PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 11, November 2011, Pages 3985–3997
S 0002-9939(2011)10803-7
Article electronically published on March 17, 2011

FACTOR MAP, DIAMOND AND DENSITY OF PRESSURE FUNCTIONS

JUNG-CHAO BAN AND CHIH-HUNG CHANG

(Communicated by Yingfei Yi)

ABSTRACT. Letting $\pi : X \to Y$ be a one-block factor map and Φ be an almost-additive potential function on X, we prove that if π has diamond, then the pressure $P(X, \Phi)$ is strictly larger than $P(Y, \pi\Phi)$. Furthermore, if we define the ratio $\rho(\Phi) = P(X, \Phi)/P(Y, \pi\Phi)$, then $\rho(\Phi) > 1$ and it can be proved that there exists a family of pairs $\{(\pi_i, X_i)\}_{i=1}^k$ such that $\pi_i : X_i \to Y$ is a factor map between X_i and Y, $X_i \subseteq X$ is a subshift of finite type such that $\rho(\pi_i, \Phi|_{X_i})$ (the ratio of the pressure function for $P(X_i, \Phi|_{X_i})$ and $P(Y, \pi\Phi)$) is dense in $[1, \rho(\Phi)]$. This extends the result of Quas and Trow for the entropy case.

1. Introduction

The present paper is devoted to studying the topic that for a given one-block factor map, how the existence of diamond and different kinds of potential functions affect the pressure function, and what is the *density* of the pressure. This is mainly motivated by the related works concerning entropy [2] and the dense entropy property [3]. Before formulating our results, we give some notation and background first. Let $\pi : X \to Y$ be a 1-block factor map between two one-dimensional mixing subshifts of finite type X and Y. Then the following result is well-known:

Theorem 1.1 (Theorem 4.1.7 of [4]). *Suppose $\pi : X \to Y$ is a one-block factor map between mixing subshifts of finite type (SFTs for short) and that X has positive entropy. Then either*

(1) $\pi : X \to Y$ *is uniformly bounded-to-one,*
(2) π *has no diamond,*
(3) $h_{top}(X) = h_{top}(Y)$

Received by the editors May 3, 2010 and, in revised form, September 19, 2010.
2010 *Mathematics Subject Classification.* Primary 37D35; Secondary 37B10, 37A35, 28A78.
Key words and phrases. Factor map, diamond, **a**-weighted thermodynamic formalism, density of pressure.
The first author is partially supported by the National Science Council, ROC (Contract No. NSC 98-2628-M-259-001), National Center for Theoretical Sciences (NCTS) and CMPT (Center for Mathematics and Theoretical Physics) in National Central University.
The second author wishes to express his gratitude to Professor Cheng-Hsiung Hsu for his valuable comments and thanks the National Central University for financial support.

or

(4) $\pi : X \to Y$ *is uncountable-to-one on some points,*
(5) π *has diamond,*
(6) $h_{top}(X) > h_{top}(Y)$.

We remark here that Theorem 1.1 also holds for higher-dimensional SFTs (Theorem 3.6 of [2]). However, unlike the one-dimensional case, some stronger specification property is needed for the higher-dimensional case. Let $\Phi = (\log \phi_n)_{n=1}^{\infty}$ be a real-valued potential function on X, i.e., $\log \phi_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$. We define the *push-forward* potential in Y by

$$\pi\Phi(y) = \left(\max_{x \in X:\ \pi(x)=y} \log \pi\phi_n(x) \right)_{n=1}^{\infty} = \left(\max_{x \in X:\ \pi(x)=y} \log \phi_n \circ \pi^{-1}(x) \right)_{n=1}^{\infty}, \tag{1.1}$$

and define the pressure function on X by

$$P(X, \Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{I \in X_n} \sup_{x \in [I]} \phi_n(x) \tag{1.2}$$

whenever the limit exists and X_n stands for the collection of n-cylinders in X. It is of interest to know whether Theorem 1.1 holds for the pressure function. Precisely, we consider the following.

Problem 1.2. If $\pi : X \to Y$ is a one-block factor map with diamond between X and Y, which potential functions $\Phi = (\log \phi_n)_{n=1}^{\infty}$ on X make $P(X, \Phi) > P(Y, \pi\Phi)$?

Problem 1.3. Under the same assumption of Problem 1.2, what is the difference $P(X, \Phi) - P(Y, \pi\Phi)$?

In this investigation, we have the following results:

Theorem A. *Let $\pi : X \to Y$ be a one-block factor map between two mixing shift spaces X and Y. Assume $\Phi = (\log \phi_n) \in \mathbf{C}_{aa}(X, T)$ (defined in (2.2)) and satisfies the bounded distortion property (defined in (2.4)). Then either*

(1) $P(X, \Phi) > P(Y, \pi\Phi)$,
(2) π *has diamonds,*

or

(3) $P(X, \Phi) = P(Y, \pi\Phi)$,
(4) π *has no diamond.*

For the case π has diamond, Theorem A shows that $P(X, \Phi) - P(Y, \pi\Phi) > 0$ if and only if π has diamond. This extends Theorem 1.1 to pressure for $\Phi \in \mathbf{C}_{aa}(X, T)$. For the difference $P(X, \Phi) - P(Y, \pi\Phi)$ of Problem 1.3, we have the following result.

Theorem B. *Under the same assumption of Theorem A, let $\nu \in \mathcal{M}(Y, S)$ be the equilibrium measure on Y with respect to the push-forward potential $\pi\Phi(y) = (\max_{x \in X:\ \pi(x)=y} \log \phi_n \circ \pi^{-1}(x))_{n=1}^{\infty}$ and $\mu \in \mathcal{M}(X, T)$ be the conditional equilibrium state of Φ with respect to ν (see (2.13)) and Proposition 2.6). Then*

$$P(X, \Phi) - P(Y, \pi\Phi) = h_\mu(T) - h_\nu(S). \tag{1.3}$$

Theorem B indicates that the difference of $P(X, \Phi) - P(Y, \pi\Phi)$ equals $h_\mu(T) - h_\nu(S)$, and it is useful for characterizing the positivity of $P(X, \Phi) - P(Y, \pi\Phi)$ by showing $h_\mu(T) > h_\nu(S)$ (see Theorem 3.1).

On the other hand, for a dynamical system (X,T), it is natural to ask what are the subsystems of X and what are the possible values of the entropies (resp. pressure) of the subsystems of X. If X is an n-dimensional SFT for $n \in \mathbb{N}$, Quas and Trow [3] show that for $\varepsilon > 0$, there exists a proper subshift $\hat{X}$ of X which is also an SFT with the property that

$$h_{top}(X) - \varepsilon < h_{top}(\hat{X}) < h_{top}(X).$$

If $\pi : X \to Y$ is a one-block factor map with diamond and $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathbf{C}_{aa}(X,T)$ with the bounded distortion property, we define the ratio of $P(X,\Phi)$ and $P(Y,\pi\Phi)$ by

$$\rho(\pi,\Phi) = P(X,\Phi)/P(Y,\pi\Phi) \text{ if } P(Y,\pi\Phi) \neq 0. \tag{1.4}$$

It follows from Theorem A that we have $\rho(\pi,\Phi) > 1$. We ask the following questions:

Problem 1.4. Under the same assumptions of Theorem A, does there exist a family $\pi_i : X_i \to Y$ where X_i is a subsystem of X and $\pi_i = \pi|_{X_i}$ is a one-block factor for all $i \in \mathbb{N}$ such that

$$\rho(\pi_i, \Phi|_{X_i}) = P(X_i, \Phi|_{X_i})/P(Y,\pi\Phi)$$

is dense in $[1, \rho(\pi,\Phi)]$, where $\Phi|_{X_i}$ stands for the restriction of Φ to X_i?

For Problem 1.4, we have the following result.

Theorem C. *Under the same assumptions of Theorem A, there exists a family of pairs* $\{(\pi_i, X_i)\}_{i=1}^{\infty}$ *such that*

(1) X_i *is a subsystem of* X, $\forall i \in \mathbb{N}$;
(2) X_i *is an SFT*, $\forall i \in \mathbb{N}$;
(3) $\pi_i : X_i \to Y$ *is a one-block factor map for all* $i \in \mathbb{N}$ *such that* $\rho(\pi_i, \Phi|_{X_i}) \neq 0$ *are dense in* $[1, \rho(\Phi)]$. *That is, for* $\varepsilon > 0$, *there exists an integer* $k = k(\varepsilon)$ *and a monotone decreasing sequence* $\{P(X_i, \Phi|_{X_i})\}_{i=1}^{k}$ *such that*
$$P(X_i, \Phi|_{X_i}) - P(X_{i+1}, \Phi|_{X_{i+1}}) < \varepsilon,$$
and, for all $p \in [P(Y,\pi\Phi), P(X,\Phi)]$, *there exists a* $1 \leq j \leq k$ *with*
$$P(X_{j+1}, \Phi|_{X_{j+1}}) < p < P(X_j, \Phi|_{X_j}).$$

The content of this paper is the following. In Section 2, we introduce the so-called **a**-*weighted thermodynamic formalism* developed recently by Barral and Feng [1]. This tool is useful for the proofs of Theorem B and Theorem A, and we leave their proofs to Section 3 and give the proof of Theorem C in Section 4.

2. Preliminaries and a-weighted thermodynamic formalism

For the reader's convenience we recall some definitions and known results in this section.

2.1. Sub-additive thermodynamic formalism.

The following definitions and notation come from the recent works of Barral and Feng [1].

Definition 2.1. (1) We say that $\Phi = (\log \phi_n)_{n=1}^{\infty}$ is *sub-additive* on X and write $\Phi \in \mathbf{C}_s(X,T)$ if there exists $C_1 > 0$ such that

$$\phi_{n+m}(x) \leq C_1 \phi_n(x) \phi_m(T^n x) \ \forall x \in X \text{ and } n, m \in \mathbb{N}. \tag{2.1}$$

(2) We say that $\Phi = (\log \phi_n)_{n=1}^{\infty}$ is *asymptotically sub-additive* on X and write $\Phi \in \mathbf{C}_{ass}(X,T)$ if for any $\varepsilon > 0$ there exists a sub-additive potential $\Psi = (\log \psi_n(x))_{n=1}^{\infty}$ on X such that

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} |\log \phi_n(x) - \log \psi_n(x)| \le \varepsilon. \tag{2.2}$$

(3) $\Phi = (\log \phi_n)_{n=1}^{\infty}$ is called *almost additive* on X and we write $\Phi \in \mathbf{C}_{aa}(X,T)$ if ϕ_n is positive and continuous on X for all $n \in \mathbb{N}$ and there exists $C_2 > 0$ such that

$$C_2^{-1}\phi_n(x)\phi_m(T^n(x)) \le \phi_{n+m}(x) \le C_2\phi_n(x)\phi_m(T^n(x)), \tag{2.3}$$

$\forall x \in X$ and $n, m \in \mathbb{N}$.

(4) $\Phi = (\log \phi_n)_{n=1}^{\infty}$ is called the *bounded distortion property* if there exists a constant $C_3 > 0$ such that

$$C_3^{-1}\phi_n(y) \le \phi_n(x) \le C_3\phi_n(y) \ \forall x, y \in I \in X_n. \tag{2.4}$$

We introduce the following result of the variational principle for the asymptotic sub-additive potential Φ on X.

Theorem 2.2 (Feng and Huang, [9])**.** *Let $\Phi \in \mathbf{C}_{ass}(X,T)$ and $T : X \to X$ be a mixing continuous transformation. Then*

$$P(X,\Phi) = \sup\{h_\eta(T) + \Phi_*(\eta) : \eta \in \mathcal{M}(X,T)\}, \tag{2.5}$$

where $\mathcal{M}(X,T)$ denotes the collection of T-invariant probability measures on X endowed with the weak-star topology, $h_\eta(T)$ denote the measure-theoretic entropies of η and $\Phi_*(\eta)$ is given by

$$\Phi_*(\eta) = \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n(x) d\eta(x). \tag{2.6}$$

A measure $\mu \in \mathcal{M}(X,T)$ attaining the supremum of (2.5) is called the *equilibrium measure* of Φ. A measure $\mu \in \mathcal{M}(X,T)$ is called a *Gibbs measure* with respect to Φ if there exists a $Q_1 > 0$ such that

$$Q_1^{-1} \le \frac{\mu([I])}{\exp(-nP(X,\Phi))\phi_n([I])} \le Q_1, \ \forall I \in X_n, \ n \in \mathbb{N}, \tag{2.7}$$

where

$$\phi_n([I]) = \sup_{x\in[I]} \phi_n(x). \tag{2.8}$$

It follows from Theorem 2.2 that we can construct the variational principle for $P(X,\Phi)$ and $P(Y,\pi\Phi)$.

Proposition 2.3. *Let $\Phi \in \mathbf{C}_{ass}(X,T)$ and let $\pi\Phi$ be defined as in* (1.1) *on Y. Then:*

(1) $\pi\Phi \in \mathbf{C}_{ass}(Y,S)$.

(2) *The two variational principles hold:*

$$P(X,\Phi) = \sup\{h_\eta(S) + \Phi_*(\eta) : \eta \in \mathcal{M}(X,T)\}, \tag{2.9}$$

$$P(Y,\pi\Phi) = \sup\{h_\xi(S) + (\pi\Phi)_*(\xi) : \xi \in \mathcal{M}(Y,S)\}. \tag{2.10}$$

Furthermore, if we assume $\Phi \in \mathbf{C}_{aa}(X,T)$ and satisfies the bounded distortion property, then

(3) $\pi\Phi \in \mathbf{C}_{aa}(Y,S)$.

(4) *There exist unique equilibrium measures* $\mu \in \mathcal{M}(X,T)$, $\nu \in \mathcal{M}(Y,S)$ *attaining the supremums of* (2.9) *and* (2.10) *respectively.*
(5) *Both* μ *and* ν *satisfy the Gibbs property; i.e.,* (2.7) *holds for* μ *and* ν.

2.2. **a-weighted thermodynamic formalism.** Let (X,T) and (Y,S) be mixing shift spaces. Assume $\Phi \in \mathbf{C}_{ass}(X,T)$ and $\mathbf{a} = (a,b) \in \mathbb{R}^2$ so that $a > 0$ and $b \geq 0$. Barral and Feng [1] introduce the $\mathbf{a}$-*weighted topological pressure* of Φ:

$$P^{\mathbf{a}}(X,\Phi) = \sup\left\{\Phi_*(\eta) + ah_\eta(T) + bh_{\eta\circ\pi^{-1}}(S) : \eta \in \mathcal{M}(X,T)\right\}. \tag{2.11}$$

A measure $\mu \in \mathcal{M}(X,T)$ attaining the supremum of (2.11) is called the $\mathbf{a}$-*weighted equilibrium state* of Φ. Let $\Phi = (\log\phi_n)_{n=1}^{\infty} \in \mathbf{C}_{ass}(X,T)$, define a sequence $\Psi = (\log\psi_n)_{n=1}^{\infty}$ of potentials on Y by

$$\psi_n(y) = \sum_{I\in X_n:\ [I]\cap\pi^{-1}(y)\neq\emptyset} \sup_{x\in[I]\cap\pi^{-1}(y)} \phi_n(x)^{\frac{1}{a}},\ y \in Y, \tag{2.12}$$

and set $\frac{a}{a+b}\Psi = (\log\left(\psi_n^{\frac{a}{a+b}}\right))_{n=1}^{\infty}$. For $\nu \in \mathcal{M}(Y,S)$, a measure $\mu \in \mathcal{M}(X,T)$ is called a *conditional equilibrium state* of Φ *with respect to* ν if $\mu\circ\pi^{-1} = \nu$ and

$$\Phi_*(\mu)+h_\mu(T)-h_\nu(S) = \sup\left\{\Phi_*(\eta) + h_\eta(T) - h_\nu(S) : \eta \in \mathcal{M}(X,T), \eta\circ\pi^{-1} = \nu\right\}. \tag{2.13}$$

Barral and Feng [1] developed the following results.

Theorem 2.4. *Let* $\mathbf{a} = (a,b) \in \mathbb{R}^2$ *so that* $a > 0$ *and* $b \geq 0$. *If* $\Phi \in \mathbf{C}_{ass}(X,T)$ *(resp.* $\Phi \in \mathbf{C}_{aa}(X,T)$*), then*

(1) Ψ *and* $\frac{a}{a+b}\Psi \in \mathbf{C}_{ass}(X,T)$ *(resp.* Ψ *and* $\frac{a}{a+b}\Psi \in \mathbf{C}_{aa}(X,T)$*);*
(2) $P^{\mathbf{a}}(X,\Phi) = (a+b)P(Y,\frac{a}{a+b}\Psi)$ *(*$P(Y,\Psi)$ *is defined in* (1.2)*);*
(3) μ *is an* $\mathbf{a}$*-weighted equilibrium state of* Φ *iff* $\nu = \mu\circ\pi^{-1}$ *is an equilibrium state of* $\frac{a}{a+b}\Psi$ *and* μ *is a conditional equilibrium state of* $\frac{1}{a}\Phi$ *with respect to* ν, *where* $\frac{1}{a}\Phi = (\log(\phi_n^{\frac{1}{a}}))_{n=1}^{\infty}$.

Letting $\mathbf{a} = (a,b) \in \mathbb{R}^2$ so that $a > 0$ and $b \geq 0$, a measure $\mu \in \mathcal{M}(X,T)$ is called an $\mathbf{a}$-*weighted Gibbs measure* if there exists $Q_2 > 0$ such that

$$Q_2^{-1} \leq \frac{\mu([I])\psi_n(\pi[I])^{\frac{b}{a+b}}}{\exp(-\frac{n}{a+b}P^{\mathbf{a}}(X,\Phi))\phi_n(I)^{\frac{1}{a}}} \leq Q_2, \tag{2.14}$$

where

$$\psi(J) = \sum_{I\in X_n:\ \pi I = J} \phi_n([I])^{\frac{1}{a}},\ \forall J \in Y_n. \tag{2.15}$$

The following theorem was also proved in [1]. It shows that the **a-**weighted Gibbs measure exists uniquely for $\Phi \in \mathbf{C}_{aa}(X,T)$ with the bounded distortion property.

Theorem 2.5. *Let* $\pi : X \to Y$ *be a one-block factor. Let* $\mathbf{a} = (a,b) \in \mathbb{R}^2$ *so that* $a > 0$ *and* $b \geq 0$. *Let* $\Phi \in \mathbf{C}_{aa}(X,T)$ *and satisfy the bounded distortion property. Then*

(1) Φ *has a unique* $\mathbf{a}$*-weighted equilibrium measure, say* μ*;*
(2) μ *is also the unique* $\mathbf{a}$*-weighted Gibbs measure of* Φ*;*

(3) *if we define* $\nu = \mu \circ \pi^{-1}$, *then there exists* Q_3 *and* $Q_4 > 0$ *such that for all* $J \in Y_n, n \in \mathbb{N}$ *and* $I = \pi(J)$,

$$Q_3^{-1} \leq \frac{\nu([J])}{\exp(-\frac{n}{a+b}P^{\mathbf{a}}(X,\Phi))\psi_n([J])^{\frac{a}{a+b}}} \leq Q_3 \tag{2.16}$$

and

$$Q_4^{-1} \leq \frac{\mu([I])^a \nu(\pi[J])^b}{\exp(-nP^{\mathbf{a}}(X,\Phi))\phi_n([I])} \leq Q_4. \tag{2.17}$$

Combining Theorem 2.4 and Theorem 2.5 we have the following.

Proposition 2.6. *Let* $\Phi \in \mathbf{C}_{aa}(X,T)$ *and satisfy the bounded distortion property and let* $\nu \in \mathcal{M}(Y,S)$ *be the Gibbs measure of* $\Psi = (\log \psi_n)_{n=1}^{\infty}$ *as defined in* (2.22)*; it is thus an equilibrium measure of* Ψ. *Then* Φ *has a unique conditional equilibrium measure* μ *with respect to* ν *and there exists a constant* $C_4 > 0$ *such that*

$$C_4^{-1} \leq \frac{\mu([I])\,\psi_n(\pi([I]))}{\nu(\pi([I]))\,\phi_n([I])} \leq C_4 \ \forall I \in X_n \text{ and } n \in \mathbb{N}, \tag{2.18}$$

where $\psi(J) = \sum_{I \in X_n : \pi(I) = J} \phi_n([I])$ *and* $\phi_n([I])$ *is defined in* (2.8)*).*

Next we show that $P(X,\Phi) = P(Y,\Psi)$.

Proposition 2.7. *Let* $\pi : X \to Y$ *be a one-block factor map and* $\Phi \in \mathbf{C}_{ass}(X,T)$. *Let* $\Psi \in \mathbf{C}_{ass}(Y,S)$ *be defined in* (2.22). *Then*

$$P(X,\Phi) = P(Y,\Psi). \tag{2.19}$$

Proof. Taking $\mathbf{a} = (1,0) \in \mathbb{R}^2$, it follows from (2.11) and Theorem 2.4 that

$$\begin{aligned} P^{\mathbf{a}}(X,\Phi) &= \sup\{\Phi_*(\eta) + h_\eta(T) : \eta \in \mathcal{M}(X,T)\} \\ &= P(Y,\Psi). \end{aligned} \tag{2.20}$$

Combining (2.20) and Theorem 2.2 with the fact that $\Phi \in \mathbf{C}_{ass}(X,T)$ yields

$$P(X,\Phi) = P^{\mathbf{a}}(X,\Phi) = \sup\{\Phi_*(\eta) + h_\eta(T) : \eta \in \mathcal{M}(X,T)\} = P(Y,\Psi).$$

This completes the proof. □

We end this subsection by introducing the *relativised variational principle*, which was developed by Ledrappier, Walters, Cao, Zhao, Feng and Huang (cf. [5], [8], [9] and [1]). This will be useful in the study of the relationship between $P(X,\Phi)$ and $P(Y,\pi\Phi)$.

Proposition 2.8 (Lemma 3.1 of [1])**.** *Let* $\Phi \in \mathbf{C}_{aa}(X,T)$ *and satisfy the bounded distortion property. If* $\nu \in \mathcal{M}(Y,S)$, *then:*

(1) *The relativised variational principle holds:*

$$\int_Y P(X,\Phi,\pi^{-1}(y))d\nu(y) = \sup\{h_\mu(T) - h_\nu(S) + \Phi_*(\mu)\}, \tag{2.21}$$

where the supremum is taken over all $\mu \in \mathcal{M}(X,T)$ *with* $\mu \circ \pi^{-1} = \nu \in \mathcal{M}(Y,S)$.

(2) *Define* $\Psi = (\log \psi_n)_{n=1}^{\infty}$ *on* Y, *where*

$$\psi_n(y) = \sum_{I\in X_n:\ [I]\cap\pi^{-1}(y)\neq\emptyset} \sup_{x\in[I]\cap\pi^{-1}(y)} \phi_n(x). \tag{2.22}$$

Then

$$\begin{aligned}\int_Y P(X,\Phi,\pi^{-1}(y))d\nu(y) &= \sup\{h_\mu(T) - h_\nu(S) + \Phi_*(\mu)\}\\ &= \Psi_*(\nu),\end{aligned}$$

where the supremum is taken over all $\mu \in \mathcal{M}(X,T)$ *with* $\mu\circ\pi^{-1} = \nu \in \mathcal{M}(Y,S)$ *and* $\Psi_*(\nu)$ *is defined in* (2.6).

3. Proofs of Theorem A and Theorem B

In this subsection we give the proofs of Theorem B and Theorem A. In the following, we assume $\Psi = (\log \psi_n)_{n=1}^{\infty}$, as defined in (2.22).

Proof of Theorem B. Take $\mathbf{a} = (1,0) \in \mathbb{R}^2$. It follows from Theorem 2.4 and Proposition 2.3 that we have $\pi\Phi$, $\Psi \in \mathbf{C}_{aa}(Y,S)$. Define the partition functions

$$Z_n(X,\Phi) = \sum_{I\in X_n} \phi_n(I),\ Z_n(Y,\Psi) = \sum_{J\in Y_n} \psi_n(J), \tag{3.1}$$

and

$$Z_n(Y,\pi\Phi) = \sum_{J\in Y_n} (\pi\phi)_n(J). \tag{3.2}$$

One can easily check that $(\log Z_n(X,\Phi))_{n=1}^{\infty} \in \mathbf{C}_{ass}(X,T)$, and $(\log Z_n(Y,\Psi))_{n=1}^{\infty}$, $(\log Z_n(Y,\pi\Phi))_{n=1}^{\infty} \in \mathbf{C}_{ass}(Y,S)$. By the standard argument, we conclude that $P(X,\Phi)$, $P(Y,\Psi)$ and $P(Y,\pi\Phi)$ exist. Since $P(X,\Phi) = P(Y,\Psi)$ (Proposition 2.7), for $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ we have

$$\exp(-n\varepsilon) \leq \frac{Z_n(X,\Phi)}{Z_n(Y,\Psi)} \leq \exp(n\varepsilon). \tag{3.3}$$

Let $\nu \in \mathcal{M}(Y,S)$ be the Gibbs measure of Ψ as in Proposition 2.3 and $\mu \in \mathcal{M}(X,T)$ be its unique conditional equilibrium measure according to Proposition 2.6. Then there exist Q_3 and $Q_4 > 0$ such that for all $I \in X_n$ and $J = \pi I \in Y_n$ with $n \in \mathbb{N}$,

$$Q_3^{-1}\exp(-nP(X,\Phi))\phi_n([I]) \leq \mu([I]) \leq Q_3\exp(-nP(X,\Phi))\phi_n([I]) \tag{3.4}$$

and

$$Q_4^{-1}\exp(-nP(Y,\Psi))\psi_n([J]) \leq \nu([J]) \leq Q_4\exp(-nP(Y,\Psi))\psi_n([J]). \tag{3.5}$$

Since μ and ν are Gibbs on X and Y, they are also ergodic. It follows from the Shannon-McMillian-Brieman Theorem, for $\varepsilon > 0$ there exists $N_2 > 0$ such that if $n \geq N_2$ and μ-a.e. $I \in X_n$ with $J = \pi([I]) \in Y_n$,

$$\exp(-n(h_\mu(T)+\varepsilon)) \leq \mu([I]) \leq \exp(-n(h_\mu(T)-\varepsilon)) \tag{3.6}$$

and

$$\exp(-n(h_\nu(S)+\varepsilon)) \leq \nu([J]) \leq \exp(-n(h_\nu(S)-\varepsilon)). \tag{3.7}$$

Let $n \geq \max\{N_1, N_2\}$. Combining (3.3), (3.4), (3.5), (3.6) and (3.7) with the fact that $P(Y,\Psi) = P(X,\Phi)$ we have

$$\begin{aligned}
Z_n(X,\Phi) &\leq Z_n(Y,\Psi)\exp(n\varepsilon) \\
&= \exp(n\varepsilon)\sum_{J\in Y_n}\psi_n([J]) = \exp(n\varepsilon)\sum_{J\in Y_n}\psi_n([J])\,\phi_n([I])^{-1}(\pi\phi)_n([J]) \\
&\leq Q_3Q_4\exp(n\varepsilon)\exp(n(P(Y,\Psi)-P(X,\Phi)))\sum_{J\in Y_n}\nu([J])\mu^{-1}([I])\,(\pi\phi)_n([J]) \\
&\leq Q_3Q_4\exp(n(h_\mu(T)-h_\nu(S)+3\varepsilon))\sum_{J\in Y_n}(\pi\phi)_n([J]). \qquad (3.8)
\end{aligned}$$

For the opposite inequality, we have

$$\begin{aligned}
Z_n(X,\Phi) &\geq \exp(-n\varepsilon)Z_n(Y,\Psi) = \exp(-n\varepsilon)\sum_{J\in Y_n}\psi_n([J])\,\phi_n([I])^{-1}(\pi\phi)_n([J]) \\
&\geq Q_3^{-1}Q_4^{-1}\exp(-n\varepsilon)\exp(n(P(Y,\Psi)-P(X,\Phi))) \\
&\quad\times\sum_{J\in Y_n}\nu([J])\mu^{-1}([I])\,(\pi\phi)_n([J]) \\
&\geq Q_3^{-1}Q_4^{-1}\exp(n(h_\mu(T)-h_\nu(S)-3\varepsilon))\sum_{J\in Y_n}(\pi\phi)_n([J]). \qquad (3.9)
\end{aligned}$$

Then (1.3) follows by dividing both sides of (3.8) and (3.9) with n and taking n to infinity. This completes the proof of Theorem B. □

For the proof of Theorem A, we need the following results. They show that under the same assumption of Theorem A with the fact that π has diamond, $P(X,\Phi)$ is strictly larger than $P(Y,\pi\Phi)$.

Theorem 3.1. *Let $\pi : X \to Y$ be a one-block factor with diamond. Let $\Phi \in \mathbf{C}_{aa}(X,T)$ and satisfy the bounded distortion property. If $\nu \in \mathcal{M}(Y,S)$ is the equilibrium state $\Psi = (\log\psi_n)_{n=1}^\infty$ as defined in (2.22) and μ is the conditional equilibrium state of Φ with respect to ν, then*

$$h_\mu(T) > h_\nu(S). \qquad (3.10)$$

Furthermore, $P(X,\Phi) > P(Y,\pi\Phi)$.

Proof. Since π has diamond, then by Theorem 3.6 of [2] there exists $y \in Y$, $C_5 > 0$ and $C_6 > 1$ such that

$$\#\left\{I \in X_n : I \cap \pi^{-1}(y) \neq \emptyset\right\} \geq C_5C_6^n. \qquad (3.11)$$

It follows from Proposition 2.6 that there exists $C_4 > 0$ such that $\forall I \in X_n$ and $J = \pi([I]) \in Y_n$,

$$C_4^{-1}\frac{\phi_n([I])}{\psi_n([J])} \leq \frac{\mu([I])}{\nu([J])} \leq C_4\frac{\phi_n([I])}{\psi_n([J])}. \qquad (3.12)$$

For μ-a.e. $I \in X_n$ with $y \in J = \pi(I)$, it follows from (3.11), (3.12) and the Shannon-McMillian-Brieman Theorem that

$$\begin{aligned}
h_\mu(T) &= \lim_{n\to\infty} \frac{-1}{n} \log \mu([I]) \\
&\geq \lim_{n\to\infty} \frac{\log C_4}{n} + \lim_{n\to\infty} \frac{-1}{n} \log \left\{ \frac{\phi_n([I])}{\psi_n([J])} \nu\left([J]\right) \right\} \qquad \text{(by (3.12))} \\
&= \lim_{n\to\infty} \frac{-1}{n} \left[\log \frac{\phi_n([I])}{\psi_n([J])} \right] - \frac{1}{n} \log \nu\left([J]\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\log \sum_{I \in X_n : I \cap \pi^{-1}(y) \neq \emptyset} \phi_n([I]) - \log \phi_n([I]) \right] + \lim_{n\to\infty} \frac{-1}{n} \log \nu\left([J]\right) \\
&\geq \lim_{n\to\infty} \frac{1}{n} \left[\log C_3^{-1} C_5 C_6^n \phi_n\left([I]\right) - \log \phi_n(I) \right] + \lim_{n\to\infty} \frac{-1}{n} \log \nu\left([J]\right) \\
&= \log C_6 + h_\nu(S).
\end{aligned}$$

The constant C_3 comes from the bounded distortion property for Φ ((2.4) in Definition 2.1). Since $C_6 > 1$, we have $h_\mu(T) > h_\nu(S)$. Combining Theorem B and (3.10) we have $P(X, \Phi) > P(Y, \pi\Phi)$, and the proof is completed. □

We continue the proof of Theorem A.

Proof of Theorem A. By the variational principle of $P(X, \Phi)$ and $P(Y, \pi\Phi)$ and Theorem 3.1, we only need to show that if $P(X, \Phi) > P(Y, \pi\Phi)$, then π has diamond. Assume π has no diamond. Then by Theorem 3.6 of [2], $h_{top}(X) = h_{top}(Y)$. Then $\pi : X \to Y$ is almost everywhere bounded-to-one. Using the identical argument of Theorem 3.1, we can also derive that $P(X, \Phi) = P(Y, \pi\Phi)$, a contradiction. This completes the proof of Theorem A □

4. Proof of Theorem C

Let $\rho(\pi, \Phi)$ be as defined in (1.4). It follows from Theorem A that we have

Proposition 4.1. *Under the same assumptions of Theorem A:*

1. *If π has no diamond, then $\rho(\pi, \Phi) = 1$.*
2. *If π has diamond, then $\rho(\pi, \Phi) > 1$.*

We are ready to give the proof of Theorem C. Some auxiliary results are needed. First we define $X \backslash I = X \backslash \bigcup_{i \in \mathbb{N}} T^{-i}(I)$. The following result comes from [3].

Theorem 4.2 (Theorem 2.9 of [3])**.** *Let X be an SFT with positive topological entropy and let $\mu \in \mathcal{M}(X, T)$ be the measure of maximal entropy. Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ and $I \in X_n$, then*

$$h_\mu(X) - \varepsilon \leq h_\mu(X \backslash I) \leq h_\mu(X). \tag{4.1}$$

Remark 4.3. We remark here that in [3], Quas and Trow derived that

$$h_{top}(X) - \varepsilon \leq h_{top}(X \backslash I) \leq h_{top}(X).$$

It is not hard to extend this result to (4.1) for μ is Gibbs for some potential Φ from their proof.

We now deduce the pressure from Theorem 4.2.

Theorem 4.4. *Let X be a mixing SFT, and let $\Phi \in \mathbf{C}_{aa}(X,T)$ and satisfy the bounded distortion property. For $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$ and for all $I \in X_n$ where $X\backslash I$ is mixing, we have*

$$P(X,\Phi) - \varepsilon \leq P(X\backslash I, \Phi|_{X\backslash I}) < P(X,\Phi).$$

Proof. Since $\Phi \in \mathbf{C}_{aa}(X,T)$ and satisfies the bounded distortion property, it follows from Proposition 2.3 that we may assume μ is the Gibbs measure for Φ on X. Let $\frac{\varepsilon}{2} > 0$ and $I_1 \in X_n$. According to the variational principle and Theorem 4.2,

$$P(X,\Phi) - \Phi_*(\mu) - \varepsilon/2 = h_\mu(X) - \varepsilon/2 \leq h_\mu(X\backslash I_1). \tag{4.2}$$

This shows that

$$P(X,\Phi) - \varepsilon/2 \leq h_\mu(X\backslash I_1) + \Phi_*(\mu). \tag{4.3}$$

We claim that $\Phi_*(\mu) \leq \left(\Phi|_{X\backslash I_1}\right)_*(\mu) + \varepsilon/2$. Indeed, since $\Phi \in \mathbf{C}_{aa}(X,T)$ we have

$$\begin{aligned}
\Phi_*(\mu) &= \lim_{n\to\infty} \int_X \frac{1}{n} \log \phi_n(x) d\mu(x) \leq \int_X \frac{1}{n} \log \phi_n(x) d\mu(x) \\
&\leq \sum_{I\in X_n} \int_I \frac{1}{n} \log \phi_n(x) d\mu(x) \leq \sum_{I\in X_n} \frac{1}{n} \log \phi_n([I])\, \mu([I]) \\
&\leq \sum_{I\in (X\backslash I_1)_n} \frac{1}{n} \log \phi_n([I])\, \mu([I]) + \frac{1}{n} \log \phi_n([I_1])\, \mu([I_1]) \\
&\leq \left(\Phi|_{X\backslash I_1}\right)_*(\mu) + \Phi_*(\mu)\, \mu([I_1]) + \delta,
\end{aligned}$$

for some small $\delta > 0$. We note here that the 4^{th} inequality follows from the Birkhoff ergodic theorem and the fact that $\Phi \in \mathbf{C}_{aa}(X,T)$. Then the claim follows by taking $\mu([I_1])$ small enough and the fact that $\delta \to 0$ as $n \to \infty$. Therefore, it follows from (4.3) that we have

$$\begin{aligned}
P(X,\Phi) &\leq h_\mu(X\backslash I_1) + \Phi_*(\mu) + \varepsilon/2 \\
&\leq h_\mu(X\backslash I_1) + \left(\Phi|_{X\backslash I_1}\right)_*(\mu) + \varepsilon. \\
&\leq P(X\backslash I_1, \Phi|_{X\backslash I_1}) + \varepsilon.
\end{aligned}$$

This completes the proof. □

Lemma 4.5. *Under the same assumptions of Theorem A, Theorem C holds if and only if for all $\varepsilon > 0$ there exists a family of pairs $\{(\pi_i, X_i)\}_{i=1}^k$ such that*

(1) *X_i is a subsystem of X, $\forall i = 1, \cdots, k$,*
(2) *X_i is an SFT, $\forall i = 1, \cdots, k$, and*
(3) *$\{B(P(X_i, \Phi|_{X_i}), \varepsilon)\}_{i=1}^k$ forms an ε-cover of $[P(Y,\pi\Phi), P(X,\Phi)]$.*

Proof. Let $\rho \in [1, \rho(\pi,\Phi)]$ and let $P(Y,\pi\Phi) = p > 0$. For $\varepsilon p > 0$ we assume that there exists a family of pairs $\{(\pi_i, X_i)\}_{i=1}^k$ where $\pi_i : X_i \to Y$ is a factor map and X_i is an SFT $\forall i = 1, \cdots, k$, and $\{B(P(T_i, \Phi|_{X_i}), \varepsilon p)\}_{i=1}^k$ forms an εp-cover of $[P(Y,\pi\Phi), P(X,\Phi)]$. Since $\rho \in [1, \rho(\pi,\Phi)]$ and

$$P(Y,\pi\Phi) = p \leq \rho p \leq \rho(\pi,\Phi) P(Y,\pi\Phi) = P(X,\Phi),$$

i.e., $\rho p \in [P(Y,\pi\Phi), P(X,\Phi)]$, there exists an $1 \leq i \leq k$ such that $\rho p \in B(P(X_i, \Phi|_{X_i}), \varepsilon p)$, i.e.,

$$|\rho p - P(T_i, \Phi|_{X_i})| \leq \varepsilon p.$$

This means that $|\rho - \rho(\pi_i, \Phi|_{X_i})| \leq \varepsilon$. On the other hand, for $\varepsilon > 0$, take a sequence $1 \leq \rho_1 = 1, \rho_2, \cdots, \rho_{k-1}, \rho_k = \rho(\pi, \Phi) \leq \rho(\pi, \Phi)$ with

$$\frac{\varepsilon}{2p} \leq |\rho_{i+1} - \rho_i| \leq \frac{\varepsilon}{p} \text{ for all } i \in 1, \cdots, k-1. \tag{4.4}$$

Define $I_i = [\rho_i, \rho_{i+1}]$ for $i \in 1, \cdots, k-1$. It follows from Theorem C that there exists a sequence $\{(\pi_i, X_i)\}_{i=1}^k$ where π_i is a factor from X_i to Y, $X_i \subseteq X$ is an SFT and $\rho(\pi_i, \Phi|_{X_i}) \in [\rho_i, \rho_{i+1}]$. If $i = k$, $P(X_k, \Phi|_{X_k}) \leq \rho_k P(Y, \pi\Phi) = P(T, \Phi)$, and if $i = 1$, $P(X_1, \Phi|_{X_1}) \geq \rho_1 P(Y, \pi\Phi) = P(Y, \pi\Phi)$. If $P(Y, \pi\Phi) \leq q \leq P(X, \Phi)$, then $1 \leq \frac{q}{p} \leq \rho(\pi, \Phi)$; thus $\frac{q}{p} \in I_i$ for some i. Therefore

$$\left| \frac{q}{p} - \rho(\pi_i, X_i) \right| \leq \varepsilon.$$

This implies that $|q - P(X_i, \Phi|_{X_i})| \leq p\varepsilon$, and this means that $\{B(P(X_i, \Phi|_{X_i}), p\varepsilon)\}_{i=1}^k$ forms a $p\varepsilon$-cover of $[P(Y, \pi\Phi), P(X\Phi)]$. This completes the proof. $\square$

Lemma 4.6. *Let $\pi : X \to Y$ be a one-block factor with diamond. Then properties (1), (2) and (3) of Lemma* 4.5 *hold.*

Proof. Without loss of generality we assume that X is full shift. Since $\Phi \in \mathbf{C}_{aa}(X, T)$, we conclude that $Z_n(X, \Phi) = \sum_{I \in X_n} \phi_n([I])$ is sub-additive. Then we have

$$P(X, \Phi) \leq \frac{1}{n} \log Z_n(X, \Phi), \ \forall n \in \mathbb{N}, \tag{4.5}$$

and for $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then

$$P(X, \Phi) \geq \frac{1}{n} \log Z_n(X, \Phi) - \varepsilon. \tag{4.6}$$

By Theorem 4.4 we choose $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ and for all $I \in X_n$, $X \backslash I$ is mixing, we have

$$P(X, \Phi) - \varepsilon \leq P(X \backslash I, \Phi|_{X \backslash I}) < P(X, \Phi).$$

Since $\pi : X \to Y$ has diamond, then for all $J \in Y_n$ with $n \in \mathbb{N}$,

$$\# \{I \in X_n : \pi(I) = J\} \geq 1.$$

We define $S_J = \{I \in X_n : \pi(I) = J\}$ and arrange $\{J : J \in Y_n\}$ in the lexicographic order with $J_1 < J_2 < \cdots < J_m$ and define $S_1 = S_{J_1}, S_2 = S_{J_2}, \cdots, S_m = S_{J_m}$. We also arrange S_i in the lexicographic order, i.e.,

$$S_i = \left\{ I_j^{(i)} \right\}_{j=1}^{|S_i|}, \text{ for } i = 1, \ldots, m,$$

where $|A|$ denotes the number of elements of A. For all i, define

$$\hat{S}_i = S_i \backslash I_1^{(i)};$$

i.e., drop the first pattern in S_i for all $i = 1, \cdots, m$. Therefore $\left|\hat{S}_i\right| = |S_i| - 1$ for $i = 1, \ldots, m$. Letting $\left|\hat{S}_1\right| + \left|\hat{S}_2\right| + \cdots + \left|\hat{S}_m\right| = r(m)$, we put all elements of

$\hat{S}_1, \cdots, \hat{S}_m$ together in order and renumber all its elements by $\{I_j\}_{j=1}^{r(m)}$, i.e.,

$$\left\{\hat{S}_1, \cdots, \hat{S}_m\right\} = \left\{\left\{I_2^{(1)}, \cdots, I_{|\hat{S}_1|}^{(1)}\right\}, \cdots, \left\{I_2^{(1)}, \cdots, I_{|\hat{S}_1|}^{(1)}\right\}\right\}$$
$$= \left\{I_1, I_2, \cdots, I_{r(m)}\right\}.$$

Define $S = \left\{I_1, I_2, \cdots, I_{r(m)}\right\}$. We construct a family of subsystems of X as follows:

(1) Let $X_0 = X$ and $X_1 = X \backslash I_1$.
(2) $X_j = X_{j-1} \backslash I_j$ for all $1 \leq j \leq r(m)$.
(3) Finally, $X_{r(m)} = X_0 \backslash \bigcup_{j=1}^{r(m)} I_j$.

For $1 \leq j \leq r(m)$, define $\pi_i = \pi|_{X_i} : X_i \to Y$. According to the construction, it can be easily checked that π_i is a factor for all $i \in [1, r(m)]$, and it follows from Theorem 4.4 that

$$P(X, \Phi) > P(X_1, \Phi|_{X_1}) > \cdots > P(X_{r(m)}, \Phi|_{X_{r(m)}}) \tag{4.7}$$

and

$$P(X_{i+1}, \Phi|_{X_{i+1}}) \geq P(X_i, \Phi|_{X_i}) - \varepsilon \text{ for all } i \in [1, r(m) - 1]. \tag{4.8}$$

Finally, we claim that

$$P(X_{r(m)}, \Phi|_{X_{r(m)}}) \leq P(Y, \pi\Phi) + \varepsilon. \tag{4.9}$$

Indeed, it follows from (4.6) and Proposition 2.7 that

$$P(Y, \Psi) = P(X, \Phi) \geq \frac{1}{n} \log Z_n(X, \Phi) - \varepsilon. \tag{4.10}$$

Since

$$Z_n(Y, \Psi) = \sum_{J \in Y_n} \sum_{I : \pi(I) = J} \phi_n(I)$$

is also sub-additive, there exists $N_3 \in \mathbb{N}$ such that if $n \geq N_3$, then

$$P(Y, \Psi) \geq \frac{1}{n} \log Z_n(Y, \Psi) - \varepsilon. \tag{4.11}$$

By the construction of $X_{r(m)}$, we have

$$Z_n(Y, \Psi) = Z_n(Y, \pi\Phi) = Z_n(X_{r(m)}, \Phi), \tag{4.12}$$

Combining (4.11), (4.5) and (4.12) we have that if $n \geq \max\{N_1, N_2, N_3\}$, then

$$P(Y, \pi\Phi) \geq \frac{1}{n} \log Z_n(Y, \pi\Phi) - \varepsilon = \frac{1}{n} \log Z_n(Y, \Psi) - \varepsilon$$
$$= \frac{1}{n} \log Z_n(X_{r(m)}, \Phi|_{X_{r(m)}}) - \varepsilon \geq P(X_{r(m)}, \Phi|_{X_{r(m)}}) - \varepsilon.$$

Thus (4.9) holds, and it follows from (4.7), (4.8) and (4.9) that

$$\bigcup_{i=1}^{r(m)} B(P(X_i, \Phi|_{X_i}), \varepsilon)$$

forms an ε-cover of $[P(Y, \pi\Phi), P(X, \Phi)]$. The proof is completed. $\square$

Finally, we finish the proof of Theorem C.

Proof of Theorem C. The proof is obtained by combining Lemma 4.5 and Lemma 4.6. The proof is completed. $\square$

Acknowledgement

The authors would like to thank the referees for their suggestions, which led to a great improvement of this paper.

References

[1] J. Barral and D. J. Feng, *Weighted thermodynamic formalism and applications* (2009), arXiv:0909.4247v1.

[2] R. Meester and J. Steif, *Higher-dimensional subshifts of finite type, factor maps and measures of maximal entropy*, Pacific J. Math. **2** (2001), 497–510. MR1868699 (2002j:37028)

[3] A. Quas and P. Trow, *Subshifts of multi-dimensional shifts of finite type*, Ergodic Theory Dynam. Systems **20** (2000), 859–874. MR1764932 (2001d:37011)

[4] B. Kitchen, *Symbolic dynamics, one-sided, two-sided and countable state Markov shifts.* Springer-Verlag (1998). MR1484730 (98k:58079)

[5] F. Ledrappier and P. Walters, *A relativised variational principle for continuous transformations*, J. London Math. Soc. **16** (1977), 568–576. MR0476995 (57:16540)

[6] P. Walters, *An introduction to ergodic theory.* Springer-Verlag (1982). MR648108 (84e:28017)

[7] Y. Zhao and Y. L. Cao, *On the topological pressure of random bundle transformations in sub-additive case*, J. Math. Anal. Appl. **342** (2008), 715–725. MR2440833 (2010d:37060)

[8] Y. L. Cao, D. J. Feng and W. Huang, *The thermodynamic formalism for sub-additive potentials*, Discrete Contin. Dyn. Syst. **20** (2008), 639–657. MR2373208 (2008k:37072)

[9] D. J. Feng and W. Huang, *Lyapunov spectrum of asymptotically sub-additive potentials*, Comm. Math. Phys. **297** (2010), 1–43. MR2645746

Department of Mathematics, National Dong Hwa University, Hualien 970003, Taiwan
E-mail address: jcban@mail.ndhu.edu.tw

Department of Mathematics, National Central University, Taoyuan 32001, Taiwan
E-mail address: chchang@mx.math.ncu.edu.tw