Analytic Formulae for Valuing Guaranteed Minimum Withdrawal Benefits in a Multi-Asset Framework

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Abstract

This study attempts to find the analytic formulae for valuing guaranteed minimum withdrawal benefits (GMWB) in a multi-asset framework. With the assumption that each equity process follows a geometric Brownian motion, we obtain the analytic formulae for GMWB by approximating the sum of the correlated lognormal variables using the averaging concept of reciprocal gamma and lognormal distributions. Numerical experiments show that analytic formulae of the averaging concept are accuracy, even for long-duration policies. Thus, the analytic formulae provide the significant advantage and efficiency in valuing GMWB.

Key words: Guaranteed minimum withdrawal benefits, analytic formulae, averaging concept, reciprocal gamma distribution, lognormal distribution

I. Introduction

Variable annuities (VA) are an insurance product sold in the retirement market. The policyholder makes a single or periodic payment into a fund and the fund value accumulates in accordance with the underlying investment portfolio. Offering an investment guarantee is an innovative design for the VA. If the

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market performs badly, the policyholders can still receive a certain guaranteed fund value. Guaranteed minimum death benefits (GMDB), guaranteed minimum maturity benefits (GMMB), guaranteed minimum income benefits (GMIB), and guaranteed minimum withdrawal benefits (GMWB) are the common types of investment guarantees. The investment guarantee feature of VA has received great attention in the retirement market. According to LIMRA's Election Tracking Survey, the combinations of these investment guarantees comprise 61% of the VA sales in 2015 (LIMRA Secure Retirement Institute (2016)).

Granting investment guarantees means that VA products are embedded financial options. The pricing issues of these embedded options for insurance contracts have been discussed for a long time (Brennan and Schwartz (1976, 1979), Milevsky and Posner (2001), Boyle and Hardy (2003), Ballotta and Haberman (2003, 2006), and Bauer, Kling, and Russ (2008)). The above literature focuses more on the GMDB, GMMB, and GMIB guarantees. GMWB is the latest type of investment guarantee. Under a GMWB contract, the policyholder is entitled to make a withdrawal periodically for a contractually specified amount for the duration of the contract, regardless of the performance of the underlying investment portfolio. Thus, the payoff of the GMWB is quite complex and hard to value, because the embedded options contain the feature of arithmetic Asian options, whose payoffs depend on the average price of the underlying asset. The challenge in pricing arithmetic Asian options arises because the arithmetic average is not lognormally distributed when the underlying asset price follows a standard lognormal process, especially in long-duration contracts (Liu (2008)). The analytic closed-form solution is especially difficult to obtain for pricing GMWB and we attempt to find the closed-form solution for valuing GMWB in this research.

Prior financial literature proposes several approximation techniques to solve this problem, which consist of three main categories: (1) Monte Carlo simulation, (2) lattice or finite difference methods, and (3) analytical approximations. The first method offers a high level of accuracy for pricing Asian options, but in terms of the computing time required, it is not efficient, because the standard simulation provides convergence rates of (number of paths)$^{-1/2}$. As a technique to improve this performance, the variance-reduction technique uses geometric Asian options as control variates (Kemna and Vorst (1990)). Bacinello, Millossovich, and Montealegre (2016) provide a dynamic programming algorithm for pricing GMWBs under a general Lévy processes framework by simulation. The second approach provides a very flexible and efficient means for pricing Asian options; this method even remains viable when early exercise is allowed. For example, Hull and White (1993) propose an extended binomial method that can efficiently value American-style Asian options, and Dewynne and Wilmott (1993) develop a finite difference method to value Asian options. The third approach, which approximates the density function of the arithmetic average with a lognormal random density (Turnbull and Wakeman (1991), Levy (1992), and Bouaziz, Briys, and Crouhy (1994)) or reciprocal gamma density (Milevsky and Posner (1998b)), is generally more preferred in practice if the approximate analytical solutions can be performed quickly and accurately. Boyle and Potapchik (2008) provide additional details in their survey of methods for pricing Asian options.

To date, the partial differential equation (PDE) approach has remained the primary technique for dealing with GMWB (Milevsky and Salisbury (2006), Chen, Vetzal, and Forsyth (2008), and Donnelly, Jaimungal, and Rubisov (2014)). Milevsky and Salisbury (2006) treat embedded options of GMWBs as Quanto Asian puts and use the PDE approach to find a fair charge. Chen, Vetzal, and Forsyth (2008) also consider an optimal withdrawal strategy and the jump effect of the underlying risky asset to value the no-arbitrage fee for GMWBs using the PDE approach. Donnelly, Jaimungal, and Rubisov (2014) develop an efficient method to value GMWBs by PDE method under stochastic volatility and stochastic interest rates environment. In addition, Yang and Dai (2013) and Moenig and Bauer (2016) use a tree method to consider surrender options and tax structure for GMWBs, respectively. To achieve our objective of finding analytic formulae to value GMWBs, we utilize the reciprocal gamma distribution to approximate the probability density function of the sum of the correlated lognormal random variables. It is different from the idea provided by Peng, Leung, and Kwok (2012) and Huang and Kwok (2014). Peng, Leung, and Kwok (2012) derive both the lower and upper bounds on the price functions of GMWBs. Huang and Kwok (2014) derive the integral equations for the determination of a pair of optimal withdrawal boundaries for GMWBs. But we use the reciprocal gamma distribution introduced first by Milevsky and Posner (1998a, 1998b) to deal with arithmetic Asian options. Due to the similar features of arithmetic Asian options with GMWBs, it encourages us to try this approximation for GMWB contracts. To derive the parameters underlying the approximate reciprocal gamma distribution for the GMWB contract, we use the moment matching technique and derive corresponding first and second moments. For comparison, we also derive the analytic formula using a lognormal distribution approximation and examine the comparative accuracy of these formulae, based on reciprocal gamma, lognormal distribution and the average concept of them, with Monte Carlo simulations.

Unlike Milevsky and Posner (1998a, 1998b), we extend the valuation framework to deal with insurance contracts. Our assessment captures the realistic features of a VA product, including management fees, charges, and withdrawals; we also derive the valuation formulae on the basis of a multi-asset framework. In practice, VA products are usually linked to an investment portfolio with both risky and riskless assets. Therefore, a fair charge for GMWB contracts depends on the underlying investment portfolio, because the corresponding risk profile for the VA product differs. Our valuation formulae can benefit insurers to determine the fair charges for different investment portfolios, an issue that has not been addressed in valuation frameworks that appear in previous works pertaining to GMWB (e.g., Milevsky and Salisbury (2006), Bauer, Kling, and Russ (2008), Dai, Kwok, and Zong (2008), Liu (2008), Peng, Leung, and Kwok (2012), Yang and Dai (2013), Donnelly, Jaimungal, and Rubisov (2014), Huang and Kwok (2014), Bacinello, Millossovich, and Montealegre (2016), and Moenig and Bauer (2016)). In addition, the existing approximation formulae for arithmetic Asian options normally have been derived to value financial contracts (Lo, Palmer, and Yu (2014)). The accuracy and efficiency of the valuation approach sometimes are limited for short-duration contracts (Liu (2008)). In our numerical analysis, we demonstrate that the proposed analytic formulae, based on the average value of lognormal and reciprocal gamma distribution, perform well for not only short- but also long-duration contracts, such as 30-year GMWB contracts.

In Section II, we describe the valuation framework for GMWB embedded in a VA product and present the underlying financial model. In Section III, we derive analytic solutions for GMWB using reciprocal gamma and lognormal density functions, respectively. We continue in Section IV with the numerical results and sensitivity analysis for fair charges for GMWB in different investment portfolios. Section V provides our conclusions.

II. Valuation Framework for GMWB

We assume a single-premium variable annuity product associated with GMWB with maturity date T. The policyholder is guaranteed to withdraw a certain amount each year during the deferred period, and the guaranteed withdrawal amounts are a fixed percentage of the premium. The account value of the VA product depends on the performance of the underlying investment portfolio. We consider an investment portfolio with n risky assets ($S_1,\ldots,S_n$) and one risk-free asset (B). The dynamics of the market prices for the risky assets under the risk-neutral probability measure Q are

$$dS_{it} = rS_{it}dt + \sigma_i S_{it} \cdot dZ_t^Q, \quad i = 1,\ldots,n, \tag{1}$$

where Z_t^Q denotes a n-dimensional standard Brownian motion; the parameter r is the constant risk-free rate; and σ_i represents a n-dimensional volatility function of the i^{th} risky asset return. As a result, the volatility of the i^{th} risky asset is equal to $\tilde{\sigma}_i = |\sigma_i|$, where $|\cdot|$ denotes the Euclidean norm in R^n. The covariance between the i^{th} risky asset and the j^{th} risky asset equals $\rho_{ij}\tilde{\sigma}_i\tilde{\sigma}_j$, where ρ_{ij} is their correlation coefficient. The dynamic of the riskless asset is

$$dB_t = rB_t dt. \tag{2}$$

Let w_o denote the initial single premium and W_t denote the account value at time t with $W_o = w_o$. The account value of GMWB changes according to the return on the underlying investment portfolio, less the withdrawal and payments of the guaranteed charges. In the realistic framework we consider, the accumulation of account value depends on an investment portfolio with a proportion of λ_i invested in the i^{th} risky asset and the rest, or $\eta = 1 - \sum_{i=1}^{n} \lambda_i$, in the risk-free asset. Thus, the corresponding stochastic differential equation (SDE) of W_t can be expressed as

$$dW_t = \sum_{i=1}^{n} \lambda_i \frac{W_t}{S_{it}} dS_{it} + \eta \frac{W_t}{S_t} dB_t - G_t dt - cW_t dt, \tag{3}$$

where c is the proportional charge, based on the account value, and G_t represents the guaranteed minimum withdrawal benefit at time t. Regardless of the performance of the underlying risky investment, we assume the GMWB promises policyholders could receive at least the entire original investment. Thus, the guaranteed minimum withdrawal benefit is a fixed amount denoted G for each withdrawal. According to the financial settings of the asset dynamics in Equations (1) and (2), we also can express the SDE of W_t in Equation (3) as

$$dW_t = ((r-c)W_t - G)dt + \sigma_\lambda W_t \cdot dZ_t^Q, \tag{4}$$

where $\sigma_\lambda = \sum_{i=1}^{n} \lambda_i \sigma_i$; therefore, the variance of W_t is $\tilde{\sigma}_\lambda^2 = |\sigma_\lambda|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j \rho_{ij} \tilde{\sigma}_i \tilde{\sigma}_j$. For a VA product with GMWB, the account value might eventually fall to zero as a result of poor investment performance, but the policyholder is still guaranteed a certain amount to withdraw. To calculate the future dynamics of account value W_t, we must consider a situation in which account value declines all the way to zero. For this purpose, following the similar idea of Peng, Leung, and Kwok (2012), we define a pseudo-asset U as follows:

$$dU_t = (r-c)U_t dt + \sigma_\lambda U_t \cdot dZ_t^Q, \quad \text{with} \quad U_0 = 1. \tag{5}$$

In turn, we can solve Equation (4) and obtain the expression for the account value W_t at time $t \in [0, T]$, which is

$$W_t = Max\left(U_t\left(w_0 - G\int_0^t U_s^{-1}\, ds\right), 0\right). \tag{6}$$

See Appendix A for the derivation of Equation (6).

The total benefits that the policyholder receives from the VA product with GMWB include two parts: the terminal account value (W_T) and total withdrawal benefits. For a fixed guaranteed withdrawal benefit (G), the total benefits at the end of the deferred period (time T) are

$$W_T + \int_0^T e^{r(T-u)} G du, \tag{7}$$

where $\int_0^T e^{r(T-u)} G du$ denotes the total withdrawal benefits accumulating by maturity time T.

Then let $V(W_0, T)$ denote the no-arbitrage value at inception for the GMWB with maturity date T. On the basis of the risk-neutral valuation, the no-arbitrage value of $V(W_0, T)$ can be expressed as

$$\begin{aligned} V(W_0, T) &= e^{-rT} E_Q\left(W_T + \int_0^T e^{r(T-u)} G du\right) \\ &= e^{-rT} E^Q\left(Max\left(U_T\left(w_0 - G\int_0^{\mathrm{T}} U_s^{-1}\, ds\right), 0\right)\right) + \frac{G}{r}\left(1 - e^{-rT}\right), \end{aligned} \tag{8}$$

where E_Q is the expectation according to the risk-neutral measure Q. The first term on the right-hand side of Equation (8) represents a reciprocal Asian option, denoted by $J(W_0, T)$. For this research, we derive an analytic valuation formula for valuing the embedded GMWB option using the reciprocal gamma distribution approximation. For comparison purposes, we also derive the valuation formulae using lognormal approximation.

III. Derivation of Analytic Formula

According to the dynamics of U_t in Equation (5), the no-arbitrage value of the option term ($J(W_0, T)$) at inception can be rewritten a

$$\begin{aligned} J(W_0, T) &= e^{-rT} E^Q\left(U_T\, Max\left(w_0 - G\int_0^T U_s^{-1}\, ds, 0\right)\right) \\ &= e^{-rT} E^Q\left(\exp\left(\left(r - c - |\sigma_\lambda|^2/2\right)T + \sigma_\lambda \cdot Z_T^Q\right) Max\left(w_0 - G\int_0^T U_s^{-1}\, ds, 0\right)\right). \end{aligned} \tag{9}$$

The expectation calculation procedure in Equation (9) can be simplified using a new probability measure Q with the Radon-Nikodym derivative

$$\xi_T = \frac{d\tilde{Q}}{dQ} = \exp\left(\sigma_\lambda \cdot Z_T^Q - \tilde{\sigma}_\lambda^2 T/2\right). \tag{10}$$

By the Girsanov Theorem, $\tilde{Z}_t^Q$ refers to $\tilde{Q}$-Brownian motion in which Z_t^Q and $\tilde{Z}_t^Q$ are related by $dZ_t^Q = d\tilde{Z}_t^Q + \sigma_\lambda dt$. By changing the measure from Q to $\tilde{Q}$, Equation (9) becomes

$$\begin{aligned} J(W_0,T) &= e^{-cT} E^Q\left(\xi_T \, Max\left(w_0 - G\int_0^T U_s^{-1}\, ds, 0\right)\right) \\ &= e^{-cT} E^{\tilde{Q}}\left(Max\left(w_0 - G\int_0^T U_s^{-1}\, ds, 0\right)\right), \end{aligned} \tag{11}$$

where U_s, according to the probability measure $\tilde{Q}$, is of the form

$$U_s = \exp\left(\left(r - c + \tilde{\sigma}_\lambda^2/2\right)s + \sigma_\lambda \cdot Z_s^{\tilde{Q}}\right). \tag{12}$$

The problem involved in dealing with the valuation framework in Equation (11) is the integral $\int_0^T U_s^{-1}\, ds$, the average reciprocal price of pseudo-asset *U*. We approximate this term using the first and second moments of the integral to match those of two distributions—namely, a reciprocal gamma distribution and a lognormal distribution—to derive the analytic solutions. As a crucial element for deriving the approximation solutions, we provide the first and second moments of the integral in the following proposition.

PROPOSITION 1 Under the probability measure $\tilde{Q}$, the first and second moments of the integral $\int_0^T U_s^{-1}\, ds$, where U_s satisfies Equation (12), are as follows:

$$M_1 = \begin{cases} \dfrac{1 - e^{-(r-c)T}}{(r-c)}, & \text{if } r \neq c \\ T & \text{if } r = c, \end{cases} \tag{13}$$

and

$$M_2=\begin{cases} 2\left[\dfrac{e^{-\left(2r-2c-\tilde{\sigma}_\lambda^2\right)T}}{\left(r-c-\tilde{\sigma}_\lambda^2\right)\left(2r-2c-\tilde{\sigma}_\lambda^2\right)}+\dfrac{1}{(r-c)}\left(\dfrac{1}{\left(2r-2c-\tilde{\sigma}_\lambda^2\right)}-\dfrac{e^{-(r-c)T}}{\left(r-c-\tilde{\sigma}_\lambda^2\right)}\right)\right] & \text{if } r-c\neq 0,\ r-c-\tilde{\sigma}_\lambda^2\neq 0 \text{ and } 2r-2c-\tilde{\sigma}_\lambda^2\neq 0 \\ \left[\dfrac{2\left(1-e^{-(r-c)T}\right)-T\tilde{\sigma}_\lambda^2}{(r-c)\left(r-c-\tilde{\sigma}_\lambda^2\right)}\right] & \text{if } r-c\neq 0,\ r-c-\tilde{\sigma}_\lambda^2\neq 0 \text{ and } 2r-2c-\tilde{\sigma}_\lambda^2=0 \\ 2\left[\dfrac{1}{(r-c)^2}-\dfrac{T}{(r-c)}e^{-(r-c)T}-\dfrac{1}{(r-c)^2}e^{-(r-c)T}\right] & \text{if } r-c\neq 0,\ r-c-\tilde{\sigma}_\lambda^2=0 \text{ and } 2r-2c-\tilde{\sigma}_\lambda^2\neq 0 \\ 2\left[\dfrac{e^{\tilde{\sigma}_\lambda^2 T}-1-\tilde{\sigma}_\lambda^2 T}{\tilde{\sigma}_\lambda^4}\right] & \text{if } r-c=0,\ r-c-\tilde{\sigma}_\lambda^2\neq 0 \text{ and } 2r-2c-\tilde{\sigma}_\lambda^2\neq 0 \end{cases} \tag{14}$$

The calculations of the first two moments appear in Appendix B.

Approximating the integral $\int_0^T U_s^{-1}\,ds$ by moment matching the reciprocal gamma distribution, we can apply seminal work by Milevsky (1997), which demonstrates that the inverse of the present value of a stochastic perpetuity is gamma distributed under Wiener returns. Specifically, we define the integral

$$I_T=\int_0^T \exp\left(-\left(\mu t+\sigma Z_t\right)\right)dt \tag{15}$$

as the present value of a stochastic perpetuity of \$1, exposed to an instantaneous force of mean μ and volatility parameter σ that is driven by a Brownian motion. The reciprocal of I_∞ obeys a gamma distribution, namely,

$$I_\infty^{-1}\sim g\left(x\,|\,\frac{2\mu}{\sigma^2},\frac{\sigma^2}{2}\right), \tag{16}$$

and the gamma distribution with parameter (α,β) is defined by the following probability density function:

$$g(x\,|\,\alpha,\beta)=\frac{\beta^{-\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)},\qquad \forall x>0,\alpha>0,\beta>0. \tag{17}$$

By virtue of Equations (12) and (15), substituting $\mu = r - c + \tilde{\sigma}_\lambda^2/2$ and $\sigma = \tilde{\sigma}_\lambda$ yields

$$\left(\int_0^\infty U_s^{-1}\,ds\right)^{-1} \sim g\left(x \mid \alpha_U, \beta_U\right), \tag{18}$$

where $\alpha_U = 1 + \frac{2(r-c)}{\tilde{\sigma}_\lambda^2}$, and $\beta_U = \tilde{\sigma}_\lambda^2/2.$. Thus, Equation (18) implies that the sum of correlated lognormal variables $\int_0^T U_s^{-1}\,ds$ converges to the reciprocal gamma distribution when $T \to \infty$ (Milevsky and Posner (1998b)). Because we deal with VA products with long times to maturity, their duration, often more than 30 years, is much longer than that of financial options. Therefore, approximating $\int_0^T U_s^{-1}\,ds$ with a reciprocal gamma distribution is applicable to insurance contracts, and by matching the first and second moments of $\int_0^T U_s^{-1}\,ds$ to the reciprocal gamma distribution, we can derive the analytical solution of account value, as we summarize in the following proposition.

PROPOSITION 2 Using Proposition 1, in an investment portfolio with a proportion of λ_j invested in the j^{th} risky asset, if $\int_0^T U_s^{-1}\,ds$ is approximated by a reciprocal gamma distribution, the analytical solution of the reciprocal Asian option embedded in a GMWB takes the form:

$$J_{RG}\left(W_0, T\right) = w_0 e^{-cT}\left(1 - \tilde{G}\left(\frac{1}{T} \mid \alpha_J, \beta_J\right) - \frac{M_1}{T}\left(1 - \tilde{G}\left(\frac{1}{T} \mid \alpha_J - 1, \beta_J\right)\right)\right), \tag{19}$$

where

$$\alpha_J = \frac{2M_2 - M_1^2}{M_2 - M_1^2}, \quad \text{and} \quad \beta_J = \frac{M_2 - M_1^2}{M_2 M_1}. \tag{20}$$

and $\tilde{G}\left(x \mid \alpha_J, \beta_J\right)$ is the cumulative gamma density function with the parameters α_J and β_J.

The derivation of Proposition 2 appears in Appendix C.

COROLLARY 1 Using Proposition 2, with a 100% investment in a single risky asset (i.e., $\lambda_i = 100\%, \lambda_j = 0,$ where $j \neq i$), the initial market value of the option term embedded with a GMWB is virtually the same as that in Equation (19), except that $\tilde{\sigma}_\lambda = \tilde{\sigma}_i$.

To approximate the term $\int_0^T U_s^{-1}\,ds$ with a lognormal distribution, we make the first two moments of the lognormal distribution equal to M_1 and M_2, which provides the closed-form solutions of reciprocal Asian options in Proposition 3.

PROPOSITION 3 Using Proposition 1, in an investment portfolio with a proportion of λ_j invested in the j^{th} risky asset, if $\int_0^T U_s^{-1}\,ds$ is approximated by a lognormal distribution, the analytical solution of the reciprocal Asian option embedded in a GMWB takes the form:

$$J_{LN}\left(W_0,T\right)=w_0e^{-cT}\left(N\left(\ln T\middle|\mu_{LN},\sigma_{LN}\right)-\frac{M_1}{T}N\left(\ln T\middle|\mu_{LN}+\sigma_{LN}^2,\sigma_{LN}\right)\right), \quad (21)$$

where $N\left(x\middle|\mu_{LN},\sigma_{LN}\right)$ is the cumulative normal density function with mean μ_{LN} and volatility σ_{LN}; and the parameters of μ_{LN} and σ_{LN} are calculated using the first two moments matching, namely, $\mu_{LN}=2\ln M_1-0.5\ln M_2$ and $\sigma_{LN}=\sqrt{\ln M_2-2\ln M_1}$.

The derivation of Proposition 3 is in Appendix D.

COROLLARY 2 Using Proposition 3, with 100% investment in one risky asset (i.e., $\lambda_i=100\%, \lambda_j=0$, where $j\neq i$), the initial market value of the option term embedded with GMWBs is virtually the same as that in Equation (21), except that $\tilde{\sigma}_\lambda=\tilde{\sigma}_i$.

With Propositions 1-3, we find the analytic closed-form solutions for the option term in valuing GMWB according to both reciprocal gamma distribution and lognormal distribution approximations. According to Milevsky and Posner (1998a), they say the traditional lognormal method overprices the options and the reciprocal gamma method underprices them. So we also consider the average value of $J_{RG}\left(W_0,T\right)$ and $J_{LN}\left(W_0,T\right)$ as $J\left(W_0,T\right)$. In practice, the insurer charges the policyholder a guarantee fee to reflect the value of embedded option, and to obtain a fair value of this charge, the insurer might set the initial premium equal to the expected present value of the total benefits from the VA product with GMWB, that is

$$w_0=J(W_0,T)+\frac{G}{r}\left(1-e^{-rT}\right). \quad (22)$$

IV. Numerical Analysis

We investigate fair charges for GMWB in this section. Assume the policyholder invests a single premium of 100, which also is the guaranteed withdrawal amount for the policy duration of the contract. For illustration purpose, we demonstrate the results based on a three-asset investment portfolio that contains two risky assets and one riskless asset. In the base illustration case, we consider an investment portfolio with 60% invested in higher risky asset, 20% in less risky asset, and 20% in riskless asset. The volatility of the more and less risky assets are 0.3 and 0.1, respectively. The correlation (ρ_{12}) of these two risky assets are assumed to be 0.5 or -0.5. The interest rate is 0.02. For a robustness check, we also conduct sensitivity analyses in which we study the effect of parameter changes and investment proportion on the value of fair charge for GMWB.

We first study the accuracy of our derived formulae by comparing the simulation results for the value of the embedded option with the analytical results derived in Equations (19) and (21) and the average value of them, which include the outcomes of both the approximate lognormal and reciprocal gamma distributions. The Monte Carlo simulation is based on 100,000 paths with 252 time partitions per year. In Table I and Table II, we list the option value and the relative error for different policy durations, according to the analytical solutions and simulation results when asset correlations are 0.5 or -0.5 separately. The relative errors between the simulated values and closed-form solutions are small, especially in the average value of the approximate lognormal and reciprocal gamma distributions. In Table I for the policy duration less than one year, the relative errors are less than 0.15% using lognormal distribution approximation, -0.06% using reciprocal gamma distribution approximation, and 0.05% using the average value of lognormal and reciprocal gamma distribution approximation; for a long-duration policy such as 30 years, the relative errors for both approximations are less than 1.98% using lognormal distribution approximation, -1.81% using reciprocal gamma distribution approximation, and 0.09% using the average value of lognormal and reciprocal gamma distribution approximation. The estimated method of averaging lognormal and reciprocal gamma distribution approximation has also well performance in Table II. Therefore, the average value of both formulae provides a precise and an efficient way to value the embedded option with GMWB contracts, even for a long-duration policy.

The corresponding fair charges of the GMWB contract are calculated in Table III. For simplicity, we only show the results using the analytic formula based on averaging lognormal and reciprocal gamma distribution approximation. When the correlation coefficient is 0.5, the guaranteed charge is 0.0274 per annum for a 10-year policy and decreases to 0.0070 per annum for a 30-year policy. Similarly, when the correlation coefficient is -0.5, the guaranteed charge is 0.0229 for a 10-year policy and decreases to 0.0056 per annum for a 30-year policy. It decreases as the policy duration gets longer because the guaranteed withdrawal amount is assumed to be the same for different duration policy.

Table I

Accuracy of the Option Value for GMWBs: Analytic Formulae vs. Simulation (ρ_{12} =0.5)

Value_MC=Option value using Monte Carlo simulation; *MC_std*=Standard deviation of Monte Carlo simulation; *Value_RG*=Option value based on reciprocal gamma distribution; *Value_LN*=Option value based on lognormal distribution; *Value_Avg*=Option value based on the average of *Value_RG* and *Value_LN*; *Rel_er_LN*=Relative error based on lognormal distribution; *Rel_er_RG*=Relative error based on reciprocal gamma distribution; *Rel_er_Avg*=Relative error based on *Value_Avg*.

T	*Value_MC*	*MC_std*	*Value_LN*	*Value_RG*	*Value_Avg*	*Rel_er_LN*	*Rel_er_RG*	*Rel_er_Avg*
1	4.726477	0.008743	4.733740	4.723841	4.728791	0.15%	-0.06%	0.05%
3	8.504071	0.012693	8.538481	8.488657	8.513569	0.40%	-0.18%	0.11%
6	12.342150	0.014380	12.423130	12.288770	12.355950	0.66%	-0.43%	0.11%
10	16.147070	0.014266	16.316160	16.044840	16.180500	1.05%	-0.63%	0.21%
20	22.816090	0.011118	23.152330	22.496820	22.824575	1.47%	-1.40%	0.04%
30	27.211760	0.009241	27.750570	26.720270	27.235420	1.98%	-1.81%	0.09%

Table II

Accuracy of the Option Value for GMWBs: Analytic Formulae vs. Simulation (ρ_{12} =-0.5)

T	*Value_MC*	*MC_std*	*Value_LN*	*Value_RG*	*Value_Avg*	*Rel_er_LN*	*Rel_er_RG*	*Rel_er_Avg*
1	4.274794	0.007955	4.279984	4.273834	4.276909	0.12%	-0.02%	0.05%
3	7.744888	0.011702	7.768813	7.737904	7.753359	0.31%	-0.09%	0.11%
6	11.314776	0.013454	11.368123	11.284962	11.326543	0.47%	-0.26%	0.10%
10	14.897237	0.013570	15.009410	14.841984	14.925697	0.75%	-0.37%	0.19%
20	21.281004	0.010843	21.484172	21.082816	21.283494	0.95%	-0.93%	0.01%
30	25.559943	0.008711	25.903349	25.277626	25.590488	1.34%	-1.10%	0.12%

Table III
Fair Charges for Different Investment Portfolios

T	ρ_{12}=0.5	ρ_{12}=-0.5
1	0.213085	0.183092
3	0.087637	0.074535
6	0.046326	0.039046
10	0.027395	0.022886
20	0.012095	0.009928
30	0.006961	0.005627

In the following figures and tables, we also provide the findings from the sensitivity analyses conducted for the key parameters. To highlight the effects of these key parameters, we only illustrate the results based on averaging lognormal and reciprocal gamma distribution approximation. The effects of investment portfolio on the fair charges are investigated first. We illustrate the result of the investment portfolios with proportions of the higher risky assets ranging from 20% to 100%. The patterns of fair charges for the various policy durations are shown in Figure 1. Intuitively, the more risky the investment portfolio, the higher the fair charge, even the weights in low risky and riskless assets are not equal. In particular, this effect is more significant for short-duration policies. For example, in the based illustration case, the fair charge is around 0.046 per annum for a 6-year policy, but it falls to 0.007 per annum for a 30-year policy; in the most risky case, i.e. 100% invested in high risky asset, the fair charge is 0.087 per annum for a 6-year policy, but it falls to less than 0.014 per annum for a 30-year policy.

The effects of the volatility of risky assets on the fair charges, as we present in Table IV, reflect the comparison of different volatilities underlying the investment portfolio, including $\tilde{\sigma}_{\lambda}$=0.1, 0.2, and 0.3. The fair charge increases with the volatility of the investment portfolio, though again, the pattern is more significant for shorter-duration policies. For example, for an investment portfolio in which volatility increases from 0.2 to 0.3, the fair charge increases from 0.050 to 0.087 per annum for a policy with a 6-year duration; it increases from 0.008 to 0.014 per annum for a policy with a 30-year duration. Therefore, the value of the fair charge is very sensitive to volatility.

We assume a constant interest rate to derive the analytic formulae, but we also investigate the effects of the interest rate on the fair charge in Table V. All fair charge values clearly are decreasing functions of the interest rate. The fair charges are related to the probabilities which account value declines to 0. The higher the probabilities which account value declines to 0, the higher the fair charge.

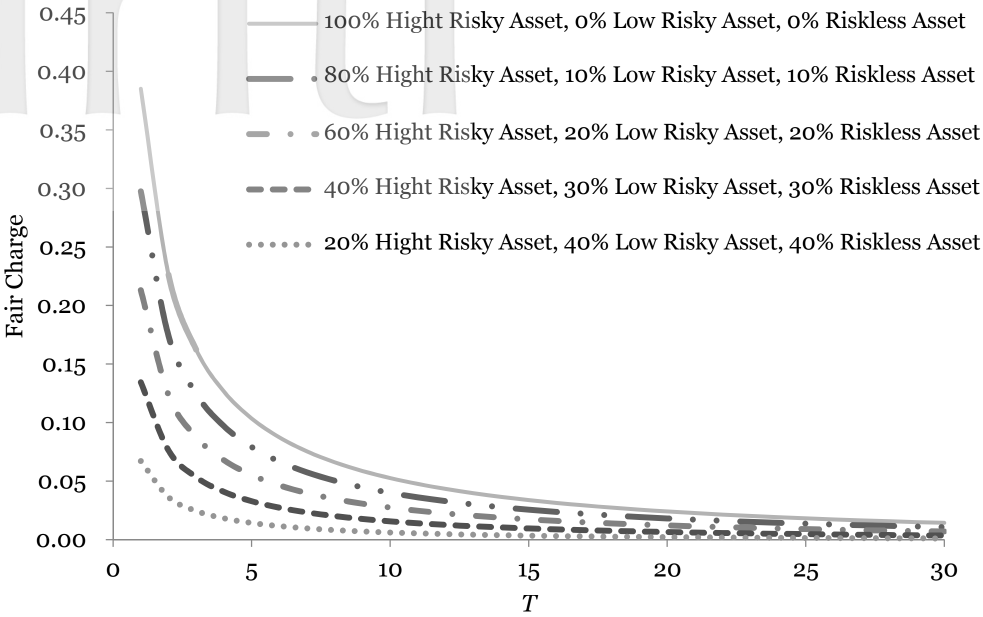

Figure 1. Fair Charge for Different Investment Strategies (ρ_{12}=0.5)

Table IV

Fair Charge for Different Volatility Levels in an Investment Portfolio (ρ_{12}=0.5)

T	$\tilde{\sigma}_{\lambda}$=0.1	$\tilde{\sigma}_{\lambda}$=0.2	$\tilde{\sigma}_{\lambda}$=0.3
1	0.083364	0.227161	0.385120
3	0.031336	0.093784	0.162191
6	0.015230	0.049739	0.087362
10	0.008271	0.029509	0.052591
20	0.003078	0.013111	0.024060
30	0.001525	0.007589	0.014308

Table V

Fair Charge for Different Interest Rate: ($\tilde{\sigma}_{\lambda}$ =0.2; ρ_{12}=0.5)

T	r=2%	r=4%	r=6%
1	0.227161	0.159184	0.120335
3	0.093784	0.057264	0.037998
6	0.049739	0.026546	0.015370
10	0.029509	0.013715	0.006845
20	0.013111	0.004640	0.001688
30	0.007589	0.002137	0.000585

V. Conclusions

In recent years, variable annuities have emerged as key components of the retirement income system. To transfer part of the investment risk inherent in VA products, guarantees commonly are embedded in them, which means that the way the guarantee is valued is critical for the insurer. We tackle a GMWB contract that contains features similar to an arithmetic Asian option and derive appropriate analytic formulae. We extend the lognormal distribution approximation, reciprocal gamma distribution approximation and average value of them to GMWB contracts and derive the analytic formulae in a multi-asset valuation framework. Closed-form solutions for valuing GMWB guarantees offer several benefits, including succinctness and decreased computing time. Because the duration for VA products is usually longer than that of financial options, a simulation framework requires far more time to value the guarantees than would the closed-form solution.

In our numerical analysis, we demonstrate the accuracy of our closed-form solutions in comparison with the simulated results. The result based on the average value of lognormal and reciprocal gamma distribution performs well, even for long-duration policies. In addition, the analytic closed-form solutions based on average value of lognormal and reciprocal gamma are more accurate than those approximated by lognormal and reciprocal gamma distribution. The numerical analysis offers great insight into ways to determine fair charges for different investment portfolios. The multi-asset valuation formulae not only benefit the insurer in its efforts to price GMWB more efficiently but also allow for a more realistic valuation.

According to our analysis in this paper, we point out some fields for further research. First, we do not investigate the issue of stochastic interest rates. Second, we ignore mortality in the valuation framework. Third, the hedging strategy that an insurer uses to deal with the guarantee risk is critical. Fourth, the fair charge under a dynamics investment strategy is also interesting for further investigation. These four areas are worthy of additional research to explore.

Appendix A

Proof of Equation (6)

The following technique, presented by Klebaner (2012), provides the solution of W_t. Consider the general linear stochastic differential equation (SDE):

$$dW_t = \left(\alpha(t) + \beta(t)W_t\right)dt + \left(\gamma(t) + \varphi(t)W_t\right) \cdot dZ_t^Q, \tag{A1}$$

where functions α, β, γ, and φ are continuous functions of *t*. Therefore, W_t takes the form:

$$W_t = X_t\left[W_0 + \int_0^t \frac{\alpha(u) - \varphi(u)\cdot\gamma(u)}{X_u}du + \int_0^t \frac{\gamma(u)}{X_u}\cdot dZ_u^Q\right], \tag{A2}$$

where X_t is given by

$$dX_t = \beta(t)X_t\,dt + \varphi(t)X_t \cdot dZ_t^Q. \tag{A3}$$

Comparing Equation (4) with Equation (A1), we obtain Equation (A2) by setting $\alpha(t) = -G$, $\beta(t) = r - c$, $\gamma(t) = 0$, and $\varphi(t) = \sigma_\lambda$. As a result, the dynamic of X_t is the same as that of U_t, and W_t can be expressed as follows:

$$W_t = X_t\left(W_0 - G\int_0^t \frac{1}{X_u}du\right) = U_t\left(w_0 - G\int_0^t U_u^{-1}du\right) \tag{A4}$$

Using Ito's lemma, we obtain

$$U_t = \exp\left(\left(r - c - \tilde{\sigma}_\lambda^2/2\right)t + \sigma_\lambda \cdot Z_t^Q\right). \tag{A5}$$

In view of Equation (A5), U_t is definitely positive, and $w_0 - G\int_0^t U_u^{-1}du$ is a nonincreasing function of time *t*. Let $\tau_0 = \inf\{t : W_t = 0\}$ be a stopping time at which the account value reaches zero. If the account value reaches zero at time τ_0 during the deferred period, $w_0 - G\int_0^t U_u^{-1}du$ will be less than zero for $t \geq \tau_0$, which leads to a negative value of W_t for $t \geq \tau_0$. To ensure the nonnegative account value, W_t is assumed to be 0 after time τ_0. Equivalently, W_t can attain a maximum value of zero and $U_t\left(w_0 - G\int_0^t U_u^{-1}du\right)$. This completes the proof of Equation (6).

Appendix B

Proof of Proposition 1

To compute the first moment of the integral $\int_0^T U_s^{-1}\,ds$, using Equation (12), we use

$$M_1 = E^{\tilde{Q}}\left(\int_0^T U_s^{-1}\,ds\right) = \int_0^T E^{\tilde{Q}}\left(U_s^{-1}\right)ds$$
$$= \int_0^T \exp\left(-\left(r-c+\tilde{\sigma}_\lambda^2/2\right)s\right)E^{\tilde{Q}}\left(\exp\left(-\sigma_\lambda \cdot Z_s^{\tilde{Q}}\right)\right)ds$$
$$= \int_0^T \exp\left(-\left(r-c+\tilde{\sigma}_\lambda^2/2\right)s\right)\exp\left(\tilde{\sigma}_\lambda^2\, s/2\right)ds$$
$$= \begin{cases} \dfrac{1-e^{-(r-c)T}}{(r-c)} & \text{if } r \neq c \\ T & \text{if } r = c. \end{cases} \quad \text{(B1)}$$

To compute the second moment of the integral, we recognize

$$M_2 = E^{\tilde{Q}}\left[\left(\int_0^T U_s^{-1}\,ds\right)^2\right] = \int_0^T\int_0^T E^{\tilde{Q}}\left(U_u^{-1}U_s^{-1}\right)duds$$
$$= \int_0^T\int_0^T \exp\left(-\left(r-c+\tilde{\sigma}_\lambda^2/2\right)(u+s)\right)E^{\tilde{Q}}\left(\exp\left(-\sigma_\lambda \cdot (Z_u^{\tilde{Q}}+Z_s^{\tilde{Q}})\right)\right)duds$$
$$= \int_0^T\int_0^s \exp\left(-\left(r-c+\tilde{\sigma}_\lambda^2/2\right)(u+s)\right)E\left(\exp\left(-\sigma_\lambda \cdot (2Z_u^{\tilde{Q}}+Z_{s-u}^{\tilde{Q}})\right)\right)duds$$
$$+ \int_0^T\int_s^T \exp\left(-\left(r-c+\tilde{\sigma}_\lambda^2/2\right)(u+s)\right)E\left(\exp\left(-\sigma_\lambda \cdot (2Z_s^{\tilde{Q}}+Z_{u-s}^{\tilde{Q}})\right)\right)duds$$
$$= M_{2A} + M_{2B}. \quad \text{(B2)}$$

To compute M_{2A}, because $Z_u^{\tilde{Q}}$ and $Z_{s-u}^{\tilde{Q}}$ are independent, we consider

$$M_{2A} = \int_0^T\int_0^s \exp\left(-\left(r-c+\tilde{\sigma}_\lambda^2/2\right)(u+s)\right)E\left(\exp\left(-2\sigma_\lambda \cdot Z_u^{\tilde{Q}}\right)\right)E\left(\exp\left(-\sigma_\lambda \cdot Z_{s-u}^{\tilde{Q}}\right)\right)duds$$
$$= \int_0^T\int_0^s \exp\left(-\left(r-c+\tilde{\sigma}_\lambda^2/2\right)(u+s)\right)\exp\left(2\tilde{\sigma}_\lambda^2 u\right)\exp\left(\tilde{\sigma}_\lambda^2(s-u)/2\right)duds$$
$$= \int_0^T\int_0^s \exp\left(-(r-c)s\right)\exp\left(-(r-c-\tilde{\sigma}_\lambda^2)u\right)duds. \quad \text{(B3)}$$

Following a similar procedure, we can obtain M_{2B} as follows:

$$M_{2B} = \int_0^T \int_s^T \exp\left(-\left(r-c\right)u\right)\exp\left(-\left(r-c-\tilde{\sigma}_\lambda^2\right)s\right)duds. \tag{B4}$$

when $r-c \neq 0$, $r-c-\tilde{\sigma}_\lambda^2 \neq 0$, and $2r-2c-\tilde{\sigma}_\lambda^2 \neq 0$, we have

$$\begin{aligned}
M_2 &= M_{2A} + M_{2B} \\
&= \int_0^T \left(\frac{e^{-(r-c)s} - e^{-\left(2r-2c-\tilde{\sigma}_\lambda^2\right)s}}{r-c-\tilde{\sigma}_\lambda^2} + \frac{e^{-\left(2r-2c-\tilde{\sigma}_\lambda^2\right)s} - e^{-(r-c)T}e^{-\left(r-c-\tilde{\sigma}_\lambda^2\right)s}}{r-c} \right) ds \\
&= \frac{1-e^{-(r-c)T}}{(r-c)\left(r-c-\tilde{\sigma}_\lambda^2\right)} - \frac{1-e^{-\left(2r-2c-\tilde{\sigma}_\lambda^2\right)T}}{\left(r-c-\tilde{\sigma}_\lambda^2\right)\left(2r-2c-\tilde{\sigma}_\lambda^2\right)} \\
&\quad + \frac{1-e^{-\left(2r-2c-\tilde{\sigma}_\lambda^2\right)T}}{(r-c)\left(2r-2c-\tilde{\sigma}_\lambda^2\right)} - \frac{e^{-(r-c)T} - e^{-\left(2r-2c-\tilde{\sigma}_\lambda^2\right)T}}{(r-c)\left(r-c-\tilde{\sigma}_\lambda^2\right)}. \\
&= 2\left[\begin{array}{l} \dfrac{e^{-\left(2r-2c-\tilde{\sigma}_\lambda^2\right)T}}{\left(r-c-\tilde{\sigma}_\lambda^2\right)\left(2r-2c-\tilde{\sigma}_\lambda^2\right)} \\ + \dfrac{1}{(r-c)}\left(\dfrac{1}{\left(2r-2c-\tilde{\sigma}_\lambda^2\right)} - \dfrac{e^{-(r-c)T}}{\left(r-c-\tilde{\sigma}_\lambda^2\right)} \right) \end{array} \right].
\end{aligned} \tag{B5}$$

Following an analogous procedure, we can obtain the second moment of the integral. This completes the proof of Proposition 1.

Appendix C

Proof of Proposition 2

Let X follow a gamma distribution with parameters α and β. If the random variable $Y=1/X$, Y is the reciprocal gamma distribution, whose moments are

$$E\left(Y^i\right)=E\left(\frac{1}{X^i}\right)=\frac{1}{\beta^i\left(\alpha-1\right)\left(\alpha-2\right)\cdots\left(\alpha-i\right)},\quad i=1,\ldots,n. \tag{C1}$$

The cumulative density function of Y, denoted by $G_R\left(y\,|\,\alpha,\beta\right)$, is of the form

$$G_R\left(y\,|\,\alpha,\beta\right)=\Pr\left(Y\le y\right)=\Pr\left(\frac{1}{Y}>\frac{1}{y}\right)=1-\tilde{G}\left(\frac{1}{y}\,|\,\alpha,\beta\right), \tag{C2}$$

where $\tilde{G}\left(y\,|\,\alpha,\beta\right)$ is a cumulative gamma density distribution with parameters α and β. Therefore, the probability density function of Y satisfies

$$g_R\left(y\,|\,\alpha,\beta\right)\equiv\frac{1}{y^2}g\left(\frac{1}{y}\,|\,\alpha,\beta\right),\quad \forall y>0. \tag{C3}$$

Thus, we approximate $Y\equiv\int_0^T U_s^{-1}\,ds$ to the reciprocal gamma distribution with parameters α_J and β_J, choosing the parameters to make the first and second moments of the reciprocal gamma distribution equal to M_1 and M_2 of Appendix B, which are

$$E\left(Y\right)=\frac{1}{\beta_J\left(\alpha_J-1\right)}\equiv M_1,\quad E\left(Y^2\right)=\frac{1}{\beta_J^2\left(\alpha_J-1\right)\left(\alpha_J-2\right)}\equiv M_2. \tag{C4}$$

Solving Equation (C4) yields

$$\alpha_J=\frac{2M_2-M_1^2}{M_2-M_1^2},\quad\text{and}\quad \beta_J=\frac{M_2-M_1^2}{M_2M_1}. \tag{C5}$$

Thus, the valuation of the reciprocal Asian option can be calculated as

$$\begin{aligned}J_{RG}\left(W_0,T\right)&=e^{-cT}E^{\tilde{Q}}\left(Max\left(w_0-G\int_0^T U_s^{-1}\,ds,0\right)\right)\\&=w_0e^{-cT}E^{\tilde{Q}}\left(Max\left(1-\frac{Y}{T},0\right)\right)=w_0e^{-cT}\int_0^T\left(1-\frac{y}{T}\right)g_R\left(y\,|\,\alpha_J,\beta_J\right)dy,\end{aligned} \tag{C6}$$

where $g_R\left(y|\alpha_J,\beta_J\right)$ is the probability density function of Y. Using Equation (C3), we obtain

$$
\begin{aligned}
J_{RG}\left(W_0,T\right) &= w_0 e^{-cT}\int_0^T\left(1-\frac{y}{T}\right)\frac{1}{y^2}g\left(\frac{1}{y}|\alpha_J,\beta_J\right)dy \\
&= w_0 e^{-cT}\int_{\frac{1}{T}}^{\infty}\left(1-\frac{1}{xT}\right)g\left(x|\alpha_J,\beta_J\right)dx \\
&= w_0 e^{-cT}\left(\int_{\frac{1}{T}}^{\infty}g\left(x|\alpha_J,\beta_J\right)dx-\frac{1}{T}\int_{\frac{1}{T}}^{\infty}\frac{1}{x}g\left(x|\alpha_J,\beta_J\right)dx\right) \\
&= w_0 e^{-cT}\left(\left(1-\int_0^{\frac{1}{T}}g\left(x|\alpha_J,\beta_J\right)dx\right)-\frac{1}{T}\int_{\frac{1}{T}}^{\infty}\frac{1}{\beta_J\left(\alpha_J-1\right)}g\left(x|\alpha_J-1,\beta_J\right)dx\right) \\
&= w_0 e^{-cT}\left(1-\tilde{G}\left(\frac{1}{T}|\alpha_J,\beta_J\right)-\frac{1}{T\beta_J\left(\alpha_J-1\right)}\left(1-\tilde{G}\left(\frac{1}{T}|\alpha_J-1,\beta_J\right)\right)\right) \\
&= w_0 e^{-cT}\left(1-\tilde{G}\left(\frac{1}{T}|\alpha_J,\beta_J\right)-\frac{M_1}{T}\left(1-\tilde{G}\left(\frac{1}{T}|\alpha_J-1,\beta_J\right)\right)\right),
\end{aligned}
\tag{C7}
$$

where we use $g\left(x|\alpha,\beta\right)=\frac{x}{\beta\left(\alpha-1\right)}g\left(x|\alpha-1,\beta\right)$. This completes the proof of Proposition 2.

Appendix D

Proof of Proposition 3

Let Y_{LN} follow a lognormal distribution with parameters μ_{LN} and σ_{LN}. Therefore, we approximate $Y_{LN} \equiv \int_0^T U_s^{-1}\, ds$ to the lognormal distribution with parameters μ_{LN} and σ_{LN}, choosing parameters to make the first and second moments of the lognormal distribution equal to M_1 and M_2 of Appendix B, that is,

$$E\left(Y_{LG}\right) = \exp\left(\mu_{LN} + 0.5\sigma_{LN}^2\right) \equiv M_1, E\left(Y_{LG}^2\right) = \exp\left(2\mu_{LN} + 2\sigma_{LN}^2\right) \equiv M_2. \qquad \text{(D1)}$$

Solving Equation (D1) yields

$$\mu_{LN} = 2\ln M_1 - 0.5\ln M_2,\ \ \sigma_{LN} = \sqrt{\ln M_2 - 2\ln M_1}. \qquad \text{(D2)}$$

Thus, the valuation of the reciprocal Asian option can be calculated as

$$\begin{aligned}
J_{LN}\left(W_0, T\right) &= e^{-cT} E^{\tilde{Q}}\left(Max\left(w_0 - G\int_0^T U_s^{-1}\, ds, 0\right)\right) \\
&= w_0 e^{-cT} E^{\tilde{Q}}\left(Max\left(1 - \frac{1}{T}\int_0^T U_s^{-1}\, ds, 0\right)\right) \\
&= w_0 e^{-cT} E^{\tilde{Q}}\left(Max\left(1 - \frac{1}{T} e^{\ln Y_{LG}}, 0\right)\right) \\
&= w_0 e^{-cT} \int_{-\infty}^{\ln T} \left(1 - \frac{1}{T}e^x\right) \frac{1}{\sqrt{2\pi}\sigma_{LN}} \exp\left(-\frac{\left(x-\mu_{LN}\right)^2}{2\sigma_{LN}^2}\right) dx \\
&= w_0 e^{-cT} \left[\begin{array}{l} \displaystyle\int_{-\infty}^{\ln T} \frac{1}{\sqrt{2\pi}\sigma_{LN}} \exp\left(-\frac{\left(x-\mu_{LN}\right)^2}{2\sigma_{LN}^2}\right) dx \\ \displaystyle -\frac{1}{T}\left(e^{\mu_{LN}+0.5\sigma_{LN}^2}\right)\int_{-\infty}^{\ln T} \frac{1}{\sqrt{2\pi}\sigma_{LN}} \exp\left(-\frac{\left(x-\left(\mu_{LN}+\sigma_{LN}^2\right)\right)^2}{2\sigma_{LN}^2}\right) dx \end{array} \right] \\
&= w_0 e^{-cT} \left[N\left(\ln T \middle| \mu_{LN}, \sigma_{LN}\right) - \frac{M_1}{T} N\left(\ln T \middle| \mu_{LN} + \sigma_{LN}^2, \sigma_{LN}\right)\right]. \qquad \text{(D3)}
\end{aligned}$$

This completes the proof of Proposition 3.

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評價保證最低提領給付於多資產架構下之解析解公式

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摘 要

本文試圖找到於多資產架構下保證最低提領給付（GMWB）之解析解公式。假設每個資產過程皆依據幾何布朗運動，藉運用反咖瑪與對數常態分配之平均值概念近似有相關性的對數常態隨機變數之和，我們得到 GMWB 之解析解公式。其數值顯示平均值概念之解析解公式表現精確，用於長期保單時亦然。評價 GMWB 時，此解析解公式具有顯著優勢與效率。

關鍵詞：保證最低提領給付、解析解公式、平均值概念、反咖瑪分配、對數常態分配

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