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Pattern generation problems arising in multiplicative integer systems

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Abstract. This study investigates a multiplicative integer system, an invariant subset of the full shift under the action of the semigroup of multiplicative integers, by using a method that was developed for studying pattern generation problems. The spatial entropy and the Minkowski dimensions of general multiplicative systems can thus be computed. A coupled system is the intersection of a multiplicative integer system and the golden mean shift, which can be decoupled by removing the multiplicative relation set and then performing procedures similar to those applied to a decoupled system. The spatial entropy can be obtained after the remaining error term is shown to approach zero.

1. *Introduction*

Multiplicative integer systems have been intensively studied in recent years; see [**13**–**15**, **23**, **25**, **26**, **33**, **34**, **36**] and the references therein. One of the main issues is to compute the Minkowski (box) dimension and Hausdorff dimension of such systems and to compare them. These two dimensions are equal in a shift space [**16**], a closed translation-invariant subset of a full shift. However, for most known examples of multiplicative integer systems, they are different. Since the computations of these two dimensions are difficult, effective methods for computing them for general multiplicative systems must be developed.

This investigation is motivated directly by the work of Kenyon *et al* [**26**], who utilized a variational method to obtain the results on

$$\mathbb{X}_2^0 = \{(x_1, x_2, x_3, \ldots) \in \{0, 1\}^{\mathbb{N}} \mid x_k x_{2k} = 0 \text{ for all } k \geq 1\}, \tag{1.1}$$

and also pointed out that the method fails for the system

$$\mathbb{X}^0_{2,3} = \{(x_1, x_2, x_3, \ldots) \in \{0, 1\}^{\mathbb{N}} \mid x_k x_{2k} x_{3k} = 0 \text{ for all } k \geq 1\}. \tag{1.2}$$

This paper will consider the following two classes of systems:

(i) multi-dimensional decoupled systems like (1.2);

(ii) one-dimensional coupled systems like

$$\begin{aligned}\mathbb{X}^G_2 &\equiv \mathbb{X}^0_2 \cap \Sigma_G \\ &= \{(x_1, x_2, \ldots) \in \{0, 1\}^{\mathbb{N}} \mid x_k x_{2k} = 0 \text{ and } x_k x_{k+1} = 0 \text{ for all } k \geq 1\},\end{aligned} \tag{1.3}$$

where Σ_G is the golden mean shift.

In (i), this work provides an approach to the computation of the Minkowski dimension of general multiplicative systems through computing spatial entropy, including (1.1) and (1.2). In (ii), a sequence of lower and upper bounds is obtained to approach spatial entropy of general one-dimensional coupled systems.

Firstly, the notion of Minkowski dimension is recalled. For any subset $Z \subset \mathbb{R}^m$, the upper and lower Minkowski dimension of Z can respectively be defined by

$$\overline{\dim}_M(Z) = \limsup_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{\log(1/\varepsilon)} \tag{1.4}$$

and

$$\underline{\dim}_M(Z) = \liminf_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{\log(1/\varepsilon)}, \tag{1.5}$$

where $N(Z, \varepsilon)$ is the smallest number of balls of radius ε needed to cover Z [**35**]. When the upper and lower Minkowski dimension are equal, denote the Minkowski dimension of Z by $\dim_M(Z) = \overline{\dim}_M(Z) = \underline{\dim}_M(Z)$.

Let $\mathcal{S}_N = \{0, 1, \ldots, N-1\}$ be a set of symbols, $N \geq 2$. The full shift is denoted by

$$\mathcal{S}^{\mathbb{N}}_N = \{x = (x_i)_{i=1}^{\infty} \mid x_i \in \mathcal{S}_N \text{ for all } i \in \mathbb{N}\}.$$

Consider the map $\varphi : \mathcal{S}^{\mathbb{N}}_N \to [0, 1]$ defined by $\varphi(x) = \sum_{k=1}^{\infty} (x_k/N^k)$. For any subset $\mathbb{X} \subseteq \mathcal{S}^{\mathbb{N}}_N$, let the corresponding set $\widetilde{\mathbb{X}} = \varphi(\mathbb{X}) \subseteq [0, 1]$. Then the upper and lower Minkowski dimension $\overline{\dim}_M(\widetilde{\mathbb{X}})$ and $\underline{\dim}_M(\widetilde{\mathbb{X}})$ can be defined.

Next, we turn to the notion of the spatial entropy of $\mathbb{X} \subseteq \mathcal{S}^{\mathbb{N}}_N$, introduced to study the complexity of symbolic dynamical systems $\mathbb{X} \subset \mathcal{S}^{\mathbb{Z}^n}_N$ and the applications to lattice dynamical systems [**1**, **2**, **5–7**, **12**, **18–20**, **24**, **29**, **31**, **32**, **37**, **38**]. The set of positive integers $\mathbb{N}$ is now considered as a half of $\mathbb{Z}^1$, the one-dimensional lattice. The set of positive integers $\mathbb{N}$ becomes a static discrete space, a geometric object. $i \in \mathbb{N}$ is the ith site. The distance between i and $i+1$ is unity. Therefore, in this paper, $\mathbb{N}$ is regarded as a spatial object instead of a temporal object.

Let $\mathbb{Z}_m = \{1, 2, \ldots, m\} \subset \mathbb{N}$ be the finite lattice of length m. For $\mathbb{X} \subseteq \mathcal{S}^{\mathbb{N}}_N$, let $X_m = X_m(\mathbb{X})$ be the set of the m-sequences of $\mathbb{X}$ on $\mathbb{Z}_m$, that is,

$$X_m = \{x|_{\mathbb{Z}_m} : x \in \mathbb{X}\}. \tag{1.6}$$

Let $|X_m|$ be the cardinality of X_m. The upper spatial entropy is defined by

$$\overline{h}(\mathbb{X}) = \limsup_{m \to \infty} \frac{\log |X_m|}{m} \tag{1.7}$$

and the lower spatial entropy is defined by

$$\underline{h}(\mathbb{X}) = \liminf_{m\to\infty} \frac{\log |X_m|}{m}. \tag{1.8}$$

When $\overline{h}(\mathbb{X}) = \underline{h}(\mathbb{X})$, $h(\mathbb{X}) = \overline{h}(\mathbb{X}) = \underline{h}(\mathbb{X})$ is called the spatial entropy of $\mathbb{X}$.

Then, using covers of cylinder sets of $\widetilde{\mathbb{X}}$, the Minkowski dimensions of $\widetilde{\mathbb{X}}$ are given by

$$\overline{\dim}_M(\widetilde{\mathbb{X}}) = \frac{1}{\log N}\overline{h}(\mathbb{X}), \tag{1.9}$$

$$\underline{\dim}_M(\widetilde{\mathbb{X}}) = \frac{1}{\log N}\underline{h}(\mathbb{X}) \tag{1.10}$$

and

$$\dim_M(\widetilde{\mathbb{X}}) = \frac{1}{\log N}h(\mathbb{X}) \tag{1.11}$$

whenever the spatial entropy exists.

This study emphasizes the computation of the spatial entropy $h(\mathbb{X})$ of the multiplicative system $\mathbb{X}$.

It is known that the spatial entropy $h(\mathbb{X})$ always exists when $\mathbb{X}$ is a shift space due to the subadditivity of $\log |X_m|$ in m. The spatial entropy is equal to the topological entropy $h(\mathbb{X}, \sigma)$, where σ is the shift map; see [**31**]. However, for a general subset of the full shift, the spatial entropy may not exist, and when it does may not be equal to Bowen's definition of topological entropy [**8**]. In this paper, we show that the spatial entropy always exists when $\mathbb{X}$ is in the form as in (1.1)–(1.3).

The multi-dimensional shifts of finite type have been studied intensively; see [**9–12**, **17–20**, **27**, **28**, **30–32**, **37**, **38**, **40**, **41**] and the references therein. The authors have studied pattern generation problems on multi-dimensional shifts of finite type and developed some efficient means of studying the generation of admissible patterns, and then computing the spatial entropy; see [**1–7**, **21**, **22**, **24**, **29**]. This study shows that these methods can be used to study multi-dimensional decoupled systems, including $\mathbb{X}^0_{2,3}$, and one-dimensional coupled systems of multiplicative integers.

To illustrate our method, equation (1.1) is investigated first. The spatial entropy $h(\mathbb{X}^0_2)$ has been shown to be

$$h(\mathbb{X}^0_2) = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \log a_k, \tag{1.12}$$

where a_k is a Fibonacci number with $a_1 = 2$, $a_2 = 3$ and $a_{k+1} = a_k + a_{k-1}$ for all $k \geq 2$ [**14**]. The derivation of (1.12) in [**14**] is through the computation of Minkowski dimension and (1.9). The derivation of (1.12) in this paper is as follows. Denote by $\mathbb{M}_2$ the multiplicative relation set of the integers which are powers of 2:

$$\mathbb{M}_2 = \{1, 2, 4, 8, 16, 32, \ldots, 2^n, \ldots\}. \tag{1.13}$$

Denote by $\mathcal{I}_2$ the complementary index set of $\mathbb{X}^0_2$ that contains all positive odd integers:

$$\mathcal{I}_2 = \{n \in \mathbb{N} \mid 2 \nmid n\} = \{2k+1\}_{k=0}^{\infty}. \tag{1.14}$$

The set of all natural number $\mathbb{N}$ can now be rearranged into

$$\mathbb{N} = \bigcup_{i\in\mathcal{I}_2} i\mathbb{M}_2, \tag{1.15}$$

TABLE 1. Arrangement for positive integers by $\mathbb{M}_2$ and $\mathcal{I}_2$.

$\mathcal{I}_2 \uparrow$ / $\mathbb{M}_2 \rightarrow$					
⋮					
17					
15	30				
13	26				
11	22	44			
9	18	36			
7	14	28			
5	10	20	40		
3	6	12	24	48	
1	2	4	8	16	32 ···

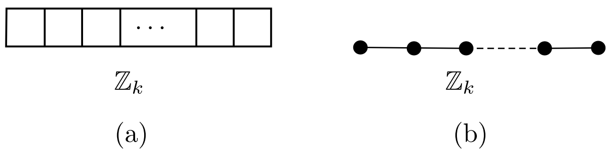


FIGURE 1. Representations for $\mathbb{Z}_k$.

where $i\mathbb{M}_2 = \{i, 2i, 2^2 i, \ldots, 2^n i, \ldots\}$. Clearly,

$$i\mathbb{M}_2 \cap j\mathbb{M}_2 = \emptyset \tag{1.16}$$

if $i, j \in \mathcal{I}_2$ and $i \neq j$. Furthermore, the right-hand side of (1.15) can be regarded as the first quarter of the two-dimensional lattice $\mathbb{Z}^2 = \mathbb{Z}^1 \times \mathbb{Z}^1$; see Table 1.

On integer lattice $\mathbb{Z}^1$, for $k \geq 1$, a k-lattice $\mathbb{Z}_k$ can be represented by k-cells as in Figure 1(a) for drawing numbers or k-vertices as in Figure 1(b) for drawing graphs below.

Let M_k and iM_k be the numbered lattices of the first k elements in $\mathbb{M}_2$ and in $i\mathbb{M}_2$ on the $\mathbb{Z}_k$, respectively.

Let

$$\mathcal{N}(m) = \{k \in \mathbb{N} \mid 1 \leq k \leq m\} \tag{1.17}$$

be the set of natural numbers that are less than or equal to m. For each $n \geq 1$ and $1 \leq i \leq 2^n$, let

$$q_n(i) = \max\{q \mid i \cdot 2^{q-1} \leq 2^n\} = \log_2 \left\lfloor \frac{2^{n+1}}{i} \right\rfloor, \tag{1.18}$$

where $\lfloor x \rfloor$ is the largest integer that is less than or equal to x.

Then, from Table 1, it is clear that

$$\mathcal{N}(2^n) = \bigcup_{i \in \mathcal{I}_2, 1 \leq i \leq 2^n} iM_{q_n(i)}. \tag{1.19}$$

For example, for $n = 4$,

$$\mathcal{N}(2^4) = M_5 \cup (3M_3) \cup (5M_2) \cup (7M_2) \cup (9M_1) \cup (11M_1) \cup (13M_1) \cup (15M_1). \tag{1.20}$$

In terms of blank lattices, the numbers in $\mathcal{N}(2^4)$ lie on one copy of $\mathbb{Z}_5$, one copy of $\mathbb{Z}_3$, two copies of $\mathbb{Z}_2$ and 2^2 copies of $\mathbb{Z}_1$.

To study $\mathcal{N}(2^n)$, let

$$I_n(k) = |\{i \in \mathcal{I}_2 \mid q_n(i) = k,\ 1 \le i \le 2^n\}|, \tag{1.21}$$

$k \ge 1$. It can be verified that

$$I_n(k) = \begin{cases} 2^{n-1-k} & \text{for } 1 \le k \le n-1, \\ 0 & \text{for } k = n, \\ 1 & \text{for } k = n+1, \\ 0 & \text{for } k \ge n+2. \end{cases}$$

Therefore, from (1.19), the result for $\mathcal{N}(2^n)$ in the following proposition can be proven. For general $Q \ge 2$, a similar result for $\mathcal{N}(Q^n)$ also holds.

PROPOSITION 1.1. *For integers $Q \ge 2$ and $n \ge 1$,*

$$Q^n = (n+1) + n(Q-2) + (Q-1)^2 \sum_{k=1}^{n-1} k \cdot Q^{n-1-k}. \tag{1.22}$$

In particular,

$$2^n = (n+1) + \sum_{k=1}^{n-1} k \cdot 2^{n-1-k}. \tag{1.23}$$

Therefore, equation (1.23) states that the numbers in $\mathcal{N}(2^n)$ are spread out on blank lattices with one copy of $\mathbb{Z}_{n+1}$ and 2^{n-1-k} copies of $\mathbb{Z}_k$, $1 \le k \le n-1$. In particular, setting $n = 4$ in (1.23) yields (1.20).

Now, consider again system $\mathbb{X}_2^0$ and the target formula (1.12). For any $n \ge 1$, let X_n be the set of all admissible n-sequences in $\mathbb{X}_2^0$:

$$X_n = \{(x_1, x_2, \ldots, x_n) \in \{0, 1\}^{\mathbb{Z}_n} \mid x_k x_{2k} = 0 \text{ for all } k \ge 1 \text{ and } 2k \le n\}. \tag{1.24}$$

Our purpose is to compute $|X_n|$, which is the number of elements in (1.11). The spatial entropy $h(\mathbb{X}_2^0)$ follows from

$$h(\mathbb{X}_2^0) = \lim_{n\to\infty} \frac{1}{n} \log |X_n|. \tag{1.25}$$

The constraint

$$x_k x_{2k} = 0 \tag{1.26}$$

in (1.24) is the admissible condition of the golden mean shift on $\mathbb{M}_2$, and it states that symbol 1 is not allowed to follow symbol 1 immediately. Then the forbidden set on $\mathbb{Z}_2$ is $\{\boxed{1}\boxed{1}\}$. The transition matrix is

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1.27}$$

Let Σ_k be the set of all admissible patterns on $\mathbb{Z}_k$ with respect to (1.27); then

$$|\Sigma_k| = a_k, \tag{1.28}$$

FIGURE 2. Numbered lattices M_k and iM_k.

which is the kth Fibonacci number. Since the constraint (1.26) applies to each $i\mathbb{M}_2$ independently for $i \in \mathcal{I}_2$,

$$|X_{2^n}| = |\Sigma_{n+1}| \prod_{k=1}^{n-1} |\Sigma_k|^{2^{n-1-k}}, \tag{1.29}$$

which implies

$$\frac{1}{2^n} \log |X_{2^n}| = \frac{1}{2^n} a_{n+1} + \sum_{k=1}^{n-1} \frac{1}{2^{k+1}} \log a_k. \tag{1.30}$$

Hence, equation (1.12) can be shown by carefully computing $|X_m|$ when $2^n < m < 2^{n+1}$.

By a similar argument, equation (1.22) of Proposition 1.1 also recovers the following results [**14**].

THEOREM 1.2. *For any $Q \geq 2$, denote the multiplicative integer system*

$$\mathbb{X}_Q^0 = \{(x_1, x_2, \ldots) \in \{0, 1\}^{\mathbb{N}} \mid x_k x_{Qk} = 0 \textit{ for all } k \geq 1\}. \tag{1.31}$$

Then

$$h(\mathbb{X}_Q^0) = (Q-1)^2 \sum_{k=1}^{\infty} \frac{1}{Q^{k+1}} \log a_k. \tag{1.32}$$

Consideration of the above reveals the following three main parts of our study of $\mathbb{X}_2^0$.

(I) Identify the numbered lattice M_k and the admissible blank lattice $\mathbb{Z}_k$ from the given system; see Figures 1 and 2.

(II) Compute the numbers of copies of independent admissible lattices of the same length; see formulae (1.22) and (1.23).

(III) Determine the set of all admissible patterns Σ_k, which can be generated on $\mathbb{Z}_k$, and compute the number $|\Sigma_k|$.

Notably, step (III) in the study of $\mathbb{X}_2^0$ is the classical one-dimensional pattern generation problem; see [**31**].

First, consider multi-dimensional decoupled systems, including (1.2). Let

$$1 < \gamma_1 < \gamma_2 < \cdots < \gamma_d \tag{1.33}$$

be natural numbers, $d \geq 2$, such that γ_i and γ_j are relatively prime for all $i < j$, that is,

$$(\gamma_i, \gamma_j) = 1 \tag{1.34}$$

for all $1 \leq i < j \leq d$, where (a, b) is the greatest common divisor of natural numbers a and b.

Denote

$$\Gamma \equiv \Gamma_d = \{\gamma_1, \gamma_2, \ldots, \gamma_d\} \tag{1.35}$$

and

$$\begin{aligned}\mathbb{X}_\Gamma^0 &\equiv \mathbb{X}^0_{\gamma_1,\gamma_2,\ldots,\gamma_d} \\ &= \{(x_1, x_2, x_3, \ldots) \in \{0,1\}^{\mathbb{N}} \mid x_k x_{\gamma_1 k} x_{\gamma_2 k} \cdots x_{\gamma_d k} = 0 \text{ for all } k \geq 1\}.\end{aligned} \tag{1.36}$$

Let

$$\begin{aligned}\mathbb{M}_\Gamma &\equiv \{\gamma_1^{m_1}\gamma_2^{m_2}\cdots\gamma_d^{m_d} \mid m_j \geq 0, 1 \leq j \leq d\} \\ &= \{q_k\}_{k=1}^{\infty},\end{aligned} \tag{1.37}$$

with $q_k < q_j$ if $k < j$. Then equation (1.37) defines a sequence of d-dimensional numbered lattices M_k of k cells. The blank lattices L_k are defined analogously; see (2.8). The complementary index set $\mathcal{I}_\Gamma$ of $\mathbb{M}_\Gamma$ is defined by

$$\mathcal{I}_\Gamma = \{n \in \mathbb{N} \mid \gamma_j \nmid n, 1 \leq j \leq d\}. \tag{1.38}$$

Hence,

$$\mathbb{N} = \bigcup_{i \in \mathcal{I}_\Gamma} i\mathbb{M}_\Gamma. \tag{1.39}$$

The following theorem for multi-dimensional decoupled system will be proven in §2.

THEOREM 1.3. *Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_d\}$ satisfy (1.33) and (1.34). Then the spatial entropy of $\mathbb{X}_\Gamma^0$ is given by*

$$h(\mathbb{X}_\Gamma^0) = \sum_{k=1}^{\infty} \beta_\Gamma \left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) \log|\Sigma_k|, \tag{1.40}$$

where

$$\beta_\Gamma = \frac{\sharp(\mathcal{I}_\Gamma \cap [1, \gamma_1\gamma_2\cdots\gamma_d])}{\gamma_1\gamma_2\cdots\gamma_d}. \tag{1.41}$$

Finally, consider the one-dimensional coupled system

$$\mathbb{X}_Q^G \equiv \mathbb{X}_Q^0 \cap \Sigma_G, \tag{1.42}$$

where Σ_G is the golden mean shift. Denote by $L_{Q;k}$ the degree-k blank lattice of the admissible numbered lattice $M_{Q;k}(l)$; see Figure 6 for $Q = 2$ and Figure 7 for $Q = 3$. The following theorem will be proven in §3.

THEOREM 1.4. *For any $Q \geq 2$ and $k \geq 2$,*

$$\frac{Q-1}{Q(Q^k-1)} \log|\Sigma_{Q,G;k}| \leq h(\mathbb{X}_Q^G) \leq \frac{Q-1}{Q(Q^k-1)}(\log|\Sigma_{Q,G;k}| + k\log 2) \tag{1.43}$$

and

$$h(\mathbb{X}_Q^G) = \lim_{k\to\infty} \frac{Q-1}{Q(Q^k-1)} \log|\Sigma_{Q,G;k}|, \tag{1.44}$$

where $\Sigma_{Q,G;k}$ is the set of all admissible patterns on $L_{Q;k}$.

TABLE 2. Representation for $\mathbb{M}_{2,3}$.

243	486	972	1944	3888	7776
81	162	324	648	1296	2592
27	54	108	216	432	864
9	18	36	72	144	288
3	6	12	24	48	96
1	2	4	8	16	32

q_{27}	q_{33}	q_{40}	q_{47}	q_{55}	q_{64}
q_{19}	q_{24}	q_{30}	q_{36}	q_{43}	q_{51}
q_{12}	q_{16}	q_{21}	q_{26}	q_{32}	q_{39}
q_7	q_{10}	q_{14}	q_{18}	q_{23}	q_{29}
q_3	q_5	q_8	q_{11}	q_{15}	q_{20}
q_1	q_2	q_4	q_6	q_9	q_{13}

After we completed our study of (i) and (ii), we became aware of the work of Peres *et al* [**33**] on (1.2). These authors obtained the same results as ours for (1.2). Our approach for studying (i) differs from theirs by using the results from an investigation of pattern generation problems in computing spatial entropy, and involving the three specified steps (I)–(III). Moreover, a modification of these procedures enables us to study the one-dimensional coupled system (ii).

Multi-dimensional coupled systems like

$$\begin{aligned}\mathbb{X}^{G}_{2,3} &\equiv \mathbb{X}^{0}_{2,3} \cap \Sigma_G \\ &= \{(x_1, x_2, \ldots) \in \{0, 1\}^{\mathbb{N}} \mid x_k x_{2k} x_{3k} = 0 \text{ and } x_k x_{k+1} \text{ for all } k \geq 1\} \end{aligned} \tag{1.45}$$

are much more delicate. The problem of (1.45) is still not solved by using our method which works well in studying the one-dimensional coupled system (1.3); see Remark 3.7.

The rest of this paper is arranged as follows. Section 2 studies multi-dimensional systems and proves Theorem 1.3. Section 3 studies one-dimensional coupled systems and proves Theorem 1.4.

2. *Multi-dimensional systems*

This section concerns multi-dimensional decoupled systems and proves Theorem 1.3. First, $\mathbb{X}^0_\Gamma$ satisfying (1.33) and (1.34) is considered. Recall that

$$\mathbb{X}^0_\Gamma \equiv \{(x_1, x_2, x_3, \ldots) \in \{0, 1\}^{\mathbb{N}} \mid x_k x_{\gamma_1 k} x_{\gamma_2 k} \cdots x_{\gamma_d k} = 0 \text{ for all } k \geq 1\}. \tag{2.1}$$

Before the three main steps, let $\mathbb{M}_\Gamma$ be the set of the numbers that are multiples of powers of $\gamma_1, \gamma_2, \ldots, \gamma_n$:

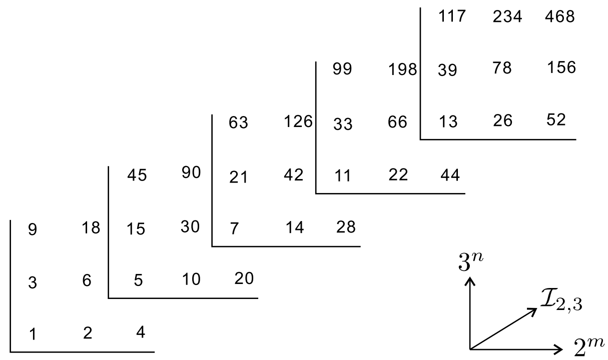
$$\mathbb{M}_\Gamma \equiv \{\gamma_1^{m_1} \gamma_2^{m_2} \cdots \gamma_d^{m_d} \mid m_j \geq 0\} \equiv \{q_k\}_{k=1}^{\infty}, \tag{2.2}$$

with $q_k < q_j$ if $k < j$. Notably, $\mathbb{M}_\Gamma$ can be arranged in a d-dimensional lattice with the coordinate axes: powers of γ_1, powers of γ_2, ... and powers of γ_n; see Table 2 for $\mathbb{M}_{2,3}$.

The complementary index set $\mathcal{I}_\Gamma$ of $\mathbb{X}^0_\Gamma$ is the set of all natural numbers that cannot be divided by γ_j, $1 \leq j \leq d$, as defined in (1.38). The set $\mathbb{N}$ of natural numbers can be rearranged into the first octant of $(d+1)$-dimensional space as

$$\mathbb{N} = \bigcup_{i \in \mathcal{I}_\Gamma} i\mathbb{M}_\Gamma, \tag{2.3}$$

TABLE 3. Arrangement for positive integers by $\mathbb{M}_{2,3}$ and $\mathcal{I}_{2,3}$.



and for $i, j \in \mathcal{I}_\Gamma$ with $i \neq j$,

$$i\mathbb{M}_\Gamma \cap j\mathbb{M}_\Gamma = \emptyset. \tag{2.4}$$

Indeed, the $d+1$ coordinate axes are $\mathcal{I}_\Gamma$ and powers of γ_j, $1 \leq j \leq d$; see Table 3 for $\mathbb{X}_{2,3}$.

In the following, $\mathbb{X}^0_{2,3}$ is used to illustrate above. Let $\mathbb{M}_{2,3}$ be the set of the numbers that are multiples of powers of 2 and powers of 3:

$$\mathbb{M}_{2,3} = \{2^k 3^l \mid k, l \geq 0\} = \{q_k\}_{k=1}^{\infty}. \tag{2.5}$$

In Table 2, $\mathbb{M}_{2,3}$ can be expressed as a quarter of $\mathbb{Z}^2$.

The complementary index set $\mathcal{I}_{2,3}$ of $\mathbb{X}^0_{2,3}$ is

$$\mathcal{I}_{2,3} = \{n \in \mathbb{N} \mid 2 \nmid n \text{ and } 3 \nmid n\} = \{6k+1, 6k+5\}_{k=0}^{\infty}. \tag{2.6}$$

Therefore, the set $\mathbb{N}$ of natural numbers can be rearranged into the first octant of three-dimensional space as in Table 3.

Now, for step (I), the admissible numbered and blank lattices, determined by the constraint

$$x_k x_{\gamma_1 k} x_{\gamma_2 k} \cdots x_{\gamma_d k} = 0, \tag{2.7}$$

must be identified in $\mathbb{X}^0_\Gamma$.

Since $\mathbb{M}_\Gamma$ can be arranged in a d-dimensional lattice, for any $k \geq 1$, the L-shaped k-cell numbered lattice M_k that contains $\{q_1, q_2, \ldots, q_k\}$ can be defined; see Figure 3 for M_k of $\mathbb{X}^0_{2,3}$, $1 \leq k \leq 8$. Then, the k-cell (blank) lattice L_k can also be defined by deleting the numbers of M_k. Indeed, for any $k \geq 1$,

$$L_k = \{(i_1, i_2, \ldots, i_d) \in \mathbb{Z}^d \mid \gamma_1^{i_1}\gamma_2^{i_2}\cdots\gamma_d^{i_d} \leq q_k \text{ for } i_j \geq 0, 1 \leq j \leq d\}. \tag{2.8}$$

Therefore, for step (I), the numbered lattice M_k and the admissible blank lattice L_k are obtained. In contrast to $\mathbb{X}^0_2$, the lattice L_k is now n-dimensional.

Turning to step (II), let $\mathbb{Z}_m = \{1, 2, \ldots, m\} \subset \mathbb{N}$ be the finite lattice of length n. From (2.3), the lattice $\mathbb{Z}_m$ can be arranged to be a finite $(d+1)$-dimensional lattice with size m. Firstly, some necessary notation is introduced. Let

$$J_{\Gamma;m} = \{i \in \mathcal{I}_\Gamma \mid i \leq m\} \tag{2.9}$$

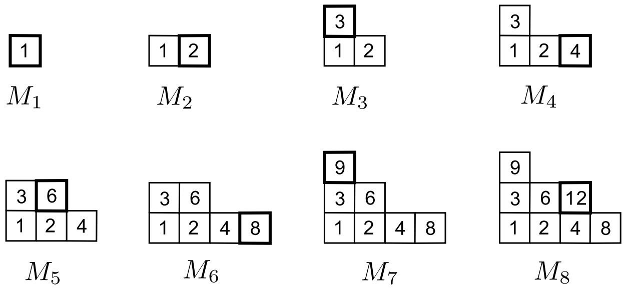


FIGURE 3. Numbered lattices M_k of $\mathbb{X}^0_{2,3}$.

be the intersection of $\mathcal{I}_\Gamma$ and $\mathbb{Z}_m$. For each $i \in J_{\Gamma;m}$, let

$$q_{\Gamma;m}(i) = \max\{q \in \mathbb{M}_\Gamma \mid iq \le m\}. \tag{2.10}$$

Then, for $1 \le k \le m$, denote by

$$I_\Gamma(k;\, m) = \{i \in J_{\Gamma;m} \mid q_{\Gamma;m}(i) = q_k\} \tag{2.11}$$

the subset of $J_{\Gamma;m}$ such that, for $i \in J_{\Gamma;m}$, $|i\mathbb{M}_\Gamma \cap \mathbb{Z}_m| = k$. Let

$$\alpha_\Gamma(k;\, m) = |I_\Gamma(k;\, m)|. \tag{2.12}$$

It can be proven immediately that, for $m \ge 1$,

$$\mathbb{Z}_m = \bigcup_{k=1}^{m} \bigcup_{i \in I_\Gamma(k;m)} iM_k \tag{2.13}$$

and

$$m = \sum_{k=1}^{m} \alpha_\Gamma(k;\, m) \cdot k. \tag{2.14}$$

Then, after rearranging by (2.13), $\mathbb{Z}_m$ is decomposed into $\alpha_\Gamma(k;\, m)$ copies of k-cell (blank) lattice L_k, $1 \le k \le m$.

Let us use $\mathbb{X}^0_{2,3}$ to illustrates (2.11) and (2.13). For $m = 36$, it is easy to verify that

$$\begin{cases} I_{2,3}(1;\, 14) = \{19,\, 23,\, 25,\, 29,\, 31,\, 35\}, \\ I_{2,3}(2;\, 14) = \{13,\, 17\}, \\ I_{2,3}(3;\, 14) = \{11\}, \\ I_{2,3}(4;\, 14) = \{7\}, \\ I_{2,3}(5;\, 14)(14) = \{5\}, \\ I_{2,3}(14;\, 14)(14) = \{1\}. \end{cases}$$

The others are empty. Therefore,

$$\mathbb{Z}_{36} = \left(\bigcup_{i \in \{19,23,25,29,31,35\}} iM_1 \right) \cup 17M_2 \cup 13M_2 \cup 11M_3 \cup 7M_4 \cup 5M_5 \cup M_{14}$$

and

$$m = 36 = 6 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 1 \cdot 5 + 1 \cdot 14.$$

The following proposition shows the limiting density of $\alpha_\Gamma(k;\, m)$, the number of the copies of L_k, as m tends to infinity, which is crucial in computing the spatial entropy.

PROPOSITION 2.1. *On $\mathbb{X}_\Gamma^0$, for $k \geq 1$,*

$$\lim_{m\to\infty} \frac{\alpha_\Gamma(k; m)}{m} = \beta_\Gamma\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right), \tag{2.15}$$

where

$$\beta_\Gamma = \frac{\sharp(\mathcal{I}_\Gamma \cap [1, \gamma_1\gamma_2\cdots\gamma_d])}{\gamma_1\gamma_2\cdots\gamma_d}. \tag{2.16}$$

Proof. Fix $k \geq 1$. For each $i \in J_{\Gamma;m}$, it is clear that if $q_{\Gamma;m}(i) = q_k$, then

$$iq_k \leq m < iq_{k+1}.$$

Then

$$I_\Gamma(k; m) = \left(\frac{m}{q_{k+1}}, \frac{m}{q_k}\right] \cap \mathcal{I}_\Gamma. \tag{2.17}$$

Therefore,

$$\begin{aligned}\lim_{m\to\infty} \frac{\alpha_\Gamma(k; m)}{m} &= \lim_{m\to\infty} \frac{\beta_\Gamma\left(\frac{m}{q_k} - \frac{m}{q_{k+1}}\right)}{m}\\ &= \beta_\Gamma\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right).\end{aligned} \tag{2.18}$$

The proof is complete. □

Having obtained the limiting density (2.15) of the copies of L_k, step (II) is complete. Y=The final step (III) is to compute the admissible patterns on the d-dimensional lattice L_k for all $k \geq 1$.

The basic lattice $\mathbb{L}_\Gamma$ of $\mathbb{X}_\Gamma^0$ is defined by

$$\mathbb{L}_\Gamma = \left\{(i_1, i_2, \ldots, i_d) \in \mathbb{Z}^d \;\middle|\; 0 \leq \sum_{k=1}^{d} i_k \leq 1 \text{ for } i_k \geq 0,\ 1 \leq k \leq d\right\}, \tag{2.19}$$

the d-dimensional L-shaped lattice with the origin $(0, 0, \ldots, 0)$ as the corner vertex. From the constraint

$$x_k x_{\gamma_1 k} x_{\gamma_2 k} \cdots x_{\gamma_d k} = 0 \tag{2.20}$$

and M_Γ can be arranged in a d-dimensional lattice, with the coordinate axes: powers of γ_j, $1 \leq j \leq d$, we define the forbidden set

$$\mathcal{F}_\Gamma = \{U = (u_{i_1,i_2,\ldots,i_d}) \in \{0, 1\}^{\mathbb{L}_\Gamma} \mid u_{i_1,i_2,\ldots,i_d} = 1 \text{ for all } (i_1, i_2, \ldots, i_d) \in \mathbb{L}_\Gamma\}.$$

Then the basic set of admissible patterns is defined by

$$\mathcal{B}_\Gamma = \{0, 1\}^{\mathbb{L}_\Gamma} \setminus \mathcal{F}_\Gamma, \tag{2.21}$$

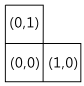
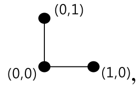
which induces a d-dimensional shift of finite type $\Sigma(\mathcal{B}_\Gamma)$ on the d-dimensional lattice space $\mathbb{Z}^d$. Denote by $\Sigma_k = \Sigma_k(\mathcal{B}_\Gamma)$ the set of all admissible patterns determined by $\mathcal{B}_\Gamma$ on L_k. Indeed,

$$\begin{aligned}\Sigma_k &= \Sigma_k(\mathcal{B}_\Gamma)\\ &= \{U \in \{0, 1\}^{L_k} : U|_L \in \mathcal{B}_\Gamma \text{ for all } L = \mathbb{L}_\Gamma + \mathbf{v} \subseteq L_k \text{ with some } \mathbf{v} \in \mathbb{Z}^d\}.\end{aligned} \tag{2.22}$$

For example, consider $\mathbb{X}^0_{2,3}$. The basic lattice $\mathbb{L}_{2,3}$ is defined by

$$\mathbb{L}_{2,3} = \{(i, j) \in \mathbb{Z}^2 \mid 0 \le i + j \le 1 \text{ for } i, j \ge 0\} = \{(0, 0), (1, 0), (0, 1)\}, \tag{2.23}$$

that is,

$\mathbb{L}_{2,3} =$  or ,

the L-shaped lattice with origin (0, 0) as the corner vertex. The forbidden local pattern on $\mathbb{L}_{2,3}$ is

$$\left\{ \begin{matrix} 1 & \\ 1 & 1 \end{matrix} \right\}. \tag{2.24}$$

Therefore, the basic set of admissible patterns is

$$\mathcal{B}_{2,3} = \left\{ \begin{matrix} 0 & \\ 0 & 0 \end{matrix}, \begin{matrix} 1 & \\ 0 & 0 \end{matrix}, \begin{matrix} 0 & \\ 0 & 1 \end{matrix}, \begin{matrix} 1 & \\ 0 & 1 \end{matrix}, \begin{matrix} 0 & \\ 1 & 0 \end{matrix}, \begin{matrix} 1 & \\ 1 & 0 \end{matrix}, \begin{matrix} 0 & \\ 1 & 1 \end{matrix} \right\}; \tag{2.25}$$

see [**5**, **6**]. For $k \ge 1$, $\Sigma_k = \Sigma_k(\mathcal{B}_{2,3})$ is the set of all admissible patterns on L_k that are determined by $\mathcal{B}_{2,3}$. Previously, two-dimensional pattern generation problems on L-shaped lattices have been studied by Lin and Yang in [**29**].

Recall that $X_{\Gamma;n}$ is the set of the n-sequences of X_Γ on $\mathbb{Z}_n$, that is,

$$X_{\Gamma;n} = \{(x_1, x_2, \dots, x_n) \in \{0, 1\}^{\mathbb{Z}_n} \mid x_k x_{\gamma_1 k} \cdots x_{\gamma_d k} = 0 \text{ for all } k \ge 1 \text{ and } \gamma_d k \le n\}$$

and

$$h(\mathbb{X}^0_\Gamma) = \lim_{n\to\infty} \frac{1}{n} \log |X_{\Gamma;n}|.$$

Having completed the above procedures, we will now prove Theorem 1.3, which says that the spatial entropy of $\mathbb{X}^0_\Gamma$ that satisfies (1.33) and (1.34) is equal to

$$h(\mathbb{X}^0_\Gamma) = \sum_{k=1}^{\infty} \beta_\Gamma \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log |\Sigma_k|.$$

Proof of Theorem 1.3. First, for any $n \ge 1$, let $X_{\Gamma;n}$ be the set of all admissible n-sequences on $\mathbb{Z}_n$ in $\mathbb{X}^0_\Gamma$. From condition (2.20), by (2.3), it is easy to see that for any two $i_1, i_2 \in \mathcal{I}_\Gamma$, the admissible patterns on $i_1\mathbb{M}_\Gamma$ and the admissible patterns on $i_2\mathbb{M}_\Gamma$ are mutually independent.

Then, by (2.13), we have that for any $n \ge 1$,

$$|X_{\Gamma;n}| = \prod_{k=1}^{n} |\Sigma_k|^{\alpha_\Gamma(k;n)}.$$

Therefore, from Proposition 2.1,

$$\begin{aligned} h(\mathbb{X}^0_\Gamma) &= \lim_{n\to\infty} \frac{\log |X_{\Gamma;n}|}{n} \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \alpha_\Gamma(k; n) \log |\Sigma_k| \right) \Big/ n \\ &= \sum_{k=1}^{\infty} \beta_\Gamma \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log |\Sigma_k|. \end{aligned}$$

The proof is complete. □

TABLE 4. b_k of $\mathbb{X}^0_{2,3}$.

k	5	14	26	43
b_k	14	3722	5 434 757	172 749 984 030

Remark 2.2.

(i) Consider $\mathbb{X}^0_{2,3}$. Denote by

$$|\Sigma_k| = b_k \tag{2.26}$$

the number of the patterns in Σ_k. Since no exact formula relates $2^m 3^n$ to q_i for $\mathbb{M}_{2,3}$ in Table 2, unlike for a Fibonacci number, no recursive formula exists for b_k; see [**39**]. This fact creates the most difficulty in computing spatial entropy for a multi-dimensional system; see [**5**, **6**, **29**]. However, for relatively small k, b_k can be computed using the transition matrices developed in [**5**, **6**]. Table 4 presents a few cases for $q_k = 6^l$, $1 \le l \le 4$.

(ii) Define

$$r_k = |\Sigma_k|/|\Sigma_{k-1}| = b_k/b_{k-1} \tag{2.27}$$

for $k \ge 2$. From Table 2 and Figure 3, it is easy to verify that

$$r_k = 2 \quad \text{if } q_k = 2^n, \tag{2.28}$$

for some $n \ge 1$. On the other hand, it can be shown that there exists $C \le \frac{31}{16}$ such that

$$r_k \le C \quad \text{for } q_k \ne 2^n \text{ for all } n \ge 2. \tag{2.29}$$

Therefore, $\{r_k\}$ cannot have a limit as k tends to ∞, unlike the Fibonacci sequence which has the limit $(1+\sqrt{5})/2$. Further study of $\{r_k\}$ and b_k is needed.

In the following, an approximation of (1.40) is given. For $n \ge 1$, let

$$h^{(n)}(\mathbb{X}^0_\Gamma) = \sum_{k=1}^{n} \beta_\Gamma \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log |\Sigma_k|. \tag{2.30}$$

Clearly, from Theorem 1.3, $h^{(n)}(\mathbb{X}^0_\Gamma)$ is a lower bound of $h(\mathbb{X}^0_\Gamma)$, and $h^{(n)}(\mathbb{X}^0_\Gamma)$ increasingly approaches $h(\mathbb{X}^0_\Gamma)$ as n tends to infinity. Furthermore, let

$$\begin{aligned} E^{(n)}(\mathbb{X}^0_\Gamma) &\equiv \sum_{k=n+1}^{\infty} \beta_\Gamma \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log 2^k \\ &= \sum_{k=1}^{\infty} \beta_\Gamma \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log 2^k - \sum_{k=1}^{n} \beta_\Gamma \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log 2^k \\ &= \beta_\Gamma \left(\sum_{k=1}^{\infty} \frac{1}{q_k} - \sum_{k=1}^{n} \frac{1}{q_k} + \frac{n}{q_{n+1}} \right) \log 2, \end{aligned} \tag{2.31}$$

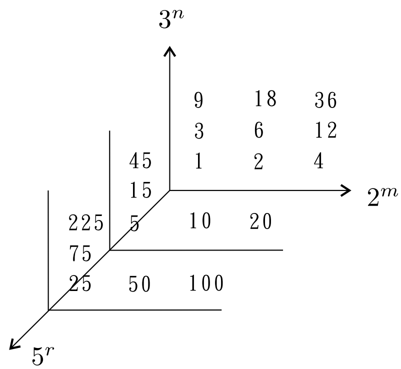
where

$$\sum_{k=1}^{\infty} \frac{1}{q_k} = \prod_{j=1}^{d} \frac{1}{1 - 1/\gamma_j}.$$

TABLE 5. $h^{(n)}(\mathbb{X}^0_{2,3})$ for $\mathbb{X}^0_{2,3}$.

n	4	13	25	42
$h^{(n)}(\mathbb{X}^0_{2,3})$	0.319901	0.537229	0.620707	0.645733

TABLE 6. Representation for $\mathbb{M}_{2,3,5}$.



Hence,

$$h(\mathbb{X}^0_\Gamma) - h^{(n)}(\mathbb{X}^0_\Gamma) \le E^{(n)}(\mathbb{X}^0_\Gamma). \tag{2.32}$$

For example, consider $\mathbb{X}^0_{2,3}$. Table 5 presents cases for n with $q_{n+1} = 6^l$ and $1 \le l \le 4$. Moreover, $h^{(152)}(\mathbb{X}^0_{2,3}) \approx 0.654303$ and $E^{(152)}(\mathbb{X}^0_{2,3}) \approx 0.0000238741$.

The following example considers the three-dimensional system $\mathbb{X}^0_{2,3,5}$.

Example 2.3. For $d = 3$, consider

$$\mathbb{X}^0_{2,3,5} = \{(x_1, x_2, x_3, \ldots) \in \{0, 1\}^{\mathbb{N}} \mid x_k x_{2k} x_{3k} x_{5k} = 0 \text{ for all } k \ge 1\}.$$

Then

$$\mathbb{M}_{2,3,5} = \{2^m 3^n 5^r \mid m, n, r \ge 0\}$$

can be arranged into the first octant of $\mathbb{Z}^3$; see Table 6. Clearly,

$$\mathcal{I}_{2,3,5} = \{30k + j \mid j \in \{1, 7, 11, 13, 17, 19, 23, 29\} \text{ and } k \ge 0\}.$$

The first five numbered lattices are listed as shown in Figure 4.

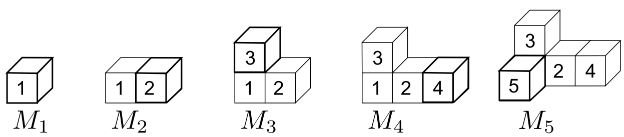


FIGURE 4. Numbered lattices M_k of $\mathbb{X}^0_{2,3,5}$.

The basic lattice is

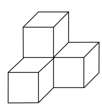
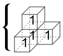
$\mathbb{L}_{2,3,5} =$  .

TABLE 7. $|\Sigma_n(2, 3, 5)|$ for $\mathbb{X}^0_{2,3,5}$.

n	5	10	15	20	25
$\lvert\Sigma_n(2, 3, 5)\rvert$	30	904	25 720	738 816	19 959 552

Therefore, it can be verified that

$$h(\mathbb{X}^0_{2,3,5}) = \sum_{k=1}^{\infty} \beta_{2,3,5}\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) \log|\Sigma_k(2, 3, 5)|, \tag{2.33}$$

where $\beta_{2,3,5} = \frac{4}{15}$ and the forbidden set of $\Sigma_k(2, 3, 5)$ is $\Big\{$  $\Big\}$. Moreover, the nth-order approximation of (2.33) is as follows. For $n \geq 1$, let

$$h^{(n)}(\mathbb{X}^0_{2,3,5}) = \sum_{k=1}^{n} \beta_{2,3,5}\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) \log|\Sigma_k(2, 3, 5)|.$$

In Table 7, some cases for $|\Sigma_n(2, 3, 5)|$ in $1 \leq n \leq 25$ are listed. Moreover, $h^{(25)}(\mathbb{X}^0_{2,3,5}) \approx 0.548837$.

The previous idea also applies to the system $\mathbb{X}^0_\Gamma$ that does not satisfy conditions (1.33) and (1.34), where $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_d\}$. Without conditions (1.33) and (1.34), $\mathbb{M}_\Gamma$ cannot be arranged in a d-dimensional lattice. Consider, then, the prime factors of γ_j, $1 \leq j \leq d$. Indeed, let C^* be the least common multiple of $\gamma_1, \gamma_2, \ldots, \gamma_d$. Denote by

$$\Gamma^* = \{p_1, p_2, \ldots, p_Q\} \tag{2.34}$$

the set of prime factors of C^* with $p_1 < p_2 < \cdots < p_Q$, $Q \geq 1$. Clearly, Γ^* satisfies both (1.33) and (1.34), and then $\mathbb{M}_{\Gamma^*}$ can be arranged in a Q-dimensional lattice.

Furthermore, $\mathbb{X}^0_\Gamma$ can be studied by using $\mathbb{M}_{\Gamma^*}$ and $\mathcal{I}_{\Gamma^*}$. Denote

$$\mathbb{M}_{\Gamma^*} \equiv \{p_1^{m_1} p_2^{m_2} \cdots p_Q^{m_Q} \mid m_j \geq 0, 1 \leq j \leq Q\} = \{q^*_k\}_{k=1}^{\infty}, \tag{2.35}$$

with $q^*_k < q^*_j$ if $k < j$. The complementary index set $\mathcal{I}_{\Gamma^*}$ of $\mathbb{M}_{\Gamma^*}$ is defined by

$$\mathcal{I}_{\Gamma^*} = \{n \in \mathbb{N} \mid p_j \nmid n, 1 \leq j \leq Q\}. \tag{2.36}$$

For $k \geq 1$, let the k-cell lattice L^*_k of $\mathbb{X}^0_\Gamma$ be

$$L^*_k = \{(i_1, i_2, \ldots, i_Q) \in \mathbb{Z}^Q \mid p_1^{i_1} p_2^{i_2} \cdots p_Q^{i_Q} \leq q^*_k \text{ for } i_q \geq 0, 1 \leq q \leq Q\}. \tag{2.37}$$

Now the constraint

$$x_k x_{\gamma_1 k} x_{\gamma_2 k} \cdots x_{\gamma_d k} = 0 \tag{2.38}$$

can be expressed in terms of Γ^*. Indeed, define the basic lattice $\mathbb{L}_\Gamma$ of $\mathbb{X}^0_\Gamma$ by

$$\mathbb{L}_\Gamma = \{(i_1, i_2, \ldots, i_Q) \in \mathbb{Z}^Q \mid p_1^{i_1} p_2^{i_2} \cdots p_Q^{i_Q} \in \{1, \gamma_1, \gamma_2, \ldots, \gamma_d\}\}. \tag{2.39}$$

Then the forbidden set $\mathcal{F}_\Gamma$ is given by

$$\mathcal{F}_\Gamma = \{U = (u_{i_1,i_2,\ldots,i_Q}) \in \{0, 1\}^{\mathbb{L}_\Gamma} \mid u_{i_1,i_2,\ldots,i_Q} = 1 \text{ for all } (i_1, i_2, \ldots, i_Q) \in \mathbb{L}_\Gamma\}.$$

Therefore, the basic set of admissible patterns can be written

$$\mathcal{B}_\Gamma = \{0, 1\}^{\mathbb{L}_\Gamma} \setminus \mathcal{F}_\Gamma. \tag{2.40}$$

Notably, $\mathcal{B}_\Gamma$ induces a Q-dimensional shift of finite type $\Sigma(\mathcal{B}_\Gamma)$.

Let $\Sigma_k(\mathcal{B}_\Gamma)$ be the set of all admissible patterns that can be determined by $\mathcal{B}_\Gamma$ on L_k^*, $k \geq 1$. In the following, Theorem 1.3 is generalized for $\mathbb{X}_\Gamma^0$ without conditions (1.33) and (1.34).

THEOREM 2.4. *Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_d\}$. Then the spatial entropy of $\mathbb{X}_\Gamma^0$ is given by*

$$h(\mathbb{X}_\Gamma^0) = \sum_{k=1}^{\infty} \beta_{\Gamma^*}\left(\frac{1}{q_k^*} - \frac{1}{q_{k+1}^*}\right) \log |\Sigma_k(\mathcal{B}_\Gamma)|, \tag{2.41}$$

where

$$\beta_{\Gamma^*} = \frac{\sharp(\mathcal{I}_{\Gamma^*} \cap [1, p_1 p_2 \cdots p_Q])}{p_1 p_2 \cdots p_Q}, \tag{2.42}$$

and Γ^*, $\mathbb{M}_{\Gamma^*}$ and $\mathcal{I}_{\Gamma^*}$ are given by (2.34), (2.35) and (2.36), respectively.

Proof. First, from the construction of (2.34)–(2.36), it is clear that

$$\mathbb{N} = \bigcup_{i \in \mathcal{I}_{\Gamma^*}} i\mathbb{M}_{\Gamma^*}$$

and for $i, j \in \mathcal{I}_{\Gamma^*}$ with $i \neq j$,

$$i\mathbb{M}_{\Gamma^*} \cap j\mathbb{M}_{\Gamma^*} = \emptyset.$$

It is easy to see that $\gamma_q \in \mathbb{M}_{\Gamma^*}$ for $1 \leq q \leq d$. Moreover, if $n \in i\mathbb{M}_{\Gamma^*}$ for some $i \in \mathcal{I}_{\Gamma^*}$, then $\gamma_q n \in i\mathbb{M}_{\Gamma^*}$ for all $1 \leq q \leq d$. Hence, from constraint (2.38), the admissible patterns on $i_1\mathbb{M}_{\Gamma^*}$ and on $i_2\mathbb{M}_{\Gamma^*}$ are mutually independent for $i_1 \neq i_2 \in \mathcal{I}_{\Gamma^*}$.

As in the proof of Theorem 1.3, we can define $J_{\Gamma^*;n}$, $q_{\Gamma^*;n}(i)$, $I_{\Gamma^*}(k; n)$ and $\alpha_{\Gamma^*}(k; n)$. It can be proven that

$$I_{\Gamma^*}(k; n) = \left(\frac{n}{q_{k+1}^*}, \frac{n}{q_k^*}\right] \bigcap \mathcal{I}_{\Gamma^*}$$

and

$$\lim_{n\to\infty} \frac{\alpha_\Gamma^*(k; n)}{n} = \beta_{\Gamma^*}\left(\frac{1}{q_k^*} - \frac{1}{q_{k+1}^*}\right).$$

Next, theconstraint (2.38) and the construction of $\mathcal{B}_\Gamma$ imply that the admissible patterns on $i\mathbb{M}_{\Gamma^*}$, $i \in \mathcal{I}_{\Gamma^*}$, are completely determined by $\mathcal{B}_\Gamma$. Hence,

$$|X_{\Gamma;n}| = \prod_{k=1}^{n} |\Sigma_k(\mathcal{B}_\Gamma)|^{\alpha_{\Gamma^*}(k;n)},$$

where $\Sigma_k(\mathcal{B}_\Gamma)$ is the set of all admissible patterns determined by $\mathcal{B}_\Gamma$ on L_k^*. Therefore, equation (2.41) follows. The proof is complete. □

The following example illustrates Theorem 2.4.

Example 2.5. Consider $\mathbb{X}^0_{2,8}$. It is easy to see that $\Gamma^* = \{2\}$. From (2.39), the basic lattice $\mathbb{L}_{2,8} = \{0, 1, 3\} = \square\square\cdots\square$, where the third cell is deleted. The forbidden set $\mathcal{F}_{2,8}$ is $\{\boxed{1}\boxed{1}\cdots\boxed{1}\}$ and $\mathcal{B}_{2,8} = \{0, 1\}^{\mathbb{L}_{2,8}} \setminus \mathcal{F}_{2,8}$. Define the associated transition matrix

$$A(\mathcal{B}_{2,8}) = \begin{array}{c} \\ \boxed{0|0|0} \\ \boxed{0|0|1} \\ \boxed{0|1|0} \\ \boxed{0|1|1} \\ \boxed{1|0|0} \\ \boxed{1|0|1} \\ \boxed{1|1|0} \\ \boxed{1|1|1} \end{array} \begin{array}{c} \begin{array}{cccccccc} \boxed{0|0|0} & \boxed{0|0|1} & \boxed{0|1|0} & \boxed{0|1|1} & \boxed{1|0|0} & \boxed{1|0|1} & \boxed{1|1|0} & \boxed{1|1|1} \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

Then

$$h(\mathbb{X}^0_{2,8}) = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \log |\Sigma_k(\mathcal{B}_{2,8})|,$$

where $|\Sigma_m(\mathcal{B}_{2,8})| = 2^m$ and $|\Sigma_n(\mathcal{B}_{2,8})| = |A(2, 8)^{n-3}|$ for $1 \leq m \leq 3$ and $n \geq 4$.

In the remainder of this section, constraint (2.38) is further relaxed. Therefore, we can study more general cases than $\mathbb{X}^0_\Gamma$. For simplicity, only Γ satisfying conditions (1.33) and (1.34) is studied.

For any $N \geq 2$, consider a multiplicative system of N symbols, $\{0, 1, 2, \ldots, N-1\}$. For any $d \geq 1$, let the constraint set $\mathcal{C}$ be a subset of $\{0, 1, \ldots, (N-1)^d\}$. Denote by $\mathbb{X}_\Gamma(N, \mathcal{C})$ the multiplicative integer system with constraint set $\mathcal{C}$:

$$\mathbb{X}_\Gamma(N, \mathcal{C}) = \{(x_1, x_2, \ldots) \in \{0, 1, \ldots, N-1\}^{\mathbb{N}} \mid x_k x_{\gamma_1 k} \cdots x_{\gamma_d k} \in \mathcal{C} \text{ for } k \geq 1\}. \quad (2.43)$$

Then the basic set $\mathcal{B}_\Gamma(N, \mathcal{C})$ of admissible patterns on $\mathbb{L}_\Gamma$ is given by

$$\mathcal{B}_\Gamma(N, \mathcal{C}) = \left\{ U = (u_{i_1,i_2,\ldots,i_d}) \in \{0, 1, \ldots, N-1\}^{\mathbb{L}_\Gamma} \,\middle|\, \prod_{(i_1,i_2,\ldots,i_d)\in\mathbb{L}_\Gamma} u_{i_1,i_2,\ldots,i_d} \in \mathcal{C} \right\}. \quad (2.44)$$

The following theorem can be proven similarly Theorem 1.3.

THEOREM 2.6. *Let* $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_d\}$ *satisfy (1.33) and (1.34) and* $\mathcal{C} \subseteq \{0, 1, \ldots, (N-1)^d\}$. *The spatial entropy of* $\mathbb{X}_\Gamma(N, \mathcal{C})$ *is given by*

$$h(\mathbb{X}_\Gamma(N, \mathcal{C})) = \sum_{k=1}^{\infty} \beta_\Gamma \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log |\Sigma_k(\mathcal{B}_\Gamma(N, \mathcal{C}))|, \quad (2.45)$$

where $\Sigma_k(\mathcal{B}_\Gamma(N, \mathcal{C}))$ *is the set of* d*-dimensional admissible local patterns that can be generated by* $\mathcal{B}_\Gamma(N, \mathcal{C})$ *on* L_k.

Proof. The only difference between $\mathbb{X}^0_\Gamma$ and $\mathbb{X}_\Gamma(N, \mathcal{C})$ is their constraints. By (2.43), it is easy to see that the basic set $\mathcal{B}_\Gamma(N, \mathcal{C})$ can completely determine the patterns on $i\mathbb{M}_\Gamma$ for $i \in \mathcal{I}_\Gamma$. Therefore, the result follows. $\square$

The following example illustrates Theorem 2.6.

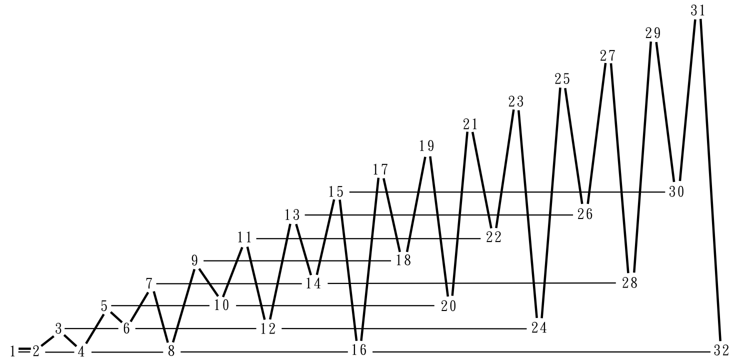


FIGURE 5. Representation for $\mathbb{X}_2^G$.

Example 2.7. Let $N = 3$ and $\mathcal{C} = \{0, 2\}$. Then

$$\mathbb{X}_2(3, \mathcal{C}) = \{(x_1, x_2, x_3, \ldots) \in \{0, 1, 2\}^{\mathbb{N}} \mid x_k x_{2k} \in \{0, 2\} \text{ for all } k \geq 1\}.$$

The basic set of admissible local patterns is now given by

$$\mathcal{B}_2(3, \mathcal{C}) = \{\boxed{0}\boxed{0},\ \boxed{0}\boxed{1},\ \boxed{0}\boxed{2},\ \boxed{1}\boxed{0},\ \boxed{1}\boxed{2},\ \boxed{2}\boxed{0},\ \boxed{2}\boxed{1}\}.$$

The associated transition matrix is

$$A(2; 3, \mathcal{C}) = \begin{array}{c} \\ \boxed{0} \\ \boxed{1} \\ \boxed{2} \end{array} \begin{array}{c} \begin{array}{ccc} \boxed{0} & \boxed{1} & \boxed{2} \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right] \end{array}.$$

Therefore, as in Theorem 1.2,

$$h(\mathbb{X}_2(3, \mathcal{C})) = 4 \sum_{k=1}^{\infty} \frac{1}{3^{k+1}} \log a_k(2; 3, \mathcal{C}),$$

where $a_1(2; 3, \mathcal{C}) = 3$ and $a_k(2; 3, \mathcal{C}) = |A(2; 3, \mathcal{C})^{k-1}|$ for all $k \geq 2$.

3. *One-dimensional coupled systems*

This section investigates the one-dimensional coupled system which is an intersection of the multiplicative integer system $\mathbb{X}_Q^0$, $Q \geq 2$, and the golden mean shift Σ_G, that is,

$$\begin{aligned} \mathbb{X}_Q^G &\equiv \mathbb{X}_Q^0 \cap \Sigma_G \\ &= \{(x_1, x_2, x_3, \ldots) \in \{0, 1\}^{\mathbb{N}} \mid x_k x_{Qk} = 0 \text{ and } x_k x_{k+1} = 0 \text{ for all } k \geq 1\}. \end{aligned} \quad (3.1)$$

To incorporate the effect of Σ_G in $\mathbb{X}_Q^G$, Table 1 for $\mathbb{X}_2^0$ is replaced by Figure 5 for $\mathbb{X}_2^G$. As in Table 1, the horizontal lines in Figure 5 connect the integers in $i\mathbb{M}_2$ for each $i \in \mathcal{I}_2$; the effect comes from $\mathbb{X}_2^0$. On the other hand, the bold zigzag line in Figure 5 connecting all natural integers comes from Σ_G. Therefore, for any $i \neq j$ in $\mathcal{I}_2$, $i\mathbb{M}_2$ and $j\mathbb{M}_2$ are no longer mutually independent. In fact, they are all coupled through the relation set $\mathbb{M}_2$. Therefore, (3.1) is regarded as a coupled system.

In this section, the main steps for the study of the spatial entropy of $\mathbb{X}_Q^G$ are as follows.

(I)$_c$ Identify the lattice $M_{Q;k}$ (and $L_{Q;k}$), which is the maximal connected graph of degree k.

(II)$_c$ In the region $[1, m] \cap \mathbb{N}$, find the almost maximal number of disjoint copies of mutually independent $M_{Q;k}$, and compute the number of unused vertices.

(III)$_c$ Compute the number $\Sigma_{Q,G;k}$ of admissible patterns on L_k.

Step (I)$_c$ gives the admissible lattices of $h(\mathbb{X}_Q^G)$, and (II)$_c$ gives lower and upper bounds of $h(\mathbb{X}_Q^G)$. Then $h(\mathbb{X}_Q^G)$ follows if the error term in (II)$_c$ approaches zero as n tends to infinity. To minimize the error in (II)$_c$, the amount of unused lattices should be as small as possible. Therefore, the choice of $M_{Q;k}$ or the graph $L_{Q;k}$ in (I)$_c$ should be as large as possible as far as they are decoupled.

Before the system $\mathbb{X}_Q^G$ is decoupled, the following definition is needed.

Definition 3.1. Two sets of integers of M and M' are mutually independent in $\mathbb{X}_Q^A$ if

$$M \cap M' = \emptyset \tag{3.2}$$

and any numbers m in M and m' in M' are not consecutive and also not consecutive as powers of Q, that is, if $m = Q^n$ for some n then $m' \neq Q^{n+1}$ and Q^{n-1}.

Then the following lemma can be obtained.

LEMMA 3.2. *Suppose M and M' are mutually independent in $\mathbb{X}_Q^G$. Then*

$$|\Sigma(M \cup M')| = |\Sigma(M)||\Sigma(M')|, \tag{3.3}$$

where $\Sigma(M)$ is the set of all admissible patterns on lattice M, and $\Sigma(M')$ and $\Sigma(M \cup M')$ are defined analogously.

Proof. Since M and M' are decoupled in $\mathbb{X}_Q^G$, the patterns in $\Sigma(M)$ are independent of the patterns in $\Sigma(M')$. Therefore, the result holds. □

In step (I)$_c$, as in the decoupled system $\mathbb{X}_Q^0$, the admissible numbered lattice $M_{Q;k}$ is first picked up. First, $\mathbb{X}_2^G$ is considered. In Figure 6, some $M_{2;k}(l)$ are drawn for $1 \leq k \leq 4$.

The choice of $M_{2;k}$ is recursive and robotic. The basic idea is that any number can produce the next generation through $\mathbb{X}_2$ or Σ_G. More precisely, for each number n, if $n \in \mathcal{I}_2$, then n can produce the next generation $2n \in n\mathbb{M}_2$. If $n \notin \mathcal{I}_2$ with $n = i2^m$, $m \geq 1$, then n can produce $2n = i2^{m+1} \in i\mathbb{M}_2$ through $\mathbb{X}_2$ and $n \pm 1 = i2^m \pm 1 \in \mathcal{I}_2$ through Σ_G. In summary, a complete production cycle is as follows. If $n \in \mathcal{I}_2$, then n produces $2n$ and then $2n \pm 1$. If $n = i2^m$, $m \geq 1$, then n produces $i2^{m+1}$ and then $i2^{m+1} \pm 1$.

For example, $M_{2;1}(3)$ has one cell, and the number 3 is regarded as the first-generation number. $M_{2;2}(3)$ is constructed from $M_{2;1}(3)$ by producing number 6 from 3 through $3\mathbb{M}_2$. Immediately, 6 creates numbers 5 and 7 as descendants through Σ_A. $M_{2;2}(3)$ is of degree 2 since there are two numbers {3, 6} on the horizontal line.

The construction of $M_{2;3}$ from $M_{2;2}$ is performed similarly: the number 6 yields number 12 in $3\mathbb{M}_2$. At the same time, the numbers 5 and 7 yield the numbers 10 and 14 in $5\mathbb{M}_2$ and $7\mathbb{M}_2$, respectively. Next, the numbers 10, 14 and 12 yield their descendants 9, 11, 13, 15 and 11, 13 in $\mathcal{I}_2$ through Σ_G, as presented in Figure 6(e). Now, the three numbers 3,

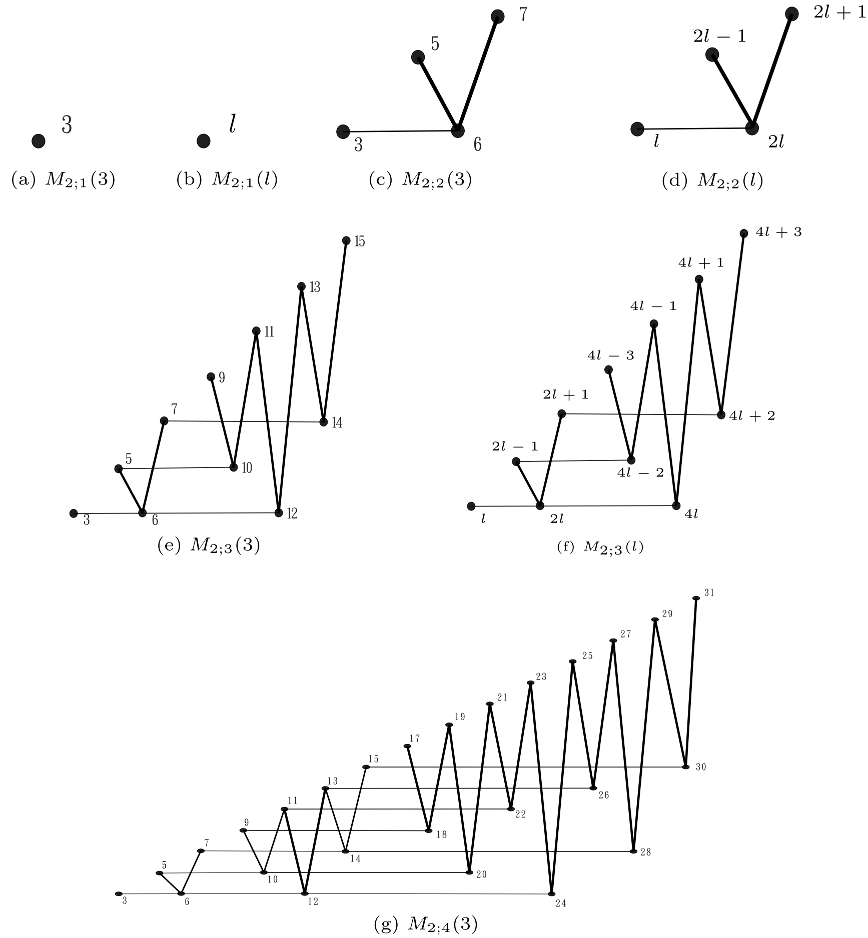


FIGURE 6. Admissible numbered lattices $M_{2;k}(l)$.

6 and 12 are in the lowest horizontal direction, and $M_{2;3}(3)$ is therefore of degree 3. On $M_{2;k}(i)$, the maximal number of cells in the horizontal direction is k, and $M_{2;k}(i)$ is of degree k.

Now, for general $\mathbb{X}_Q^G$, $M_{Q;k}(l)$ can be defined formally as follows.

Definition 3.3. For each $l \in \{1 + jQ \mid j \geq 1\} \subset \mathcal{I}_Q$, let $V_{Q;1}(l) = \{l, l+1, \ldots, l+Q-2\}$, and for $k \geq 1$, define

$$V_{Q;k+1}(l) = \{Qi,\ Qi \pm 1,\ Qi \pm 2,\ \ldots,\ Qi \pm (Q-1) \mid i \in V_{Q;k}(l)\}.$$

Then, let $M_{Q;1}(l) = V_{Q;1}(l)$ and for $k \geq 2$, define

$$M_{Q;k+1}(l) \equiv M_{Q;k}(l) \cup V_{Q;k+1}(l).$$

Moreover, the blank lattice $L_{Q;k}$ of degree k is defined by deleting the numbers of $M_{Q;k}(l)$.

Notably, $M_{Q;k}(l)$ and $M_{Q;k}(l')$ are mutually independent when l is not in $M_{Q;k}(l')$. See Figure 7 for $M_{3;2}(4)$, $M_{3;2}(7)$ and $M_{3;3}(4)$.

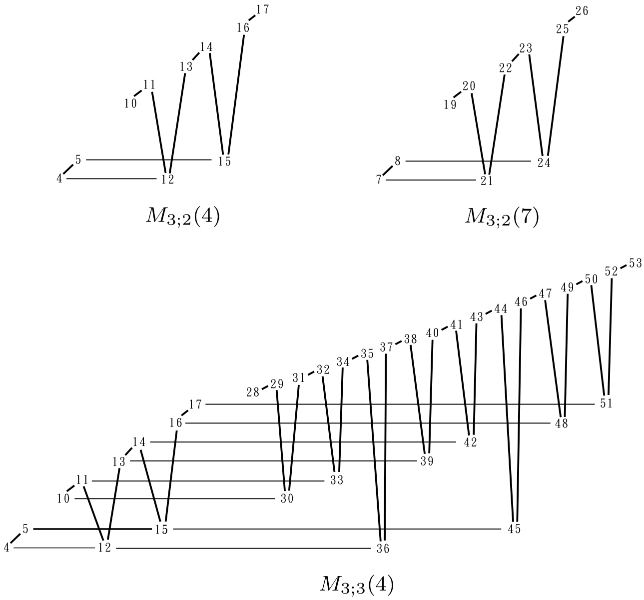


FIGURE 7. Admissible numbered lattices $M_{3;k}(l)$.

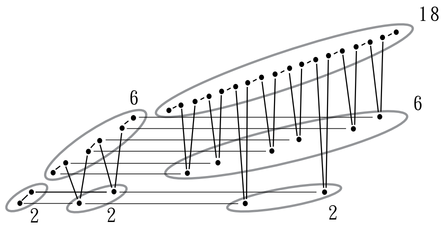


FIGURE 8. Illustration for computing $L_{3;3}$.

The following lemma gives the number of vertices of $L_{Q;k}$.

LEMMA 3.4. *For any $Q \geq 2$ and $k \geq 2$, the number $|L_{Q;k}|$ of vertices of $L_{Q;k}$ is*

$$|L_{Q;k}| = \frac{Q(Q^k - 1)}{Q - 1} - k. \tag{3.4}$$

Proof. First, equation (3.4) is proven for the case $Q = 3$. The other cases can be treated analogously.

For $Q = 3$, let

$$a_{3,n} = 2 \cdot 3^n.$$

The blank lattice $L_{3;2}$ can be obtained from $M_{3;2}(4)$ in Figure 7, and

$$|L_{3;2}| = a_{3,1} + (a_{3,1} + a_{3,2}) = 10.$$

Now $L_{3;3}$ is obtained from $M_{3;3}(4)$ in Figure 7 and can be grouped as shown in Figure 8.

In Figure 8, it is easy to see that the three numbers in the bottom layer are equal to 2, the two numbers in the middle layer are equal to $2 \cdot 3$ and the number in the top layer is

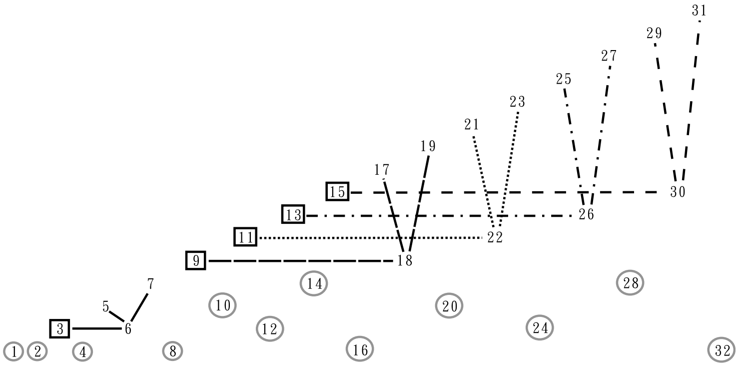


FIGURE 9. Mutually independent $M_{2;2}(l)$ in $\mathcal{N}(32)$.

equal to $2 \cdot 3^2$. Then,

$$\begin{aligned}|L_{3;3}| &= \sum_{l=1}^{3}\sum_{j=1}^{l} a_{3,j} \\ &= a_{3,1} + (a_{3,1} + a_{3,2}) + (a_{3,1} + a_{3,2} + a_{3,3}) \\ &= |L_{3;2}| + 26.\end{aligned}$$

By induction, it can be proven that

$$|L_{3;k}| = |L_{3;k-1}| + 3^k - 1 \tag{3.5}$$

for all $k \geq 2$. Therefore, equation (3.4) follows for $Q = 3$.

Now, for general $Q \geq 2$,

$$a_{Q,n} = (Q-1)Q^{n-1}.$$

Similarly, it can be proven that

$$|L_{Q;k}| = |L_{Q;k-1}| + Q^k - 1, \tag{3.6}$$

for all $k \geq 2$. Therefore, (3.4) holds. The proof is complete. □

Having identified the lattices $M_{Q;k}$ and $L_{Q;k}$, in a given range

$$\mathcal{N}(m) \equiv \{j \mid 1 \leq j \leq m\},$$

step (II) is then to be carried out, that is, the number of disjoint copies of mutually independent $M_{Q;k}(l) \subset \mathcal{N}(m)$ with $l \in \{1 + jQ \mid j \geq 1\} \subset \mathcal{I}_Q$ is computed.

For example, for $\mathbb{X}_2^G$, in Figure 9, $\mathcal{N}(32)$ can be decoupled by $M_{2;2}(3)$, $M_{2;2}(9)$, $M_{2;2}(11)$, $M_{2;2}(13)$, $M_{2;2}(15)$ and the numbers in {1, 2, 4, 8, 10, 12, 14, 16, 20, 24, 28, 32} are not used. There is one copy $M_{2;2}(3)$ in $\mathcal{N}(2^3) \setminus \mathcal{N}(2)$ and four copies $M_{2;2}(9)$, $M_{2;2}(11)$, $M_{2;2}(13)$, $M_{2;2}(15)$ in $\mathcal{N}(2^5) \setminus \mathcal{N}(2^3)$; see Figure 9.

Based on the above idea, we compute the number of disjoint copies of mutually independent $M_{Q;k}(l) \subset \mathcal{N}(m)$ as follows. Clearly, for $m \geq 1$, the $\mathcal{N}(m)$ can be decomposed as

$$\mathcal{N}(m) \setminus \{1\} = \bigcup_{j=1}^{J} \mathcal{N}_{Q,k}(m, j)$$

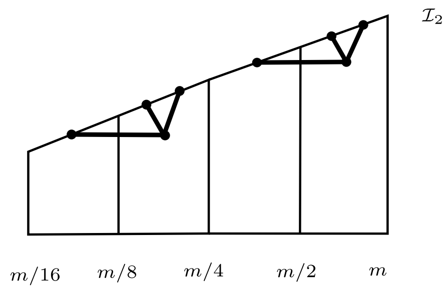


FIGURE 10. Illustration for computing $\alpha_{2,2}(2,1)$ and $\alpha_{2,2}(2,2)$.

where

$$\mathcal{N}_{Q,k}(m,j) = (m/Q^{kj}, m/Q^{k(j-1)}] \cap \mathbb{N} \tag{3.7}$$

and

$$J = \max\left\{ j \geq 1 \;\middle|\; \frac{m}{Q^j} > 1 \right\}. \tag{3.8}$$

It is easy to see that

$$\mathcal{N}_{Q,k}(m,j) = \bigcup_{l=0}^{k-1} (m/Q^{jk-l}, m/Q^{jk-l-1}] \cap \mathbb{N}. \tag{3.9}$$

For $1 \leq j \leq J$, let

$$\mathcal{N}^*_{Q,k}(m,j) = (m/Q^{jk}, m/Q^{jk-1}] \cap \{1 + pQ \mid p \geq 1\}. \tag{3.10}$$

It can easily be proved that if $l \in \mathcal{N}^*_{Q,k}(m,j)$, then $M_{Q;k}(l) \subset \mathcal{N}_{Q,k}(m,j)$. Clearly, for $l, l' \in \mathcal{N}^*_{Q,k}(m,j)$ with $l \neq l'$, $M_{Q;k}(l)$ and $M_{Q;k}(l')$ are mutually independent.

Therefore, there are

$$\alpha_{Q,k}(m,j) \equiv \sharp(\mathcal{N}^*_{Q,k}(m,j)) \tag{3.11}$$

disjoint copies of mutually independent $M_{Q;k}(l) \subset \mathcal{N}_{Q,k}(m,j)$. Moreover, there are

$$\alpha_{Q,k}(m) \equiv \sum_{j=1}^{J} \alpha_{Q,k}(m,j) \tag{3.12}$$

disjoint copies of mutually independent $M_{Q;k}(l)$ in $\mathcal{N}(m)$.

For example, consider $\mathbb{X}^G_2$. There are

$$\alpha_{2,2}(2,1) = \sharp((m/4, m/2] \cap \{1 + 2p \mid p \geq 1\})$$

disjoint copies of mutually independent $M_{2;2}(l)$ in $\mathcal{N}_{2,2}(m,1) = (m/4, m] \cap \mathbb{N}$, and there are

$$\alpha_{2,2}(2,2) = \sharp((m/16, m/8] \cap \{1 + 2p \mid p \geq 1\})$$

disjoint copies of mutually independent $M_{2;2}(l)$ in $\mathcal{N}_{2,2}(m,2) = (m/16, m/4] \cap \mathbb{N}$; see Figure 10.

For step (II), the following lemma shows the limiting density of $\alpha_{Q,k}(m)$, the number of disjoint copies of mutually independent $M_{Q;k}(l)$ in $\mathcal{N}(m)$, as m tends to infinity.

LEMMA 3.5. *If $Q \geq 2$ and $k \geq 1$, then*

$$\lim_{m\to\infty} \frac{\alpha_{Q,k}(m)}{m} = \frac{Q-1}{Q(Q^k-1)}. \tag{3.13}$$

Furthermore, let

$$\beta_{Q,k}(m) \equiv m - \alpha_{Q,k}(m) \cdot |L_{Q;k}| \tag{3.14}$$

be the number of vertices that are not used in choosing the $\alpha_{Q,k}(m)$ disjoint copies of mutually independent $M_{Q;k}(l)$ in $\mathcal{N}(m)$. Then

$$\lim_{m\to\infty} \frac{\beta_{Q,k}(m)}{m} = \frac{(Q-1)k}{Q(Q^k-1)}. \tag{3.15}$$

Proof. From (3.10)–(3.12),

$$\begin{aligned}
\lim_{m\to\infty} \frac{\alpha_{Q,k}(m)}{m} &= \lim_{m\to\infty} \sum_{j=1}^{J} \alpha_{Q,k}(m, j)/m \\
&= \lim_{m\to\infty} \left[\sum_{j=1}^{J} \left(\frac{m}{Q^{jk-1}} - \frac{m}{Q^{jk}} \right) \frac{1}{Q} \right] \Big/ m \\
&= \sum_{j=1}^{\infty} \left(\frac{1}{Q^{jk-1}} - \frac{1}{Q^{jk}} \right) \frac{1}{Q} \\
&= \frac{Q-1}{Q(Q^k-1)}.
\end{aligned}$$

Next, by Lemma 3.4 and (3.14),

$$\begin{aligned}
\lim_{m\to\infty} \frac{\beta_{Q,k}(m)}{m} &= \lim_{m\to\infty} 1 - \frac{\alpha_{Q,k}(m)}{m} |L_{Q;k}| \\
&= 1 - \frac{Q-1}{Q(Q^k-1)} \left(\frac{Q(Q^k-1)}{Q-1} - k \right) \\
&= \frac{(Q-1)k}{Q(Q^k-1)}.
\end{aligned}$$

The proof is complete. □

Finally, for step (III), denote by $\Sigma_{Q,G;k}$ the admissible patterns of $\mathbb{X}_Q^G$ on $L_{Q;k}$. In the following, we will now prove Theorem 1.4 which says that $\mathbb{X}_Q^G$ has two sequences of lower and upper bounds that can approach the spatial entropy $h(\mathbb{X}_Q^G)$. Indeed,

$$\frac{Q-1}{Q(Q^k-1)} \log |\Sigma_{Q,G;k}| \leq h(\mathbb{X}_Q^G) \leq \frac{Q-1}{Q(Q^k-1)} (\log |\Sigma_{Q,G;k}| + k \log 2), \tag{3.16}$$

and

$$h(\mathbb{X}_Q^G) = \lim_{k\to\infty} \frac{Q-1}{Q(Q^k-1)} \log |\Sigma_{Q,G;k}|. \tag{3.17}$$

Proof of Theorem 1.4. First, from the above procedure to find $\alpha_{Q,k}(m)$ copies $M_{Q;k}(l)$ in $\mathcal{N}(m)$, all $M_{Q;k}(l)$ are mutually independent. In $\mathcal{N}(m)$, since 0 is a safe symbol of $\mathbb{X}_Q^G$, we

TABLE 8. Lower and upper bounds for $\mathbb{X}_2^G$.

n	2	3	4
$\lvert\Sigma_{2,G;n}\rvert$	9	237	213 624
$h^{(n)}(\mathbb{X}_2^G)$	0.366204	0.390576	0.409066
$\bar{h}^{(n)}(\mathbb{X}_2^G)$	0.597253	0.539107	0.501485

put 0 on the vertices that are not used in the procedure. Then the lower bound of $|X_{Q,G;m}|$ can be obtained:

$$|X_{Q,G;m}| \geq |\Sigma_{Q,G;m}|^{\alpha_{Q,k}(m)}. \tag{3.18}$$

For the upper bound of $|X_{Q,G;m}|$, since two symbols 0 and 1 may appear on the vertices that are not used in the procedure, it is clear that

$$|X_{Q,G;m}| \leq |\Sigma_{Q,G;m}|^{\alpha_{Q,k}(m)} \cdot 2^{\beta_{Q,k}(m)}. \tag{3.19}$$

Therefore, by Lemma 3.5, equation (3.16) holds. Furthermore, from (3.16), (3.17) follows immediately. The proof is complete. □

Example 3.6. Consider the one-dimensional couple system $\mathbb{X}_2^G \equiv \mathbb{X}_2^0 \bigcap \Sigma_G$. Table 8 presents a numerical approximation of (3.16). For $n \geq 1$, let

$$h^{(n)}(\mathbb{X}_2^G) = \frac{1}{2(2^n-1)} \log |\Sigma_{2,G;n}| \quad \text{and} \quad \bar{h}^{(n)}(\mathbb{X}_2^G) = h^{(n)}(\mathbb{X}_2^G) + \frac{n}{2(2^n-1)} \log 2.$$

Then it is clear that

$$h^{(n)}(\mathbb{X}_2^G) < h(\mathbb{X}_2^G) < \bar{h}^{(n)}(\mathbb{X}_2^G).$$

Remark 3.7. In studying the one-dimensional coupled system $\mathbb{X}_Q^G$, for each $k \geq 1$, we split the natural numbers $\mathbb{N}$ into two parts,

$$\mathbb{N} = U_k \cup W_k, \tag{3.20}$$

where U_k is used to select the mutually independent admissible numbered lattice $M_{Q;k}(l)$ for approximating the entropy, and W_k is the set of cells removed from $\mathbb{N}$ to achieve independence of $M_{Q;k}(l)$ in U_k. A good splitting requires

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{|W_k \cap [1,n]|}{|U_k \cap [1,n]|} = 0; \tag{3.21}$$

see Lemma 3.5 and Theorem 1.4.

However, for the two-dimensional coupled system $\mathbb{X}_{2,3}^G$, this paper does not find any means to split $\mathbb{N}$ into two parts such that (3.21) holds. The connection between $\mathbb{X}_{2,3}^0$ and Σ_G is quite complicated. A better understanding of the connection is required before dealing with $\mathbb{X}_{2,3}^G$.

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