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Industry equilibrium with random exit or default

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Abstract

An industry consisting of a large number of small price taking firms subject to idiosyncratic productivity shocks is considered. At the moment of entry, a firm takes on debt. We show that in a competitive equilibrium, some firms exit and pay out their debt while others choose to default. The outcome depends on the realization of firm-specific shocks. The paper demonstrates that if the firms self-select between exit with debt repayment and default, then the default region is disconnected from the exit region. The methodological contribution of the paper is the analytical characterization of the long-run equilibrium for two scenarios of the initial distribution of productivity shocks. We consider two public policy mechanisms—contract enforcement and creditor protection. Our policy recommendation is that regulators need to reduce the contract enforcement if they want to decrease the long-run default rate.

KEYWORDS

endogenous default, endogenous exit, heterogeneous firms, industry equilibrium

JEL CLASSIFICATION

C61; D81; G31; H32

1 | INTRODUCTION

1.1 | Motivation

Small businesses are an important part of the economy of virtually every nation. The importance of small firms' growth for economic development has been recognized in policy-oriented literature (see, e.g., Beck, Demirguc-Kunt, Laeven, & Levine, 2008). According to the US Small Business Administration, small firms produce more than half of US nonfarm output, employ 50% of workers, and pay 45% of total private payroll. Small firms represent 99.7% of all employer firms. Small firms are an important source of income for the poor in developing countries. Ninety nine percent of the firms in many poor countries have 10 or fewer workers (see, McKenzie, 2017). Many micro enterprises in developing countries have high returns to capital, but also face risky revenue streams (see de Mel, McKenzie, & Woodruff, 2019). In principle, equity offers several advantages over debt when financing investments of this nature, but the use of equity in practice has been largely limited to investments in much larger firms. Although there is a lot of optimism about the power of finance for small-scale business development, a growing literature (see, e.g., de Mel et al., 2019; Inekwe, 2019; McKenzie & Paffhausen, 2018) shows that success cannot be taken for granted and may critically depend both on the entrepreneur's personal characteristics and on institutions.

The goal of this paper is to understand how small firms' entry and exit decisions are affected by industry-wide and firm-specific exogenous parameters and procedures and to provide policy recommendations. We leave for the future analysis of the optimal business tax policy as in Becker and Schneider (2017).

Small firms rely heavily on external finance. Picard and Rusli (2018) emphasize the importance of private debt financing in reducing government transfers and information costs. Small firms have historically faced significant difficulties in accessing funding for positive net present value (NPV) projects due to lack of credible information about them by potential providers of funds. Maybe due to creditors' cautiousness, bankruptcy rate (at least in the United States) is low: The average annual default rate on small business administration (SBA) loans is 3.5% according to Glennon and Nigro (2005). Hillegeist, Keating, Cram, and Lundstedt (2004) document that the average annual bankruptcy rate for US firms was 1% from 1980 to 2000. Mester (1997) estimates the annual default rate for business loans as ranging between 1% and 3%. Herranz, Krasa, and Villamil (2015) use the Survey of Small Business Finances administered by The Board of Governors of the Federal Reserve System and the US Small Business Administration, data and find a large percentage of entrepreneurs who inject personal funds to keep their firms alive. This may seem puzzling because incorporated firms are protected by limited liability in case of bankruptcy. Negative equity and low default rates indicate that bankruptcy is a strategic decision.

In the United States, businesses can file Chapter 7 or 11 for bankruptcy. Chapter 11 is designed for large firms, or corporations negotiating debt restructuring while continuing their operations. Chapter 7 is primarily intended as a bankruptcy procedure for consumers, but it is also de facto a bankruptcy procedure for small firms. After filing for Chapter 7, an entrepreneur "gets a fresh start" in the sense that debts are discharged and all future earnings are exempt from debt payments.

Death or exit rates of small firms are fairly high. McKenzie and Paffhausen (2018) find that small firms die at an average rate of 8.2% per year. Due to high exit rates of small firms, creditors face not only default risk, but also prepayment risk, that is why it is really important to understand why some small businesses partially financed by debt choose to exit and pay out their debt, while others file for bankruptcy as well as creditor protection and contract enforcement mechanisms.

1.2 | Contribution and results

The main contribution of this paper is to derive an equilibrium model for a competitive industry that, due to technological reasons, is composed of small firms and to demonstrate how small firms self-select between default and exit with debt prepayment. Our numerical results produce the exit rate at least two times higher than the default rate, which agrees with empirical findings listed above. Our model also shows that when exit and default are strategic decisions, firms are making negative profits before liquidation. Hence, the entrepreneurs have to inject their own money to cover business losses. This agrees with empirical findings of Herranz et al. (2015).

We derive analytical expressions for equilibrium output price, debt coupon rate, entry rate of new firms, and stationary distribution of active firms. We also describe analytically the following four state space regions.

1. The good luck zone (the upper tail of the initial distribution of shocks): Firms in this zone are the most productive ones. Eventually, they will exit, but they never default.
2. The exit zone (adjacent to the left boundary of the good luck zone): The entrepreneur finds out that her own prospects are not too good but the investment project has sufficiently high scrap value, so that it makes sense to sell the firm's assets and pay back the debt.
3. The default zone (the lower tail of the initial distribution of shocks): The entrepreneur observes that the future is too bleak and defaults immediately.
4. The Buridan zone (an interval that separates the default and exit zones): The entrepreneur observes that her prospects are not great, but it is optimal to remain active, so she goes on producing. If eventually the shock enters the default zone first, the firm defaults. If instead the shock enters the exit zone first, then the firm pays the debt and exits. While the entrepreneur is indecisive, her behavior resembles the behavior of the famous animal placed between two piles of hay.

The novelty of prediction of our dynamic model is that the default zone is disconnected from the exit zone, so that there are relatively unproductive firms that remain active for some time. The dynamics of the firms inside the Buridan zone suggests that the firms may leave the industry after a sequence of favorable productivity shocks. The reader must keep in mind that the initial productivity shock of a firm in the Buridan zone is rather low. Had the entrepreneur invested only her own assets in the project, she would have sold the firm immediately and exited the industry. Since the project is partially financed by debt, the entrepreneur cannot sell the firm immediately, because the value she will recover is insufficient to pay back the debt. At the same time, the entrepreneur is too good to file for bankruptcy. She can always do this later if her productivity worsens. So she needs a sequence of successful realizations of revenues to be able to sell the firm at the scrap value that will at least be high enough to pay back the debt.

Exit from the Buridan zone with debt repayment can be interpreted as an intraindustry liquidation. Fleming and Moon (1995) find that many liquidating firm assets are sold to firms operating in the same industry. They describe such voluntary liquidations as an interesting example of efficient and orderly asset reallocation. Since a voluntary liquidation is conducted at management's discretion, managers choose to liquidate when financial factors make it value-increasing for the firm. Fleming and Moon (1995) also document that liquidating firms experience positive abnormal returns in the period preceding liquidation announcement. They suggest that the market anticipates the firm's liquidation and responds to this value enhancing action.

Being able to characterize equilibrium values analytically is very important, because comparative statics analysis becomes possible. In particular, we study how equilibrium values depend on parameters that comprise contract enforcement and creditor protection mechanisms. The contract enforcement is captured in our model by a penalty for debt prepayment, and creditor protection is modeled as a fraction of the firm's assets which creditors get after bankruptcy and liquidation costs have been paid. We find that the coupon rate decreases both when contract enforcement is stronger and when the creditors can acquire a larger portion of the firm's assets in case of default.

The equilibrium price is much less sensitive to changes in the institutional parameters than the coupon rate. The price decreases when the debtor protection increases: Firms have to pay lower interest on their debt and therefore can sell their output at a lower price. The dependence of the price on the credit enforcement parameter is less evident and can be nonmonotone. We explain this nonmonotone effect as follows. Better contract enforcement reduces the coupon rates, which in turn leads to lower prices. At the same time, the option value of exit (and hence the total value of the equity of an active firm) decreases, the number of new entrants decreases as well, which makes it possible for firms to rise prices to increase the equity values. If the contract enforcement parameter is relatively small, the first effect dominates, for larger values of this parameter, the second effect dominates.[1]

Since the aggregate demand for the firms' output is decreasing in the price of output and market clearing holds, we may conclude that the aggregate output increases with the creditor protection, which agrees with Levine (1999). If we study the dependence of the average output (per entering firm) on credit protection and enforcement we observe that the average output increases in both parameters, which is also in accord with Levine (1999). A monotone dependence on the contract enforcement parameter is due to the fact that the number of entrants decreases, because entrepreneurs become more cautious when the penalty for debt prepayment is higher.

We also find that when the exit option becomes less attractive due to higher cost of debt prepayment, the exit zone shrinks, and both Buridan zone and the default zone become wider, thus the exit rate decreases more sharply than the default rate. Moreover, no matter how small the Buridan zone is, its existence is necessary for the firms to self-select between exit and default. We show that if the Buridan zone does not exist, all the firms will default eventually. Thus, we conclude that regulators have to reduce the contract enforcement if they want to decrease the long-run default rate.

One of the ways regulators could improve creditor protection is reduction of bankruptcy and liquidation costs, so that debt holders could capture a larger fraction of the firm's assets. Another way is introduction of so-called debt covenants. We leave for the future examining how equilibrium values depend of debt covenants. We believe that our dynamic model can be used as a benchmark for a model with debt covenants. Suppose that the firm is declared bankrupt if its operating profit drops below a certain level (equivalently the productivity shock drops below a certain threshold). Clearly, such debt covenant makes no sense if the threshold is in the good luck or exit zone since the firm will pay back the debt eventually, and bankruptcy procedures are costly. The threshold in the default zone is useless because the firm will default strategically earlier than the profit drops below the contracted level. Hence the contract has to specify the bankruptcy threshold which falls in the Buridan zone.

1.3 | Related literature

This paper relates to the literature of financing-production decisions and industry equilibrium models in a dynamic setting. This literature bridges the framework of dynamic contingent

[1]Krasa, Sharma, and Villamil (2008) also find nonlinear effects of credit enforcement and protection though they measure them in a different way.

claims analysis and the framework developed by Hopenhayn (1992) and Hopenhayn and Rogerson (1993) where the concept of stationary equilibrium is introduced to analyze industry dynamics. Cooley and Quadrini (2001) introduce financing decisions in a model of competitive industry based on the framework developed by Hopenhayn (1992), Hopenhayn and Rogerson (1993), and study how financial frictions can explain empirical regularities of firm dynamics. Miao (2005) presents an equilibrium model of a competitive industry where firms make financing, entry, investment, and default decisions under uncertainty created by idiosyncratic productivity shocks. In all these models with mixed financing, exit (not related to bankruptcy) happens as a result of some catastrophic event (exogenous death), so firms never consider an option to exit with debt prepayment. There are also models of competitive industry dynamics with endogenous entry and exit decisions for all equity financed firms. See, for example Leahy (1993), Caballero and Pindyck (1996), and Dixit and Pindyck (1996).

The aforementioned papers of industry dynamics use the real options approach to decision-making under uncertainty in continuous time. This approach recognizes the option value of waiting and spells out why the naive NPV rule makes wrong policy recommendations. The real options approach was introduced in the seminal paper by McDonald and Siegel (1986). Later, this approach together with a large number of extensions and applications, including industry equilibrium, were summarized by Dixit and Pindyck (1996).

The traditional literature in industrial organization develops and analyzes strategic models of industry dynamics in discrete time, with endogenous entry and exit and thus variable number of firms in oligopolistic industries (see, e.g., Amir & Lambson, 2003; Doraszelski & Satterthwaite, 2010; Ericson & Pakes, 1995). See also Doraszelski and Pakes (2007) for a survey. Unlike industry equilibrium models that admit analytical solutions as, for example, our model, computational models such as, for example, Doraszelski and Satterthwaite (2010), make it difficult (or impossible) to perform comparative statics analysis. Also, it is not clear whether Doraszelski and Satterthwaite's (2010) model remains computable if in each time period, a firm has to chose among three possible actions: continue, default, or exit with debt prepayment.

1.4 | Outline

The rest of the paper is organized as follows. In Section 2, key features of our model are presented and values of firms' assets are derived. In Section 3, we write down and solve the problem of the firm which takes the output price P and coupon rate ρ as given. We derive sufficient conditions for the existence of the Buridan zone and calculate the default and exit thresholds, which define the boundaries of the Buridan zone and good luck zone, and the value functions of an active firm in these zones.

We obtain explicit analytical formulas for the exit threshold and value function in the good luck zone, but the solution of the problem in the Buridan zone reduces to an algebraic equation for the ratio of exit and default thresholds. This equation has a unique solution. Solving the equation numerically, we derive analytical expressions for the thresholds and value function in terms of the above ratio, P, ρ, and exogenous parameters of the model.

The relationship between firm size, growth rate, and endogenous entry and exit decisions has attracted significant attention both in theoretical and empirical studies. The Gibrat law states that firm size and growth are independent. In Section 4, we discuss empirically documented departures from the Gibrat law as well as theoretical attempts to explain these departures. We show that the Gibrat law does not hold in our model. In Section 5, we define the competitive industry equilibrium. We find equilibrium output price and coupon rate from free entry conditions for entrepreneurs and lenders, and use the market clearing condition to find the equilibrium entry rate. A numerical example with comparative statics exercise is also presented in Section 5. Section 6 concludes. Technical details are presented in appendix. In appendix, we also demonstrate how to we calculate the steady-state distributions of active firms in the Buridan zone and good luck zone. In addition, we compute equilibrium entry rate and rates of exit and default using an exogenously given demand function and the market clearing condition.

2 | MODEL DESCRIPTION

2.1 | Key features of our model

Our model combines features of Dixit and Pindyck (1996) model of a competitive industry dynamics, and Miao (2005). We add two key ingredients to the Dixit and Pindyck (1996) model (DP-model): a fixed operating cost and external financing.

Ex ante, firms in our model are identical—their initial productivity shocks are drawn from the same distribution, and will follow the same stochastic process. Ex post, firms differ in the realization of productivity shocks. As in DP-model, shocks of different firms are independent, and the industry aggregates are nonrandom due to the law of large numbers (for rigorous justification, see, e.g, Judd, 1985).

An entrepreneur draws her initial productivity shock only after the investment had been made. An active firm has to pay interest on the debt and operating costs; therefore, if the shock is not a good one, the entrepreneur may decide to exit the industry immediately. On the other hand, since shocks are firm specific, the entrepreneur knows that, on average, her bad luck is not shared by her competitors; therefore it is not always necessary to exit at once. Instead, the entrepreneur can wait and see if this bad luck is transitory. At the same time, even if the initial shock is a good one, it may become optimal to leave the industry eventually if the stochastic factor falls too low. If the entrepreneur finds it optimal to discontinue the firm's operations, she has two options: (a) sell the firm's assets at a scrap value, pay back the debt, and exit the industry; (b) file for bankruptcy. We call the first option exit with debt repayment, or simply exit, and the second option—exit through default, or simply default. The fixed operating cost together with the recoverability of some of the firm's assets makes exit of the firm optimal. External financing makes default optimal.

Unlike Cooley and Quadrini (2001) and Miao (2005), we ignore growth opportunities of the firms and restrict our attention to exogenous leverage only. The model generalizes straightforwardly to the case of frictionless choice of inputs of production as in the Miao (2005).

The investment is partially financed by debt. The entrepreneurs in our model borrow as much as they are allowed to. According to Temkin and Kormendi (2003), small business lenders typically require the borrower to place between 25% and 30% equity in the transaction so as to exploit the tax advantages of debt. We leave for the future research the problem of the optimal capital structure with endogenous exit and default. The main difference of our model from Cooley and Quadrini (2001) and Miao (2005) is that both default and exit with debt payout are options for an active firm.

Deriving an industry equilibrium, Miao (2005) considers only demand and supply markets for industry output and ignores the market for loans. So, in his model with exogenous leverage, the coupon payment is a parameter, but not an endogenous variable. As opposed to this, in our model, the equilibrium interest rate (and hence the coupon rate) is determined under the assumption that lenders' profit is zero (i.e., lenders are competitive).

We assume that both in case of exit and default, the entrepreneur can re-enter the industry as a new entrant. A possibility of re-entry after exit agrees with McKenzie and Paffhausen (2018) who find that firm death need not mean permanent exit from self-employment for the firm owner. A possibility of re-entry after default corresponds to the core concept of Chapter 7.

All agents are risk-neutral and discount the future at rate $r > 0$. A potential entrepreneur has to invest a fixed size capital into a project (production technology) that will produce output flow, Y, where Y is a firm-specific productivity shock that follows the geometric Brownian motion. The initial value of the shock is revealed after the investment had been made. The output good is sold at the market price, P; therefore the firm's operating revenue is YP.

We normalize the debt amount to one and express the output price, P, as a fraction of the debt. We understand that in real life, the ability to borrow depends on the initial wealth and credit history of the entrepreneur as well as the amount of the loan. For simplicity, we assume that all these factors are reflected in a fixed instantaneous contractual coupon payment ρ; and ρ is determined in equilibrium from the zero profit condition for the lenders. Zero profit condition for the lenders is a standard assumption given that the credit market is competitive and free of arbitrage. We rule out any dynamic adjustments to leverage assuming that such adjustments are prohibitively costly. Let L be the leverage, that is, the ratio of the debt to the total capital invested in the project. Then, the size of investment is $1/L$. The part $1/L - 1$ of the investment is the entrepreneur's own resources. In addition to the coupon payment ρ, the active firm suffers the operating cost ν also measured as a fraction of the initial debt.

Both potential managers and active firms are subject to the exogenous death which is modeled as a Poisson process with the parameter $\lambda > 0$. This assumption (also used by Dixit & Pindyck, 1996, p. 275; Miao, 2005) ensures the existence of a stationary distribution of firms if the drift of the underlying Brownian motion is nonnegative. If the drift is negative, it is not necessary to introduce the exogenous death. See Section A.5 for details and derivation of the stationary distribution. In the event of the exogenous death of an active firm, no value can be recovered.

2.2 | Value of assets

Let idiosyncratic productivity shocks be specified as $Y_t = e^{X_t},\ \ t \geq 0$. $X = \{X_t\}$ is the Brownian motion, that is, X_t satisfies the following stochastic differential equation:

$$dX_t = \mu dt + \sigma dW_t, \tag{1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are, respectively, the drift and volatility of X, and dW_t is the increment of the standard Wiener process.

Let $Y = e^x$ be the current realization of the productivity shock. As a starting point, we define the value of assets of the firm as the expected present value (EPV) of revenues which the firm would generate if it never exits the industry except for the case of exogenous death:

$$V_{\rm as}(Y) = E^Y\left[\int_0^{+\infty} e^{-(r+\lambda)t} P Y_t\, dt\right]. \tag{2}$$

Here and below, $E^Y[f(Y_t)] := E[f(Y_t)|\ Y_0 = Y]$ denotes the conditional expectation operator. It is possible to use a more natural definition of $V_{\rm as}$ as the value of the unlevered firm with an option to exit. However, this leads to additional complication in the structure of the model, and does not add any new insight.

The value of the assets satisfies the following differential equation (the Euler equation):

$$\left(r + \lambda - \frac{\sigma^2}{2} Y^2 \frac{\partial^2}{\partial Y^2} - \left(\mu + \frac{\sigma^2}{2}\right) Y \frac{\partial}{\partial Y}\right) V_{as}(Y) = PY \quad \forall \;\; Y > 0. \tag{3}$$

We introduce the notation $\Psi(\beta) = (\sigma^2/2)\beta^2 + \mu\beta$. The value of assets $V_{as}(Y)$ is finite if and only if the following no-bubble condition holds:

$$r + \lambda - \Psi(1) > 0. \tag{4}$$

In Section A.1, we show that

$$V_{as}(Y) = \frac{PY}{r + \lambda - \Psi(1)}. \tag{5}$$

The value of the equity of the firm which commits itself to production and coupon payments until the moment of the Poisson random death is

$$V_0(Y) = V_{as}(Y) - (\rho + \nu)/(r + \lambda) = PY/(r + \lambda - \Psi(1)) - (\rho + \nu)/(r + \lambda) = bY - a_1, \tag{6}$$

where

$$b = b(P) = P/(r + \lambda - \Psi(1)), \quad a_1 = a_1(\rho) = (\rho + \nu)/(r + \lambda).$$

Assume that in case of exit with prepayment, the entrepreneur recovers a fraction $\alpha_1 \in [0, 1)$ of the firm's assets. In the literature on partially irreversible investment, it is common to assume that the liquidation value is fixed. We believe that the liquidation value changes as market prices change, hence it depends on the value of the firm's assets. This also agrees with Doraszelski and Satterthwaite (2010) who introduce random scrap values into their model. Also, the firm returns α_3 to the lender, where $\alpha_3 \geq 1$. If $\alpha_3 > 1$, then the contract specifies a penalty for prepayment. We interpret α_3 as a contract enforcement parameter. If the entrepreneur files for bankruptcy, she gets nothing.

3 | FIRM'S PROBLEM

The firm chooses the optimal time of default and optimal time of exit, denoted τ_d and τ_e, which maximize the value of the equity:

$$\begin{aligned} V(Y) \;\; = \;\; \sup_{\tau_d, \tau_e} \Big\{ & E^Y \left[\int_0^{\tau_d \wedge \tau_e} e^{-(r+\lambda)t} (PY_t - \rho - \nu) dt \right] \\ & + E^Y \left[\mathbf{1}_{\tau_e < \tau_d} e^{-\tau_e (r+\lambda)} (\alpha_1 V_{as}(Y_{\tau_e}) - \alpha_3) \right] \Big\}. \end{aligned}$$

The first term on the RHS of the last equation is the EPV of profits until exit or default (whichever happens first), and the second term is the net gain for the entrepreneur in the event of exit. Notice that there is no term responsible for the option value of reentering the industry: In the competitive case which we consider, this value is zero.

To find optimal τ_d and τ_e, we represent the value of the equity as

$$\begin{aligned} V(Y) \;=\; & E^Y\Big[\int_0^{+\infty} \mathrm{e}^{-(r+\lambda)t}(PY_t - \rho - v)dt\Big] \\ & + \sup_{\tau_d,\tau_e}\Big\{E^Y\Big[\int_{\tau_d\wedge\tau_e}^{+\infty} \mathrm{e}^{-(r+\lambda)t}(\rho + v - PY_t)dt\Big] \\ & + E^Y\left[\mathbf{1}_{\tau_e<\tau_d}\mathrm{e}^{-\tau_e(r+\lambda)}(\alpha_1 V_{\mathrm{as}}(Y_{\tau_e}) - \alpha_3)\right]\Big\}. \end{aligned} \tag{7}$$

The first term on the RHS of the last equation is $V_0(Y)$, the term under the supremum is the option value to leave the industry; therefore we write

$$V(Y) = V_0(Y) + V_{\mathrm{opt}}(Y). \tag{8}$$

In Section A.2, we show that

$$\begin{aligned} V_{\mathrm{opt}}(Y) \;=\; & \sup_{\tau_d,\tau_e}\{E^Y\left[\mathbf{1}_{\tau_d<\tau_e}\mathrm{e}^{-(r+\lambda)\tau_d}(a_1 - V_{\mathrm{as}}(Y_{\tau_d}))\right] \\ & + E^Y\left[\mathbf{1}_{\tau_e<\tau_d}\mathrm{e}^{-(r+\lambda)\tau_e}(a_1 - \alpha_3 - (1-\alpha_1)V_{\mathrm{as}}(Y_{\tau_e}))\right]\} \\ =\; & \sup_{\tau_d,\tau_e}\left\{E^Y\left[\mathbf{1}_{\tau_d<\tau_e}\mathrm{e}^{-(r+\lambda)\tau_d}G_d(Y_{\tau_d})\right] + E^Y\left[\mathbf{1}_{\tau_e<\tau_d}\mathrm{e}^{-(r+\lambda)\tau_e}G_e(Y_{\tau_e})\right]\right\}, \end{aligned} \tag{9}$$

where

$$G_d(Y) = a_1 - bY, \tag{10}$$

$$G_e(Y) = a_2 - (1-\alpha_1)bY \tag{11}$$

are the payoffs in the event of default and exit, respectively. Here $a_2 = a_2(\rho) = a_1(\rho) - \alpha_3$. If the penalty for debt prepayment is too high then no firm will exit the industry with debt repayment. Therefore, we assume that $\alpha_3 < \rho/(r+\lambda)$, which implies $a_2 > 0$.

3.1 | Exit options

Payoff functions G_d and G_e are linear and decreasing. The line $G_e(Y)$ has smaller (in absolute value) slope than $G_d(Y)$ because the exiting firm can partially recover the value of its assets. Since $a_1 > a_2$, $G_e(Y)$ and $G_d(Y)$ intersect. Let H_b be a (unique) solution to $G_e(H_b) = G_d(H_b)$. Straightforward calculations show that $H_b = \alpha_3/(\alpha_1 b)$.

We are interested in the case when the intersection happens in the positive cone, that is, $G_d(H_b) > 0$. Equivalently,

$$a_1 - bH_b = a_1 - \frac{\alpha_3}{\alpha_1} > 0.$$

It is easy to check that the last inequality holds iff

$$a := \frac{a_1}{a_2} < (1 - \alpha_1)^{-1}. \tag{12}$$

If Condition (12) does not hold, the value of default is higher than the value of exit for all values of Y s.t. $G_d(Y) > 0$. In other words, if (12) is not satisfied, the penalty for prepayment is too high and/or the scrap value of the firm is too low, so that it is never optimal to exit without default.

In the rest of the paper, we assume that (12) holds. Since $a_{1,2}$ depend on ρ, Condition (12) involves ρ, which is an endogenous variable, so we will need to check it after we solve the model. For low realizations of the idiosyncratic shock ($Y < H_b$), the value of default is higher than the value of exit. For $Y > H_b$, the value of exit is higher than the value of default. An active firm will certainly choose the mode of exit with the higher value. Let

$$G(Y) = \max\{G_d(Y),\ G_e(Y),\ 0\}$$

be the payoff from leaving the industry. If the firm leaves the industry at a random time τ it will get the payoff $G(Y_\tau)$. We see that the firm's problem is equivalent to the problem of optimal exercise of the perpetual American option with the payoff $G(Y)$, that is,

$$V_{\text{opt}}(Y) = \sup_\tau E^Y\left[\mathrm{e}^{-(r+\lambda)\tau} G(Y_\tau)\right].$$

Notice that had we considered a static model where the uncertainty is resolved at the moment of entry, or an isomorphic problem with naive treatment of uncertainty, where upon observing the initial productivity the firms decide whether to exit or default on the now-or-never basis, the firms whose productivity shock $Y < H_b$ would have defaulted immediately, the firms with the shock level $H_b < Y < a_2/(1 - \alpha_1)/b$ would have exited immediately, and the rest of them would have remained active until a random death took place. Condition (12) would have been the necessary and sufficient condition for separation of the firms into those who exit with debt repayment and those who exit through default.

In the dynamic model, Condition (12) does not ensure that some firms may find optimal to exit without default. The necessary and sufficient conditions can be derived after possible shapes of the inaction region are analyzed. Before doing this, we will consider a benchmark case, when the firm has only one exit option available.

3.2 | The case of one exit option

In this case, $G(Y) = \max\{G_i(Y),\ 0\}$, where $i = e$ if default is not an option, and $i = d$ if exit with debt repayment is not an option. If the idiosyncratic shock is sufficiently high, it is optimal to remain active. Let H_{i0} ($i \in \{e, d\}$) be the level of the shock s.t. the firm stays in the industry if $Y > H_{i0}$ and leaves the industry the first time $Y \le H_{i0}$. The standard argument shows (see, e.g., Dixit & Pindyck, 1996) that in the inaction region $(H_{i0}, +\infty)$, the option value of exit satisfies the following equation:

$$\left(r+\lambda-\frac{\sigma^2}{2}Y^2\frac{\partial^2}{\partial Y^2}-\left(\mu+\frac{\sigma^2}{2}\right)Y\frac{\partial}{\partial Y}\right)V^i_{\mathrm{opt}}(Y)=0,\quad Y>H_{i0}. \tag{13}$$

The general solution to this equation can be written as

$$V^i_{\mathrm{opt}}(Y)=A_i^-\left(\frac{Y}{H_i}\right)^{\beta^-}+A_i^+\left(\frac{Y}{H_{i0}}\right)^{\beta^+},\quad Y>H_{i0}.$$

The option value of exit vanishes as $Y\to\infty$, therefore $A_i^+=0$. At the threshold, H_{i0}, the option value satisfies the value matching and smooth pasting conditions:

$$V^i_{\mathrm{opt}}(H_{i0})=G_i(H_{i0}),\quad (V^i_{\mathrm{opt}})'(H_{i0})=G'_i(H_{i0}).$$

The last two conditions are equivalent to

$$A_i^-=a_i-b_iH_{i0},\quad A_i^-\beta^-=-b_iH_{i0},$$

where $a_d=a_1$, $a_e=a_2$, $b_d=b$, and $b_e=(1-\alpha_1)b$. Whence we derive

$$A_i^-=\frac{a_i}{1-\beta^-}\quad\text{and}\quad H_{i0}=\frac{\beta^-}{\beta^--1}\cdot\frac{a_i}{b_i}=\kappa_-(1)\frac{a_i}{b_i}.$$

Here and below we use the notation

$$\kappa_\pm(\beta)=\frac{\beta^\pm}{\beta^\pm-\beta}. \tag{14}$$

Thus the optimal thresholds for the exit with and without default are

$$H_{d0}=\kappa_-(1)\frac{a_1}{b}\quad\text{and}\quad H_{e0}=\kappa_-(1)\frac{a_2}{(1-\alpha_1)b}.$$

Notice that

$$\frac{H_{d0}}{H_{e0}}=\frac{(1-\alpha_1)a_1}{a_2}=(1-\alpha_1)a<1$$

if (12) is satisfied.

The option value of leaving the industry is

$$V^i_{\mathrm{opt}}(Y)=\begin{cases}\dfrac{a_i}{1-\beta^-}\left(\dfrac{Y}{H_{i0}}\right)^{\beta^-} & \text{if } Y>H_{i0},\\ G_i(Y) & \text{if } Y\le H_{i0}.\end{cases}$$

Observe that

$$V^e_{\text{opt}}(H_{e0}) > V^d_{\text{opt}}(H_{e0}) \quad \Leftrightarrow \quad \left(\frac{H_{d0}}{H_{e0}}\right)^{\beta^-} > a.$$

Straightforward calculations show that the last inequality is equivalent to

$$a < (1 - \alpha_1)^{-\varkappa_-(1)}. \tag{15}$$

Later, we will prove that (15) is the necessary and sufficient condition for existence of the Buridan zone. If (15) is satisfied, Condition (12) is satisfied as well, because $\alpha_1 < 1$ and $\varkappa_-(1) < 1$.

3.3 | Characterization of inaction regions with two exit options

Now $G(Y) = \max\{G_d(Y), G_e(Y), 0\}$. It is clear that at high levels of the productivity shock, it is not optimal to exercise the option to exit or default. Denote by H_+ the minimal number such that the firm remains active at all levels $Y > H_+$. The argument in the previous subsection can be invoked to prove that in the region $(H_+, +\infty)$, the option value of exit or default is of the form

$$V^+_{\text{opt}}(Y) = A\left(\frac{Y}{H_+}\right)^{\beta^-}, \quad Y > H_+. \tag{16}$$

The constant A and the exercise boundary H_+ can be found from the value matching and smooth pasting conditions:

$$A = G(H_+), \beta^- A = G'(H_+)H_+, \tag{17}$$

whence we see that H_+ solves the equation

$$\beta^- G(H_+) = G'(H_+)H_+. \tag{18}$$

For a sufficiently large A, the curve V^+_{opt} does not touch the curve G. As A decreases, the touchdown happens eventually, and we obtain the exercise threshold at the point of tangency, H_+. Clearly, three cases are possible; we show them in the (G, Y)-plane, in Figure 1.

Case I. The curve V^+_{opt} touches the curve G at a point on the G_d portion, but not on the G_e portion. $H_+ = H_{d0}$ is the default boundary. The firm never exits without default, and it defaults the first time $Y \le H_+$.

Case II. The curve V^+_{opt} touches the curve G at a point on the G_e portion, but not on the G_d portion. Then $H_+ = H_{e0}$ is an exit boundary, and $H_+ > H_b$. Starting with the supposition that the interval (H_+, ∞) is the firm's inaction region, we will arrive at a contradiction, which will demonstrate that the inaction region is disconnected in this case. Suppose that H_+ is the unique exit threshold. Then the option value of exit is given by

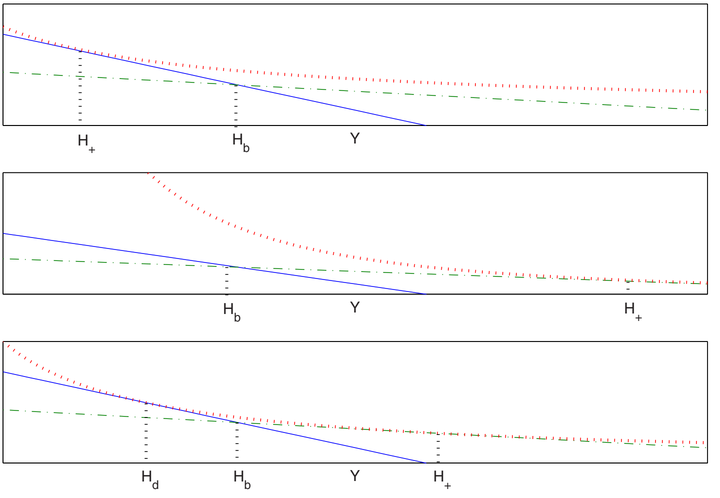

FIGURE 1 Payoffs G_d (solid line) and G_e (dashes), and the option value to exit $V^+_{\text{opt}} = A(Y/H_+)^{\beta^-}$ (dots). Upper panel: Case 1. Only default is possible, and $H_+ = H_{d0}$ is the default boundary. Middle panel: Case 2. At $H_+ = H_{e0}$, it is optimal to exit. Lower panel: Case 3. At $H_d = H_{d0}$, it is optimal to default; at $H_+ = H_{e0}$, the value of exit and waiting for default is equal

$$\tilde{V_{\text{opt}}}(Y) = \begin{cases} AY^{\beta^-}, & Y > H_+, \\ G(Y), & Y \leq H_+. \end{cases}$$

We will use the following general results of the optimal stopping theory to demonstrate that $\tilde{V_{\text{opt}}}(Y)$ cannot be the option value, and, therefore, the inaction region must be of a more complex form than a semi-infinite interval $(H_+, +\infty)$. If τ^* is an optimal stopping time, then

$$V_{\text{opt}}(Y) = E^Y\left[e^{-(r+\lambda)\tau^*}G(Y^*_\tau)\right]$$

satisfies the following two conditions (see, e.g., Theorem 2.4 in Peskir & Shiryaev, 2006):

$$V_{\text{opt}}(Y) \geq G(Y), \quad Y > 0; \tag{19}$$

$$V_{\text{opt}}(Y) \geq e^{-rt}E^Y[V_{\text{opt}}(Y_t)], \quad Y > 0, \quad t \geq 0. \tag{20}$$

The first condition says that the option value has to be at least as big as the payoff (otherwise, it would have been optimal to exercise the option earlier). The second condition tells us that the current option value is at least as big as its present value, that is, if the option is exercised optimally, it is impossible to increase its value by waiting longer.

Evidently, $\tilde{V_{\text{opt}}}(Y)$ satisfies (19). We will show now that there is a neighborhood of H_b, where $\tilde{V_{\text{opt}}}(Y)$ violates (20), hence $\tilde{V_{\text{opt}}}(Y)$ cannot be the option value. Intuitively, $\tilde{V_{\text{opt}}}(Y)$ cannot be the option value because it has a kink at H_b. Therefore, there is an additional inaction interval around H_b. To see why, divide the time into discrete periods of length Δt, and assume that in each period, the variable Y either moves up or down by an amount Δh. Let the probability that it moves up be p, and the probability that it moves down be $q = 1 - p$. Using the random walk representation of the Brownian motion (see, e.g., Dixit & Pindyck, 1996), one can show that $\Delta h = \sigma Y \sqrt{\Delta t}$ and

$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right), \quad q = \frac{1}{2}\left(1 - \frac{\mu}{\sigma}\sqrt{\Delta t}\right).$$

For a small Δt, we can write (20) as

$$V_{\text{opt}}(Y) \geq (1 - (r + \lambda)\Delta t)E^Y[V_{\text{opt}}(Y_{\Delta t})]. \tag{21}$$

Suppose that the current realization of the shock is $Y = H_b(1 - \sigma\sqrt{\Delta t}/2)$. Then

$$\tilde{V_{\text{opt}}}(Y) = G_d(Y) = G_d(H_b) + \frac{b\sigma\sqrt{\Delta t}}{2}H_b$$

(recall that $G_d(Y) > G_e(Y)$ for $Y < H_b$). With probability p,

$$Y_{\Delta t} = Y + \Delta h = H_b(1 + \sigma\sqrt{\Delta t}/2) + O(\Delta t),$$

and with probability q,

$$Y_{\Delta t} = Y - \Delta h = H_b(1 - 3\sigma\sqrt{\Delta t}/2) + O(\Delta t).$$

Hence

$$\begin{aligned} E^Y[\tilde{V_{\text{opt}}}(Y_{\Delta t})] &= [pG_e(H_b(1 + \sigma\sqrt{\Delta t}/2)) + qG_d(H_b(1 - 3\sigma\sqrt{\Delta t/2})) + O(\Delta t)] \\ &= pG_e(H_b) + qG_d(H_b) - p(1 - \alpha_1)\frac{b\sigma\sqrt{\Delta t}}{2}H_b + q\frac{3b\sigma\sqrt{\Delta t}}{2}H_b + O(\Delta t) \\ &= G_d(H_b) + \frac{b\sigma\sqrt{\Delta t}}{2}H_b + \frac{\alpha_1 b\sigma\sqrt{\Delta t}}{4}H_b + O(\Delta t), \end{aligned}$$

and

$$(1 - (r + \lambda)\Delta t)E^Y[V_{\text{opt}}(Y_{\Delta t})] = G_d(H_b) + \frac{b\sigma\sqrt{\Delta t}}{2}H_b + \frac{\alpha_1 b\sigma\sqrt{\Delta t}}{4}H_b + O(\Delta t) > \tilde{V_{\text{opt}}}(Y).$$

The case when the current realization of the shock is $Y = H_b(1 + \sigma\sqrt{\Delta t}/2)$ can be treated similarly. Hence there is a neighborhood of H_b, where $\tilde{V_{\text{opt}}}(Y)$ does not satisfy (20), therefore $\tilde{V_{\text{opt}}}(Y)$ is not the option value. Hence H_+ is not the unique exit barrier. Therefore, there must be an additional subset of the inaction region (H_d, H_-) around H_b where waiting is optimal. We will find this interval in Subsection 3.5. The firm stays alive if $Y > H_+$ or $Y \in (H_d, H_-)$. The firm exits the first time the shock Y enters the interval $[H_-, H_+]$, and defaults the first time $Y \le H_d$. Hence the interval $(0, H_d]$ is the default zone, (H_d, H_-) is the Buridan zone, $[H_-, H_+]$ is the exit zone, and $(H_+, +\infty)$ is the good luck zone, described in the Introduction. See Figure 2 for illustration.

Notice that for a general payoff function, the inaction region may be a union of intervals $(0, a), (b, +\infty)$, and a number of intervals of the form (c, d), where $0 < a < c < d < b$. In this model, the inaction zone adjacent to zero is impossible, because firms with very low productivity shocks will not stay in the industry, hence we can rule out an interval of the form $(0, a)$ as a candidate for a subset of the inaction region. We have already established that a semi-infinite inaction interval of the form $(b, +\infty)$, namely, $(H_+, +\infty)$, exists. It remains to consider an inaction interval of the form (c, d). In Subsection 3.5 below, we show that the boundaries of such an interval are uniquely determined in terms of their ratio $R = d/c > 1$, and derive an algebraic equation for R. After that, we prove that this equation has a unique solution on $(1, +\infty)$, which satisfies the necessary condition $d = d(R) < H_+$. Hence, the Buridan zone is unique.

Case III. The curve V^+_{opt} touches both curves G_d and G_e. This case can be regarded as the degenerate case of Case 2, with $H_- = H_+$. An active firm defaults the first time $Y \le H_d = H_{d0}$. At $Y = H_+ = H_{e0}$, the firm is indifferent between exiting and staying alive.

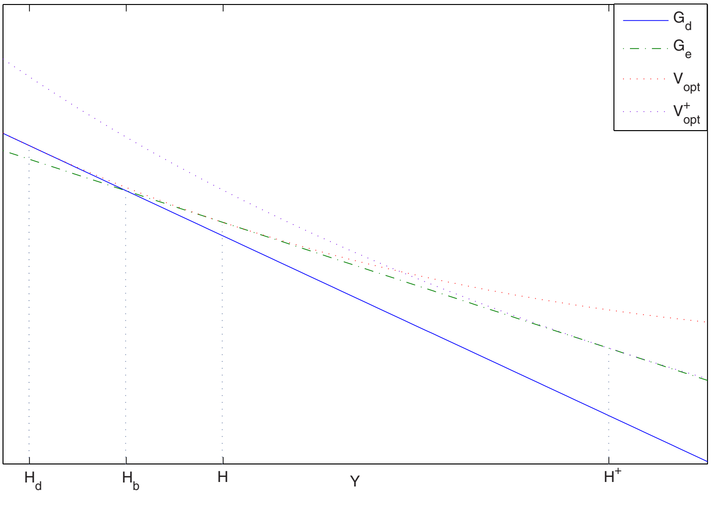

FIGURE 2 Case 2: exit and default. Graphs of G_d and G_e, and option values V^-_{opt} and V^+_{opt} in the Buridan zone and good luck zone. H_d is the default boundary, and $H_\pm$ are the exit boundaries

3.4 | Existence of the Buridan zone

To derive explicit conditions for Cases 1, 2, and 3, consider the option with the payoff G_d and the one with the payoff G_e. On the strength of (12), $H_{d0} < H_{e0}$. Obviously,

- Case 1 ⇔ ${V_{\text{opt}}}^d(H_{e0}) > {V_{\text{opt}}}^e(H_{e0})$.
- Case 2 ⇔ ${V_{\text{opt}}}^d(H_{e0}) < {V_{\text{opt}}}^e(H_{e0})$.
- Case 3 ⇔ ${V_{\text{opt}}}^d(H_{e0}) = {V_{\text{opt}}}^e(H_{e0})$.

We are interested in Case 2 because it generates both endogenous exit without default and default. In the Subsection 3.2, we derived that $V^d_{\text{opt}}(H_{e0}) < V^e_{\text{opt}}(H_{e0})$ iff (15) holds. Hence (15) is the necessary and sufficient condition for existence of the Buridan zone. Condition (15) involves one endogenous variable ρ, and therefore, after the model with exit and default is solved, we need to check (15) for consistency.

The results obtained so far are summarized in the following theorem:

Theorem 3.1. *Assume that (15) holds. Then*

(i) *the firm remains active in a region of the form* $(H_d, H_-) \cup (H_+, +\infty)$, *where* $H_d < H_- < H_+$;
(ii) H_d *is the default boundary, and* H_-, H_+ *are exit boundaries;*
(iii) H_+ *is given by*

$$H_+ = H_+(P, \rho) = \varkappa_-(1)\frac{a_2(\rho)}{(1-\alpha_1)b(P)}; \tag{22}$$

(iv) *for the firm in the good luck zone,* $(H_+, +\infty)$, *the option value of exit is given by*

$$V^+_{\text{opt}}(Y) = G_e(H_+)\left(\frac{Y}{H_+}\right)^{\beta^-} = \frac{a_2}{1-\beta^-}\left(\frac{Y}{H_+}\right)^{\beta^-}, \tag{23}$$

and the value of the equity in the good luck zone is

$$V^+(Y) = bY - a_1 + \frac{a_2}{1-\beta^-}\left(\frac{Y}{H_+}\right)^{\beta^-}. \tag{24}$$

It remains to find H_d, H_-, V^-_{opt}, and V^- —the default and exit thresholds from the Buridan zone, the option value of leaving the industry, and value of the equity in the Buridan zone.

3.5 | Boundaries of the Buridan zone

The option value of leaving the industry satisfies the following differential equation in the Buridan zone:

$$\left(r+\lambda-\frac{\sigma^2}{2}Y^2\frac{\partial^2}{\partial Y^2}-\left(\mu+\frac{\sigma^2}{2}\right)Y\frac{\partial}{\partial Y}\right)V^-_{\mathrm{opt}}(Y)=0,\quad Y\in(H_d,H_-).\tag{25}$$

We write the general solution of (25) as

$$V^-_{\mathrm{opt}}(Y)=A^-\left(\frac{Y}{H_d}\right)^{\beta^-}+A^+\left(\frac{Y}{H_d}\right)^{\beta^+}.\tag{26}$$

The constants A^+, A^- and optimal thresholds H_d, H_- are found from the value matching and smooth pasting conditions (see, e.g., Section 9.1 in Peskir & Shiryaev, 2006):

$$V^-_{\mathrm{opt}}(H_d)=G_d(H_d),\quad (V^-_{\mathrm{opt}})'(H_d)=G'_d(H_d),\tag{27}$$

$$V^-_{\mathrm{opt}}(H_-)=G_e(H_-),\quad (V^-_{\mathrm{opt}})'(H_-)=G'_e(H_-).\tag{28}$$

Using (10), (11), and (26), we can write (27) and (28) as

$$A^++A^-=a_1-bH_d,\tag{29}$$

$$\beta^+A^++\beta^-A^-=-bH_d,\tag{30}$$

$$A^+R^{\beta^+}+A^-R^{\beta^-}=a_2-(1-\alpha_1)bH_-,\tag{31}$$

$$\beta^+A^+R^{\beta^+}+\beta^-A^-R^{\beta^-}=-(1-\alpha_1)bH_-,\tag{32}$$

where $R=H_-/H_d$. Set

$$\Delta=\Delta(R)=R^{\beta^+}-R^{\beta^-},\quad \Delta_1=\Delta_1(R)=\frac{aR^{\beta^+}-1}{\kappa_+(1)}+\frac{1-aR^{\beta^-}}{\kappa_-(1)},\tag{33}$$

$$\Delta_2=\Delta_2(R)=\frac{aR^{\beta^+}-1}{\kappa_+(1)}R^{\beta^-}+\frac{1-aR^{\beta^-}}{\kappa_-(1)}R^{\beta^+},\tag{34}$$

$$B_1=B_1(R)=\frac{\Delta_1(R)}{a\Delta(R)},\quad B_2=B_2(R)=\frac{\Delta_2(R)}{\Delta(R)},\tag{35}$$

and consider the equation

$$(1-\alpha_1)R=\frac{\Delta_2(R)}{\Delta_1(R)}:=F(R).\tag{36}$$

Define R_1 as a zero of $\Delta_1(R)$ on $(1,a^{-1/\beta^-})$ and R_2 as a zero of $\Delta_2(R)$ on $(R_1,a^{-1/\beta^-})$. In appendix, we prove the following:

Lemma 3.2. *Let R be a solution to (36) on* $\left(R_2, a^{-1/\beta^-}\right)$. *Then*

a) *the solution of Systems (29)–(32) is given by*

$$H_d = \frac{\varkappa_+(1)\varkappa_-(1)a_1 B_1}{b} = \frac{(\rho + \nu)B_1}{P}, \tag{37}$$

$$H_- = RH_d, \tag{38}$$

$$A^+ = \varkappa_-(\beta^+)\left(a_1 - \frac{b}{\varkappa_-(1)}H_d\right), \tag{39}$$

$$A^- = \varkappa_+(\beta^-)\left(a_1 - \frac{b}{\varkappa_+(1)}H_d\right). \tag{40}$$

b) $H_d < H_- < H_+$.

Now we make the following statement.

Theorem 3.3. *Assume that Condition (15) is satisfied. Then*

a) *Equation* (36) *has a solution on* $\left(R_2, a^{-1/\beta^-}\right)$, *and the Buridan zone exists;*
b) *the boundary points of the Buridan zone are defined by* (37) *and* (38);
c) *the value of the equity in the Buridan zone is*

$$V^-(Y) = bY - a_1 + A^-\left(\frac{Y}{H_d}\right)^{\beta^-} + A^+\left(\frac{Y}{H_d}\right)^{\beta^+}, \tag{41}$$

where A^+ *and* A^- *are given by* (39) *and* (40);
d) *the option value of leaving the industry is*

$$V_{\text{opt}}(Y) = \begin{cases} a_1 - bY & \text{if } Y \le H_d, \\ A^-\left(\frac{Y}{H_d}\right)^{\beta^-} + A^+\left(\frac{Y}{H_d}\right)^{\beta^+} & \text{if } Y \in [H_d, H_-], \\ a_2 - (1 - \alpha_1)bY & \text{if } Y \in [H_-, H_+], \\ \frac{a_2}{1 - \beta^-}\left(\frac{Y}{H_+}\right)^{\beta^-} & \text{if } Y \ge H_+. \end{cases} \tag{42}$$

Finally, we can write the value of the equity of the firm whose current shock is Y as

$$V(Y) = V_0(Y) + V_{\text{opt}}(Y) = \begin{cases} 0 & \text{if } Y \le H_d, \\ V^-(Y) & \text{if } Y \in [H_d, H_-], \\ \alpha_1 bY - \alpha_3 & \text{if } Y \in [H_-, H_+], \\ V^+(Y) & \text{if } Y \ge H_+. \end{cases} \tag{43}$$

4 | SIZE DEPENDENCE OF GROWTH RATE AND VOLATILITY OF GROWTH

The initial conclusion of the Gibrat law which states that firm size and growth are independent was rejected by a number of empirical studies. See extensive discussion of empirical fundings and theoretical models that tried to explain those findings in Cooley and Quadrini (2001). Cooley and Quadrini (2001) was the first model that accounts simultaneously for the conditional size and age dependence. The main idea in Cooley and Quadrini (2001) is that to reproduce all stylized facts of the firm dynamics, it is necessary to have two dimensions of heterogeneity across firms. In a frictionless market, the exogenous productivity shock fully determines the size and dynamics of the firm. Cooley and Quadrini (2001) introduce financial market frictions in a basic model of industry dynamics with persistent idiosyncratic productivity shocks and instantly adjustable capital and labor. With financial frictions, the size of the firm also depends on its equity. Cooley and Quadrini (2001) only found the sought after effects for a very small class of dynamics (a two-state Markov process). Correct age dependence in that model is driven by assumptions that new entrants are more productive, and productivity shocks are highly persistent. This driving assumption contradicts an empirical fact presented in MacKay and Phillips (2005): "...entrants begin less capital-intensive and less profitable than incumbents..." The latter fact agrees with Williams' (1995) prediction that entering firms must rely on less efficient, more labor-intensive technologies because of the limited access to capital markets.

Boyarchenko (2006) shows that both conditional age and size dynamics of firms can be naturally explained even in the absence of financial frictions, though undoubtedly the latter are important. Instead, Boyarchenko (2006) introduces a different friction, which is one of the key components of the real options theory. Namely, she assumes that investment is irreversible, or reversible at a cost, and constructs a model of a competitive industry equilibrium refining the model in Dixit and Pindyck (1996) and demonstrates size and age dependence that agree with empirical findings.

To be able to derive age dependence, one needs to be able to compare firms from different cohorts. To this end, Boyarchenko (2006) introduces two stages of a firm development, where in the second stage, firms have an option to adopt a better technology at a sunk cost. In the current paper, we cannot compare firms from different cohorts, but we can compare firms of different equity size.

As in Cooley and Quadrini (2001), we measure the size of the firm by its equity. First, we consider a firm in the Buridan zone with the value function $V^{-}(Y)$ given by (41). Applying the Itô lemma to V^{-}, we obtain

$$dV^{-} = \left(\frac{\sigma^2}{2}Y^2\frac{\partial^2}{\partial Y^2} + \left(\mu + \frac{\sigma^2}{2}\right)Y\frac{\partial}{\partial Y}\right)V^{-}dt + Y(V^{-}(Y))'\sigma dW_t. \tag{44}$$

Using (25), (26), and (41), we derive

$$\left(\frac{\sigma^2}{2}Y^2\frac{\partial^2}{\partial Y^2} + \left(\mu + \frac{\sigma^2}{2}\right)Y\frac{\partial}{\partial Y}\right)V^{-} = \psi(1)bY + (r+\lambda)V^{-}_{\mathrm{opt}}(Y). \tag{45}$$

A typical assumption in the real options literature, which guarantees that the no-bubble condition holds for any $r > 0$, is $\psi(1) = 0$, hence the first term on the RHS in (45) is zero. Substituting (45) into (44) and dividing both sides of Equation (44) by V^-, we arrive at

$$\frac{dV^-}{V^-} = \frac{(r+\lambda)V^-_{\mathrm{opt}}}{V^-}dt + \frac{Y(V^-(Y))'}{V^-}\sigma dW_t, \tag{46}$$

where the factor at dt is the rate of growth, and the factor at dW_t is the volatility of growth. Since V^-_{opt} is a decreasing function, and V^- is an increasing function of Y, the growth rate is a decreasing function of Y. Furthermore, it is a decreasing function of V^-, because V^- is increasing in Y. For the volatility of growth $Y(V^-(Y))'/V^-$, the situation is less clear because both the numerator and the denominator are increasing in Y (recall that V^- is convex). Numerical examples show that, for reasonable parameter values, $Y(V^-(Y))'/V^-$ increases as a function of Y, and hence as a function of V^-. See Figure 3 for illustration.

Similarly, for the firm born in the good luck zone,

$$\frac{dV^+}{V^+} = \frac{(r+\lambda)V^+_{\mathrm{opt}}}{V^+}dt + \frac{Y(V^+(Y))'}{V^+}\sigma dW_t. \tag{47}$$

Repeating the same argument as above, we conclude that the rate of growth $(r+\lambda)V^+_{\mathrm{opt}}/V^+$ is a decreasing function of V^+. As Figure 4 shows, the volatility of growth may be a nonmonotone function of V^+.

Thus, our results agree with empirical findings that the growth rate of a firm is decreasing in its size. This holds for the firms born both in the Buridan zone and in the good luck zone. For the firms born in the Buridan zone, the volatility of growth decreases as the firm becomes larger, which is quite natural, because the larger the firm is, the farther it is away from the default boundary. Hence, for the larger firm it becomes more and more likely that it will exit with the debt prepayment. Interestingly, numerical experiments show that for the firms born in the good luck zone, the volatility of growth may be a nonmonotone function of the firm's equity size. For a firm born close to the exit threshold, volatility increases as the firm grows until it reaches the peak. After that, volatility decreases in the firm's size. The nonmonotone behavior

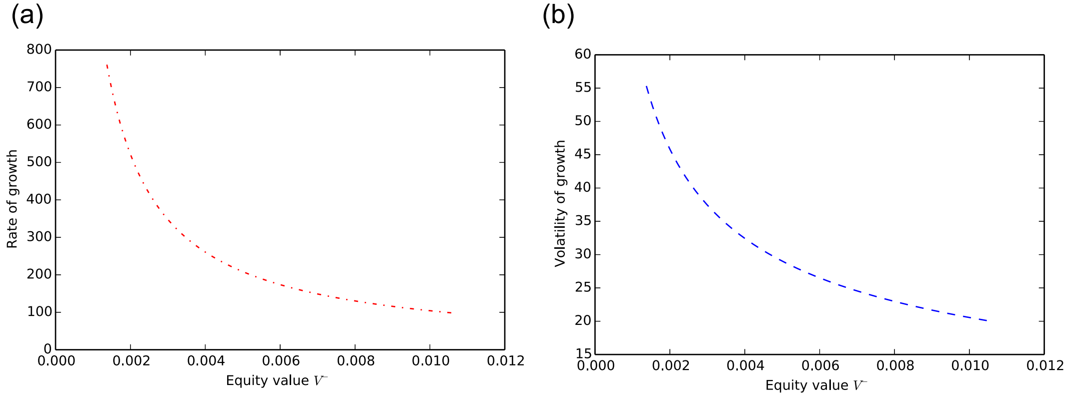

FIGURE 3 Rate of growth (a) and volatility of growth (b) of a firm in the Buridan zone. Parameters: $r = 0.04$, $\lambda = 0.01$, $\sigma = 0.2$, $\alpha_1 = 0.25$, $\alpha_2 = 0.5$, $\alpha_3 = 1.05$, $\nu = 1.2$, $P = 0.01$, $\rho = 0.053$

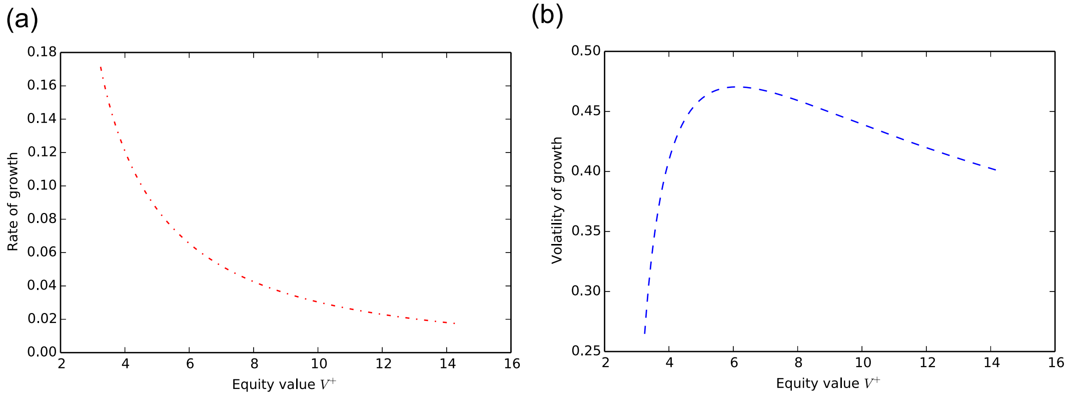

FIGURE 4 Rate of growth (a) and volatility of growth (b) of a firm in the good luck zone. Parameters: $r = 0.04$, $\lambda = 0.01$, $\sigma = 0.2$, $\alpha_1 = 0.25$, $\alpha_2 = 0.5$, $\alpha_3 = 1.05$, $\nu = 1.2$, $P = 0.01$, $\rho = 0.053$

can be explained as follows. If the idiosyncratic shock is close to the exit boundary, it is very likely that the firm will exit the industry soon, so the volatility is relatively low. As the firm starts growing, its growth may continue for a while, or it may be transitory and the firm will exit, so the volatility increases. After the size of the firm reaches some critical value, it becomes less and less likely that it will exit in the nearest future, therefore the volatility starts decreasing.

5 | EQUILIBRIUM PRICE OF OUTPUT AND COUPON RATE

5.1 | Industry equilibrium

We define a long-run industry equilibrium as a list $(P, \rho, N, g_{act}, H_d, H_-, H_+)$, where P is the equilibrium price, ρ is the coupon rate, N is the entry rate, and g_{act} is the stationary distribution of active firms, s.t. (a) firms choose the exit and/or default thresholds to maximize their NPV, (b) free entry conditions are satisfied, and (c) the good market clears.

So far, we calculated the exit and default thresholds, value of the firm's equity taking the endogenous variables P and ρ as given. In this section, we derive a system of two equations for P and ρ using the following two free entry conditions: The expected value of a new entrant is equal to the size of the investment, and lenders earn zero profit.

Assume that initial shocks Y are distributed on an interval $[Y_{min}, Y_{max}]$ with $Y_{min} < H_d < H_-$ and $H_+ < Y_{max}$. Let $g(Y)$ be the p.d.f. of Y.

Recall that the entrepreneur has to invest $1/L - 1$ at the moment of entry and after observing the productivity shock, gets $V(Y)$, given by (43). Therefore, the free entry condition for entrepreneurs is

$$1/L - 1 = \int_{H_d}^{H_-} V^-(Y)g(Y)dY + \int_{H_-}^{H_+} (\alpha_1 bY - \alpha_3)g(Y)dY + \int_{H_+}^{Y_{max}} V^+(Y)g(Y)dY, \quad (48)$$

where V^+ and V^- are given by (24) and (41), respectively. The first term on the RHS of (48) is the expected value of active firms in the Buridan zone, the second term is the expected value of the firms that exit immediately and pay back the debt, and the last term is the expected value of active firms in the good luck zone.

The free entry condition for the lenders (which is also a no-arbitrage condition) is

$$\begin{aligned} 1 &= \alpha_2 b \int_{Y_{\min}}^{H_d} Yg(Y)dY + \int_{H_d}^{H_-} V_{\mathscr{D}}^{-}(Y)g(Y)dY \\ &+ \alpha_3 \int_{H_-}^{H_+} g(Y)dY + \int_{H_+}^{Y_{\max}} V_{\mathscr{D}}^{+}(Y)g(Y)dY, \end{aligned} \tag{49}$$

where $V_{\mathscr{D}}^{-}$ and $V_{\mathscr{D}}^{+}$ are the values of the debt of an active firm in the Buridan zone and good luck zone, respectively. The terms on the RHS of (49) represent contributions of the firms in the default zone, Buridan zone, exit zone, and good luck zone, respectively.

To evaluate the integrals in (49), we need to determine $V_{\mathscr{D}}^{+}$ and $V_{\mathscr{D}}^{-}$.

5.2 | Good luck zone

While the firm is in the good luck zone, the debt holders receive coupon payment, ρ. When the firm liquidates, the creditors receive α_3. Therefore, the value of the debt of a firm in the good luck zone, $V_{\mathscr{D}}^{+}$, solves the following boundary problem:

$$\left(r + \lambda - \frac{\sigma^2}{2}Y^2\frac{\partial^2}{\partial Y^2} - \left(\mu + \frac{\sigma^2}{2}\right)Y\frac{\partial}{\partial Y}\right)V_{\mathscr{D}}^{+}(Y) = \rho,\ Y > H_+, \tag{50}$$

$$V_{\mathscr{D}}^{+}(H_+) = \alpha_3. \tag{51}$$

Also, $V_{\mathscr{D}}^{+}$ is bounded. Therefore, $V_{\mathscr{D}}^{+}$ is of the form

$$V_{\mathscr{D}}^{+}(Y) = \frac{\rho}{r+\lambda} + A\left(\frac{Y}{H_+}\right)^{\beta^-},$$

where the constant A is easily found from the boundary condition (51):

$$A = \alpha_3 - \frac{\rho}{r+\lambda}.$$

Thus, the value of the debt of the firm in the good luck zone is

$$V_{\mathscr{D}}^{+}(Y) = \frac{\rho}{r+\lambda} - \left(\frac{\rho}{r+\lambda} - \alpha_3\right)\left(\frac{Y}{H_+}\right)^{\beta^-}. \tag{52}$$

The first term in (52) is the present value of the stream of coupons paid by a firm that never exits voluntary, the second term is the net loss incurred in case of exit.

5.3 | Buridan zone

A firm in the Buridan zone pays the coupon until exit or default. As in Leland (1994, 1998), Leland and Toft (1996) assume that in case of default, the creditors capture the fraction α_2 of the firm's assets. The remaining portion is lost as the cost of bankruptcy procedures. Parameter α_2 is a proxy for creditor protection in our model. The value of the debt of the firm in the Buridan zone, $V^-_{\mathscr{D}}$, solves the following boundary problem

$$\begin{aligned}\left(r+\lambda-\frac{\sigma^2}{2}Y^2\frac{\partial^2}{\partial Y^2}-\left(\mu+\frac{\sigma^2}{2}\right)Y\frac{\partial}{\partial Y}\right)V^-_{\mathscr{D}}(Y) &= \rho, \quad H_d<Y<H_-,\\ V^-_{\mathscr{D}}(H_-) &= \alpha_3,\\ V^-_{\mathscr{D}}(H_d) &= \alpha_2 bH_d.\end{aligned}$$

The general solution is

$$V^-_{\mathscr{D}}(Y)=\frac{\rho}{r+\lambda}+A^+_{\mathscr{D}}\left(\frac{Y}{H_d}\right)^{\beta^+}+A^-_{\mathscr{D}}\left(\frac{Y}{H_d}\right)^{\beta^-}. \tag{53}$$

Substituting the general solution into the boundary conditions, we obtain a system of two equations with two unknowns $A^+_{\mathscr{D}}$ and $A^-_{\mathscr{D}}$:

$$\begin{aligned}A^+_{\mathscr{D}}R^{\beta^+}+A^-_{\mathscr{D}}R^{\beta^-} &= \alpha_3-\frac{\rho}{r+\lambda},\\ A^+_{\mathscr{D}}+A^-_{\mathscr{D}} &= \alpha_2 bH_d-\frac{\rho}{r+\lambda},\end{aligned}$$

where $R=H_-/H_d$, as before. Applying the Cramer theorem, we find that the value of the debt of the firm in the Buridan zone is given by (53), where

$$\begin{aligned}A^+_{\mathscr{D}} &= \Delta_+/\Delta, \quad A^-_{\mathscr{D}}=\Delta_-/\Delta,\\ \Delta_+ &= \alpha_3-\frac{\rho}{r+\lambda}-\left(\alpha_2 bH_d\frac{\rho}{r+\lambda}\right)R^{\beta^-},\\ \Delta_- &= \left(\alpha_2 bH_d-\frac{\rho}{r+\lambda}\right)R^{\beta^+}-\alpha_3+\frac{\rho}{r+\lambda},\end{aligned} \tag{54}$$

and Δ is given by (33).

5.4 | Numerical example

In this subsection, we calculate the equilibrium assuming that the p.d.f. of the initial shocks is concentrated in a very small neighborhood of two points $X_- \in (H_d, H_-)$ and $X_+ > H_+$:

$$g=c^-g_-+c^+g_+,$$

where $g_\pm$ is concentrated around $X_\pm$ ($c_\pm>0,\ c^++c^-=1$). The goal of this example is to study how equilibrium variables depend on contract enforcement parameter, α_3, and creditor protection parameter, α_2. We vary $\alpha_3 \in [1, 1.05]$ (where $\alpha_3=1$ is the case when there is no penalty for debt prepayment) and $\alpha_2 \in [0.45, 0.55]$. Leland (1994) uses $\alpha_2=0.5$. The variance and drift of the underlying Brownian motion are $\sigma=0.2$ and $\mu=-\sigma^2/2$, which are typical values in the real options literature (see, e.g., Dixit & Pindyck, 1996). We set the annual riskless rate $r = 0.04$, which is a typical choice in macro and finance literature, and the exogenous death rate $\lambda = 0.01$, which is the minimal estimated default rate for business loans (see Hillegeist et al., 2004; Mester, 1997). The leverage is $L = 0.5$, which corresponds to the average leverage for small firms (see Herranz et al., 2015). We set $c^+ = 0.8$, $c^- = 0.2$, $X_+ = 20$, and $X_- = 8$, which generates realistic default rates between 2.5% and 2.7%. To find the equilibrium entry rate, we use an isoelastic demand curve $Q = P^{-\epsilon}$ with $\epsilon = 10$, which is a typical choice in macro models. Recall that we derived the industry equilibrium assuming that $\rho > \alpha_3(r + \lambda)$ and the Buridan zone exists, that is, Condition (15) is satisfied. It is straightforward to check that Condition (15) is equivalent to

$$\rho > \frac{\alpha_3(r+\lambda)}{1-(1-\alpha_1)^{\kappa_-(1)}} - \nu.$$

Hence, we must find the equilibrium coupon rate, ρ, s.t.

$$\rho > \max\left\{\alpha_3(r+\lambda),\ \frac{\alpha_3(r+\lambda)}{1-(1-\alpha_1)^{\kappa_-(1)}} - \nu\right\}.$$

Notice that if $\nu > \alpha_3(r+\lambda)(1-\alpha_1)^{\kappa_-(1)}/\left(1-(1-\alpha_1)^{\kappa_-(1)}\right)$, then the Buridan zone exists for any $\rho > \alpha_3(r+\lambda)$. We set $\nu = 0.5$, to ensure existence of the Buridan zone.

In Figure 5a, we plot the dependence of the equilibrium coupon rate on the contract enforcement parameter, α_3, for three different levels of the creditor protection parameter, α_2. The coupon rate decreases both when contract enforcement is stronger and when the creditors can acquire a larger portion of the firm's assets in case of default, which is quite intuitive. The dependence of the equilibrium price on α_2 and α_3 is shown in Figure 5b. Evidently, the price is much less sensitive to changes in the institutional parameters than the coupon rate. The price decreases when the debtor protection increases: Firms have to pay lower interest on their debt and therefore can sell their output at a lower price. The dependence of the price on the credit enforcement parameter is less evident. We see that the price slightly goes down for small levels of α_3, but then starts growing. Please see the Introduction for an explanation of this phenomenon. Since the aggregate demand for the firms' output is $P^{-\epsilon}$ and market clearing

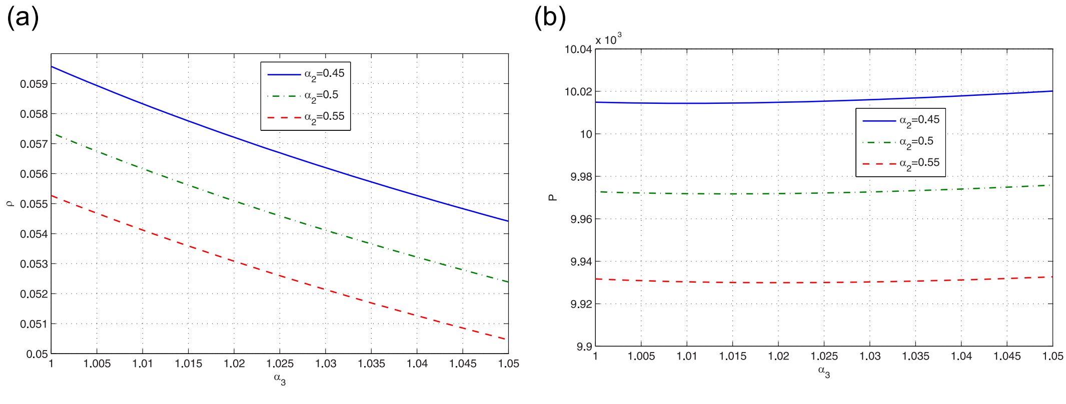

FIGURE 5 (a) Dependence of equilibrium coupon rate on α_2 and α_3; (b) dependence of equilibrium price on α_2 and α_3

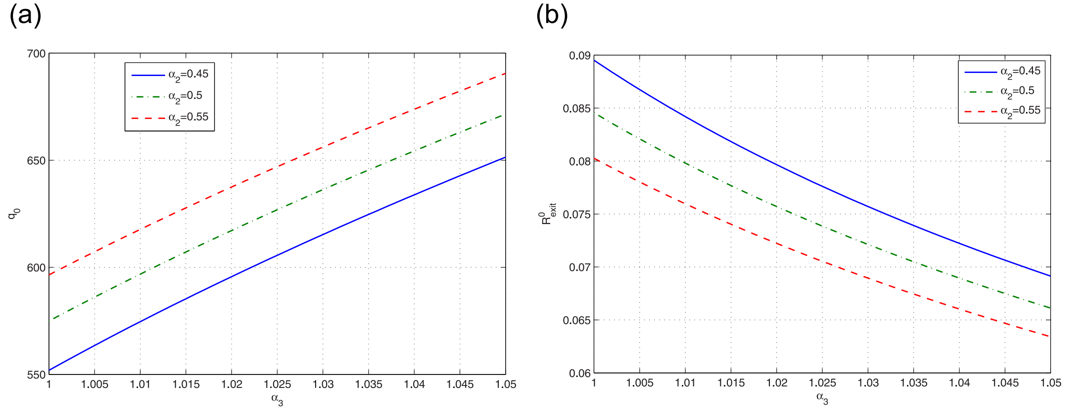

FIGURE 6 (a) Dependence of average output on α_2 and α_3; (b) dependence of exit rates on α_3 and α_2

holds, we may conclude that the aggregate output increases with the creditor protection. If we look at the dependence of the average output (per entering firm) on credit protection and enforcement, plotted in Figure 6a, we see that the average output increases in both parameters.

Figures 6b and 7a demonstrate how equilibrium exit and default rates (scaled by the number of active firms) depend on parameters α_2 and α_3. Both rates decrease in α_2 and α_3. This is due to the fact that the coupon rate becomes smaller as these two parameters increase. Observe that exit rates drop more sharply than default rates when α_3 increases, because, in addition to lower interest payments on the debt, exit option becomes less attractive. Notice that all equilibrium values, except for the price, change more due to changes in the penalty for prepayment than to changes in the creditor protector parameter, α_2.

Finally, in Figure 7b, we plot the operating profit levels that trigger default, $\pi_d^*= PH_d - \rho - \nu$, and exit, $\pi_\pm^* = PH_\pm - \rho - \nu$, as functions of α_3. For comparison, we also show $\pi_\pm = PX_\pm - \rho - \nu$—the operating profits at the moment of entry. Observe that both at the exit and default thresholds, profits are negative. This agrees with Herranz et al. (2015): Despite limited liability, entrepreneurs inject their personal funds to keep firms alive for some time.

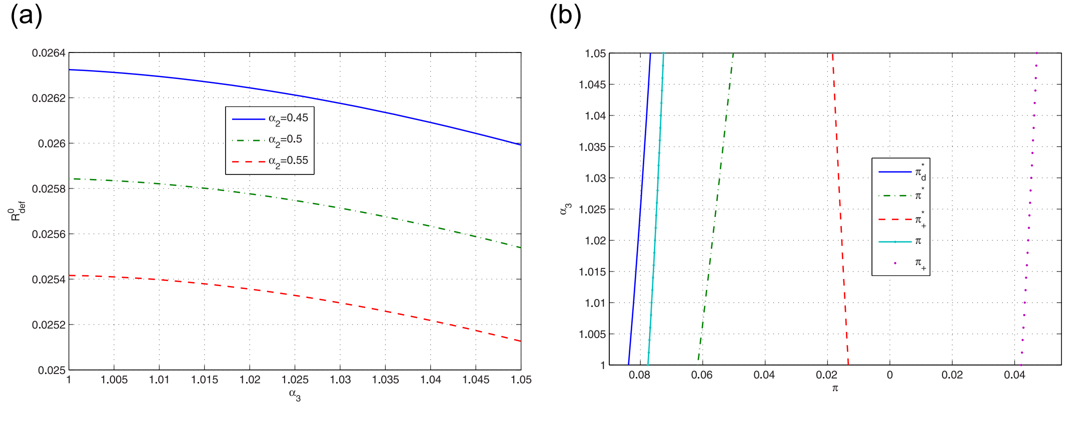

FIGURE 7 (a) Dependence of default rates on α_3 and α_2; (b) dependence of default and exit thresholds in terms of operating profits on α_3

6 | CONCLUSION

We have constructed a model of competitive industry equilibrium that generates endogenous exit with debt payout and default for the firms partially financed by debt. The outcome of investment for each entrepreneur depends on the initial draw of the firm-specific productivity shock. Some of the entrepreneurs enter the industry with such a high level of productivity shock, that they will exit eventually, but default is never optimal for them. Other firms, though start producing, will file for bankruptcy if their productivity deteriorates, and exit paying out the debt if the productivity shocks are favorable. The model allows one to analyze how equilibrium coupon rate, price, output, exit, and default rates depend on the exogenous variables, in particular, on contract enforcement and creditor protection.

One of the immediate extensions of the model is to endogenize the amount of the debt and study the optimal capital structure. In the current model, there is no information asymmetry; however, modeling active lenders allows one to extend the model by introducing asymmetric information and to consider firm dynamics with optimal lending contracts similar to Clementi and Hopenhayn (2006) and Albuquerque and Hopenhayn (2004). Following the line of research suggested in Evans and Jovanovic (1989), it would be interesting to distinguish entrepreneurs not only by their productivity shocks, but by the initial wealth distribution as well. Finally, as it was mentioned in the Introduction, this model can be used as a benchmark case for a model with debt covenants.

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APPENDIX A: TECHNICAL CALCULATIONS

A.1 | Proof of (2.5)

Consider the characteristic equation

$$r + \lambda - \Psi(\beta) = 0. \tag{A1}$$

Denote by $\beta^- < 0 < \beta^+$ the roots of Equation (A1). Under the no-bubble condition (4), $\beta^+ > 1$, and it can be proved that $V_{\text{as}}(Y) \leq CY$, where $C > 0$ is a constant. The general solution to Euler equation (3) is

$$V_{\text{as}}(Y) = A^+ Y^{\beta^+} + A^- Y^{\beta^-} + AY.$$

Since the value of the assets is bounded by CY, we conclude that $A^\pm = 0$. It is easy to find the remaining constant, A, substituting AY for $V_{\text{as}}(Y)$ in (3). As a result, we derive (5).

A.2 | Proof of (3.3)

It follows from (7) that

$$\begin{aligned} V_{\text{opt}}(Y) \;=\; & \sup_{\tau_d, \tau_e} \Big\{ E^Y \Big[\int_{\tau_d \wedge \tau_e}^{+\infty} \mathrm{e}^{-(r+\lambda)t} (\rho + \nu - PY_t) dt \Big] \\ & + \; E^Y \left[\mathbf{1}_{\tau_e < \tau_d} \mathrm{e}^{-\tau_e (r+\lambda)} (\alpha_1 V_{\text{as}}(Y_{\tau_e}) - \alpha_3) \right] \Big\}. \end{aligned}$$

First, using the strong Markov property of the Brownian motion, we change the variable $t \mapsto t + \tau_d \wedge \tau_e$ and calculate

$$\begin{aligned}
&E^Y\left[\int_{\tau_d\wedge\tau_e}^{+\infty} \mathrm{e}^{-(r+\lambda)t}(\rho + \nu - PY_t)dt\right]\\
&= E^Y\left[\int_0^{+\infty} \mathrm{e}^{-(r+\lambda)(t+\tau_d\wedge\tau_e)}(\rho + \nu - PY_{t+\tau_d\wedge\tau_e})\, dt\right]\\
&= E^Y\left[\mathrm{e}^{-(r+\lambda)\tau_d\wedge\tau_e}\left(\frac{\rho+\nu}{r+\lambda} - \int_0^{+\infty} \mathrm{e}^{-(r+\lambda)t}PY_{t+\tau_d\wedge\tau_e}\right)dt)\right],
\end{aligned}$$

using the law of iterated expectations, and we proceed

$$\begin{aligned}
&= E^Y\left[\mathrm{e}^{-(r+\lambda)\tau_d\wedge\tau_e}\left(\frac{\rho+\nu}{r+\lambda} - E^{Y_{\tau_d\wedge\tau_e}}\left[\int_0^{+\infty} \mathrm{e}^{-(r+\lambda)t}PY_{t+\tau_d\wedge\tau_e}\right)dt]\right)\right]\\
&= E^Y\left[\mathrm{e}^{-(r+\lambda)\tau_d\wedge\tau_e}\left(a_1 - V_{\mathrm{as}}(Y_{\tau_d\wedge\tau_e})\right)\right].
\end{aligned}$$

Now we can write

$$\begin{aligned}
V_{\mathrm{opt}}(Y) &= \sup_{\tau_d,\tau_e}\{E^Y\left[\mathrm{e}^{-(r+\lambda)\tau_d\wedge\tau_e}\left(a_1 - V_{\mathrm{as}}(Y_{\tau_d\wedge\tau_e})\right)\right]\\
&\quad + E^Y\left[\mathbf{1}_{\tau_e<\tau_d}\mathrm{e}^{-\tau_e(r+\lambda)}(\alpha_1 V_{\mathrm{as}}(Y_{\tau_e}) - \alpha_3)\right]\}\\
&= \sup_{\tau_d,\tau_e}\{E^Y\left[\mathbf{1}_{\tau_d<\tau_e}\mathrm{e}^{-\tau_d(r+\lambda)}(a_1 - V_{\mathrm{as}}(Y_{\tau_d}))\right]\\
&\quad + E^Y\left[\mathbf{1}_{\tau_e<\tau_d}\mathrm{e}^{-\tau_e(r+\lambda)}(a_1 - V_{\mathrm{as}}(Y_{\tau_e}) + \alpha_1 V_{\mathrm{as}}(Y_{\tau_e}) - \alpha_3)\right]\},
\end{aligned}$$

and (9) follows.

A.3 | Proof of Lemma 3.2

First, we solve (29) and (30) for A^+, A^- and derive

$$\begin{aligned}
A^+ &= \kappa_-(\beta^+)\left(a_1 - \frac{b}{\kappa_-(1)}H_d\right),\\
A^- &= \kappa_+(\beta^-)\left(a_1 - \frac{b}{\kappa_+(1)}H_d\right),
\end{aligned}$$

which are (39) and (40), respectively. Recall that $H_{d0} = \kappa_-(1)a_1/b$, hence $b/a_1 = \kappa_-(1)/H_{d0}$, and we can write

$$\begin{aligned}
A^+ &= \kappa_-(\beta^+)a_1\left(1 - \frac{H_d}{H_{d0}}\right),\\
A^- &= \kappa_+(\beta^-)a_1\left(1 - \frac{\kappa_-(1)H_d}{\kappa_+(1)H_{d0}}\right).
\end{aligned}$$

Finally, we introduce

$$B_1 = \frac{H_d}{H_{d0}\varkappa_+(1)}, \tag{A2}$$

and write

$$A^+ = \varkappa_-(\beta^+)a_1(1 - \varkappa_+(1)B_1), \tag{A3}$$

$$A^- = \varkappa_+(\beta^-)a_1(1 - \varkappa_-(1)B_1). \tag{A4}$$

Next, we solve (31) and (32) for $A^+R^{\beta^+}$ and $A^-R^{\beta^-}$ and derive

$$\begin{aligned} A^+R^{\beta^+} &= \varkappa_-(\beta^+)\left(a_2 - \frac{(1-\alpha_1)b}{\varkappa_-(1)}H_-\right), \\ A^-R^{\beta^-} &= \varkappa_+(\beta^-)\left(a_2 - \frac{(1-\alpha_1)b}{\varkappa_+(1)}H_-\right). \end{aligned}$$

Recall that $H_{e0} = H_+ = \varkappa_-(1)a_2/b/(1-\alpha_1)$, hence $(1-\alpha_1)b/a_2 = \varkappa_-(1)/H_+$, and we can write

$$\begin{aligned} A^+R^{\beta^+} &= \varkappa_-(\beta^+)a_2\left(1 - \frac{H_-}{H_+}\right), \\ A^-R^{\beta^-} &= \varkappa_+(\beta^-)a_2\left(1 - \frac{\varkappa_-(1)H_-}{\varkappa_+(1)H_+}\right). \end{aligned}$$

Set

$$B_2 = \frac{H_-}{H_+\varkappa_+(1)}. \tag{A5}$$

Notice that $H_- < H_+$ iff $B_2 < 1/\varkappa_+(1)$. Finally, we write

$$A^+R^{\beta^+} = \varkappa_-(\beta^+)a_2(1 - \varkappa_+(1)B_2), \tag{A6}$$

$$A^-R^{\beta^-} = \varkappa_+(\beta^-)a_2(1 - \varkappa_-(1)B_2). \tag{A7}$$

Divide (A6) by (A3) and (A7) by (A4) and recall that $a_1/a_2 = a$:

$$\begin{aligned} aR^{\beta^+} &= \frac{1 - \varkappa_+(1)B_2}{1 - \varkappa_+(1)B_1}, \\ aR^{\beta^-} &= \frac{1 - \varkappa_-(1)B_2}{1 - \varkappa_-(1)B_1}. \end{aligned}$$

For a given R, the system above is a system of linear equations w.r.t. B_1, B_2 which can be written as

$$\begin{aligned} aR^{\beta^+}B_1 - B_2 &= \frac{aR^{\beta^+} - 1}{\varkappa_+(1)}, \\ -aR^{\beta^-}B_1 + B_2 &= \frac{1 - aR^{\beta^-}}{\varkappa_-(1)}. \end{aligned}$$

Applying the Cramer theorem, we find

$$B_1 = \frac{\Delta_1}{a\Delta}, \quad B_2 = \frac{\Delta_2}{\Delta},$$

where Δ, Δ_1, and Δ_2 are the functions of R given by (33) and (34). Observe that B_1, B_2 derived above are the same as in (35). It follows from (A2) and (A5) that $B_1 > 0$ and $B_2 > 0$. By definition, $R = H_-/H_d > 1$, therefore $\Delta(R) > 0$. Notice that

$$\Delta_1(1) = (a-1)\left(\frac{1}{\varkappa_+(1)} - \frac{1}{\varkappa_-(1)}\right) < 0$$

and

$$\Delta'_1(R) = \frac{a\beta^+R^{\beta^+-1}}{\varkappa_+(1)} - \frac{a\beta^-R^{\beta^--1}}{\varkappa_-(1)} > 0,$$

because $\beta^- < 0 < 1 < \beta^+$. As $R \to \infty$, $\Delta_1(R) \to \infty$ as well, hence there exists $R_1 > 1$ s.t. $\Delta_1(R) > 0$ for all $R > R_1$. Clearly, R_1 is a solution to $\Delta_1(R) = 0$, or, equivalently to

$$\frac{aR^{\beta^+} - 1}{\varkappa_+(1)} = -\frac{1 - aR^{\beta^-}}{\varkappa_-(1)},$$

and we can conclude that $1 < R_1 < a^{-1/\beta^-}$.

Next, we notice that $\Delta_2(1) = \Delta_1(1) < 0$. Write $\Delta_2(R)$ as

$$\begin{aligned} \Delta_2(R) &= R^{\beta^-}\left(\frac{aR^{\beta^+} - 1}{\varkappa_+(1)} + \frac{1 - aR^{\beta^-}}{\varkappa_-(1)}R^{\beta^+-\beta^-}\right) \\ &= R^{\beta^-}\left(\Delta_1(R) + \frac{1 - aR^{\beta^-}}{\varkappa_-(1)}\left(R^{\beta^+-\beta^-} - 1\right)\right). \end{aligned}$$

We see that $\Delta_2(R) < \Delta_1(R)$ for all $R < a^{-1/\beta^-}$, and

$$\Delta_2\left(a^{-1/\beta^-}\right) = \frac{\Delta_1\left(a^{-1/\beta^-}\right)}{a} > 0.$$

Hence there exists $R_2 \in (R_1, a^{-1/\beta^-})$, s.t., $\Delta_2(R_2) = 0$. In the general case, it is difficult to establish uniqueness of R_2. We will concentrate on the case of shock dynamics which is common in the literature: assume that the process for the shocks is a martingale, that is

$$E^x[e^{X_t}] = e^x. \tag{A8}$$

Since the moment generating function of a random variable $y \sim N(b, \sigma^2)$ is

$$E[e^{\beta y}] = e^{\Psi(\beta)},$$

and X_t, conditioned on $X_0 = x$, is distributed as a normal variable with the mean $bt + x$ and variance $\sigma^2 t$, we derive

$$E^x\left[e^{\beta X_t}\right] = e^{\beta x + t\Psi(\beta)}. \tag{A9}$$

Therefore (A8) is satisfied iff $\Psi(1) = 0$, equivalently, iff $\mu = -\sigma^2/2$. For this value of the drift, μ, the roots of the characteristic equation, β^+, β^-, satisfy $\beta^- + \beta^+ = 1$. Straightforward calculations show that in this case,

$$\begin{aligned}\Delta''_2(R) &= -\frac{\beta^-(\beta^- - 1)R^{\beta^- -2}}{\kappa_+(1)} + \frac{\beta^+(\beta^+ - 1)R^{\beta^+ -2}}{\kappa_-(1)}\\ &= -\frac{\beta^{-2}R^{\beta^- -2}}{\kappa_+(1)\kappa_-(1)} + \frac{\beta^{+2}R^{\beta^+ -2}}{\kappa_+(1)\kappa_-(1)}\\ &= \frac{\beta^{-2}R^{\beta^- -2}}{\kappa_+(1)\kappa_-(1)}\left((\frac{\beta^+}{\beta^-})^2 R^{\beta^+ - \beta^-} - 1\right) > 0 \text{ for } R > 1.\end{aligned}$$

Hence Δ_2 is convex for $R > 1$. Therefore R_2 is a unique zero of $\Delta_2(R)$ on (R_1, ∞), hence $\Delta_2(R) > 0$ for $R > R_2$.

Recall that we need $B_2 < 1/\kappa_+(1)$. On the strength of (34), the last inequality can be written as

$$\kappa_+(1)\Delta_2 < \Delta \;\Leftrightarrow\; R^{\beta^+}(1 - aR^{\beta^-}) > \frac{\kappa_+(1)}{\kappa_-(1)}R^{\beta^+}(1 - aR^{\beta^-}).$$

Since $\kappa_+(1)/\kappa_-(1) > 1$, the last inequality can hold iff $1 - aR^{\beta^-} < 0$, that is, iff $R < a^{-1/\beta^-}$.

Consider the ratio

$$\frac{B_2}{B_1} = \frac{a\Delta_2(R)}{\Delta_1(R)}. \tag{A10}$$

We can also write

$$\frac{B_2}{B_1} = \frac{H_- H_{d0}}{H_+ H_d} = R\frac{H_{d0}}{H_{e0}} = R(1 - \alpha_1)a. \tag{A11}$$

From (A10) and (A11), we obtain Equation (36).

Let R be a solution to (36) on $\left(R_2,\, a^{-1/\beta^-}\right)$. Then $B_1 > 0$, $0 < B_2 < 1/\kappa_+(1)$, $H_- = RH_d$, and $H_d < H_- < H_+$. From (A2), we derive

$$H_d = \kappa_+(1)B_1H_{d0} = \kappa_+(1)B_1\kappa_-(1)a_1/b,$$

and (37) obtains. From (A5),

$$H_- = RH_d = \kappa_+(1)B_2H_+.$$

This concludes the proof of Lemma 3.2.

A.4 | Proof of Theorem 3.3

(a) Since $\Delta_2(R_2) = 0$, and $\Delta_1(R_2) > 0$, we have $F(R_2) = 0$, and $(1 - \alpha_1)R_2 > 0$. Hence $(1 - \alpha_1)R_2 > F(R_2)$. To prove existence of a solution to (36) on $\left(R_2,\, a^{-1/\beta^-}\right)$, it suffices to show that $F\left(a^{-1/\beta^-}\right) > (1 - \alpha_1)a^{-1/\beta^-}$. We have $F\left(a^{-1/\beta^-}\right) = a^{-1}$, therefore we need to show that

$$a^{-1} > (1 - \alpha_1)a^{-1/\beta^-} \;\Leftrightarrow\; \frac{1}{1 - \alpha_1} > a^{(\beta^- - 1)/\beta^-} \;\Leftrightarrow a < (1 - \alpha_1)^{-\kappa_-(1)}.$$

The last inequality is (15). Hence (36) has a solution on $\left(R_2,\, a^{-1/\beta^-}\right)$ iff (15) is satisfied. Numerical examples for a wide range of parameters indicate that F is concave for $R \in \left(R_2,\, a^{-1/\beta^-}\right)$; therefore (36) has a unique solution.

(b) Follows from Lemma 3.2.

(c) Follows from the definition of the value of the equity.

(d) By construction, the value function (42) satisfies Equations (13) on (H_+, ∞), (25) on (H_d, H_-) and value matching and smooth pasting conditions at $Y = H_+, H_-, H_d$. Hence (42) is the option value and $\tau_+ = \inf(t \geq 0 | Y_t \leq H_+)$, $\tau_- = \inf(t \geq 0 | Y_t \geq H_-)$, and $\tau_d = \inf(t \geq 0 | Y_t \leq H_d)$ are the optimal stopping times (see, e.g., Section 9 in Peskir & Shiryaev, 2006). Moreover, since (36) has a unique solution, and H_+, H_-, and H_d are uniquely defined by (22), (38), and (37), respectively, there are no other optimal stopping times in the class of hitting times, and the inaction region is uniquely described as $(H_d, H_-) \cup (H_+, \infty)$.

A.5 | Long-run distribution of active firms

Let N and g_{act} be the equilibrium entry rate and stationary distribution of active firms, respectively. We represent g_{act} in the form $g_{\text{act}} = Ng^0$. The distribution of entrants is Ng, and each entrant leaves the industry instantly unless the initial shock is either in the Buridan zone or good luck zone. It is more convenient to derive the long-run distributions in terms of $x = \log X$. We will also use the notation: $h_d = \log H_d$, $h_- = \log H_-$, and $h_+ = \log H_+$. Following the argument similar to the one in Dixit and Pindyck (1996), Chapter 8, Section 4C, we obtain the following boundary problems for $g^{0,-}$ and $g^{0,+}$, the restrictions of g^0 to $[h_d, h_-]$ and $[h_+, +\infty)$, respectively:

$$\left(\lambda - \frac{\sigma^2}{2}\partial_x^2 + \mu\partial_x\right)g^{0,-}(x) = g(x), \quad h_d < x < h_-, \tag{A12}$$

$$g^{0,-}(h_d) = g^{0,-}(h_-) = 0, \tag{A13}$$

and

$$\left(\lambda - \frac{\sigma^2}{2}\partial_x^2 + \mu\partial_x\right)g^{0,+}(x) = g(x), \quad h_+ < x, \tag{A14}$$

$$g^{0,+}(h_+) = g^{0,+}(+\infty) = 0. \tag{A15}$$

After $g^{0,-}$ and $g^{0,+}$ are calculated, we find the rates of default and exit (we refer the reader to Dixit & Pindyck, 1996, Chapter 8, Section 4C):

$$R_{\text{def}} = \frac{\sigma^2}{2}N(g^{0,-})'(h_d), \quad R_{\text{exit}} = \frac{\sigma^2}{2}N[-(g^{0,-})'(h_-) + (g^{0,+})'(h_+)]. \tag{A16}$$

Notice that the rate of default in (A16) is the rate of endogenous default only, and to account for the total default rate, one needs to add the exogenous default rate to R_{def}. The exogenous default happens due to the exogenous death; and the rate is λN_{act}, where N_{act} is the total number of active firms which can be calculated as

$$N_{\text{act}} = N\left[\int_{h_d}^{h_-} g^{0,-}(x)dx + \int_{h_+}^{+\infty} g^{0,+}(x)dx\right]. \tag{A17}$$

It is evident that whatever the analytical expressions for $g^{0,-}$ and $g^{0,+}$ are, both functions are independent of N, and therefore, we can use them to find N from the last equilibrium condition. Given N, the aggregate output is

$$q = N\left[\int_{h_d}^{h_-} e^x g^{0,-}(x)dx + \int_{h_+}^{+\infty} e^x g^{0,+}(x)dx\right].$$

If the demand function is Q, then using the market clearing condition, $q = Q(P)$, we obtain

$$N = Q(P)/\left[\int_{h_d}^{h_-} e^x g^{0,-}(x)dx + \int_{h_+}^{+\infty} e^x g^{0,+}(x)dx\right]. \tag{A18}$$

Finally, the distribution of the active firms is ${g_{\text{act}}}^-(x) = Ng^{0,-}(x)$, for x in the Buridan zone, and ${g_{\text{act}}}^+(x) = Ng^{0,+}(x)$, for x in the good luck zone.

In the case of an exponential g, explicit calculations are straightforward. In the next subsection, we present calculations for the case of a bimodal distribution concentrated around two points.

A.6 | The case of a bimodal distribution concentrated around two points

In this subsection, we calculate explicitly (albeit approximately) the equilibrium distribution of firms assuming that the p.d.f. of the initial shocks is concentrated in a very small neighborhood of two points $x_- \in (h_d, h_-)$ and $x_+ > h_+$. This situation is possible in equilibrium with $\alpha_3 = 1$ as well as with $\alpha_3 > 1$. Note that if $\alpha_3 = 1$, then one of the points must be in the good luck zone, and the other below the exit zone $[h_-, h_+]$. Assuming $g = c^- g_- + c^+ g_+$, where $g_\pm$ is concentrated around $x_\pm$ ($c_\pm > 0$, $c^+ + c^- = 1$), and taking into account that the process is a diffusion, we conclude that $g^0 = c^- g^{0,-} + c^+ g^{0,+}$, where $g^{0,-}$ is the solution to problems (A12) and (A13) with $g = g_-$, and $g^{0,+}$ is the solution to problems (A14) and (A15) with $g = g_+$. We derive an approximate formula for $g^{0,-}$ replacing g_- with the Dirac delta function supported at x_-.

Denote by $\tilde{\beta}^- < 0 < \tilde{\beta}^+$ the roots of the characteristic equation

$$\lambda - \frac{\sigma^2}{2}\beta^2 + \mu\beta = 0.$$

A particular solution to (A12) can be found as

$$g_p(x) = \frac{-\tilde{\beta}^-\tilde{\beta}^+}{\lambda}\int_0^{+\infty} e^{-\tilde{\beta}^+ y}\mathbf{1}_{(x_-,+\infty)}(x+y)e^{\tilde{\beta}^-(x+y-x_-)}dy.$$

If $x < x_-$, then

$$\begin{aligned} g_p(x) &= \frac{-\tilde{\beta}^-\tilde{\beta}^+}{\lambda}\int_{x_- - x}^{+\infty} e^{\tilde{\beta}^-(x-x_-)-(\tilde{\beta}^+-\tilde{\beta}^-)y}dy \\ &= \frac{-\tilde{\beta}^-\tilde{\beta}^+}{\lambda(\tilde{\beta}^+ - \tilde{\beta}^-)}e^{\tilde{\beta}^-(x-x_-)-(\tilde{\beta}^+-\tilde{\beta}^-)(x-x_-)} = \frac{-\tilde{\beta}^-\tilde{\beta}^+}{\lambda(\tilde{\beta}^+ - \tilde{\beta}^-)}e^{\tilde{\beta}^+(x-x_-)}, \end{aligned}$$

and if $x \geq x_-$, then

$$g_p(x) = \frac{-\tilde{\beta}^-\tilde{\beta}^+}{\lambda}\int_0^{+\infty} e^{\tilde{\beta}^-(x-x_-)-(\tilde{\beta}^+-\tilde{\beta}^-)y}dy = \frac{-\tilde{\beta}^-\tilde{\beta}^+}{\lambda(\tilde{\beta}^+ - \tilde{\beta}^-)}e^{\tilde{\beta}^-(x-x_-)}.$$

Thus,

$$g_p(x) = \frac{-\tilde{\beta}^-\tilde{\beta}^+}{\lambda(\tilde{\beta}^+ - \tilde{\beta}^-)}\left\{e^{\tilde{\beta}^+(x-x_-)}\mathbf{1}_{(-\infty,x_-)}(x) + e^{\tilde{\beta}^-(x-x_-)}\mathbf{1}_{[x_-,+\infty)}(x)\right\}.$$

The general solution is

$$g^{0,-}(x) = C^+ e^{\tilde{\beta}^+ x} + C^- e^{\tilde{\beta}^- x} + g_p(x),$$

where $C^\pm$ are found from the zero boundary conditions:

$$
\begin{aligned}
C^{+}e^{\tilde{\beta}^{+}h_d} + C^{-}e^{\tilde{\beta}^{-}h_d} + \frac{-\tilde{\beta}^{-}\tilde{\beta}^{+}}{\lambda(\tilde{\beta}^{+}-\tilde{\beta}^{-})}e^{\tilde{\beta}^{+}(h_d-x_-)} &= 0,\\
C^{+}e^{\tilde{\beta}^{+}h_-} + C^{-}e^{\tilde{\beta}^{-}h_-} + \frac{-\tilde{\beta}^{-}\tilde{\beta}^{+}}{\lambda(\tilde{\beta}^{+}-\tilde{\beta}^{-})}e^{\tilde{\beta}^{-}(h_--x_-)} &= 0.
\end{aligned}
$$

Using the Cramer theorem, we obtain

$$
C^{+} = \frac{\tilde{\beta}^{-}\tilde{\beta}^{+}\Delta^{+}}{\lambda(\tilde{\beta}^{+}-\tilde{\beta}^{-})\Delta}, \quad C^{-} = \frac{\tilde{\beta}^{-}\tilde{\beta}^{+}\Delta^{-}}{\lambda(\tilde{\beta}^{+}-\tilde{\beta}^{-})\Delta},
$$

where

$$
\begin{aligned}
\Delta &= e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_-} - e^{\tilde{\beta}^{+}h_-+\tilde{\beta}^{-}h_d},\\
\Delta^{+} &= e^{\tilde{\beta}^{+}(h_d-x_-)+\tilde{\beta}^{-}h_-} - e^{\tilde{\beta}^{-}(h_--x_-+h_d)},\\
\Delta^{-} &= e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}(h_--x_-)} - e^{\tilde{\beta}^{+}(h_-+h_d-x_-)}.
\end{aligned}
$$

In the result, we obtain

$$
\begin{aligned}
g^{0,-}(x) = \frac{-\tilde{\beta}^{-}\tilde{\beta}^{+}}{\lambda(\tilde{\beta}^{+}-\tilde{\beta}^{-})}\Bigg\{ & e^{\tilde{\beta}^{+}(x-x_-)}\mathbf{1}_{(-\infty,x_-)}(x) + e^{\tilde{\beta}^{-}(x-x_-)}\mathbf{1}_{[x_-,+\infty)}(x)\\
&+ \frac{e^{\tilde{\beta}^{+}(h_d-x_-)} - e^{\tilde{\beta}^{-}(h_d-x_-)}}{e^{\tilde{\beta}^{+}h_-+\tilde{\beta}^{-}h_d} - e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_-}}e^{\tilde{\beta}^{-}h_-+\tilde{\beta}^{+}x} + \frac{e^{\tilde{\beta}^{-}(h_--x_-)} - e^{\tilde{\beta}^{+}(h_--x_-)}}{e^{\tilde{\beta}^{+}h_-+\tilde{\beta}^{-}h_d} - e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_-}}e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}x}\Bigg\}.
\end{aligned} \tag{A19}
$$

Substituting (A19) and $g^{0,+}(x) = 0$ into (A16) and (A18), we can obtain explicit formulas for the rates of endogenous default and exit from the Buridan zone. The rates are

$$
\begin{aligned}
R_{\text{def}} = -c^{-}\frac{N\sigma^2}{2}\frac{\tilde{\beta}^{-}\tilde{\beta}^{+}}{\lambda(\tilde{\beta}^{+}-\tilde{\beta}^{-})}\Bigg\{ & \frac{e^{\tilde{\beta}^{+}(h_d-x_-)} - e^{\tilde{\beta}^{-}(h_d-x_-)}}{e^{\tilde{\beta}^{+}h_-+\tilde{\beta}^{-}h_d} - e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_-}}\tilde{\beta}^{+}e^{\tilde{\beta}^{-}h_-+\tilde{\beta}^{+}h_d}\\
&+ \tilde{\beta}^{+}e^{\tilde{\beta}^{+}(h_d-x_-)} + \frac{e^{\tilde{\beta}^{-}(h_--x_-)} - e^{\tilde{\beta}^{+}(h_--x_-)}}{e^{\tilde{\beta}^{+}h_-+\tilde{\beta}^{-}h_d} - e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_-}}\tilde{\beta}^{-}e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_d}\Bigg\},
\end{aligned} \tag{A20}
$$

and

$$
\begin{aligned}
R^{-}_{\text{exit}} = c^{-}\frac{N\sigma^2}{2}\frac{\tilde{\beta}^{-}\tilde{\beta}^{+}}{\lambda(\tilde{\beta}^{+}-\tilde{\beta}^{-})}\Bigg\{ & \frac{e^{\tilde{\beta}^{+}(h_d-x_-)} - e^{\tilde{\beta}^{-}(h_d-x_-)}}{e^{\tilde{\beta}^{+}h_-+\tilde{\beta}^{-}h_d} - e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_-}}\tilde{\beta}^{+}e^{\tilde{\beta}^{-}h_-+\tilde{\beta}^{+}h_-}\\
&+ \tilde{\beta}^{-}e^{\tilde{\beta}^{-}(h_--x_-)} + \frac{e^{\tilde{\beta}^{-}(h_--x_-)} - e^{\tilde{\beta}^{+}(h_--x_-)}}{e^{\tilde{\beta}^{+}h_-+\tilde{\beta}^{-}h_d} - e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_-}}\tilde{\beta}^{-}e^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_-}\Bigg\}.
\end{aligned} \tag{A21}
$$

Using (A17) and (A19), we obtain the number of active firms in the Buridan zone

$$\begin{aligned}
N_{\mathrm{act}}^{-} &= \frac{-c^{-}N\tilde{\beta}^{-}\tilde{\beta}^{+}}{\lambda(\tilde{\beta}^{+}-\tilde{\beta}^{-})}\left\{\frac{1-\mathrm{e}^{\tilde{\beta}^{+}(h_d-x_-)}}{\tilde{\beta}^{+}}+\frac{\mathrm{e}^{\tilde{\beta}^{-}(h_d-x_-)}-1}{\tilde{\beta}^{-}}\right.\\
&\quad+\frac{\mathrm{e}^{\tilde{\beta}^{+}(h_d-x_-)}-\mathrm{e}^{\tilde{\beta}^{-}(h_d-x_-)}}{\mathrm{e}^{\tilde{\beta}^{+}h_-+\tilde{\beta}^{-}h_d}-\mathrm{e}^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_-}}\mathrm{e}^{\tilde{\beta}^{-}h_-}(\mathrm{e}^{\tilde{\beta}^{+}h_-}-\mathrm{e}^{\tilde{\beta}^{+}h_d})\\
&\quad\left.+\frac{\mathrm{e}^{\tilde{\beta}^{-}(h_--x_-)}-\mathrm{e}^{\tilde{\beta}^{+}(h_--x_-)}}{\mathrm{e}^{\tilde{\beta}^{+}h_-+\tilde{\beta}^{-}h_d}-\mathrm{e}^{\tilde{\beta}^{+}h_d+\tilde{\beta}^{-}h_-}}\mathrm{e}^{\tilde{\beta}^{+}h_d}(\mathrm{e}^{\tilde{\beta}^{-}h_-}-\mathrm{e}^{\tilde{\beta}^{-}h_d})\right\}.
\end{aligned} \tag{A22}$$

Similarly to (A19), we obtain, for $x \geq h_+$,

$$\begin{aligned}
g^{0,+}(x) &= \frac{-\tilde{\beta}^{-}\tilde{\beta}^{+}}{\lambda(\tilde{\beta}^{+}-\tilde{\beta}^{-})}\{\mathrm{e}^{\tilde{\beta}^{+}(x-x_+)}\mathbf{1}_{(-\infty,x_+)}(x)\\
&\quad+\mathrm{e}^{\tilde{\beta}^{-}(x-x_+)}\mathbf{1}_{[x_+,+\infty)}(x)-\mathrm{e}^{\tilde{\beta}^{+}(h_+-x_+)-\tilde{\beta}^{-}(h_+-x)}\}.
\end{aligned} \tag{A23}$$

The rate of exit from the good luck zone is

$$\begin{aligned}
R_{\mathrm{exit}}^{+} &= c^{+}\frac{-N\sigma^2\tilde{\beta}^{-}\tilde{\beta}^{+}}{2\lambda(\tilde{\beta}^{+}-\tilde{\beta}^{-})}\{\tilde{\beta}^{+}\mathrm{e}^{\tilde{\beta}^{+}(h_+-x_+)}-\tilde{\beta}^{-}\mathrm{e}^{\tilde{\beta}^{+}(h_+-x_+)}\}\\
&= -c^{+}\frac{N\sigma^2\tilde{\beta}^{-}\tilde{\beta}^{+}}{2\lambda}\mathrm{e}^{\tilde{\beta}^{+}(h_+-x_+)}.
\end{aligned} \tag{A24}$$

The number of active firms in the good luck zone is

$$N_{\mathrm{act}}^{+} = \frac{c^{+}N}{\lambda}\left(1-\mathrm{e}^{\tilde{\beta}^{+}(h_+-x_+)}\right). \tag{A25}$$

Hence, in the case of the bimodal distribution concentrated around x_- in the Buridan zone and x_+ in the good luck zone, the proportion of active firms that default per unit of time is

$$R_{\mathrm{def}}^{0} = \frac{R_{\mathrm{def}}}{N_{\mathrm{act}}^{-}+N_{\mathrm{act}}^{+}}+\lambda, \tag{A26}$$

and the proportion of active firms that exit per unit of time is

$$R_{\mathrm{exit}}^{0} = \frac{R_{\mathrm{exit}}^{-}+R_{\mathrm{exit}}^{+}}{N_{\mathrm{act}}^{-}+N_{\mathrm{act}}^{+}}. \tag{A27}$$