Matrix Analytic Solutions for M/M/S Retrial Queues with Impatient Customers

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- Hsing Paul Luh (1) Email author (slu@nccu.edu.tw)View author's OrcID profile (View OrcID profile)
- Pei-Chun Song (1)

1. Department of Mathematical Sciences, National Chengchi University, , Taipei, Taiwan

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Abstract

In this paper, we investigate the nonhomogeneity of state space for solving retrial queues through the performance of the $M/M/S$ retrial system with impatient customers and S servers that is modeled under quasi-birth-and-death processes with level-dependent transient rates. We derive the analytic solution of multiserver retrial queues with orbit and develop an efficient method to solve this type of systems effectively. The methods proposed are based on nonhomogeneity of the state space although this queueing model was tackled by many researchers before. Under a weaker assumption in this paper, we study and provide the exact expression based on an eigenvector approach. Constructing an efficient algorithm for the stationary probability distribution by the determination of required eigenvalues with a specific accuracy, we develop streamlined matrices of state-balanced equations with the efficient implementation for computation of the performance measures.

Keywords

Quasi-birth-death process Retrial queues Matrix-geometric method Eigenvalues

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1 Introduction

In this paper, we consider a queueing system with retrials of arrivals following the human behavior that impatience users can abandon the system with certain probability after an unsuccessful retry. Retrial queues have been used to model a phenomenon in modern information and telecommunication systems that blocked customers may retry for service after a certain timeout (See [1, 9] and reference therein). Many examples of retrial queues can be found in communication networks nowadays. By an $M/M/S$ model, Do in [5] presents the effect of retrials in data transfers along Internet where in retrial queues a customer who does not receive the allocation of a server joins the orbit and later initiates a request for service. The $M/M/S$ retrial queue has been analyzed by many researchers. However, the stationary probability distribution when the number of servers is larger than two can be only obtained using approximate techniques (e.g., [1, 6]). Our goal in this study is to develop an effective method with a closed-form solution for solving this type of problems in which the number of servers is big.

The modeling of repeated attempts has been the subject of numerous investigations in queueing systems. In [7], it explains that two functional blocks are typically distinguished in models which consider retrials: a block that accommodates the servers and possibly a waiting queue, and a block where users that retry are accommodated, usually called a retrial orbit. Because the retrial rate among customers depends on the number of customers in the retrial orbit, when it is modeled by a Quasi-Birth-and-Death approach, it assumably shall build a nonhomogeneous and infinite state space. The term of nonhomogeneity of state space is used to describe the state transition probability, or the probability of increments/decrements, is not homogeneous instead it depends on the state over the studied system. When the state homogeneity condition does not hold for the case of multiserver retrial queues, the absence of closed-form solutions for the main performance characteristics is ineluctable. Either the finite truncated or generalized truncated methods may be used to replace the original infinite state space by a solvable state space, that is, a model where steady state probabilities can be computed. In this paper with an eigenvector approach, we investigate a computationally solvable with infinite state space to tackle this problem.

Falin and Templeton in [9] present necessary and sufficient conditions for ergodicity of the retrial queues with $M/M/S$. Falin in [8] presents an approximation which is based on the truncation of the state space at a sufficiently large level related to the number of customers in the orbit. Another approximation based on the homogenization of the model was pioneered by Neuts and Rao in [13], where the $M/M/S$ retrial queue is approximated by the multiserver retrial queue with the total retrial rate that does not depend on the number of customers in the orbit as long as the orbit contains the number of customers greater than the specified value N. Note that the discussion for the choice of N is presented in the book in [1] on retrial queues. With this assumption, the stationary probabilities of the $M/M/S$ retrial queue can be estimated by any algorithm of [3, 12] based on the matrix-geometric method (MGM). Domenech-Benlloch et al. [7] consider a multiserver retrial queue with the impatient phenomenon of customers waiting in the orbit. They propose two different generalized truncated methods (called HM1 and HM2) based on the homogenization of the state space when beyond the number of customers in the retrial orbit. The steady-state probabilities of the multiserver retrial queue with impatient customers are approximated with a modified retrial queue where the retrial rate beyond a certain level only depends on the conditional mean value of the number of customers in the orbit. They in [7] also compared their methods with other well-known algorithms that belong to different categories in [2], showing that the proposed HM2 method outperforms previous approaches from the aspect of accuracy at the price of increasing computation cost. Based on the HM2 algorithm, Do et al. in [6] propose an approximation that first obtains the conditional mean value $E[L|L \geq N]$ of the number L of customers in the orbit under the condition $L \geq N$ which is the simple function of both the single eigenvalue and N where N is suggested in [7].

Our contributions allow an efficient computation for the stationary probability distribution and the performance measures. The research direction is to evaluate the maximal eigenvalue of $\mathbf{R}$ without actually having computed $\mathbf{R}$. Instead, we adopted an approach based on the nonhomogeneity of the state space and provide an efficient method with the time complexity of only $O(S)$ to compute the eigenvectors of matrix $\mathbf{R}$. Then, we develop simplified equations that allow the efficient implementation of the computation of the performance measures. With a given precision level $\epsilon > 0$, we may construct an efficient computation algorithm for solving the stationary probability distribution, which guarantees a specific accuracy for the computation of performance measures.

The paper is organized as follows. A nonhomogeneous quasi-birth-and-death queueing model with orbit for impatient customers is constructed in Sect. 2. Matrix analytic derivation is presented in Sect. 3. The algorithmic solution procedures are described in Sect. 4. In Sect. 5, numerical test examples are presented for comparison with the results in [6].

2 A Queueing System with Orbit for Impatient Customers

In the system under study, we consider a queueing system with S servers. Customers arrive according to a Poisson process with rate λ and upon encountering an available server, request an exponentially distributed service time with rate μ. Without loss of generality, assume that each customer occupies one resource unit. When a new request finds all servers occupied, it joins the retrial orbit immediately. There is an infinite capacity for the retrial orbit with a random service discipline. After a random time that is exponentially distributed of rate γ this customer retries, being a successful retrial if it finds a free server. Otherwise, the customer leaves the system with probability p or returns to the retrial orbit with probability $1-p$ independently of the occupancy rate of the servers and start the retrial procedure again. Conventionally, denote by ρ the average load of the system.

The model considered here can be represented as a bi-dimensional continuous-time Markov chain (MC) whose state space is defined by the number of customers in the retrial orbit and the number of customers being served, constituting a Level Dependent Quasi-Birth-and-Death Process (LDQBD). In QBD related literature, the term level refers to a set of states with the same first coordinate. Consider a retrial queueing model with S homogeneous servers and impatient customers. Let a random variable $J(t)$ represent the number of occupied servers at time t, $0 \leq J(t) \leq S$. When $J(t) = S$, a customer joins the orbit in order to wait and retry. Let $L(t)$ be the number of customers in the orbit waiting for retrial at time t. Each customer retries with rate γ. A retrying customer either leaves the queue with probability p if all servers are busy upon the retrial or rejoins the orbit with probability $1-p$. Note that a time between subsequent retrials of a specific customer follows the exponential distribution with parameter γ. The main characteristics of this model are its infinite state space $(L(t), J(t))$ and also its space of nonhomogeneity produced by the fact that the retrial rate depends on the number of customers in the retrial orbit.

Suppose the system is stable and $\lim_{t\to\infty} Pr(L(t), J(t))$ exists. This system can be represented by two-dimensional continuous-time Markov chain (CTMC) $X = \{L(t), J(t)\}$ with state space $\{0, 1, \dots\} \times \{0, 1, \dots, S\}$. We will use a two dimensional state space description where state (ℓ, j) denotes that the number of customers in orbit equals ℓ ($\ell = 0, 1, 2, \cdots$) and that j ($j = 0, 1, 2, \dots, S$) servers are busy. Hence, the total effective retrial rate is $\ell\gamma$ when $L = \ell$. The infinitesimal generator of this process has an infinite block tridiagonal structure $\mathbf{Q}$ defined in (1). Let $m = S + 1$ and $\mathbf{e}_i$ be the row vector in which the ith component is 1, 0 elsewhere, $i = 1, 2, \cdots, m$. Denote by $\mathbf{e}_i^t$ the transpose of the vector $\mathbf{e}_i$. Construct a discrete-time and nonhomogeneously infinitesimal generator for X as the following $\mathbf{Q}$,

$$\mathbf{Q} \triangleq \begin{bmatrix} \mathbf{Q}_1^{(0)} & \mathbf{Q}_0^{(0)} & 0 & & & \cdots \\ \mathbf{Q}_2^{(1)} & \mathbf{Q}_1^{(1)} & \mathbf{Q}_0^{(1)} & 0 & & \cdots \\ 0 & \mathbf{Q}_2^{(2)} & \mathbf{Q}_1^{(2)} & \mathbf{Q}_0^{(2)} & \cdots & \\ & & & \cdots & \cdots & \cdots \end{bmatrix}$$

(1)

where

$\mathbf{Q}_2^{(\ell)} = \ell\mathcal{B}$ for $\ell = 1, 2, \cdots$

$\mathbf{Q}_1^{(\ell)} = \mathcal{A} - D^{\mathcal{A}} - \mathbf{Q}_0^{(\ell)} - D^{(\ell)}$, for $\ell = 0, 1, 2, \cdots$

$\mathbf{Q}_0^{(\ell)} = \lambda \mathbf{e}_m^t \mathbf{e}_m \quad$ for $\ell = 0, 1, 2, \cdots$

and

$$\mathcal{A} = \begin{bmatrix} 0 & \lambda & 0 & & & & \\ \mu & 0 & \lambda & 0 & & & \\ 0 & 2\mu & 0 & \lambda & & & \\ & 0 & \cdots & & & \cdots & \\ & 0 & & (S-1)\mu & 0 & \lambda & \\ & & & & S\mu & 0 & \end{bmatrix}_{m \times m}$$

$$\mathcal{B} = \begin{bmatrix} 0 & \gamma & 0 & & \\ 0 & 0 & \gamma & & \\ \vdots & & & \ddots & \\ 0 & & & & \gamma \\ 0 & & & 0 & \gamma p \end{bmatrix}_{m \times m}$$

and $D^{(\cdot)}$ denotes a diagonal matrix with the diagonal elements defined as

$$D^{\mathcal{A}}(i,i) = \textstyle\sum_{k=0}^{S} \mathcal{A}(i,k), \quad i = 0, 1, 2, \ldots, S$$

$$D^{(\ell)}(i,i) = \textstyle\sum_{k=0}^{S} \mathbf{Q}_2^{(\ell)}(i,k), \quad \ell = 0, 1, 2, \ldots$$

$$D^{(0)} = 0.$$

3 LDQBD Model Formulation

Let the steady-state probabilities of X be denoted by $\pi_{\ell,j} = \lim_{t\to\infty} \mathrm{Pr}(L(t) = \ell, J(t) = j)$. Define the row vector $\pi_\ell = [\pi_{\ell,0}, \ldots, \pi_{\ell,S}]$, $\ell = 0, 1, 2, \ldots$. Throughout the paper, we adhere to the convention, unless stated otherwise, that probability vectors are row vectors. $\mathbf{Q}$ in (1) is an irreducible stochastic matrix, its steady state probability vector associated to it is denoted by π, and we partition it as $\pi = (\pi_0, \pi_1, \pi_2 \ldots)$, where π_ℓ, $\ell \geq 0$, is an m-vector. Being a stationary probability distribution, π satisfies $\pi\mathbf{Q} = \mathbf{0}$ and $\pi\mathbf{1} = 1$, where $\mathbf{0}$ is a zero matrix and $\mathbf{1}$ is a column of all 1.

An MC is said to be positive recurrent if the mean time to return to each state for the first time after leaving it is finite. In infinite QBD MCs, this requires that the drift to higher level states be smaller than the drift to lower level states. To preserve the stability of the system, i.e., the existence of the steady state probability distribution, the MC is assumed aperiodic as well.

Theorem 1

([10]). If the LDQBD process with a transition rate matrix given by (1) is irreducible, aperiodic, and positive recurrent, then there exist matrices $\{\mathbf{R}^{(\ell)} : \ell \geq 1\}$ such that
$\pi_{(\ell+1)} = \pi_{\ell}\mathbf{R}^{(\ell+1)},\ \ \ell \geq 0$
where the sequence $\{\mathbf{R}^{(\ell)}\}$ is the minimal nonnegative solution of the set of equations given by

$$\mathbf{Q}_0^{(\ell)} + \mathbf{R}^{(\ell+1)}\mathbf{Q}_1^{(\ell+1)} + \mathbf{R}^{(\ell+1)}\mathbf{R}^{(\ell+2)}\mathbf{Q}_2^{(\ell+2)} = \mathbf{0},\ \ \ell \geq 0$$

(2)

The proof may be found in [10].

Let $||\cdot||_1$ denote a matrix norm by

$$||\mathbf{Z}||_1 = \max_{1 \leq j \leq m} \sum_{i=1}^{m} |[\mathbf{Z}]_{ij}|$$

where $\mathbf{Z}$ is an $m \times m$ matrix with its element $[\mathbf{Z}]_{ij}$ at the ith row and the jth column. With an extension of Theorem 1, we claim the following fact.

Corollary 1

If $\mathbf{Q}$ is irreducible then for any $\epsilon > 0$, there exists a number K such that for all $n > K$ we have $\frac{1}{n}\frac{||\mathbf{Q}_1^{(n)} - \mathbf{Q}_1^{(n+1)}||_1}{||\mathbf{Q}_1^{(n+1)}||_1} < \epsilon,$ if and only if $\pi_n \mathbf{1} < \epsilon$.

To extend the result from a stable queueing system, we have the following lemma in general.

Lemma 1

If $\mathbf{Q}$ and $\mathbf{A}^{(n)}$ for $n > K$ are irreducible, where $\mathbf{A}^{(n)} = \mathbf{Q}_0^{(n)} + \mathbf{Q}_1^{(n)} + \mathbf{Q}_2^{(n)}$, then $\mathbf{Q}$ is positive recurrent if and only if $\mathbf{p}(\mathbf{Q}_0^{(n)} - \mathbf{Q}_2^{(n)})\mathbf{1} < 0$, where $\mathbf{p}$ satisfies $\mathbf{p}\mathbf{A}^{(n)} = \mathbf{0}$ and $\mathbf{p}\mathbf{1} = 1$, for all $n > K$.

The proof could be done with a similar homogeneous case and can be found in [11].

Since we know for every $\ell > 0$

$$|[\mathbf{Q}_1^{(\ell)}]_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{m} |[\mathbf{Q}_1^{(\ell)}]_{ij}| \text{ for all } i = 1, 2, \ldots, m,$$

it implies that $\mathbf{Q}_1^{(\ell)}$ is invertible. Note $\sup_i |[\mathbf{Q}_1^{(\ell)}]_{ii}| < \infty$ and

$$|[\mathbf{Q}_1^{(\ell)}]_{ii}| > \sum_{j=1}^{m} |[\mathbf{Q}_2^{(\ell)}]_{ij}| \text{ for all } i = 1, 2, \ldots, m.$$

By observation, given a fixed n, if the maximal eigenvalue of $(x\mathbf{Q}_2^{(n+2)}(\mathbf{Q}_1^{(n+1)})^{-1}) < 1$, then $(\mathbf{Q}_1^{(n+1)} + x\mathbf{Q}_2^{(n+2)})$ is invertible. Define $\mathbf{I}$ as the identity matrix. Consider (2), for $n > K$, and for $0 < x < 1$, claim that matrix $(\mathbf{Q}_1^{(n+1)} + x\mathbf{Q}_2^{(n+2)})$ is invertible. Because of $(\mathbf{Q}_1^{(n+1)} + x\mathbf{Q}_2^{(n+2)}) = (\mathbf{I} + x\mathbf{Q}_2^{(n+2)}(\mathbf{Q}_1^{(n+1)})^{-1})\mathbf{Q}_1^{(n+1)}$. If $||x\mathbf{Q}_2^{(n+2)}(\mathbf{Q}_1^{(n+1)})^{-1}||_1 < 1$, then $(\mathbf{I}+x\mathbf{Q}_2^{(n+2)}(\mathbf{Q}_1^{(n+1)})^{-1})$ exists. We write $(\mathbf{I}+x\mathbf{Q}_2^{(n+2)}(\mathbf{Q}_1^{(n+1)})^{-1})^{-1}$ as a power series in $x\mathbf{Q}_2^{(n+2)}(\mathbf{Q}_1^{(n+1)})^{-1}$. This gives

$$(\mathbf{Q}_1^{(n+1)}+x\mathbf{Q}_2^{(n+2)})^{-1}=(\mathbf{Q}_1^{(n+1)})^{-1}(\mathbf{I}+x\mathbf{Q}_2^{(n+2)}(\mathbf{Q}_1^{(n+1)})^{-1})^{-1}$$
$$=(\mathbf{Q}_1^{(n+1)})^{-1}\sum_{k=0}^{\infty}(-1)^k(x\mathbf{Q}_2^{(n+2)}(\mathbf{Q}_1^{(n+1)})^{-1})^k$$

Note $\mathbf{Q}_0^{(n)}$ is rank-1 since $\mathbf{Q}_0^{(n)}=\lambda\mathbf{e}_m^t\mathbf{e}_m$. Define

$$h^{(n)}(x)\overset{\Delta}{=}-\lambda\mathbf{e}_m(\mathbf{Q}_1^{(n+1)}+x\mathbf{Q}_2^{(n+2)})^{-1}\mathbf{e}_m^t.$$

Moreover, $h^{(n)}(x)$ can be written as

$$h^{(n)}(x)=\sum_{k=0}^{\infty}c_kx^k,\quad 0<x<1,$$

where $c_k=(-1)^{k+1}\mathbf{e}_m(\mathbf{Q}_1^{(n+1)})^{-1}[\mathbf{Q}_2^{(n+2)}(\mathbf{Q}_1^{(n+1)})^{-1}]^k\,\lambda\mathbf{e}_m^t$.

Lemma 2

Given $(\mathbf{Q}_1^{(n+1)}+x\mathbf{Q}_2^{(n+2)})$ is invertible, we have $h^{(n)}(x)=x$, such that $0<x<1$ for $n>K$.

Proof

Under the stability in (2), we assume that

$\mathbf{R}^{(n)}\to\mathbf{R},$ for $n>K,$

and there exists $0<x<1$ such that

$$\det[\mathbf{Q}_0^{(n)}+x\mathbf{Q}_1^{(n+1)}+x^2\mathbf{Q}_2^{(n+2)}]=0.$$

Consider the following characteristic polynomial

$$\det[\mathbf{Q}_0^{(n)}+x\mathbf{Q}_1^{(n+1)}+x^2\mathbf{Q}_2^{(n+2)}]$$
$$=\det[\mathbf{Q}_0^{(n)}+x(\mathbf{Q}_1^{(n+1)}+x\mathbf{Q}_2^{(n+2)})]$$
$$=\det[\mathbf{Q}_0^{(n)}(\mathbf{Q}_1^{(n+1)}+x\mathbf{Q}_2^{(n+2)})^{-1}+x\mathbf{I}]\det[\mathbf{Q}_1^{(n+1)}+x\mathbf{Q}_2^{(n+2)}].$$

Since $\det[\mathbf{Q}_1^{(n+1)}+x\mathbf{Q}_2^{(n+2)}]\neq 0$, finding a zero of the characteristic polynomial is equivalent to finding a zero in the following equation,

$$\det[\mathbf{Q}_0^{(n)}(\mathbf{Q}_1^{(n+1)}+x\mathbf{Q}_2^{(n+2)})^{-1}+x\mathbf{I}]=0,$$
$$\det[1+\mathbf{e}_m(\mathbf{Q}_1^{(n+1)}+x\mathbf{Q}_2^{(n+2)})^{-1}\frac{1}{x}\lambda\mathbf{e}_m^t]=0,$$
$$-\mathbf{e}_m(\mathbf{Q}_1^{(n+1)}+x\mathbf{Q}_2^{(n+2)})^{-1}\mathbf{e}_m^t\lambda-x=0,$$
$$h^{(n)}(x)-x=0.\Box$$

Lemma 3

When $\mathbf{Q}_0^{(n)}=\lambda\mathbf{e}_m^t\mathbf{e}_m$ is rank-1, there exists uniquely *x* satisfying $x=h^{(n)}(x)$, $0<x<1$.

Proof

First, $[\mathbf{Q}_1^{(n)}]_{ii}<0$ defined by (1), for all *i*, we have $c_0=(-1)\mathbf{e}_m\,(\mathbf{Q}_1^{(n)})^{-1}\lambda\mathbf{e}_m^t>0$ and $c_1=\mathbf{e}_m(\mathbf{Q}_1^{(n)})^{-1}(\mathbf{Q}_2^{(n)}(\mathbf{Q}_1^{(n)})^{-1})\lambda\mathbf{e}_m^t>0$. By induction on *k*, we have $c_k>0$, for all $k\geq 0$. The function is thus increasing for $0<x<1$, and

$h(0) = c_0 = -\mathbf{e}_m(\mathbf{Q}_1^{(n)})^{-1}\lambda\mathbf{e}_m^t > 0$. In addition, one may check that $h'(x) > 0$, and $h''(x) > 0$. Second, claim $h(1) = 1$ in the following arguments. By Lemma 1 with $\mathbf{p}\mathbf{A}^{(n)} = \mathbf{0}$ for $n > K$, we have

$-\mathbf{p}(\mathbf{Q}_1^{(n)} + \mathbf{Q}_2^{(n)}) = \mathbf{p}\mathbf{Q}_0^{(n)} = \lambda\mathbf{p}\mathbf{e}_m^t\mathbf{e}_m$, implying that

$\mathbf{p} = -\lambda\mathbf{p}\mathbf{e}_m^t\mathbf{e}_m(\mathbf{Q}_1^{(n)} + \mathbf{Q}_2^{(n)})^{-1}$. Multiplying $\mathbf{e}_m^t$ from right on both sides, it gives

$\mathbf{p}\mathbf{e}_m^t = -\mathbf{p}\mathbf{e}_m^t\mathbf{e}_m(\mathbf{Q}_1^{(n)} + \mathbf{Q}_2^{(n)})^{-1}\mathbf{e}_m^t\lambda$.

Thus, it produces $\lambda\mathbf{e}_m(\mathbf{Q}_1^{(n)} + \mathbf{Q}_2^{(n)})^{-1}\mathbf{e}_m^t = -1$. Hence, we have that $h^{(n)}(1) = 1$. It is now clear that there is a unique solution of $x = h^{(n)}(x)$ between 0 and 1 when $n > K$. $\square$

Hence, x can be found when $n > K$ in the following equation,

$$h^{(n)}(x) = x. \quad (3)$$

Corollary 2

Suppose $\mathbf{Q}_0^{(\ell)}$ is rank-1. A fixed point of (3) is an eigenvalue of $\mathbf{R}$ and ξ is the corresponding left eigenvector.

The proof is straightforward.

Theorem 2

If $\mathbf{Q}_0^{(\ell)}$ is rank-1 and $\mathbf{Q}_2^{(\ell)}$ is nonsingular, then for any $\epsilon > 0$ there exists a K such that $\sum_{k=n}^{\infty}\pi_k < \epsilon$ for $n > K$ and solve a fixed point for $h^{(K)}(x) = x$.

The proof is easily obtained by the arguments provided in Lemma 2. For further discuss explicitly, denote that the fixed point is σ such that $h^{(K)}(\sigma) = \sigma$.

Define

$$\xi^{(\ell)}(x) \stackrel{\Delta}{=} -\lambda\mathbf{e}_m(\mathbf{Q}_1^{(\ell+1)} + x\mathbf{Q}_2^{(\ell+2)})^{-1}.$$

Since $\mathbf{Q}_0^{(\ell)} = \lambda\mathbf{e}_m^t\mathbf{e}_m$ is rank-1, let $\mathbf{R}^{(\ell)} = \mathbf{e}_m^t\xi^{(\ell)}$. Suppose x_ℓ is an eigenvalue of $\mathbf{R}^{(\ell)}$ and the corresponding left eigenvector is $\xi^{(\ell)}$, that is $\xi^{(\ell)}\mathbf{R}^{(\ell)} = x_\ell\xi^{(\ell)} = \xi^{(\ell)}\mathbf{e}_m^t\xi^{(\ell)}$. Thus, it implies $x_\ell = \xi^{(\ell)}\mathbf{e}_m^t$. Similarly, $x_{\ell+1}$ and $\xi^{(\ell+1)}$ is an eigenpair associated with $\mathbf{R}^{(\ell+1)}$.
Consider (2) again and let it be written as

$$\mathbf{Q}_0^{(\ell)} + \mathbf{e}_m^t\xi^{(\ell+1)}(\mathbf{Q}_1^{(\ell+1)} + \mathbf{e}_m^t\xi^{(\ell+2)}\mathbf{Q}_2^{(\ell+2)}) = \mathbf{0}.$$

Multiplying $\mathbf{e}_m$ on both sides of the equation above, we have

$$\begin{aligned}\mathbf{e}_m\mathbf{Q}_0^{(\ell)} + \mathbf{e}_m\mathbf{e}_m^t\xi^{(\ell+1)}(\mathbf{Q}_1^{(\ell+1)} + \mathbf{e}_m^t\xi^{(\ell+2)}\mathbf{Q}_2^{(\ell+2)}) &= \mathbf{0}\\ \lambda\mathbf{e}_m + \xi^{(\ell+1)}(\mathbf{Q}_1^{(\ell+1)} + \mathbf{e}_m^t\xi^{(\ell+2)}\mathbf{Q}_2^{(\ell+2)}) &= \mathbf{0}.\end{aligned}$$

Finally, we have

$$\xi^{(\ell+1)} = -\lambda\mathbf{e}_m(\mathbf{Q}_1^{(\ell+1)} + \mathbf{e}_m^t\xi^{(\ell+2)}\mathbf{Q}_2^{(\ell+2)})^{-1} \quad \text{for } \ell = 0, 1, \cdots, K-1. \quad (4)$$

Because $\xi^{(\ell+1)}$ is a function of x and $\xi^{(\ell+1)}\mathbf{e}_m^t = x_{\ell+1}$, we may decide $x_{\ell+1}$ by $\xi^{(\ell+1)}(x)\mathbf{e}_m^t$ with assigning a x, $0 < x < 1$.

Consider $h^{(\ell)}(x) = \xi^{\ell}(x)\mathbf{e}_m^t$ again but we are going to use it with $\xi^{(\ell)}$ for $0 < \ell < K$.

Theorem 3

If the Markov chain is positive recurrent, we have $0 < x < 1$ and $h^{(\ell)}(1) > 1$, for $0 < \ell < K$.

Proof

From (4) we know
$\mathbf{e}_m(\mathbf{Q}_1^{(\ell+1)} + (\mathbf{Q}_2^{(\ell+2)})^{-1}\mathbf{e}_m^t < \mathbf{e}_m(\mathbf{Q}_1^{(\ell+1)} + (\mathbf{Q}_2^{(\ell+1)})^{-1}\mathbf{e}_m^t,$
and
$-\mathbf{e}_m(\mathbf{Q}_1^{(\ell+1)} + (\mathbf{Q}_2^{(\ell+2)})^{-1}\mathbf{e}_m^t > -\mathbf{e}_m(\mathbf{Q}_1^{(\ell+1)} + (\mathbf{Q}_2^{(\ell+1)})^{-1}\mathbf{e}_m^t = 1.$
Thus we have $h^{(\ell)}(1) > 1$. ☐

Then it is shown in Theorem 3 that under certain irreducibility conditions, the value of the *h*(*x*) in lies (0,1), which may efficiently be solved and will be expressed by $h^{(\ell)}(x)$.

4 Deficient Matrix Approaches

In this section, we will focus on an efficient approach by taking into account π_0 at the boundary state when it solves the stationary probability π. We rewrite the system state balance equations as

$$[\,\pi_0, \pi_1, \pi_2, \cdots] \begin{bmatrix} \mathbf{Q}_1^{(0)} & \mathbf{Q}_0^{(0)} & 0 & & \mathbf{1} \\ \mathbf{Q}_2^{(1)} & \mathbf{Q}_1^{(1)} & \mathbf{Q}_0^{(1)} & \ddots & \mathbf{1} \\ 0 & \mathbf{Q}_2^{(2)} & \mathbf{Q}_1^{(2)} & \ddots & \vdots \\ & \ddots & \vdots & & \end{bmatrix} = [0, 0, \cdots, 0, 1].$$

4.1 Boundary Equations and Eigenvector Approaches

Let $\mathbf{T} = \mathbf{Q}_1^{(0)} + \mathbf{e}_m^t\xi^{(1)}\mathbf{Q}_2^{(1)}$ and $\xi^{(0)} = -\lambda\mathbf{e}_m[(\mathbf{T} + \mathbf{e}_m^t\mathbf{e}_m)^{-1} - (\mathbf{T} + a\mathbf{e}_m^t\mathbf{e}_m)^{-1}]$. We consider a general structure of $\mathbf{T}$ in the following lemma.

Lemma 4

Let $\mathbf{T}$ be an $m \times m$ matrix with rank $m-1$, and $\mathbf{T}$ has no zero row or column. There exists a rank-1 matrix $\mathbf{P}$ that satisfies $\mathbf{TP} = \mathbf{PT} = \mathbf{0}$, which is determined uniquely only up to a constant.

Proof

Let $S = m - 1$. Without losing of generality, we may assume $\{\mathbf{e}_1\mathbf{T}, \mathbf{e}_2\mathbf{T}, \cdots, \mathbf{e}_S\mathbf{T}\}$ are linearly independent. Since rank$(\mathbf{T}) = S$, there exist some constants $c_1, c_2, \cdots, c_S$ which are not all zero such that

$$c_1\mathbf{e}_1\mathbf{T} + c_2\mathbf{e}_2\mathbf{T} + \cdots + c_S\mathbf{e}_S\mathbf{T} = \mathbf{e}_m\mathbf{T}. \quad (5)$$

To prove $c_1, c_2, \cdots, c_S$ are uniquely determined. Suppose there are other numbers $d_1, d_2, \cdots, d_S$ such that $d_1\mathbf{e}_1\mathbf{T} + d_2\mathbf{e}_2\mathbf{T} + \cdots + d_S\mathbf{e}_S\mathbf{T} = \mathbf{e}_m\mathbf{T}$ which is also an expression of $\mathbf{e}_m\mathbf{T}$. By subtracting one from another, we obtain $(c_1 - d_1)\mathbf{e}_1\mathbf{T} + (c_2 - d_2)\mathbf{e}_2\mathbf{T} + \cdots + (c_S - d_S)\mathbf{e}_S\mathbf{T} = \mathbf{0}$. According to our assumption of linear independence of $\{\mathbf{e}_1\mathbf{T}, \mathbf{e}_2\mathbf{T}, \cdots, \mathbf{e}_S\mathbf{T}\}$, we have $c_i - d_i = 0$, $1 \le i \le S$. Hence, $c_i - d_i = 0$ for $1 \le i \le S$ and the expression is determined uniquely.

From (5), we have $(c_1, c_2, \cdots, c_S, -1)\mathbf{T} = 0$ which implies

$$\begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_S & -1 \\ c_1 & c_2 & c_3 & \cdots & c_S & -1 \\ & & & \cdots & \cdots & \cdots \\ c_1 & c_2 & c_3 & \cdots & c_S & -1 \end{bmatrix} \mathbf{T} = \mathbf{0}$$

Similarly, suppose $\{\mathbf{T}\mathbf{e}_1^t, \mathbf{T}\mathbf{e}_2^t, \cdots, \mathbf{T}\mathbf{e}_S^t\}$ are linearly independent and there exist some constants $a_1, a_2, \cdots, a_S$ which are not all zero such that $a_1\mathbf{T}\mathbf{e}_1^t + a_2\mathbf{T}\mathbf{e}_2^t + \cdots + a_S\mathbf{T}\mathbf{e}_S^t = \mathbf{T}\mathbf{e}_m^t$. It implies that

$$\mathbf{T} \begin{bmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & a_2 & \cdots & a_2 \\ & & & \cdots & \cdots \\ a_S & a_S & a_S & \cdots & a_S \\ -1 & -1 & -1 & \cdots & -1 \end{bmatrix} = \mathbf{0}$$

Based on two cases described, $\mathbf{P}$ may be written as

$$\mathbf{P} = c \times \begin{bmatrix} a_1c_1 & a_2c_1 & a_3c_1 & \cdots & a_Sc_1 & -c_1 \\ a_1c_2 & a_2c_2 & a_3c_2 & \cdots & a_Sc_2 & -c_2 \\ & & & \cdots & \cdots & \cdots \\ -a_1 & -a_2 & -a_3 & \cdots & -a_S & 1 \end{bmatrix}$$

where *c* is an arbitrary real number. This means $\mathbf{P}$ is uniquely determined up to a constant. It concludes that $\mathbf{TP} = \mathbf{PT} = \mathbf{0}$. $\square$

Define $\mathbf{E}$ with a nonzero row vector $\mathbf{b}$ as follows

$$\mathbf{E} \overset{\Delta}{=} \mathbf{e}_m^t\mathbf{b}.$$

Consider the matrix $\mathbf{EPE}$ with its element at the *i*th row and the *j*th column, i.e., $[\mathbf{EPE}]_{i,j}$. We write

$$\begin{aligned} [\mathbf{EPE}]_{i,j} &= \sum_{k=1}^{m} [\mathbf{E}]_{i,k} [\mathbf{PE}]_{k,j} \\ &= \sum_{k=1}^{m}\sum_{\ell=1}^{m} [\mathbf{E}]_{i,k} [\mathbf{P}]_{k,\ell} [\mathbf{E}]_{\ell,j} \\ &= \sum_{k=1}^{m} [\mathbf{E}]_{i,k} [\mathbf{P}]_{k,m} [\mathbf{E}]_{m,j} \end{aligned}$$

Since $\mathbf{E}$ is a rank-1 matrix with the first *S* rows of zeros, we know that

If $i \ne m$, $[\mathbf{EPE}]_{i,j} = 0$

If $i = m$, $[\mathbf{EPE}]_{m,j} = [\mathbf{EP}]_{m,m}[\mathbf{E}]_{m,j}$

This implies $\mathbf{EPE} = [\mathbf{EP}]_{m,m} \times \mathbf{E}$, so we choose a proper *r* with $\mathbf{P}$ such that $[\mathbf{EP}]_{m,m} = 1$ and $\mathbf{EPE} = \mathbf{E}$. It is easy to see that by given *a* one may choose a proper *r* such that $\mathbf{EPE} = a\,\mathbf{E}$.

Suppose the first S columns or rows of $\mathbf{T}$ are linear independent. It produces for $a \neq 0$ that $\mathbf{T} + a\,\mathbf{e}_m^t\mathbf{b}$ is full rank and $(\mathbf{T} + a\,\mathbf{e}_m^t\mathbf{b})^{-1}$ exists. Suppose
$(\mathbf{T} + a\,\mathbf{e}_m^t\mathbf{b})^{-1} \stackrel{\Delta}{=} \mathbf{P}_a + \mathbf{W}_a$
where $\mathbf{P}_a$ satisfies $\mathbf{TP}_a = \mathbf{P}_a\mathbf{T} = \mathbf{0}$ and $\mathbf{W}_a$ denotes a matrix for the remainders with respect to $(\mathbf{T} + a\,\mathbf{e}_m^t\mathbf{b})^{-1}$.

Lemma 5

Let $(\mathbf{T} + a\mathbf{E})^{-1} = \mathbf{P}_a + \mathbf{W}_a$, where $\mathbf{TP}_a = \mathbf{P}_a\mathbf{T} = \mathbf{0}$ and $\mathbf{EP}_a\mathbf{E} = \mathbf{E}$, then $\mathbf{EW}_a = \mathbf{W}_a\mathbf{E} = \mathbf{0}$.

Proof

Since $\mathbf{TP}_a = \mathbf{P}_a\mathbf{T} = \mathbf{0}$, we may choose a proper r such that $\mathbf{EP}_a\mathbf{E} = \mathbf{E}$. Consider

$$(-\mathbf{E})[(\mathbf{T} + a\mathbf{E})^{-1}(\mathbf{T} + a\mathbf{E})] + \mathbf{E} = \mathbf{0}$$

$$(-\mathbf{E})(\mathbf{T} + a\mathbf{E})^{-1}\mathbf{T} - \mathbf{E}(\mathbf{T} + a\mathbf{E})^{-1}\mathbf{E} + \mathbf{E} = \mathbf{0}$$

$\Rightarrow$

$$\begin{aligned}
&-\mathbf{E}(\mathbf{T} + a\,\mathbf{E})^{-1}\mathbf{T} \\
&= \mathbf{E}(\mathbf{T} + a\,\mathbf{E})^{-1}\mathbf{E} - \mathbf{E} \\
&= \mathbf{EP}_a\mathbf{E} + \mathbf{EW}_a\mathbf{E} - \mathbf{E} \\
&= \mathbf{EP}_a\mathbf{E} - \mathbf{E} + \mathbf{EW}_a\mathbf{E} \\
&= \mathbf{EW}_a\mathbf{E} \quad \text{with a properly chosen } c.
\end{aligned}$$

On the other hand, we have a similar derivation in the following,

$$\begin{aligned}
&-\mathbf{E}(\mathbf{T} + a\,\mathbf{E})^{-1}\mathbf{T} \\
&= -\mathbf{E}(\mathbf{P}_a + \mathbf{W}_a)\mathbf{T} \\
&= -\mathbf{EW}_a\mathbf{T}
\end{aligned}$$

From the two expressions above, it gives that

$$\mathbf{EW}_a\mathbf{E} = -\mathbf{EW}_a\mathbf{T}$$

$$\mathbf{EW}_a(\mathbf{T} + \mathbf{E}) = \mathbf{0}$$

$$\mathbf{EW}_a = \mathbf{0}$$

For $(\mathbf{T} + \mathbf{E})$ is of the full rank, we can similarly acquire $\mathbf{W}_a\mathbf{E} = \mathbf{0}$. $\square$

Lemma 6

Let $\mathbf{T}$ be an $m \times m$ matrix with rank $m-1$, has no zero row or column, and $\mathbf{b}$ be a row vector satisfies that $\mathbf{T} + \mathbf{e}_m^t\mathbf{b}$ is full rank, then $[(\mathbf{T} + \mathbf{e}_m^t\mathbf{b})^{-1} - (\mathbf{T} + a\,\mathbf{e}_m^t\mathbf{b})^{-1}]\mathbf{T} = \mathbf{0}$ for all $a \neq 0$.

Proof

Recall $\mathbf{E} = \mathbf{e}_m^t\mathbf{b}$ and consider

$$\mathbf{I} = (\mathbf{T} + \mathbf{E})(\mathbf{P}_1 + \mathbf{W}_1) = \mathbf{EP}_1 + \mathbf{TW}_1 \quad \text{with } a = 1,$$

$$\mathbf{I} = (\mathbf{T} + a\mathbf{E})(\frac{1}{a}\mathbf{P}_1 + \mathbf{W}_a) = \mathbf{EP}_1 + \mathbf{TW}_a$$

Combining the equations above we have

$$\mathbf{T}(\mathbf{W}_1 - \mathbf{W}_a) = \mathbf{0}.$$

Similarly, we have
$(\mathbf{W}_1 - \mathbf{W}_a)\mathbf{T} = \mathbf{0}$.
If $\mathbf{W}_1 = \mathbf{W}_a$, then we are done. Otherwise, it means $\mathbf{W}_1 - \mathbf{W}_a = \beta\mathbf{P}_1$ for some constant $\beta \neq 0$ by Lemma 4, then it produces
$\beta\mathbf{E}\mathbf{P}_1 = \mathbf{E}(\mathbf{W}_1 - \mathbf{W}_a) = \mathbf{0}$.
Since $\mathbf{E}\mathbf{P}_1 \neq \mathbf{0}$, we have $\beta = 0$, implying $\mathbf{W}_1 = \mathbf{W}_a$. □

From Lemma 12, $\mathbf{e}_m[(\mathbf{T} + \mathbf{e}_m^t\mathbf{b})^{-1} - (\mathbf{T} + a\,\mathbf{e}_m^t\mathbf{b})^{-1}]\mathbf{T} = \mathbf{0}$, it provides a way to obtain $\xi^{(0)}$, because $\mathbf{Q}_1^{(0)} + \mathbf{R}^{(1)}\mathbf{Q}_2^{(1)}$ plays the role of $\mathbf{T}$ in the equation. So we can set
$\xi^{(0)} = -\lambda\mathbf{e}_m[(\mathbf{Q}_1^{(0)} + \mathbf{e}_m^t\xi^{(1)}\mathbf{Q}_2^{(1)} + \mathbf{e}_m^t\mathbf{b})^{-1} - (\mathbf{Q}_1^{(0)} + \mathbf{e}_m^t\xi^{(1)}\mathbf{Q}_2^{(1)} + a\mathbf{e}_m^t\mathbf{b})^{-1}]$.
(6)
for any $a \neq 0$, and any row vector $\mathbf{b}$ such that rank$(\mathbf{Q}_1^{(0)} + \mathbf{e}_m^t\xi^{(1)}\mathbf{Q}_2^{(1)} + \mathbf{e}_m^t\mathbf{b}) = m$.

In our case, we set $\mathbf{b} = \mathbf{e}_m$ in (6).

4.2 LU Decomposition Approaches

In order to reduce the time complexity of matrix multiplication and inversion, we will adapt LU decomposition for computing an $m \times m$ matrix, namely

$$\mathbf{T} = \begin{bmatrix} [\mathbf{T}]_{0,0} & [\mathbf{T}]_{0,1} & & & & & & \\ [\mathbf{T}]_{1,0} & [\mathbf{T}]_{1,1} & [\mathbf{T}]_{1,2} & & & & & \\ 0 & [\mathbf{T}]_{2,1} & [\mathbf{T}]_{2,2} & [\mathbf{T}]_{2,3} & & & & \\ \vdots & 0 & \ddots & \ddots & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & [\mathbf{T}]_{S-1,S-2} & [\mathbf{T}]_{S-1,S-1} & [\mathbf{T}]_{S-1,S} \\ 0 & [\mathbf{T}]_{S,1} & [\mathbf{T}]_{S,2} & [\mathbf{T}]_{S,3} & \cdots & [\mathbf{T}]_{S,S-2} & [\mathbf{T}]_{S,S-1} & [\mathbf{T}]_{S,S} \end{bmatrix}$$
(7)
with its all diagonal elements are nonzero. Denote by $[\mathbf{T}]_{i,j}$ the element at the (i+1)th row and the (j+1)th column of matrix $\mathbf{T}$.

Let $\mathbf{L}$ and $\mathbf{U}$ be component matrices of LU decomposition of $\mathbf{T}$, where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is an unit upper triangular matrix, then $\mathbf{L}$ and $\mathbf{U}$ can be expressed as

$$\mathbf{L} = \begin{bmatrix} [\mathbf{L}]_{0,0} & & & & & & \\ [\mathbf{L}]_{1,0} & [\mathbf{L}]_{1,1} & & & & & \\ 0 & [\mathbf{L}]_{2,1} & [\mathbf{L}]_{2,2} & & & & \\ \vdots & 0 & \ddots & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \\ 0 & 0 & \cdots & 0 & [\mathbf{L}]_{S-1,S-2} & [\mathbf{L}]_{S-1,S-1} & \\ 0 & [\mathbf{L}]_{S,1} & [\mathbf{L}]_{S,2} & \cdots & [\mathbf{L}]_{S,S-2} & [\mathbf{L}]_{S,S-1} & [\mathbf{L}]_{S,S} \end{bmatrix}$$
(8)

$$\mathbf{U} = \begin{bmatrix} 1 & [\mathbf{U}]_{0,1} & 0 & 0 & \cdots & \cdots & 0 \\ & 1 & [\mathbf{U}]_{1,2} & 0 & \cdots & \cdots & 0 \\ & & 1 & [\mathbf{U}]_{2,3} & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & 1 & [\mathbf{U}]_{S-2,S-1} & 0 \\ & & & & & 1 & [\mathbf{U}]_{S-1,S} \\ & & & & & & 1 \end{bmatrix}$$

(9)

From (8), (9), and $\mathbf{T} = \mathbf{L}\mathbf{U}$, we can write the following equations.

$$\begin{cases} [\mathbf{T}]_{0,0} & = [\mathbf{L}]_{0,0} \\ [\mathbf{T}]_{0,1} & = [\mathbf{L}]_{0,0}[\mathbf{U}]_{0,1} \\ [\mathbf{T}]_{i,i-1} & = [\mathbf{L}]_{i,i-1}, \ \ 1 \le i \le S-1 \\ [\mathbf{T}]_{i,i} & = [\mathbf{L}]_{i,i} + [\mathbf{L}]_{i,i-1}[\mathbf{U}]_{i-1,i}, \ \ 1 \le i \le S-1 \\ [\mathbf{T}]_{i,i+1} & = [\mathbf{L}]_{i,i}[\mathbf{U}]_{i,i+1}, \ \ 1 \le i \le S-1 \\ [\mathbf{T}]_{S,1} & = [\mathbf{L}]_{S,1} \\ [\mathbf{T}]_{S,i} & = [\mathbf{L}]_{S,i} + [\mathbf{L}]_{S,i-1}[\mathbf{U}]_{i-1,i}, \ \ 2 \le i \le S \end{cases}$$

which induces the Algorithm 1 with time complexity $\mathcal{O}(S)$.

Algorithm 1: LU decomposition

[Step 1] $[\mathbf{L}]_{0,0} = [\mathbf{T}]_{0,0}$, $[\mathbf{U}]_{0,1} = \frac{[\mathbf{T}]_{0,1}}{[\mathbf{L}]_{0,0}}$

[Step 2] Compute recursively for $i = 1, 2, \cdots, S-1$

$$[\mathbf{L}]_{i,i-1} = [\mathbf{T}]_{i,i-1}$$

$$[\mathbf{L}]_{i,i} = [\mathbf{T}]_{i,i} - [\mathbf{L}]_{i,i-1}[\mathbf{U}]_{i-1,i}$$

$$[\mathbf{U}]_{i,i+1} = \frac{[\mathbf{T}]_{i,i+1}}{[\mathbf{L}]_{i,i}}$$

[Step 3] $[\mathbf{L}]_{S,1} = [\mathbf{T}]_{S,1}$

[Step 4] Compute recursively for $i = 2, 3, \cdots, S$

$$[\mathbf{L}]_{S,i} = [\mathbf{T}]_{S,i} - [\mathbf{L}]_{S,i-1}[\mathbf{U}]_{i-1,i}$$

Lemma 7

If $\mathbf{T}$ is of the special form as described in (7) and $\mathbf{L}, \mathbf{U}$ are the component matrices of LU decomposition of $\mathbf{T}$, then the last row of $\mathbf{T}^{-1}$ is the same as the last row of $\mathbf{L}^{-1}$, i.e. $\mathbf{e}_m\mathbf{T}^{-1} = \mathbf{e}_m\mathbf{L}^{-1}$.

Proof

It is easy to check that $\mathbf{U}^{-1}$ is modified to an upper triangular matrix with all diagonal elements of one by Gaussian elimination.

This implies that the last row of $\mathbf{U}^{-1}$ is $\mathbf{e}_m\mathbf{U}^{-1} = [0, 0, \cdots, 0, 1] = \mathbf{e}_m$

Since $\mathbf{T}^{-1} = (\mathbf{LU})^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}$, by multiply $\mathbf{e}_m$ on both sides, we obtain
$\mathbf{e}_m\mathbf{T}^{-1} = \mathbf{e}_m\mathbf{U}^{-1}\mathbf{L}^{-1} = \mathbf{e}_m\mathbf{L}^{-1}$
Hence, we have the result. □

Now, define $\mathbf{e}_m\mathbf{T}^{-1} = \mathbf{e}_m\mathbf{L}^{-1} = [\ell_0, \ell_1, \ell_2, \cdots, \ell_S]$. Since $\mathbf{e}_m\mathbf{L}^{-1}\mathbf{L} = \mathbf{e}_m$, we obtain the following equation, i.e.,

$$\begin{aligned}
&\ell_0[\mathbf{L}]_{0,0} + \ell_1[\mathbf{L}]_{1,0} &&= 0 \\
&\ell_i[\mathbf{L}]_{i,i} + \ell_{i+1}[\mathbf{L}]_{i+1,i} + \ell_S[\mathbf{L}]_{S,i} &&= 0, \quad 1 \le i \le S-2 \\
&\ell_{S-1}[\mathbf{L}]_{S-1,S-1} + \ell_S[\mathbf{L}]_{S,S-1} &&= 0 \\
&\ell_S[\mathbf{L}]_{S,S} &&= 1.
\end{aligned}$$

Algorithm 2: Compute $\mathbf{e}_m^t\mathbf{L}^{-1}$

[Step 1] $\ell_S = \frac{1}{[\mathbf{L}]_{S,S}}$
[Step 2] $\ell_{S-1} = -\frac{[\mathbf{L}]_{S,S-1}}{[\mathbf{L}]_{S-1,S-1}}$
[Step 3] Compute recursively for $i = S-2, S-1, \cdots, 1$

$$\ell_i = -\frac{[\mathbf{L}]_{i+1,i}}{[\mathbf{L}]_{i,i}}\ell_{i+1} - \frac{[\mathbf{L}]_{S,i}}{[\mathbf{L}]_{i,i}}\ell_S$$

[Step 4] $\ell_0 = -\frac{[\mathbf{L}]_{1,0}}{[\mathbf{L}]_{0,0}}$

which induces the Algorithm 2 with time complexity $\mathcal{O}(S)$.

Thus, replacing $\mathbf{T}$ by $(\mathbf{Q}_1^{(K)} + \sigma\mathbf{Q}_2^{(K)})$ or $(\mathbf{Q}_1^{(\ell)} + \mathbf{e}_m\xi^{(\ell+1)}\mathbf{Q}_2^{(\ell+1)})$, the time complexity of solving $\xi^{(\ell)}$ can be reduced from $\mathcal{O}(S^3)$ to $\mathcal{O}(S)$.

In summary, we present a general computing procedure for obtaining π in the following.

Algorithm 3: Computing stationary probabilities $\boldsymbol{\pi}$

[Step 0] Let K be determined in Corollary 1 by a preset $\epsilon > 0$
[Step 1] solve $x = -\lambda\mathbf{e}_m(\mathbf{Q}_1^{(K)} + x\mathbf{Q}_2^{(K)})^{-1}e_m^t$
and $\boldsymbol{\xi}^{(K)} = -\lambda\mathbf{e}_m(\mathbf{Q}_1^{(K)} + \sigma\mathbf{Q}_2^{(K)})^{-1}, \sigma = x$
[Step 2] $\boldsymbol{\xi}^{(k)} = -\lambda\mathbf{e}_m(\mathbf{Q}_1^{(k)} + \mathbf{e}_m^t\boldsymbol{\xi}^{(k+1)}\mathbf{Q}_2^{(k+1)})^{-1}, \quad k = K-1, K-2, \cdots, 1$
[Step 3] $\boldsymbol{\xi}^{(0)} = -\lambda\mathbf{e}_m[(\mathbf{Q}_1^{(0)} + \mathbf{e}_m^t\boldsymbol{\xi}^{(1)}\mathbf{Q}_2^{(1)} + \mathbf{e}_m^t\mathbf{e}_m)^{-1} - (\mathbf{Q}_1^{(0)} + \mathbf{e}_m^t\boldsymbol{\xi}^{(1)}\mathbf{Q}_2^{(1)} + a\mathbf{e}_m^t\mathbf{e}_m)^{-1}]$
[Step 4] $\sigma_k = \boldsymbol{\xi}^{(k)}\mathbf{e}_m^t, \quad k = 0, 1, \cdots, K-1$
[Step 5] $s_k \triangleq \prod_{i=0}^{k-1}\sigma_i, \quad s_0 = 1, \quad k = 0, 1, \cdots, K$
[Step 6] $\phi = (\sum_{k=0}^{K-1} s_k\boldsymbol{\xi}^{(k)}\mathbf{1} + \frac{s_K}{1-\sigma}\boldsymbol{\xi}^{(K)}\mathbf{1})^{-1}$
[Step 7] $\boldsymbol{\pi}_k = \phi s_k\boldsymbol{\xi}^{(k)}, \quad 0 \le k \le K$
[Step 8] $\boldsymbol{\pi}_k = (\sigma)^{k-K}\boldsymbol{\pi}_K, \quad k \ge K+1.$

The expected number of customers in the retrial orbit L_q can be determined by

$$L_q = \sum_{i=0}^{K-1} i\pi_i\mathbf{1} + \sum_{i=K}^{\infty} i\sigma^{i-K}\pi_K\mathbf{1} = \sum_{i=0}^{K-1} i\pi_i\mathbf{1} + \pi_K\mathbf{1}\{\frac{K}{1-\sigma} + \frac{\sigma}{(1-\sigma)^2}\}.$$

Define the effective retrial rate and the effective service rate as E_r and E_s, respectively. The performance measures are expressed by

$$E_r = \sum_{k=1}^{\infty} k\gamma\pi_k \mathbf{1} = L_q\gamma,$$
$$E_s = \sum_{i=1}^{S}\sum_{k=1}^{\infty} i\pi_k \mathbf{e}_i^t \mu.$$

5 Numerical Experiments

By our model, the computational effort of the suggested approach in Algorithm 3 is significantly reduced while the numerical stability associated with the computational procedure is controlled under a preset precision level. We will conduct numerical experiments on PC with Intel(R) Core(TM) i5-5200U CPU @ 2.20GHz for the proposed method with the test problems appeared in [6]. With the help of stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ of $M/M/S$ with impatient customers, we can compute the expected number of customers in the retrial queue and the expected number of customers at each level by treating ρ and S as the decision variables, e.g., $\lambda = \rho \times S \times \mu$. In experiments, we first use the test problems in [6] and comparing the results by computing N_{ret} which is denoted by L_q in our model, the blocking probability P_b, the delayed service probability P_{ds}, and the nonservice probability P_{ns}, i.e.,

$$P_b = \sum_{i=0}^{K-1} \pi_i \mathbf{e}_m^t + \pi_K \mathbf{e}_m^t \frac{1}{1-\sigma}$$
$$P_{ds} = \{L_q - \sum_{i=1}^{K} i\pi_i \mathbf{e}_m^t - \pi_K \mathbf{e}_m^t (\frac{K}{1-\sigma} + \frac{\sigma}{(1-\sigma)^2})\}\gamma/\lambda$$
$$P_{ns} = p\gamma\{\sum_{i=0}^{K} i\pi_i \mathbf{e}_m^t + \pi_K \mathbf{e}_m^t (\frac{K}{1-\sigma} + \frac{\sigma}{(1-\sigma)^2})\}/\lambda.$$

Referring to [6], the default value of number of examples are set as S = 50, 100, 200, 500, and 1000, μ = 1/180, γ = 0.01, p = 0,2, $\epsilon = 10^{-5}$, respectively. We confirm the computing procedure and robustness of Algorithm 3. The performance measures are presented in particular for S = 500, ρ = 1, μ = 1/180, γ = 0.01 in Fig. 1.

The main purpose of this paper is the development of an eigenpair approach that results in an efficient method to effectively solve retrial systems with customer impatience. This novel method is a continuing effort inspired in the previous research papers. The proposed algorithm depends on a series of eigenvalues and eigenvectors for nonhomogeneous QBD. The computational complexity is much lower because it only needs to solve an eigenvalue once and the remaining probabilities are attained by substitution. According to our experiments over 100 test problems including $S = 2000$, we found the computational complexity depends on ρ which confirms the observation in [6]. In specific, our test problems illustrate the relationship K which is denoted by N in [6] among other system parameters in Fig. 1. Therefore, we choose K by the following rule in our case rather than using Corollary 1 which only provides a rough upper bound in general.

Observation: There exists a $f(\rho)$ such that $\ln(K)$ is proportional to $f(\rho)\ln(S)$ where $f(\rho)$ may be written as

$$f(\rho) = \begin{cases} 0.3, & \text{if } \rho < 0.9 \\ 1.4\rho, & \text{if } \rho \geq 0.9. \end{cases}$$

Although this method is one of the generalized truncated methods, we believe our method can be used in many cases in engineering problems where the matrix $\mathbf{Q}_0^{(\ell)}$ has only one non-zero row of which examples are found in [4, 14, 15]. We expect that this method will outperform the previous proposals in terms of accuracy for the most common performance parameters used in retrial systems and under a wide range of scenarios in applications (Tables 1 and 2).

Table 1.

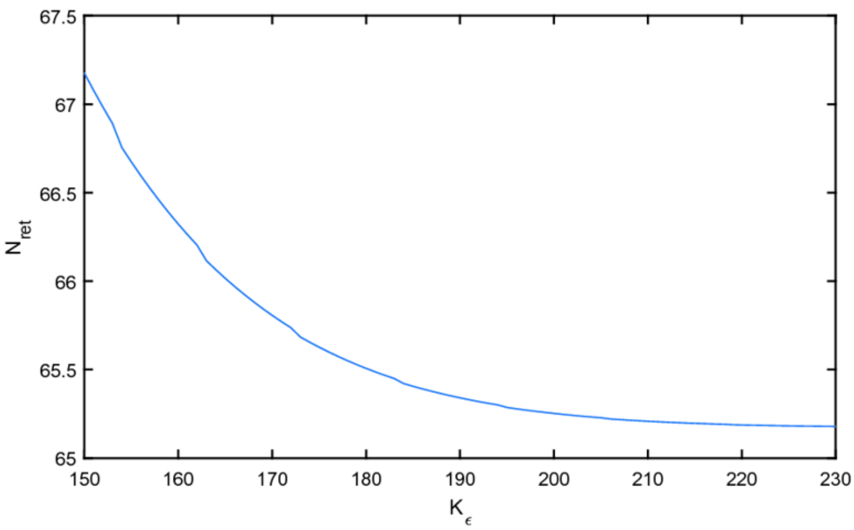
Computational Time for $\lambda = \rho(\mu S), \mu = 1/180, \gamma = 0.01, p = 0.2$

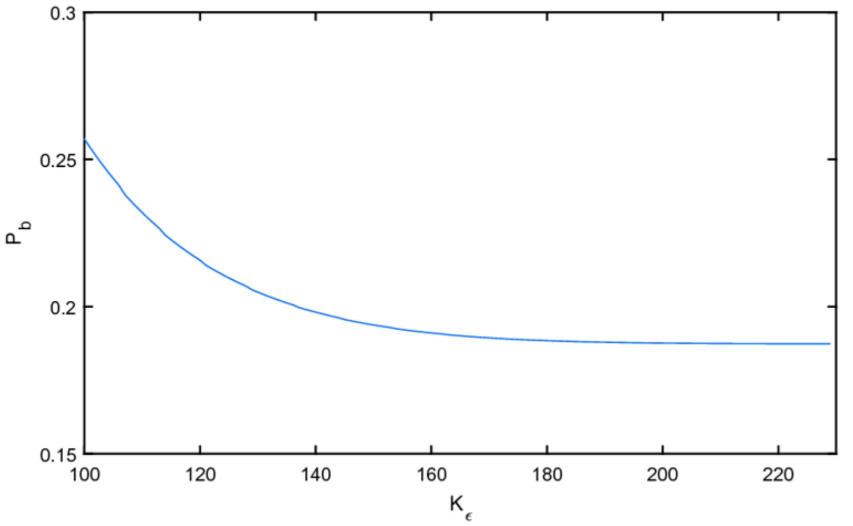
ρ	$S = 50$		$S = 100$		$S = 200$		$S = 500$		$S = 1000$	
	K	**Time (s)**	***K***	**Time (s)**	***K***	**Time (s)**	***K***	**Time (s)**	***K***	**Time (s)**
0.4	10	0.001	10	0.001	10	0.004	10	0.010	10	0.021
	20	0.001	20	0.001	20	0.006	20	0.016	20	0.016
	30	0.001	30	0.003	30	0.006	30	0.014	30	0.020
	40	0.001	40	0.005	40	0.008	40	0.014	40	0.030
0.8	30	0.002	30	0.003	30	0.005	30	0.016	30	0.033
	40	0.002	40	0.003	40	0.007	40	0.026	40	0.033
	50	0.004	50	0.004	50	0.008	50	0.026	50	0.047
	60	0.003	60	0.005	60	0.009	60	0.03	60	0.047
1.0	50	0.009	80	0.008	120	0.033	200	0.097	350	0.288
	75	0.011	100	0.009	160	0.033	250	0.102	400	0.339
	100	0.01	120	0.009	200	0.039	300	0.124	450	0.352
	125	0.014	140	0.012	240	0.05	350	0.157	500	0.396
1.4	100	0.009	200	0.015	300	0.05	800	0.279	1000	0.937
	150	0.013	250	0.02	400	0.061	900	0.332	1500	1.368
	200	0.014	300	0.018	500	0.07	1000	0.35	2000	1.648
	250	0.01	350	0.029	600	0.084	1100	0.378	2500	2.146

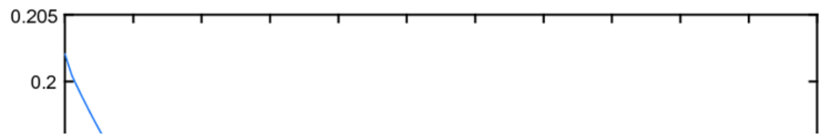
Table 2.

Computational Time for $\lambda = \rho(\mu S), \mu = 10, \gamma = 1.6, p = 0.15$

ρ	$S = 100$		$S = 200$		$S = 500$		$S = 1000$		$S = 2000$	
	K	**Time (s)**	***K***	**Time (s)**	***K***	**Time (s)**	***K***	**Time (s)**	***K***	**Time (s)**
0.8	60	0.003	50	0.01	50	0.021	40	0.027	40	0.060
0.9	170	0.01	150	0.031	100	0.009	90	0.07	80	0.124
0.95	320	0.02	350	0.055	320	0.11	270	0.19	190	0.287
1.0	560	0.043	810	0.15	1510	0.507	2390	1.853	3760	6.119







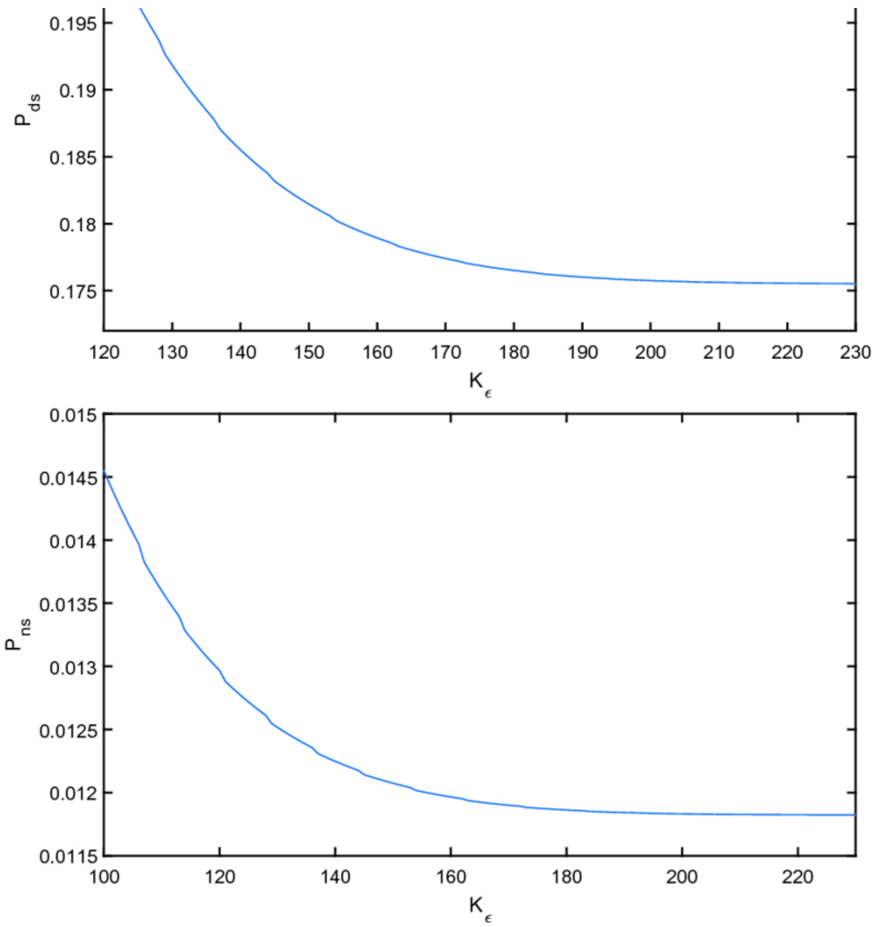


Fig. 1.

Stats v.s. K for $S = 500, \rho = 1.0, \mu = 1/180, \gamma = 0.01$

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