On the Quantitative Behavior of the Linear Cellular Automata

ON THE QUANTITATIVE BEHAVIOR OF THE LINEAR CELLULAR AUTOMATA

HASAN AKIN, JUNG-CHAO BAN, AND CHIH-HUNG CHANG

Abstract. In this paper, we study the quantitative behavior of one-dimensional linear cellular automata $T_{f[-r,r]}$, defined by local rule $f(x_{-r},\ldots,x_r)=\sum_{i=-r}^{r}\lambda_i x_i \pmod m$, acting on the space of all doubly infinite sequences with values in a finite ring $\mathbb{Z}_m$, $m\geq 2$. Once generalize the formulas given by Ban *et al.* [J. Cellular Automata **6** (2011) 385-397] for measure-theoretic entropy and topological pressure of one-dimensional cellular automata, we calculate the measure entropy and the topological pressure of the linear cellular automata with respect to the Bernoulli measure on the set $\mathbb{Z}_m^{\mathbb{Z}}$. Also, it is shown that the uniform Bernoulli measure is the unique equilibrium measure for linear cellular automata. We compare values of topological entropy and topological directional entropy by using the formula obtained by Akın [J. Computation and Appl. Math. 225 (2) (2009) 459-466]. The topological directional entropy is interpreted by means of figures. As an application, we demonstrate that the Hausdorff of the limit set of a linear cellular automaton is the unique root of Bowen's equation. Some open problems remain to be of interest.

Mathematics Subject Classification: Primary 28D20; Secondary 37A35, 37B40.
Key words: Cellular automata, measure entropy, directional entropy, maximal measure.

1. Introduction

Cellular automata (CAs for brevity), were introduced by Ulam and von Neumann, have been systematically studied by Hedlund from purely mathematical point of view [29]. The study of such dynamics called CAs has received remarkable attention in the last few years ([29, 20, 37]). CAs have been widely investigated in a lot of disciplines (e.g., mathematics, physics, computer science, and so on). In [20], it has been studied the dynamical behavior (ergodicity and topological transitivity) of D-dimensional linear CA (LCAs) over the ring $\mathbb{Z}_m$. Some open questions concerning the topological and ergodic dynamics of 1-D CAs have been addressed in [16].

It is well known that there are several notions of entropy (i.e. measure-theoretical, topological, directional and so on) of measure-preserving transformation on probability space in ergodic theory. It is important to know how these notions are related with each other. In the last years, a lot of works are devoted to this subject (see., e.g., [1]-[11], [28, 30, 35] and [40]). Recall that by the Variational Principle the topological entropy is the supremum of the entropies of invariant measures. In [3], the author has shown that the

Date: May 27, 2012.

uniform Bernoulli measure is a measure of maximal entropy for some one-dimensional (1-D) LCAs. In [7], the author has given a formula for computing the topological directional entropy of the $\mathbb{Z}^2$-actions generated by any 1-D LCA and the shift map. Also, he has proved a closed formula for the topological directional entropy of $\mathbb{Z}^2$-action generated by the pair $(T_{f[-r,r]}, \sigma)$ in the direction θ ($\theta \in [0, \pi]$) that can be efficiently and rightly computed by means of the coefficients of the local rule f as similar to [24].

The notion of entropy has been extensively studied in many disciplines (e.g., computer science, mathematics, physics, chemistry, information theory etc.) with different purposes. This notion first arose in thermodynamics as a measure of the heat absorbed (or emitted), when external work is done on a system. In probability theory, it constitutes a measure of the uncertainty. The entropy has been interpreted as a measure of the chaotic character of a dynamical system by many authors (see [16, 21, 24]), the value $h_{top}(T)$ has been in general accepted as a measure of the complexity of the dynamics of T over the space X. Some authors have stated that the topological entropy of a map is a crude global measure of the exponential complexity of the structure of the orbits of the map (see [16, 24]). Badii and Politi [12] have studied the complexity exhibited by some CAs with elementary rule by using both topological (graph-theoretical) and metric (thermodynamic) techniques. Lloyd and Pagels [31] have defined a measure complexity for the macroscopic states of physical systems. They have proved that the average complexity of a state must be proportional to Shannon entropy of the set of trajectories that experiment determines can lead to that state, $S = -\sum_i p_i \ln p_i$.

In this paper we study the measure-theoretical entropy and topological pressure of 1-D LCA $T_{f[-r,r]}$, defined by linear local rule $f(x_{-r}, \ldots, x_r) = \sum\limits_{i=-r}^{r} \lambda_i x_i \pmod m$, acting on the space of all doubly infinite sequences with values in a finite ring $\mathbb{Z}_m$, $m \geq 2$. In [3], the author has computed the measure-theoretical entropy with respect to uniform Bernoulli measure for the case $\lambda_i = 1$, for all $i \in \mathbb{Z}_m$. He also posed the question whether the maximal measure is unique. Recently, Ban *et al.* [14] have studied the complexity of permutative CA in two aspects: The viewpoints of thermodynamics and topological behavior. They [14] have also given the formulas to compute measure-theoretic entropy and topological pressure. In this paper, for both measure-theoretic entropy and topological pressure we extend results obtained in [14] to the case that $T_{f[-r,r]}$ is a LCA over the ring $\mathbb{Z}_m$ with respect to the Bernoulli measure. The formula of topological pressure of 1-D LCA actually extends D'amico *et al.*'s result [24] for the topological entropy of 1-D LCA. We also show that the uniform Bernoulli measure is the unique equilibrium measure for 1-D LCA $T_{f[-r,r]}$ with respect to some potential function. Furthermore, we interpret the topological directional entropy by means of figures. Also, some questions are raised and new open problems remain to be of interest.

One interesting phenomenon which is seen in LCA is the complexity of spatial-temporal patterns. Relations between LCA and fractals were first discussed by Wilson [41, 42, 43, 44]. Wilson showed that the sequence of the normalized spatial-temporal patterns of LCA with prime-states converges to a fractal pattern as a limit set. Based on this realization Wilson devised an efficient computation method for the Hausdorff dimension of the limit set. Takahashi has further extended Wilson's result to apply to all linear cellular automata [38, 39]. Through further research, Takahashi and Wilson went on to study cellular automata from the micro aspect perspective. Specifically, the spatial-temporal pattern is introduced by starting an initial configuration where state is one at the center and is zero otherwise. For example, consider a one-dimensional 2-states cellular automaton with the local rule $f(x_{-1}, x_0, x_1) = x_{-1}+x_1 \pmod 2$ and the initial configuration $x = (\cdots 0\ 1\ 0 \cdots)$, i.e., $x_0 = 1$ and $x_i = 0$ for $i \neq 0$. Take cells from state 1 until time $2^n - 1$ and normalize them by the factor 2^n. When n approaches infinity, the process attains a limit and is called a limit set. It is well-known that, in this case, the spatial-temporal pattern exhibits self-similarity and the limit set of this cellular automaton is known as the Sierpiński Gasket with the Hausdorff dimension $\log 3/\log 2$.

In this paper, we concentrate on the limit set $W_{f[-r,r]} = \bigcap_{i\geq 0} T^i_{f[-r,r]}(\mathbb{Z}_m^{\mathbb{Z}})$, where $T^i_{f[-r,r]} = T_{f[-r,r]} \circ T^{i-1}_{f[-r,r]}$ for $i \geq 1$. We note that self-similarity in spatial-temporal patterns may not be seen if every state is treated as distinct symbol. To be more specific let us consider two examples. Suppose the metric $d : \mathbb{Z}_m^{\mathbb{Z}} \times \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{R}$ is defined by

$$d(x, y) = \sum_{i\in\mathbb{Z}} \frac{|x_i - y_i|}{\beta^{|i|+1}}, \quad \text{for some} \quad \beta > 1.$$

Let $F, G : \mathbb{Z}_6^{\mathbb{Z}} \to \mathbb{Z}_6^{\mathbb{Z}}$ be cellular automata associate with the local rules $f(x_{-1}, x_0, x_1) = 2(x_{-1}+x_1) \pmod 6$ and $g(x_{-1}, x_0, x_1) = 2x_{-1} + 4x_1 \pmod 6$, respectively. Then the limit sets $W_f = W_g$ consist of bi-infinite strings whose state at each cell is either 0, 2 or 4. Theorem 6.1 suggests that $\dim_H(W_f) = \log 9/\log\beta$ (cf. Figure 1 and Example 6.2).

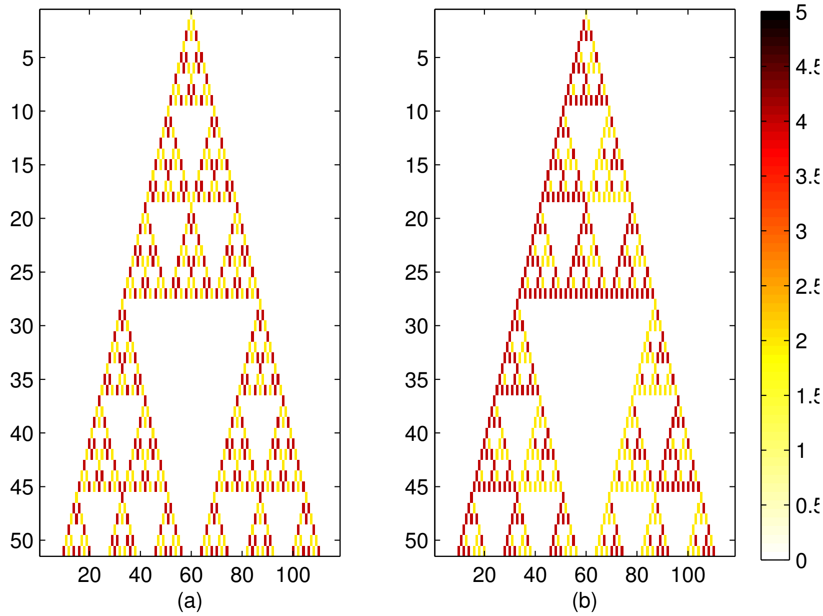

FIGURE 1. Two spatial-temporal patterns induced by 6-states cellular automaton with the same initial figuration. The local rule of Figs. (a) and (b) is $f(x_{-1}, x_0, x_1) = 2(x_{-1} + x_1)$ (mod 6) and $g(x_{-1}, x_0, x_1) = 2x_{-1} + 4x_1$ (mod 6), respectively. When we treat state-2 and state-4 as different state, (a) and (b) are two different patterns. However, these two local rule admit the same limit set under our consideration. Moreover, the self-similarity in Fig. (b) disappears.

Another example is as follows. Suppose the local rule is given by $f(x_{-2}, x_{-1}, x_0, x_1, x_2) = 18x_{-1} + 30(x_0 + x_1) + 20x_2$ (mod 90). From the micro perspective, the time-space pattern appears to be simple (cf. Figure 2). We show that, from the macro perspective, the Hausdorff dimension of $W_{f[-4,4]}$ is $2(2\log 3 + \log 5)/\log \beta$ (Example 6.3). This finding has also motivated our study of macro behavior.

In [19], Bowen discovered that the Hausdorff dimension of quasi-circles is the unique root of the pressure function $\psi(t) = P(T, t\varphi)$ with respect to some φ. This equation $\psi(t)$ is known now as Bowen's equation. As an application, we demonstrate that the Hausdorff dimension of the limit $W_{f[-r,r]} = \bigcap_{i\geq 0} T^i_{f[-r,r]}(\mathbb{Z}_m^{\mathbb{Z}})$ of a LCA is the unique root of the pressure function with respect to some potential function.

The outline of this paper is given as follows. In Section 2, the detailed definitions of the LCA will be presented. The measure-theoretical entropy of 1-D LCA with respect to the Bernoulli measure are computed in Section 3. The measure and topological entropies of LCA's are compared in the Section 4.

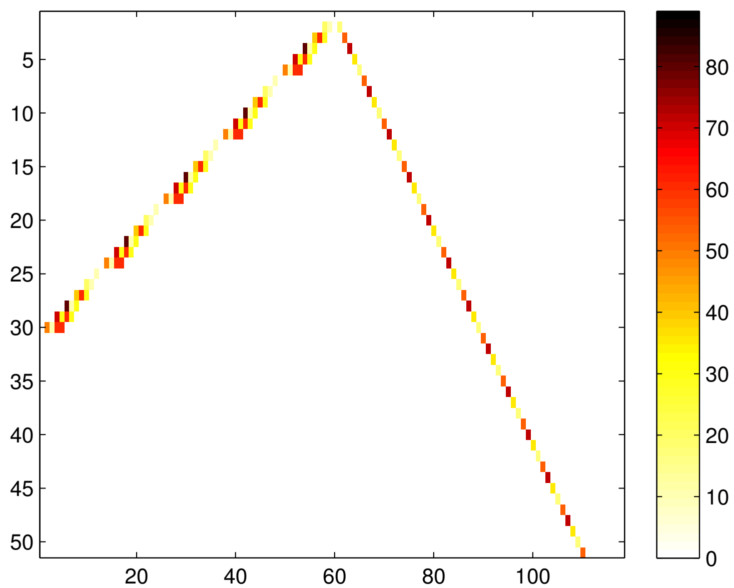

FIGURE 2. Spatial-temporal pattern determined by local rule $f(x_{-2}, x_{-1}, x_0, x_1, x_2) = 18x_{-1} + 30(x_0 + x_1) + 20x_2 \pmod{90}$ with initial configuration $(\cdots 0\ 1\ 0 \cdots)$. The self-similarity in this pattern is gone. However, the limit set still admits positive Hausdorff dimension.

Also the measure and topological directional entropies are discussed in detail in Section 4. In Section 5, we establish the existence and uniqueness of the equilibrium measure for LCA. Section 6 investigates the Hausdorff dimension of the limit set of LCA. Some conclusions and future studies are presented in the last section.

2. PRELIMINARIES

Let $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ $(m \geq 2)$ be the ring of the integers modulo m and $\mathbb{Z}_m^{\mathbb{Z}}$ be the space of all doubly-infinite sequences $x = (x_n)_{n=-\infty}^{\infty} \in \mathbb{Z}_m^{\mathbb{Z}}$ and $x_n \in \mathbb{Z}_m$. When equipped with the Tychonoff topology of a direct product, $\mathbb{Z}_m^{\mathbb{Z}}$ is a totally-disconnected compact space. The shift $\sigma : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$ defined by $(\sigma x)_i = x_{i+1}$ is a homeomorphism of the compact metric space $\mathbb{Z}_m^{\mathbb{Z}}$. A continuous map $T : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$ commuting with the shift (i.e. such that $T \circ \sigma = \sigma \circ T$) is called a cellular automaton (CA).

A 1-D CA is a continuous map $T_{f[-r,r]} : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$ defined by $(T_{f[-r,r]}x)_i = f(x_{i-r}, \ldots, x_{i+r})$, where $f : \mathbb{Z}_m^{2r+1} \to \mathbb{Z}_m$ is a given local rule or map. Favati *et al.* [27] have defined a local rule f, they have stated that a local rule f is linear (additive) if and only if it can be written as

$$f(x_{-r},\dots,x_r)=\sum_{i=-r}^{r}\lambda_i x_i \pmod m, \tag{2.1}$$

where at least one of the λ_i's is nonzero. We consider 1-D LCA $T_{f[-r,r]}$ generated by the local rule f:

$$(T_{f[-r,r]}x)=(y_n)_{n=-\infty}^{\infty}, y_n=f(x_{n-r},\dots,x_{n+r})=\sum_{i=-r}^{r}\lambda_i x_{n+i} \pmod m, \tag{2.2}$$

where $\lambda_{-r},\dots,\lambda_r\in\mathbb{Z}_m$.

We are going to use the notation $T_{f[-r,r]}$ for LCA-map defined in (2.2) to emphasize the local rule f.

Definition 2.1. The local rule defined by (2.1) is **permutative** in x_j, $-r\le j\le r$, if and only if for any given finite sequence

$$\overline{x}_{-r},\dots,\overline{x}_{j-1},\overline{x}_{j+1},\dots,\overline{x}_r\in\mathbb{Z}_m^{2r}$$

we have

$$\{f(\overline{x}_{-r},\dots,\overline{x}_{j-1},x_j,\overline{x}_{j+1},\dots,\overline{x}_r):x_j\in\mathbb{Z}_m\}=\mathbb{Z}_m.$$

The linear local rule f defined by (2.1) is permutative in the jth variable if and only if $\gcd(\lambda_j,m)=1$, where gcd denotes the greatest common divisor. Denote $R=\max\{i:\lambda_i\neq 0\}$ and $L=\min\{i:\lambda_i\neq 0\}$. A local rule f is said to be right (respectively, left) permutative, if $\gcd(\lambda_R,m)=1$ and $R>0$ (respectively, $\gcd(\lambda_L,m)=1$ and $L<0$). It is said that f is bipermutative if it is both left and right permutative (see [29, 37] for a formal definition of permutative).

3. The measure entropy of the 1-D LCA

In this section we study the measure entropy of the LCA defined in Eq. (2.2). In order to state our result, we first recall necessary definitions and Theorems. Let $(X,\mathcal{B},\mu,T)$ be a measure-theoretical dynamical system. If $\alpha=\{A_1,\dots,A_n\}$ and $\beta=\{B_1,\dots,B_m\}$ are two measurable partitions of X, then $\alpha\vee\beta=\{A_i\cap B_j:i=1,\dots,n;j=1,\dots,m\}$ is the partition of X. Also, $T^{-1}\alpha$ is the partition of X and $T^{-1}\alpha=\{T^{-1}A_1,\dots,T^{-1}A_n\}$ (see [25, 40] for the details).

Definition 3.1. Let α be a measurable partition of measure-theoretical dynamical system $(X,\mathcal{B},\mu,T)$. The partition α is called a strong generator if $\bigvee_{k=0}^{\infty}T^{-k}\alpha=\mathcal{B}$ (see [34]).

Definition 3.2. Let α be a measurable partition of X. The quantity

$$H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A)\log \mu(A)$$

is called the entropy of the partition α. The logarithm is usually taken to the base 2. Let α be a partition with finite entropy, then the quantity

$$h_\mu(T,\alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha)$$

is called the entropy of α with respect to T. The quantity

$$h_\mu(T) = \sup_\alpha \{h_\mu(T,\alpha) : \alpha \text{ is a partition with } H_\mu(\alpha) < \infty\}$$

is called the measure-theoretical entropy of $(X, \mathcal{B}, \mu, T)$, the entropy of T (with respect to μ).

Theorem 3.3 ([40, Theorem 4.18])**.** *If T is a measure-preserving transformation (but not necessary invertible) of the probability space $(X, \mathcal{B}, \mu)$ and if $\mathcal{A}$ is a finite sub-algebra of $\mathcal{B}$ with $\bigvee_{k=0}^{\infty} T^{-k}\alpha = \mathcal{B}$ then $h_\mu(T) = h_\mu(T,\alpha)$.*

Through this elucidation, we refer μ to the Bernoulli measure $(t_0, t_1, \ldots, t_{m-1})$ unless stated otherwise. Suppose μ is $T_{f[-r,r]}$-invariant. Ban *et al.* demonstrated the following result.

Theorem 3.4. *([14]) If f depends only on $x_i, \cdots, x_j$, where $i \le j$, $i, j \in \mathbb{Z}$. Denote by $L = \min\{i, 0\}$ and $R = \max\{j, 0\}$. We have the following results.*

(i) If f is left permutative, then $h_\mu(T_{f[-r,r]}) = L \sum_{k=0}^{m-1} t_k \log t_k$;

(ii) If f is right permutative, then $h_\mu(T_{f[-r,r]}) = -R \sum_{k=0}^{m-1} t_k \log t_k$;

(iii) If f is bipermutative, then $h_\mu(T_{f[-r,r]}) = -(R-L) \sum_{k=0}^{m-1} t_k \log t_k$.

A local rule f is said to be **permutative of type** 1 if f satisfies condition (i) in Theorem 3.4. Similarly, a local rule f is said to be **permutative of type** 2 if f satisfies condition (ii) in Theorem 3.4. For convenience, f is called **permutative** provided f is either bipermutative, permutative of type 1, or permutative of type 2.

Suppose $T_{f[-r,r]}$ is a one-dimensional LCA generated by a local rule f over $\mathbb{Z}_m$, and m is decomposed into $m = p_1^{k_1} \cdots p_h^{k_h}$ for some prime $p_1, \ldots, p_h$ and $h > 0$. For $i = 1, \ldots, h$, the factor map $\Phi_{p_i} : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_{p_i^{k_i}}^{\mathbb{Z}}$, defined by $(\Phi_i(x))_n = x_n \pmod{p_i^{k_i}}$ for $n \in \mathbb{Z}$, induces a push-forward measure $\mu_{p_i} = \mu \circ \Phi_{p_i}^{-1}$. The following theorem extends Theorem 3.4 to the case that $T_{f[-r,r]}$ is a LCA.

Theorem 3.5. *For $i = 1, \cdots, h$ define*

$$P_i = \{0\} \cup \{j : \gcd(\lambda_j, p_i) = 1\}, \ L_i = \min P_i, \ R_i = \max P_i.$$

If

$$\mu \cong \mu_{p_1} \times \mu_{p_2} \times \cdots \times \mu_{p_h}, \tag{3.1}$$

then

$$h_\mu(T_{f[-r,r]}) = -\sum_{i=1}^{n} (R_i - L_i) \left(\sum_{j=0}^{p_i^{k_i}-1} t_{i;j} \log t_{i;j} \right), \tag{3.2}$$

where $t_{i;j} = \mu_{p_i}(j)$, $1 \leq i \leq h, 0 \leq j \leq p_i^{k_i} - 1$.

Example 3.6. Suppose $m = 12 = 2^2 \cdot 3$ and $\mu = (t_0, t_1, \ldots, t_{11})$. Then the push forward measures of μ on $\mathbb{Z}_4^{\mathbb{Z}}$ and $\mathbb{Z}_3^{\mathbb{Z}}$ are

$$\mu_2 = (t_0 + t_4 + t_8, t_1 + t_5 + t_9, t_2 + t_6 + t_{10}, t_3 + t_7 + t_{11}),$$

$$\mu_3 = (t_0 + t_3 + t_6 + t_9, t_1 + t_4 + t_7 + t_{10}, t_2 + t_5 + t_8 + t_{11}),$$

respectively. If the local rule is given by $f(x_{-2}, x_{-1}, x_0, x_1, x_2) = 2x_{-1} + x_0 + 2x_1 + 3x_2$, it comes immediately that

$$R_1 = 2, \quad L_1 = 0, \quad R_2 = 1, \quad \text{and} \quad L_2 = -1.$$

Applying Theorem 3.5 we have

$$\begin{aligned} h_\mu(T_{f[-2,2]}) = -2(&(t_0 + t_4 + t_8) \log(t_0 + t_4 + t_8) + (t_1 + t_5 + t_9) \log(t_1 + t_5 + t_9) \\ &+ (t_2 + t_6 + t_{10}) \log(t_2 + t_6 + t_{10}) + (t_3 + t_7 + t_{11}) \log(t_3 + t_7 + t_{11}) \\ &+ (t_0 + t_3 + t_6 + t_9) \log(t_0 + t_3 + t_6 + t_9) \\ &+ (t_1 + t_4 + t_7 + t_{10}) \log(t_1 + t_4 + t_7 + t_{10}) \\ &+ (t_2 + t_5 + t_8 + t_{11}) \log(t_2 + t_5 + t_8 + t_{11})) \end{aligned}$$

provided $\mu \cong \mu_2 \times \mu_3$.

Theorem 3.5 raises a question that whether there exists an invariant measure μ which is isomorphic to the product of push-forward measures $\mu_{p_1} \times \mu_{p_2} \times \cdots \times \mu_{p_h}$. The following proposition gives an affirmative result.

Proposition 3.7. *Under the assumption of Theorem 3.5. If μ is the uniform Bernoulli measure, then $\mu \cong \mu_{p_1} \times \mu_{p_2} \times \cdots \times \mu_{p_h}$.*

The demonstration of Proposition 3.7 is straightforward, hence we omit the proof.

Example 3.8. Suppose $m = 72 = 2^3 3^2$, and μ is the uniform Bernoulli measure. Consider the local rule $f(x_{-4}, \cdots, x_4) = 6x_{-2} + 5x_{-1} + 9x_0 + 7x_1 + 33x_2 + 2x_3 + 16x_4$. It is easily seen that

$$P_1 = \{-1, 0, 1, 2\}, \quad P_2 = \{-1, 0, 1, 3, 4\}, \quad L_1 = -1, \quad L_2 = -1, \quad R_1 = 2, \quad R_2 = 4$$

Theorem 3.5 infers that

$$h_\mu(T_{f[-4,4]}) = -3 \sum_{j=0}^{7} \frac{1}{8} \log \frac{1}{8} - 5 \sum_{j=0}^{8} \frac{1}{9} \log \frac{1}{9} = 9 \log 2 + 10 \log 3$$

Proof of Theorem 3.5. We divide the proof into several steps.

Step 1. Denote by $\mathcal{L}$ the collection of linear local rules and $\mathbb{Z}_m[x, x^{-1}] = \{\Sigma_{i=n_1}^{n_2} a_i x^i, n_1, n_2 \in \mathbb{Z}\}$. Define $\chi : \mathcal{L} \to \mathbb{Z}_m[x, x^{-1}]$ by

$$\chi(\sum_{i=n_1}^{n_2} \lambda_i x_i) = \sum_{i=n_1}^{n_2} \lambda_i x^{-i}.$$

It is easily seen that χ is bijective. Moreover, let $\mathbb{Z}_m[[x, x^{-1}]]$ denote the power series generated by $\{x, x^{-1}\}$ over $\mathbb{Z}_m$. Then $\widehat{\chi} : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m[[x, x^{-1}]]$ defined by

$$\widehat{\chi}(\mathbf{b}) = \sum_{i=-\infty}^{\infty} b_i x^i, \quad \text{where} \quad \mathbf{b} = (b_i)_{i \in \mathbb{Z}} \in \mathbb{Z}_m^{\mathbb{Z}}$$

is also a bijection. Observe that, for each $\mathbf{b} = (b_i) \in \mathbb{Z}_m^{\mathbb{Z}}$,

$$\widehat{\chi}(T_{f[-r,r]}(\mathbf{b})) = \widehat{\chi}\left[(\sum_{n=-r+i}^{r+i} a_{n-i} b_n)_i\right] = \sum_{i=-\infty}^{\infty} \left(\sum_{n=-r+i}^{r+i} a_{n-i} b_n\right) x^i,$$

and

$$\mathbb{T}(\widehat{\chi}(\mathbf{b})) = \mathbb{T}\left(\sum_{i=-\infty}^{\infty} b_i x^i\right) = \sum_{n=-r}^{r} a_n x^{-n} \left(\sum_{i=-\infty}^{\infty} b_i x^i\right) = \sum_{i=-\infty}^{\infty} \left(\sum_{n=-r+i}^{r+i} a_{n-i} b_n\right) x^i,$$

where $\mathbb{T} \equiv \chi(f)$. This implements that the diagram

$$\begin{tikzcd}
\mathbb{Z}_m^{\mathbb{Z}} \arrow[r, "T"] \arrow[d, "\widehat{\chi}"'] & \mathbb{Z}_m^{\mathbb{Z}} \arrow[d, "\widehat{\chi}"] \\
\mathbb{Z}_m[[x, x^{-1}]] \arrow[r, "\mathbb{T}"'] & \mathbb{Z}_m[[x, x^{-1}]]
\end{tikzcd}$$

commutes. Moreover, yielding the Mathematical Induction we have $f^n = \chi^{-1}(\mathbb{T}^n)$ for all $n \in \mathbb{N}$, where $f^n = f \circ f^{n-1}$.

Step 2. Write $\mathbb{T}(x)$ as $\mathbb{T}(x) = \mathbb{T}_1(x) + p\mathbb{T}_2(x)$ such that $\mathbb{T}_1(x)$ is consisting of those monomials whose coefficients are coprime to p. We claim that, for all $i \in \mathbb{N}$,

$$(\mathbb{T}_1(x) + p\mathbb{T}_2(x))^{p^i} \equiv \mathbb{T}_1^{p^i}(x) \pmod{p^{i+1}}$$

It is seen that

$$\begin{aligned}
(\mathbb{T}_1(x) + p\mathbb{T}_2(x))^p &= \sum_{j=0}^{p} \binom{p}{j} (\mathbb{T}_1(x))^j (p\mathbb{T}_2(x))^{p-j} \\
&\equiv \sum_{j=p-1}^{p} \binom{p}{j} (\mathbb{T}_1(x))^j (p\mathbb{T}_2(x))^{p-j} \pmod{p^2} \\
&= p(\mathbb{T}_1(x))^{p-1}(p\mathbb{T}_2(x)) + \mathbb{T}_1^p(x) \\
&\equiv \mathbb{T}_1^p(x) \pmod{p^2}
\end{aligned}$$

Suppose that

$$(\mathbb{T}_1(x) + p\mathbb{T}_2(x))^{p^k} \equiv \mathbb{T}_1^{p^k}(x) \pmod{p^{k+1}}$$

In other words, $(\mathbb{T}_1(x) + p\mathbb{T}_2(x))^{p^k} = p^{k+1}Q(x) + \mathbb{T}_1^{p^k}(x)$ for some $Q(x)$. Therefore,

$$\begin{aligned}
(\mathbb{T}_1(x) + p\mathbb{T}_2(x))^{p^{k+1}} &= \left[(\mathbb{T}_1(x) + p\mathbb{T}_2(x))^{p^k}\right]^p \\
&= \left[p^{k+1}Q(x) + \mathbb{T}_1^{p^k}(x)\right]^p \\
&= \sum_{j=0}^{p} \binom{p}{j} (p^{k+1}Q(x))^j \left[\mathbb{T}_1^{p^k}(x)\right]^{p-j} \\
&\equiv \sum_{j=0}^{1} \binom{p}{j} (p^{k+1}Q(x))^j \left[\mathbb{T}_1^{p^k}(x)\right]^{p-j} \pmod{p^{k+2}} \\
&\equiv \mathbb{T}_1^{p^{k+1}}(x) \pmod{p^{k+2}}
\end{aligned}$$

This demonstrates our claim by the Mathematical Induction.

Step 3. Suppose $m = p^k$ for some prime number p and $k \in \mathbb{N}$. Let $n = p^{k-1}$, then $\mathbb{T}^n(x) \equiv \mathbb{T}_1^n(x)$ (mod p^k), and $f^n = \chi^{-1}(\mathbb{T}^n)$ is permutative. Denote by

$$P = \{0\} \cup \{j : (\lambda_j, p) = 1\}, \quad L = \min P, \quad \text{and} \quad R = \max P.$$

Note that $f^n(x_{-nr}, \ldots, x_{nr}) = \sum_{i=nL}^{nR} \lambda_i x_i$, where $\lambda_{nL}, \lambda_{nR}$ are coprime to p. Theorem 3.4 infers that $h_\mu(T^n_{f[-r,r]}) = -n(R-L) \sum_{i=0}^{p^k-1} t_i \log t_i$. Hence,

$$h_\mu(T_{f[-r,r]}) = \frac{h_\mu(T^n_{f[-r,r]})}{n} = -(R-L) \sum_{i=0}^{p^k-1} t_i \log t_i.$$

Step 4. Suppose $m = pq$ for some coprime factors $p, q \in \mathbb{N}$. Define $f_p : \mathbb{Z}_p^{2r+1} \to \mathbb{Z}_p$ and $f_q : \mathbb{Z}_q^{2r+1} \to \mathbb{Z}_q$ by

$$f_p(x_{-r}, \ldots, x_r) = f(x_{-r}, \ldots, x_r) \pmod p \quad \text{and} \quad f_q(x_{-r}, \ldots, x_r) = f(x_{-r}, \ldots, x_r) \pmod q,$$

respectively. Then f_p, f_q generate LCA $T_p : \mathbb{Z}_p^{\mathbb{Z}} \to \mathbb{Z}_p^{\mathbb{Z}}$ and $T_q : \mathbb{Z}_q^{\mathbb{Z}} \to \mathbb{Z}_q^{\mathbb{Z}}$. Observe that $\mathbb{Z}_m \cong \mathbb{Z}_p \times \mathbb{Z}_q$ induces an isomorphism $\Phi : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_p^{\mathbb{Z}} \times \mathbb{Z}_q^{\mathbb{Z}}$. A straightforward examination shows that the diagram

$$\begin{tikzcd} \mathbb{Z}_m^{\mathbb{Z}} \arrow[r, "T_{f[-r,r]}"] \arrow[d, "\Phi"'] & \mathbb{Z}_m^{\mathbb{Z}} \arrow[d, "\Phi"] \\ \mathbb{Z}_p^{\mathbb{Z}} \times \mathbb{Z}_q^{\mathbb{Z}} \arrow[r, "T_p \times T_q"'] & \mathbb{Z}_p^{\mathbb{Z}} \times \mathbb{Z}_q^{\mathbb{Z}} \end{tikzcd}$$

commutes. The isomorphism $\mu \cong \mu_p \times \mu_q$ indicates that $h_\mu(T_{f[-r,r]}) = h_{\mu_p \times \mu_q}(T_p \times T_q) = h_{\mu_p}(T_p) + h_{\mu_q}(T_q)$.

Step 5. From Step 4 we have the diagram

$$\begin{tikzcd} \mathbb{Z}_m^{\mathbb{Z}} \arrow[r, "T_{f[-r,r]}"] \arrow[d, "\Phi"'] & \mathbb{Z}_m^{\mathbb{Z}} \arrow[d, "\Phi"] \\ \mathbb{Z}_{p_1}^{\mathbb{Z}} \times \cdots \times \mathbb{Z}_{p_h}^{\mathbb{Z}} \arrow[r, "T_{p_1} \times \cdots \times T_{p_h}"'] & \mathbb{Z}_{p_1}^{\mathbb{Z}} \times \cdots \times \mathbb{Z}_{p_h}^{\mathbb{Z}} \end{tikzcd}$$

commutes and $\Phi = \Phi_{p_1} \times \cdots \times \Phi_{p_h}$ is an isomorphism. Since $\mu \cong \mu_{p_1} \times \mu_{p_2} \times \cdots \times \mu_{p_h}$, combine with Step 3 we have

$$h_\mu(T_{f[-r,r]}) = \sum_{i=1}^{h} h_{\mu_{p_i}}(T_{p_i}) = -\sum_{i=1}^{n} (R_i - L_i) \left(\sum_{j=0}^{p_i^{k_i}-1} t_{i;j} \log t_{i;j} \right),$$

where $t_{i;j} = \mu_{p_i}(j)$. This completes the proof. $\square$

The following proposition asserts that the uniform Bernoulli measure is invariant for any LCA $T_{f[-r,r]}$. The proof is straightforward, thus we omit it.

Proposition 3.9. *Let $T_{f[-r,r]}$ be a 1-D LCA generated by local rule f over $\mathbb{Z}_m$. Then $T_{f[-r,r]}$ is a uniform Bernoulli measure-preserving transformation.*

4. Entropies of CA's

4.1. **Comparison of Measure and topological entropies of CA's.** One of the basic invariants in ergodic theory is the notion of entropy, both topological and measure-theoretical setting. Since 1965, it has been raised the question in which cases there exists a measure μ satisfying

$$h_{top}(T) = h_{\mu}(T), \tag{4.1}$$

where $h_{top}(T)$ is the topological entropy of T (see [25, 40] for details). The measure μ satisfying Eq. (4.1) is called maximal measure.

In [3], the author studied the notion of the maximal measure for the case $\lambda_i = 1$ ($i = -r, \dots, r$). In this section we deal with this problem for 1-D LCA in the general case. We are going to study the topological entropy by means of algorithm defined by D'amico *et al.* [24].

Let $T_{f[-r,r]}$ be a 1-D LCA over $\mathbb{Z}_m$ with $f(x_{-r}, \dots, x_r) = \sum\limits_{i=-r}^{r} \lambda_i x_i \pmod{m}$, and let $m = p_1^{k_1} \dots p_h^{k_h}$ denote the prime factor decomposition of m.

Theorem 4.1 ([24, Theorem 2])**.** *Let $T_{f[-r,r]}$ be a 1-D LCA over $\mathbb{Z}_m$ with $f(x_{-r}, \dots, x_r) = \sum\limits_{i=-r}^{r} \lambda_i x_i$ $\pmod{m}$, and let $m = p_1^{k_1} \dots p_h^{k_h}$ denote the prime factor decomposition of m. Let L_i and R_i be defined as in Theorem 3.5. Then the topological entropy of the 1-D LCA $T_{f[-r,r]}$ is equal to*

$$h_{top}(T_{f[-r,r]}) = \sum_{i=1}^{h} k_i (R_i - L_i) \log p_i$$

Remark 4.2. From Theorem 4.1 it is clear that the topological entropy is dependent of the choice of λ_i.

It comes immediately from Theorems 3.5 and 4.1 that the uniform Bernoulli measure is a measure of maximal entropy for LCA.

Corollary 4.3. *Let μ be a uniform Bernoulli measure on $\mathbb{Z}_m^{\mathbb{Z}}$ with $t_i = \frac{1}{m}$, for each $i = 0, 1, \dots, m-1$, and $f(x_{-r}, \dots, x_r) = \sum_{i=-r}^{r} \lambda_i x_i \pmod m$ be a local rule. Then the uniform Bernoulli measure is a measure of maximal entropy.*

Example 4.4. Let us consider local rule

$$f(x_{-2}, x_{-1}, x_0, x_1, x_2) = 5x_{-2} + 7x_{-1} + 3x_0 + 2x_1 + 7x_2 \pmod{2^3}. \tag{4.2}$$

From the definition, it is clear that this local rule (4.2) is bipermutative. From Theorem 3.5, one can obtain that the quantity of the measure entropy of $T_{f[-2,2]}$ with respect to the uniform Bernoulli measure with $p(i) = \frac{1}{8}$ $(i = 0, 1, \dots, 7)$ is $h_\mu(T_{f[-2,2]}) = 12 \log 2$. Also, from Theorem 4.1, $h_{top}(T_{f[-2,2]}) = 12 \log 2$. Therefore, this measure is a measure of maximal entropy for the 1-D LCA generated by bipermutative local rule (4.2).

As mentioned in Introduction, the entropy has been interpreted as a measure of the chaotic character of a dynamical system by a lot of authors (see [16, 24]). Entropy and chaos are closely related. It is commonly accepted that an evidence of chaos in the sense of Li-Yorke [17] is positivity of topological entropy. Thus, we say that the dynamical system $(\mathbb{Z}_m^{\mathbb{Z}}, T_{f[-r,r]})$ is topologically chaotic if it fulfills the condition $h_{top}(T_{f[-r,r]}) > 0$ (see [21]). We conclude that the system $(\mathbb{Z}_{2^3}^{\mathbb{Z}}, T_{f[-2,2]})$ is chaotic. Also, it is well known that the topological entropy of a map is a crude global measure of the exponential complexity of the structure of the orbits of the map [24].

In the information theory, it is known that this measure carries the maximal amount of information of the system. Therefore, the value $12 \cdot \log 2 = 12$ bits is the maximal amount of information that can be carried (or the maximal measure of the amount of information obtained).

4.2. **Measure and topological directional entropies.** Let us study the topological directional entropy. The notion of the directional entropy of a $\mathbb{Z}^2$-action has first been introduced by Milnor [32]. Milnor has defined the notion of directional entropy function for $\mathbb{Z}^2$-action generated by a full shift and a block map. Since that time the directional entropy has been studied by many workers ([4, 7, 15, 18, 22, 23, 32, 33]). Park [33] has studied the continuity of directional entropy for arbitrary $\mathbb{Z}^2$-actions. Courbage an Kaminski have obtained an upper bound of the directional entropy function of any measurable CA-action with respect to an invariant Borel probability measure for the permutative ones [23].

Theorem 4.5 ([7, Theorem 4.4])**.** *Suppose that for $i = 1, \dots, h$ the rule defined by (2.1) is both left and right permutative. Let $T_{f[-r,r]}$ be a 1-D cellular automaton over $\mathbb{Z}_m^{\mathbb{Z}}$ with local rule defined by (2.1) and let $m = p_1^{k_1}.p_2^{k_2} \dots p_h^{k_h}$ be the prime factor decomposition of m. For $i = 1, \dots, h$ define*

$$P_i = \{0\} \cup \{j : gcd(\lambda_j, p_i) = 1\}, L_i = \min P_i, R_i = \max P_i.$$

$$\boldsymbol{P}_i = \{\theta_{L_i} = arccot(-L_i), \theta_{R_i} = arccot(-R_i)\}.$$

Then

$$h_\theta(\mathbb{Z}_m^{\mathbb{Z}^2}, \Phi) = \begin{cases} \sum_{i=1}^{h} k_i |\cos\theta + R_i \sin\theta| \log p_i; & \theta \in [0, \min \theta_{L_j}] \\ \sum_{i=1, i\neq j}^{h} k_i |\cos\theta + R_i \sin\theta| \log p_i +; & \theta \in [\min \theta_{L_j}, \min\{\theta_{R_{j_1}}, \theta_{L_{j_1}}\}], \\ +k_j |(R_j - L_j)\sin\theta| \log p_i; & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots ; & \cdots\cdots\cdots \\ \sum_{i=1}^{h} k_i |\cos\theta + L_i \sin\theta| \log p_i; & \theta \in [\max \theta_{R_j}, \pi] \end{cases}$$

Example 4.6 ([7, Example 4.5])**.** Let us consider the local rule

$$f(x_{-3}, \dots, x_3) = 4x_{-3} + 3x_{-2} + 24x_{-1} + 12x_0 + 3x_1 + 10x_2 + 6x_3 \pmod{2^2 \cdot 3^2} \tag{4.3}$$

From Theorem 4.5, we have

$$h_\theta(\mathbb{Z}_{2^2 3^2}^{\mathbb{Z}^2}, \Phi) = \begin{cases} \log_2 36. \cos\theta + \log_2 324. \sin\theta; & \theta \in [0, \operatorname{arccot}(3)] \\ \log_2 4. \cos\theta + \log_2 (4.9^5). \sin\theta; & \theta \in [\operatorname{arccot}(3), \operatorname{arccot}(2)] \\ \log_2 (2^6.3^{10}). \sin\theta; & \theta \in [\operatorname{arccot}(2), \frac{3\pi}{4}] \\ \log_2 4. |\cos\theta - 2\sin\theta| + \log_2 9^5. \sin\theta; & \theta \in [\frac{3\pi}{4}, \operatorname{arccot}(-2)] \\ \log_2 4. |\cos\theta - 2\sin\theta| + \log_2 9. |\cos\theta - 3\sin\theta|; & \theta \in [\operatorname{arccot}(-2), \pi]. \end{cases} \tag{4.4}$$

In Figure 3, we show the sectors relating to angles θ_{L_i} and θ_{R_i}. As shown in [22], there are three sections for 1-D LCA defined by bipermutative local rule. Otherwise, we may have more than three sectors for $\mathbb{Z}^2$-action generated by the pair $(T_{f[-r,r]}, \sigma)$ in the direction θ ($\theta \in [0, \pi]$). These sectors depend on the local rules.

It is well known that there are relations between the topological entropy and directional entropy [16]. For a vector $(n, m) \in \mathbb{N} \times \mathbb{Z}$, the topological entropy of $T^n_{f[-r,r]} \circ \sigma^m$ is the entropy in direction (n, m). In [7], the author has pointed out the relation between topological entropy of LCA $T_{f[-r,r]}$ and topological

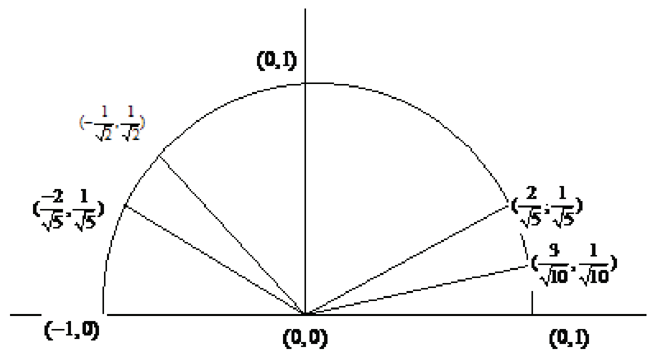

FIGURE 3. The sectors corresponding to angles θ_{L_i} and θ_{R_i}.

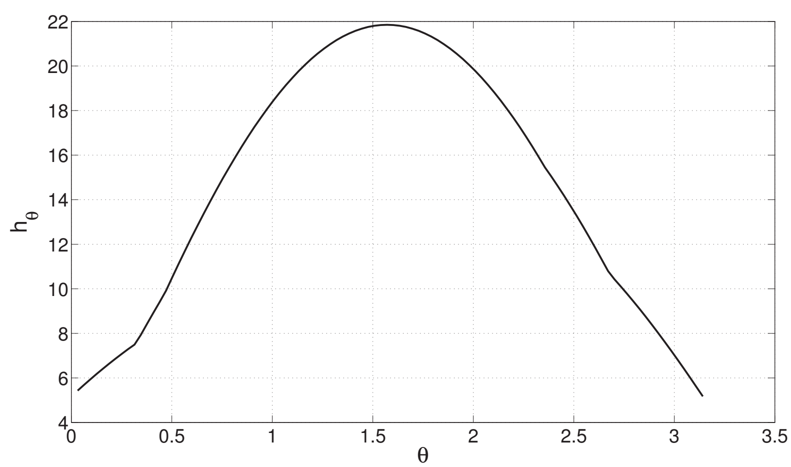

FIGURE 4. The topological directional entropy curve for LCA associated with the local rule (4.3).

directional entropy of $\mathbb{Z}^2$-action generated by the pair $(T_{f[-r,r]}, \sigma)$. As expected, Figure 4 shows that the topological directional entropy reaches the maximum value at point $\pi/2$. This value is the topological entropy of the LCA defined by the local rule (4.3).

5. The Uniqueness of the Equilibrium Measure for LCA

For a given potential function $\varphi \in C(\mathbb{Z}_m^{\mathbb{Z}}, \mathbb{R})$, the topological entropy can be generalized to the consideration of the topological pressure $P(T, \varphi)$. Suppose $a_0, a_1, \cdots, a_{m-1} \in \mathbb{R}$ are given, consider the potential function $\varphi : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{R}$ defined by $\varphi(x) = a_{x_0}$. Ban *et al.* [14] demonstrates an explicit expression of topological pressure for the case that $T_{f[-r,r]}$ is permutative.

Theorem 5.1 ([14]). *Under the same assumption of Theorem 3.4. The topological pressure of $T_{f[-r,r]}$ is $P(T_{f[-r,r]}, \varphi) = (R - L - 1) \log m + \log(e^{a_0} + e^{a_1} + \cdots + e^{a_{m-1}})$.*

For $i = 1, 2, \ldots, h$, denote by ϕ_{p_i} the canonical projection from $\mathbb{Z}_m$ to $\mathbb{Z}_{p_i^{k_i}}$. That is, $\phi_{p_i}(n) = n \pmod{p_i^{k_i}}$. The following theorem extends Theorem 5.1 to the case that $T_{f[-r,r]}$ is a LCA.

Theorem 5.2. *Suppose P_i, L_i, and R_i are the same as defined in Theorem 3.5 for $i = 1, 2, \ldots, h$. Consider the potential function φ which depends on central coordinate. In other words, $\varphi(x) = a_{x_0}$ for some $a_0, a_1, \ldots, a_{m-1}$. Then the pressure function of the LCA $T_{f[-r,r]}$ is*

$$P(T_{f[-r,r]}, \varphi) = \sum_{i=1}^{h} \left[(R_i - L_i - 1) k_i \log p_i + \log \sum_{j=0}^{p_i^{k_i}-1} e^{a_{i;j}} \right] \tag{5.1}$$

provided

$$a_\ell = \sum_{i=1}^{h} a_{i;\phi_{p_i}(\ell)} \tag{5.2}$$

for $\ell = 0, \ldots, m-1$, where $a_{i;j} = \sum_{\phi_{p_i}(k)=j} a_k$.

Proof. The proof is similar as the proof of Theorem 3.5. Here we give a brief demonstration.

Step 1. Suppose $m = p^k$ for some prime number p and positive integer k. Let P, L, and R be the same as defined in the proof of Theorem 3.5 (cf. Step 3). Define $\overline{f} = f \pmod{p}$. Let $\overline{T}_{f[-r,r]}$ be the LCA generated by the local rule $\overline{f}$. It follows that $\overline{T}_{f[-r,r]}$ is permutative, and Theorem 5.1 infers that $P(\overline{T}_{f[-r,r]}, \varphi) = (R - L - 1) k \log p + \log \sum_{i=0}^{p^k-1} e^{a_i}$.

The proof of Theorem 3.5 demonstrates that there exists $n \in \mathbb{N}$ such that $T_{f[-r,r]}^n = \overline{T}_{f[-r,r]}^n$. Observe that

$$nP(T_{f[-r,r]}, \varphi) = P(T_{f[-r,r]}^n, S_n\varphi) = P(\overline{T}_{f[-r,r]}^n, S_n\varphi) = nP(\overline{T}_{f[-r,r]}, \varphi),$$

where $S_n\varphi(x) = \sum_{i=0}^{n-1} \varphi(T^i(x))$ is the Birkhoff sum. Therefore $P(T_{f[-r,r]}, \varphi) = (R - L - 1) k \log p + \log \sum_{i=0}^{p^k-1} e^{a_i}$.

Step 2. For the case that $m = pq$ for some coprime factors $p, q \in \mathbb{N}$. Recall that the diagram

$$\begin{tikzcd} \mathbb{Z}_m^{\mathbb{Z}} \arrow[r, "T_{f[-r,r]}"] \arrow[d, "\Phi"'] & \mathbb{Z}_m^{\mathbb{Z}} \arrow[d, "\Phi"] \\ \mathbb{Z}_p^{\mathbb{Z}} \times \mathbb{Z}_q^{\mathbb{Z}} \arrow[r, "T_p \times T_q"'] & \mathbb{Z}_p^{\mathbb{Z}} \times \mathbb{Z}_q^{\mathbb{Z}} \end{tikzcd}$$

commutes, and Φ is an isomorphism. Define $\varphi_p : \mathbb{Z}_p^{\mathbb{Z}} \to \mathbb{R}$ and $\varphi_q : \mathbb{Z}_q^{\mathbb{Z}} \to \mathbb{R}$ by $\varphi_p(x) = \sum_{\phi_p(i)=x_0} a_i$ and $\varphi_q(x) = \sum_{\phi_q(i)=x_0} a_i$, respectively. Consider the potential function $\varphi_p \times \varphi_q : \mathbb{Z}_p^{\mathbb{Z}} \times \mathbb{Z}_q^{\mathbb{Z}} \to \mathbb{R}$ given by

$$\varphi_p \times \varphi_q(\mathbf{x}_p, \mathbf{x}_q) = \varphi_p(\mathbf{x}_p) + \varphi_q(\mathbf{x}_q).$$

The assumption $a_\ell = a_{p;\phi_p(\ell)} + a_{q;\phi_q(\ell)}$ asserts that the diagram

$$\begin{tikzcd} \mathbb{Z}_m^{\mathbb{Z}} \arrow[r, "\varphi"] \arrow[d, "\Phi"'] & \mathbb{R} \\ \mathbb{Z}_p^{\mathbb{Z}} \times \mathbb{Z}_q^{\mathbb{Z}} \arrow[ur, "\varphi_p \times \varphi_q"'] & \end{tikzcd}$$

commutes. This demonstrates that

$$P(T_{f[-r,r]}, \varphi) = P(T_p \times T_q, \varphi_p \times \varphi_q) = P(T_p, \varphi_p) + P(T_q, \varphi_q)$$

Step 3. In general, $m = p_1^{k_1} p_2^{k_2} \cdots p_h^{k_h}$ for some prime factors $p_1, \ldots, p_h$ and positive integers $k_1, \ldots, k_h$. Combine the above two steps we have

$$P(T_{f[-r,r]}, \varphi) = \sum_{i=1}^{h} P(T_{p_i}, \varphi_{p_i}) = \sum_{i=1}^{h} \left[(R_i - L_i - 1) k_i \log p_i + \log \sum_{j=0}^{p_i^{k_i}-1} e^{a_{i;j}} \right].$$

This completes the proof. $\square$

Remark 5.3. It is easily seen that the zero potential function, that is, $a_1 = a_2 = \cdots = a_{m-1} = 0$, satisfies the condition $a_\ell = \sum_{i=1}^{h} a_{i;\phi_{p_i}(\ell)}$ for $\ell = 0, \ldots, m-1$. In this case, the topological pressure is reduced to the topological entropy. This makes Theorem 4.1 a special case of Theorem 5.2.

An invariant measure μ is called an equilibrium measure if μ attains the supremum of the variational principle of the topological pressure. One of the most interesting issues in the thermodynamical formalism is the uniqueness of equilibrium measure. The next coming theorem demonstrates an affirmative result.

Theorem 5.4. *Along with the same assumption of Theorem 5.2. The uniform Bernoulli measure is the unique equilibrium measure for LCA.*

As the proof of Theorem 5.2, the demonstration of the uniqueness of equilibrium measure relates to the cases where m is prime. Ban *et al.* ([14]) indicated that, for a permutative cellular automaton, the number of equilibrium measures is less than or equal to one, and the uniform Bernoulli measure is the unique equilibrium measure if it exists.

Theorem 5.5 ([14])**.** *Let $a_0, a_1, \dots, a_{m-1} \in \mathbb{R}$ be given, and let the potential function $\varphi : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{R}$ be given by $\varphi(x) = a_{x_0}$ for all $x \in \mathbb{Z}_m^{\mathbb{Z}}$. If $T_{f[-r,r]}$ is permutative, then there exists equilibrium measure if and only if $a_i = a_j$ for $0 \le i, j \le m-1$. Moreover, the uniform Bernoulli measure is the equilibrium measure.*

Proof of Theorem 5.4. For $i = 1, 2, \dots, h$, since ϕ_{p_i} is a finite-to-one factor, ϕ_{p_i} maps an equilibrium measure to an equilibrium measure. Upon Theorem 5.5 and routine but cumbersome verification ensure that the equilibrium measure exists only for the case that $a_j = a_\ell$ for $0 \le j, \ell \le m-1$. Moreover, the uniform Bernoulli measure is the equilibrium measure. $\square$

6. The Hausdorff Dimension of the Limit Set of LCA

This section, as an application of the previous section, is devoted to investigating the Hausdorff dimension of the limit set $W_{f[-r,r]} = \bigcap_{i \ge 0} T^i_{f[-r,r]}(\mathbb{Z}_m^{\mathbb{Z}})$ of the LCA $T_{f[-r,r]}$. We recall some preliminaries about dimensions for the self-containing of this investigation.

Let $\beta \in \mathbb{R}$ be larger than 1. Suppose $\delta > 0$ and α is a cover of V. We say that α is of size δ if $\mathrm{diam(U)} \le \delta$ for all $U \in \alpha$, where

$$\mathrm{diam(U)} = \sup\{\mathrm{d(x,y) : x, y \in U}\} \quad \text{and} \quad \mathrm{d(x,y)} = \sum_{\mathrm{i} \in \mathbb{Z}} \frac{|\mathrm{x_i - y_i}|}{\beta^{\mathrm{i+1}}}$$

Let ν be a Borel probability measure on $\mathbb{Z}_m^{\mathbb{Z}}$, whereby the Hausdorff dimension of V is defined by

$$\dim_H V = \lim_{\delta \to 0} \inf\{d : \sum_{U \in \alpha} \mathrm{diam(U)^d} < 1, \alpha \text{ is a cover of V of size } \delta\}. \tag{6.1}$$

More specifically, denote by

$$\Gamma_\delta(V) = \inf\{card\ \alpha : \alpha \text{ is a cover of } V, \mathrm{diam(U)} = \delta \text{ for all U} \in \alpha\}.$$

The lower and upper box dimensions are defined by

$$\underline{\dim}_B V = -\liminf_{\delta\to 0} \frac{\log \Gamma_\delta(V)}{\log \delta}, \tag{6.2}$$

$$\overline{\dim}_B V = -\limsup_{\delta\to 0} \frac{\log \Gamma_\delta(V)}{\log \delta}, \tag{6.3}$$

respectively. For $x \in V$, the upper and lower pointwise dimensions of x with respect to ν is defined by

$$\overline{\dim}_\nu(x) = \limsup_{\epsilon\to 0} \frac{\log \nu(B_\epsilon(x))}{\log \epsilon}, \tag{6.4}$$

$$\underline{\dim}_\nu(x) = \liminf_{\epsilon\to 0} \frac{\log \nu(B_\epsilon(x))}{\log \epsilon}, \tag{6.5}$$

respectively, where $B_\epsilon(x)$ is the ϵ-ball centered on x.

Suppose $T_{f[-r,r]}$ is a LCA and $W_{f[-r,r]}$ is the limit set. The Hausdorff dimension $\dim_H(W_{f[-r,r]})$ is expressed as the following.

Theorem 6.1. *Let P_i, R_i, and L_i be the same as defined in Theorem 3.5. Define*

$$c_i = \begin{cases} 1, & (\lambda_0, p_i) = 1; \\ 0, & \textit{otherwise}. \end{cases}$$

for $1 \le i \le h$. Then

$$\dim_H(W_{f[-r,r]}) = \frac{2}{\log \beta} \sum_{\substack{1\le j\le h \\ R_j - L_j + c_j \ne 0}} k_j \log p_j. \tag{6.6}$$

Here we give two examples.

Example 6.2. Consider $m = 6$ and local rule as given by $f(x_{-1}, x_0, x_1) = 2(x_{-1} + x_1) \pmod 6$. It is easy to see that $W_{f[-1,1]} = \{x = (x_i)_{i\in\mathbb{Z}} : x_i = 0, 2 \text{ or } 4\}$. For $x, y \in \mathbb{Z}_m^{\mathbb{Z}}$, define

$$d(x, y) = \sum_{i\in\mathbb{Z}} \frac{|x_i - y_i|}{6^{|i|+1}}.$$

Theorem 6.1 asserts $\dim_H(W_{f[-1,1]}) = \log 9/\log 6$.

Example 6.3. Suppose $m = 90 = 2 \cdot 3^2 \cdot 5$ and the local rule is given by $f(x_{-2}, x_{-1}, x_0, x_1, x_2) = 18x_{-1} + 30(x_0 + x_1) + 20x_2 \pmod{90}$. For $x, y \in \mathbb{Z}_m^{\mathbb{Z}}$, the metric $d : \mathbb{Z}_m^{\mathbb{Z}} \times \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{R}$ is given by

$$d(x, y) = \sum_{i\in\mathbb{Z}} \frac{|x_i - y_i|}{\beta^{|i|+1}},$$

for some $\beta > 1$. The Hausdorff dimension of $W_{f[-2,2]}$ is $\dfrac{2}{\log \beta}(2\log 3 + \log 5)$.

6.1. **Fractal Dimensions on Permutative Cellular Automata.** The study of permutative cellular automata is essential for the investigation of LCA. This subsection focuses on the dimensions of the limit set of a permutative CA.

It is well-known that $\dim_H V \leq \underline{\dim}_B V \leq \overline{\dim}_B V$. The notation of pointwise dimension is an important tool to estimate Hausdorff dimension. A Borel probability measure ν supported on V is said to be **exact dimensional** provided $\overline{\dim}_\nu(x) = \underline{\dim}_\nu(x) = d$ for ν-almost all $x \in V$. If ν is exact dimensional, then $\dim_H V = \underline{\dim}_B V = \overline{\dim}_B V = d$ ([45]). In this case, we denote the dimension of V by $\dim V$.

Suppose $T_{f[-r,r]}$ is permutative, the following theorem demonstrates that the Bernoulli measure μ is exact dimensional with respect to $W_{f[-r,r]}$.

Theorem 6.4. *If $T_{f[-r,r]}$ is permutative, then μ is exact dimensional with respect to $W_{f[-r,r]}$.*

Proof. Given $\epsilon > 0$, choose

$$\overline{k}_\epsilon = \lceil \frac{\log 2 + \log 1/\epsilon}{\log \beta} \rceil - 1, \quad \underline{k}_\epsilon = \lfloor \frac{\log 2 + \log 1/\epsilon}{\log \beta} \rfloor - 1,$$

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ are upper and lower Gauss functions, respectively. Then $\beta^{-(i+1)} < \epsilon/2$ for all $i \geq \overline{k}_\epsilon$, $\beta^{-(\underline{k}_\epsilon+1)} > \epsilon/2$, and $\overline{k}_\epsilon - \underline{k}_\epsilon = 1$. For $x = (x_i)_{i\in\mathbb{Z}} \in W_{f[-r,r]}$, denote $x(a,b) = \{y \in X : y_a = x_a, \ldots, y_b = x_b, a \leq b\}$. We then find that $x(-\overline{k}_\epsilon, \overline{k}_\epsilon) \subset B_\epsilon(x) \subset x(-\underline{k}_\epsilon, \underline{k}_\epsilon)$.

Note that

$$\frac{\log \mu(x(-\underline{k}_\epsilon, \underline{k}_\epsilon))}{\log \epsilon} < \frac{\log \mu(B_\epsilon(x))}{\log \epsilon} < \frac{\log \mu(x(-\overline{k}_\epsilon, \overline{k}_\epsilon))}{\log \epsilon}$$

implies

$$\limsup \frac{\log \mu(x(-\underline{k}_\epsilon, \underline{k}_\epsilon))}{\log \epsilon} \leq \underline{dim}_\mu(x) \leq \overline{dim}_\mu(x) \leq \liminf \frac{\log \mu(x(-\overline{k}_\epsilon, \overline{k}_\epsilon))}{\log \epsilon}.$$

From this it is easily seen that

$$\limsup \frac{\log \mu(x(-\underline{k}_\epsilon, \underline{k}_\epsilon))}{\log \epsilon} = \liminf \frac{\log \mu(x(-\overline{k}_\epsilon, \overline{k}_\epsilon))}{\log \epsilon}.$$

This demonstrates $\underline{\dim}_\mu(x) = \overline{\dim}_\mu(x)$ and completes the proof. □

Suppose (X, ρ) is a compact metric space, $T : X \to X$ is a continuous map, and $\varphi : X \to \mathbb{R}$ is a negative continuous function. In [19], Bowen studied the Hausdorff dimension of quasi-circles and discovered that the Hausdorff dimension is the unique root of the pressure function $\psi(t) = P(T, t\varphi)$ with respect to some φ. This equation is known now as Bowen's equation.

Theorem 6.5. *Suppose $T_{f[-r,r]}$ is permutative. Denote $\dim(W_{f[-r,r]}) = d$. Then d is the unique root of the pressure function $\psi(t) = P(T_{f[-r,r]}, t\phi)$, where the potential function ϕ is given by $\phi(x) = \log \beta^{(L-R)/2}$ for all $x \in \mathbb{Z}_m^{\mathbb{Z}}$, and L and R are the same as defined in Theorem 3.4.*

Since μ is exact dimensional, the Hausdorff dimension of $W_{f[-r,r]}$ coincides with the box dimension of $W_{f[-r,r]}$. To prove Theorem 6.5, we recall a result.

Theorem 6.6 ([36])**.** *Let V be a subset of a metric space X and let δ_n be a monotonically decreasing sequence of positive numbers, $\delta_n \to 0$ as $n \to \infty$ and $\delta_{n+1} \geq c\delta_n$ for some $c > 0$ which is independent of n. If the limit*

$$\lim_{n\to\infty} \frac{\log \Gamma_{\delta_n}(V)}{\log 1/\delta_n} \equiv d,$$

exists, then $\underline{\dim}_B V = \overline{\dim}_B V = d$.

Proof of Theorem 6.5. Suppose f is bipermutative; that is, $R > 0$ and $L < 0$. For $k \in \mathbb{N}$, denote

$$\xi(-k, k) = \{{}_{-k}[i_{-k}, \ldots, i_k]_k : i_j \in \mathbb{Z}_m, -k \leq j \leq k\}.$$

Then $\xi(-k, k)$ is a finite partition of $\mathbb{Z}_m^{\mathbb{Z}}$ and $\bigvee_{i=0}^{n} T_{f[-r,r]}^{-i}\xi(-k, k) = \xi(-k + n\ell, k + nr)$ for $n \in \mathbb{N}$ provided k large enough ([14]). It is observed that $\mathrm{diam(C)} = 2\beta^{\mathrm{-k-1}}$ for $C \in \xi(-k, k)$. Let $\delta_k = 2\beta^{-k-1}$, then $\delta_k \to 0$ as $k \to \infty$ and $\lim_{k\to\infty} \frac{\delta_k}{\delta_{k+1}} = \beta^{-1}$. Theorem 6.6 demonstrates that $\dim(W_{f[-r,r]}) = 2\log m/\log \beta$.

Suppose the potential function $\phi : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{R}$ is given by $\phi(x) = \log \beta^{(L-R)/2}$ for $x \in \mathbb{Z}_m^{\mathbb{Z}}$. It then follows that $\phi(x)$ is negative, and $P(T_{f[-r,r]}, t\phi) = (R - L - 1)\log m + \log(m \cdot \beta^{\frac{t}{2}(L-R)}) = 0$ if and only if $t = \dim(W_{f[-r,r]})$.

Other cases can be can be treated in a similar manner. This completes the proof. □

Suppose m is prime. The local rule $f[-r, r]$ of a LCA $T_{f[-r,r]}$ is permutative if and only if $R - L > 0$. For the case that $R - L = 0$, it is easily seen that $T_{f[-r,r]}$ is a diffeomorphism if and only if the coefficient $\lambda_0 \neq 0$. The following corollary is demonstrated via applying Theorem 6.5.

Corollary 6.7. *Suppose m is prime and $T_{f[-r,r]}$ is a LCA. Then*

$$\dim(W_{f[-r,r]}) = \begin{cases} 2\log m/\log\beta, & R-L+\lambda_0>0; \\ 0, & \text{otherwise.} \end{cases} \tag{6.7}$$

6.2. **Proof of Theorem 6.1.** The proof of Theorem 6.1 is similar as the proof of Theorems 3.5 and 5.2. We divide the proof into the following steps.

Step 1. Suppose $m = p^k$ for some prime p and $k \in \mathbb{N}$. Observe that $W_{(f[-r,r])^n} = W_{f[-r,r]}$ for $n \in \mathbb{N}$. A slight modification of the proof of Theorem 6.5 demonstrates that

$$\dim_H(W_{f[-r,r]}) = \underline{\dim}_B(W_{f[-r,r]}) = \overline{\dim}_B(W_{f[-r,r]}).$$

Step 2. Next we consider the case that $m = pq$ for some $p, q \in \mathbb{N}$ and $(p,q) = 1$. Denote

$$W_{f_p} = \bigcap_{i\geq 0} T_p^i \mathbb{Z}_p^{\mathbb{Z}} \quad \text{and} \quad W_{f_q} = \bigcap_{i\geq 0} T_q^i \mathbb{Z}_q^{\mathbb{Z}},$$

where f_p, f_q, T_p, and T_q are the same as defined in the proof of Theorem 3.5. Since $T_{f[-r,r]}$ is topological conjugate to $T_p \times T_q$ (see the proof of Theorem 5.2), it follows that $W_{f[-r,r]} \cong W_{f_p} \times W_{f_q}$. In other words,

$$\dim(W_{f[-r,r]}) = \dim(W_{f_p} \times W_{f_q}), \quad \text{where} \quad \dim \in \{\dim_H, \underline{\dim}_B, \overline{\dim}_B\}.$$

The key to complete the proof of Theorem 6.1 is how to characterize $\dim(W_{f_p} \times W_{f_q})$. This has been a long-standing problem, and Falconer gave an affirmative answer.

Theorem 6.8 ([26])**.** *Suppose E, F are two bounded Borel sets. If $\dim_H(E) = \overline{\dim}_B(E)$, then $\dim_H(E \times F) = \dim_H(E) + \dim_H(F)$, herein $\times$ means the Cartesian product.*

Step 3. Let $m = p_1^{k_1} \cdots p_h^{k_h}$. Similar as the discussion in Step 2, it is seen that

$$\dim(W_{f[-r,r]}) = \dim(\prod_{i=1}^{h} W_{f_{p_i}}), \quad \text{where} \quad \dim \in \{\dim_H, \underline{\dim}_B, \overline{\dim}_B\}.$$

Combine the results of Step 1, Theorems 6.5 and 6.8 we have completed the proof.

7. Further Discussion

Suppose $T_{f[-r,r]}$ is a LCA generated by a local rule $f : \mathbb{Z}_m^{2r+1} \to \mathbb{Z}_m$, $m \geq 2$, and μ is a $T_{f[-r,r]}$-invariant Bernoulli measure. Under the assumptions (3.1) and (5.2), Theorems 3.5 and 5.2 assert the measure entropy and the topological pressure of $T_{f[-r,r]}$. Theorem 5.2 is an extension of Theorem 4.1 ([24]). It is seen that the uniform Bernoulli measure is the unique equilibrium measure. As an application, the Hausdorff dimension of the limit set $W_{f[-r,r]}$ is demonstrated as the root of Bowen's equation. With the interpretation of Theorem 4.5, we have also illustrated the topological entropy and topological directional entropy by means of figures.

Meanwhile, the following questions are in preparation.

(1) What is the measure entropy and the topological pressure of $T_{f[-r,r]}$ without the conditions (3.1) and (5.2)?

(2) Can the measure-theoretical directional entropy of $\mathbb{Z}^2$-action generated by the pair $(T_{f[-r,r]}, \sigma)$ be computed?

Acknowledgements

Ban is supported in part by the National Science Council, ROC (Contract No NSC 99-2115-M-259-003), the National Center for Theoretical Sciences and the Center for Mathematics and Theoretical Physics at National Central University. Chang is grateful for the partial support of the National Science Council, ROC (Contract No NSC 101-2115-M-035-002-).

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