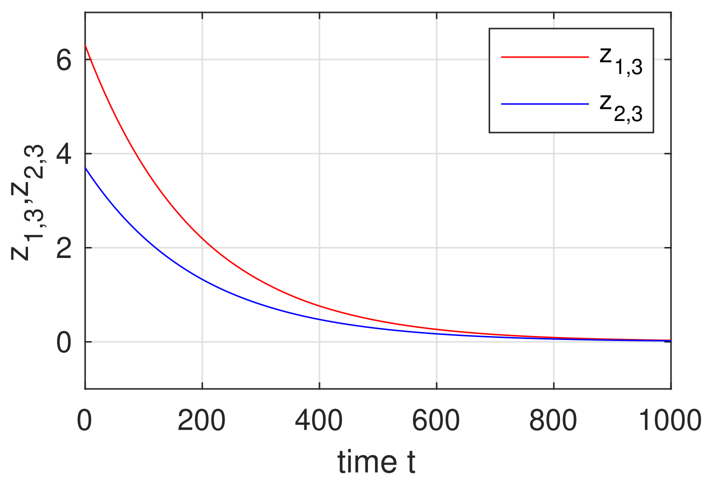


(a) $z_{i,1}$



(b) $z_{i,2}$



(c) $z_{i,3}$

Figure 4.7: Simulation for the solution of the system considered in Example 2, with $\tau = 0.00002$: evolution of components $z_{i,k}(t), i = 1, 2$, and $k = 1, 2, 3$.

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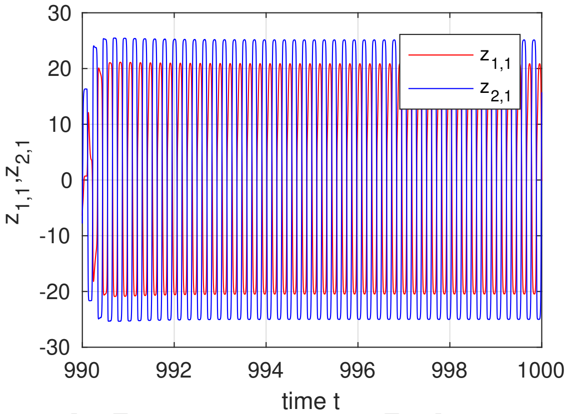
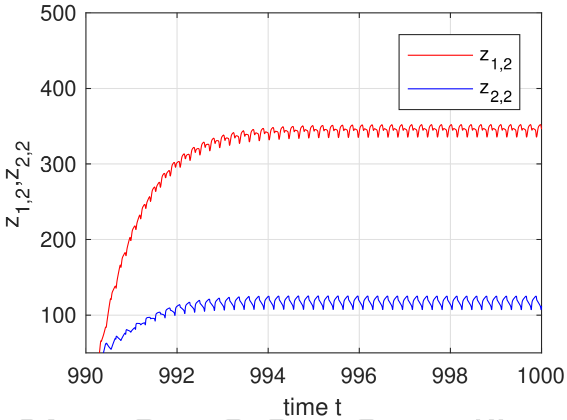
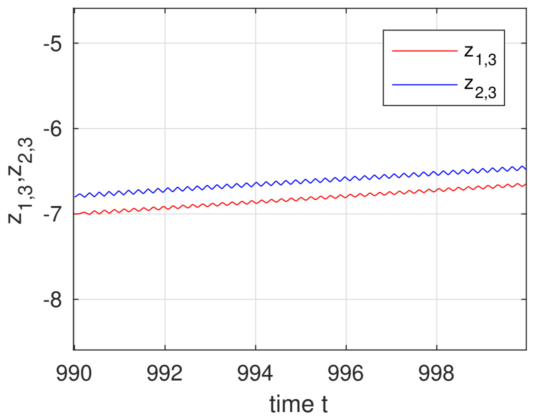


(a) $z_{i,1}$



(b) $z_{i,2}$



(c) $z_{i,3}$

Figure 4.8: Simulation for the solution of the system considered in Example 2, with $\tau = 0.1$: evolution of components $z_{i,k}(t)$, $i = 1, 2$, and $k = 1, 2, 3$

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Chapter 5

Conclusion

In the literature, there has been some investigations which addressed the global synchronization of coupled systems of Hindmarsh-Rose neurons. Among these investigations, most of them considered linear coupling functions and did not consider coupling time-delays. In addition, these studies commonly required that all nonzero and non-diagonal entries of the connection matrix have the same sign. In this thesis, we establish the global synchronization of non-linearly coupled systems of Hindmarsh-Rose neurons based on the theory in [33]. The coupling terms could be with time delays, the coupling function could be nonlinear, and the connection matrix could be with both negative and positive off-diagonal entries. By applying the synchronization criterion derived in this thesis, we can investigate the synchronization of systems of coupled Hindmarsh-Rose neurons, which cannot be treated by the previous methods, cf. Remark 4.1 and Examples 1, and 2.

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