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PATTERNS GENERATION AND TRANSITION MATRICES IN MULTI-DIMENSIONAL LATTICE MODELS

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Abstract. This work develops a general approach for investigating pattern generation problems in multi-dimensional lattice models. Let $\mathcal{S}$ be a set of p symbols or colors, $\mathbf{Z}_N$ a fixed finite rectangular sublattice of $\mathbf{Z}^d$, $d \geq 1$ and N a d-tuple of positive integers. Functions $U : \mathbf{Z}^d \to \mathcal{S}$ and $U_N : \mathbf{Z}_N \to \mathcal{S}$ are called a global pattern and a local pattern on $\mathbf{Z}_N$, respectively. An ordering matrix $\mathbf{X}_N$ is also introduced for Σ_N, the set of all local patterns on $\mathbf{Z}_N$. For a larger finite lattice $\mathbf{Z}_{\tilde{N}}$, $\tilde{N} \geq N$, A recursion formula is derived to obtain the ordering matrix $\mathbf{X}_{\tilde{N}}$ of $\Sigma_{\tilde{N}}$ from $\mathbf{X}_N$. Additionaly, the transition matrix $\mathbf{T}_N(\mathcal{B})$ is defined for a given basic admissible local patterns set $\mathcal{B} \subset \Sigma_N$. For each $\tilde{N} \geq N$ denoted by $\Sigma_{\tilde{N}}(\mathcal{B})$, the set of all local patterns can be generated from $\mathcal{B}$. The cardinal number of $\Sigma_{\tilde{N}}(\mathcal{B})$ denotes the sum of entries of the transition matrix $\mathbf{T}_{\tilde{N}}(\mathcal{B})$ which can be obtained from $\mathbf{T}_N(\mathcal{B})$ recursively. The spatial entropy $h(\mathcal{B})$ can be obtained by computing the maximum eigenvalues of a sequence of transition matrices $\mathbf{T}_n(\mathcal{B})$. Results of this study can shed further light on the set of global stationary solutions in various lattice dynamical systems and cellular neural networks.

1. Introduction

Many systems have been adopted as models for spatial pattern formation in biology, chemistry, engineering and physics. Lattices play important roles in modeling underlying spatial structures. Notable examples include models arising from biology[7, 8, 21, 22, 23, 32, 33, 34], chemical reaction and phase transitions [5, 6, 11, 12, 13, 14, 24, 40, 43], image processing and pattern recognition [11, 12, 15, 16, 17, 18, 19, 25, 39], as well as materials science[10, 20, 26]. Stationary patterns play a critical role in investigating the long time behavior of related dynamical systems. In general, multiple stationary patterns may induce complicated phenomena of such systems.

In lattice dynamical systems(LDS), especially cellular neural networks (CNN), the set of global stationary solutions (global patterns) has received considerable attention in recent years (e.g.[1, 2, 3, 27, 28, 29, 30, 31, 35, 36]). When the mutual interaction between states of a system is local, the state at each lattice point is influenced only by its finitely many neighborhood states. The admissible (or allowable) local patterns are introduced and defined on a certain finite lattice. The admissible global patterns on the entire lattice space are then glued together from

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those admissible local patterns. More precisely, let $\mathcal{S}$ be a finite set of p elements (i.e.,symbols, colors or letters of an alphabet). Where $\mathbf{Z}^d$ denotes the integer lattice on $\mathbf{R}^d$, and $d \geq 1$ is a positive integer representing the lattice dimension. Then, function $U : \mathbf{Z}^d \to \mathcal{S}$ is called a global pattern. For each $\alpha \in \mathbf{Z}^d$, $U(\alpha)$ can be written as u_α. The set of all patterns $U : \mathbf{Z}^d \to \mathcal{S}$ is denoted by

$$\Sigma_p^d \equiv \mathcal{S}^{\mathbf{Z}^d},$$

where Σ_p^d is the set of all patterns with p different colors in d-dimensional lattice. As for local patterns, i.e., functions defined on (finite) sublattices, for a given d-tuple $N = (N_1, N_2, \cdots, N_d)$ of positive integers, let

$$\mathbf{Z}_N = \{(\alpha_1, \alpha_2, \cdots, \alpha_d) : 1 \leq \alpha_k \leq N_k, 1 \leq k \leq d\}$$

be an $N_1 \times N_2 \times \cdots N_d$ finite rectangular lattice. Denoted by $\widetilde{N} \geq N$ if $\widetilde{N_k} \geq N_k$ for all $1 \leq k \leq d$, where $\widetilde{N} = (\widetilde{N_1}, \widetilde{N_2}, ..., \widetilde{N_d})$ is a d-tuple positive integers. The set of all local patterns defined on $\mathbf{Z}_N$ is denoted by

$$\Sigma_N \equiv \Sigma_{N,p} \equiv \{U|_{\mathbf{Z}_N} : U \in \Sigma_p^d\}.$$

Under many circumstances, only a(proper) subset $\mathcal{B}$ of Σ_N is admissible (allowable or feasible). In this case, local patterns in $\mathcal{B}$ are called basic patterns and $\mathcal{B}$ refers to the basic set. In a one dimensional case, $\mathcal{S}$ consists of letters of an alphabet, and $\mathcal{B}$ is also called a set of allowable words of length N.

Consider a fixed finite lattice $\mathbf{Z}_N$ and a given basic set $\mathcal{B} \subset \Sigma_N$. For a larger finite lattice $\mathbf{Z}_{\widetilde{N}} \supset \mathbf{Z}_N$, the set of all local patterns on $\mathbf{Z}_{\widetilde{N}}$ which can be generated by $\mathcal{B}$ is denoted as $\Sigma_{\widetilde{N}}(\mathcal{B})$. Indeed, as in [41], $\Sigma_{\widetilde{N}}(\mathcal{B})$ can be characterized by

$$\begin{aligned}\Sigma_{\widetilde{N}}(\mathcal{B}) \quad = \{ \quad & U \in \Sigma_{\widetilde{N}} : U_{\alpha+N} = V_N \ for\ any\ \alpha \in \mathbf{Z}^d\ with\ \mathbf{Z}_{\alpha+N} \subset \mathbf{Z}_{\widetilde{N}} \\ & and\ some\ V_N \in \mathcal{B}\},\end{aligned}$$

where

$$\alpha + N = \{(\alpha_1 + \beta_1, \cdots, \alpha_d + \beta_d) : (\beta_1, \cdots, \beta_d) \in N\},$$

and

$$U_{\alpha+N} = V_N\ means\ u_{\alpha+\beta} = v_\beta\ for\ each\ \beta \in \mathbf{Z}_N.$$

Similarly, the set of all global patterns which can be generated by $\mathcal{B}$ is denoted by

$$\Sigma(\mathcal{B}) = \{U \in \Sigma_p^d : U_{\alpha+N} = V_N\ for\ any\ \alpha \in \mathbf{Z}^d\ with\ some\ V_N \in \mathcal{B}\}.$$

The following questions arise :

(1) Can we systematically construct $\Sigma_{\widetilde{N}}(\mathcal{B})$ from $\mathcal{B}$ for $\mathbf{Z}_{\widetilde{N}} \supset \mathbf{Z}_N$?
(2) What is the complexity (or spatial entropy) of $\{\sum_{\widetilde{N}}(\mathcal{B})\}_{\widetilde{N} \geq N}$?

The spatial entropy $h(\mathcal{B})$ of $\Sigma(\mathcal{B})$ is defined as follows :
Let

$$\Gamma_{\widetilde{N}}(\mathcal{B}) = card(\Sigma_{\widetilde{N}}(\mathcal{B})), \tag{1.1}$$

the number of distinct patterns in $\Sigma_{\widetilde{N}}(\mathcal{B})$. The spatial entropy $h(\mathcal{B})$ is defined as

$$h(\mathcal{B}) = \lim_{\widetilde{N} \to \infty} \frac{1}{\widetilde{N_1} \cdots \widetilde{N_d}} \log \Gamma_{\widetilde{N}}(\mathcal{B}), \tag{1.2}$$

which is well-defined and exists (e.g. [13]). The spatial entropy, which is analogous to topological entropy in dynamical system, has been used to measure the complexity in LDS (e.g. [13], [42]).

In a one dimensional case, the above two questions can be answered by using the transition matrix. Indeed, for a given basic set $\mathcal{B}$, the transition matrix $\mathbf{T}(\mathcal{B})$ can be associated with to $\mathcal{B}$. Nest, the spatial entropy $h(\mathcal{B}) = \log \lambda$, where λ is the largest eigenvalue of $\mathbf{T}(\mathcal{B})$ (e.g. [29, 40]). On the other hand, for higher dimensional cases, constructing $\Sigma_{\widetilde{N}}(\mathcal{B})$ systematically and computing $\Gamma_{\widetilde{N}}(\mathcal{B})$ effectively for a large $\widetilde{N}$ are extremely difficult.

In the two dimensional case, Chow et al. [13] estimated lower bounds of the spatial entropy for some problems in LDS. Later, using a "building block" approach, Juang and Lin [29] studied the patterns generation and obtained lower bounds of the spatial entropy for CNN with square-cross or diagonal-cross templates. For CNN with general templates, Hsu et al [27] investigated the generation of admissible local patterns and obtained the basic set for any parameter, i.e., the first step in studying the patterns generation problem. Meanwhile, given a set of symbols $\mathcal{S}$ and a pair consisting of a horizontal transition matrix H and a vertical transition matrix V, Juang et al [30] defined m-th order transition matrices $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$ for each $m \geq 1$,thus obtaining the recursion formulae for both $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$. Furthermore, they demonstrated that $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$ have the same maximum eigenvalue λ_m and spatial entropy $h(H,V) = \lim\limits_{m\to\infty} \frac{\log \lambda_m}{m}$. For a certain class of H,V, the recursion formulae for $T^{(m)}_{H,V}$ and $\bar{T}^{(m)}_{H,V}$ yield recursion formulae for λ_m explicitly and the exact entropy. On the other hand, for the patterns generation problem, Lin and Yang [36] worked on the 3-cell L-shaped lattice, i.e., N=  . They developed an algorithm to investigate how patterns are generated on larger lattices from a smaller one. Their algorithm treated all patterns in $\Sigma_{\widetilde{N}}(\mathcal{B})$ as entries and arranged them in a "counting matrix" $M_{\widetilde{N}}(\mathcal{B})$. A good arrangement of $M_{\widetilde{N}}(\mathcal{B})$ implies an easier extension to $M_{\widetilde{\widetilde{N}}}(\mathcal{B})$ for a larger lattice $\widetilde{\widetilde{N}} \supset \widetilde{N}$ and effective counting of the number of elements in $\Sigma_{\widetilde{N}}(\mathcal{B})$. Upper and lower bounds of spatial entropy have also been obtained. Furthermore, the patterns generation problem is related to matrix shift [13], as discussed in detail in section 3.4.

Motivated by the counting matrix $M_N(\mathcal{B})$ of [36] and the recursion formulae for transition matrices in [30], this work introduces the "ordering matrix" $\mathbf{X}_2$ for $\Sigma_{2\ell\times 2\ell}$ to study the patterns generation and obtain recursion formulae for $\mathbf{X}_n$ for $\Sigma_{2\ell\times n\ell}$ where $\ell \geq 1$ is a fixed positive integer and $n \geq 2$. The recursion formulae for $\mathbf{X}_n$ imply the recursion formula for the associated transition matrices $\mathbf{T}_n(\mathcal{B})$ of $\Sigma_{2\ell\times n\ell}(\mathcal{B})$, i.e., a generalization of the recursion formulae in [30]. Notably, a different ordering matrix $\widetilde{\mathbf{X}}_2$ for $\Sigma_{2\ell\times 2\ell}$ induces different recursion formulae of $\widetilde{\mathbf{X}}_n$ for $\Sigma_{2\ell\times n\ell}$ and $\widetilde{\mathbf{T}_n}(\mathcal{B})$. Among them, $\mathbf{X}_2$ defined in (2.9) yields a simple recursion formula (3.16) and rewriting rule (3.14), allowing us to compute the maximum eigenvalue of $\mathbf{T}_n$ effectively. The computations or estimates of λ_n are interesting problems in linear algebra and numerical linear algebra. Owing to the similarity property of (3.16) or (3.14) of transition matrices $\{\mathbf{T}_n\}_{n=2}^{\infty}$, this work demonstrates that for a certain class of $\mathcal{B}$, λ_n satisfies certain recursion relations and $h(\mathcal{B})$ can be computed explicitly.

The rest of this paper is organized as follows. Section 2 describes a two dimensional case by thoroughly investigating $\Sigma_{2\times 2}$ and introducing the ordering matrix $\mathbf{X}_2$ of patterns in $\Sigma_{2\times 2}$. The ordering matrix $\mathbf{X}_n$ on $\Sigma_{2\times n}$ is then constructed from $\mathbf{X}_2$ recursively. Section 3 derives higher order transition matrices $\mathbf{T}_n$ from $\mathbf{T}_2$ and computes λ_n explicitly for a certain type of $\mathbf{T}_2$. Finally, section 4 studies a three dimensional case and explores in further detail the structure of the ordering matrix on $\Sigma_{2\times 2\times 2}$. A generalization is also made for higher dimensional cases.

2. Two Dimensional Patterns

This section describes generation of two dimensional patterns. For clarity, two symbols, i.e., $\mathcal{S} = \{0, 1\}$, are studied first. An ordering of patterns for $\Sigma_{m_1\times m_2}$ as lexicographical ordering for a one-dimensional case is then defined. On a fixed finite lattice $\mathbf{Z}_{m_1\times m_2}$, a ordering $\chi = \chi_{m_1\times m_2}$ on $\mathbf{Z}_{m_1\times m_2}$ is given by

$$\chi((\alpha_1, \alpha_2)) = m_2(\alpha_1 - 1) + \alpha_2 \ , \tag{2.1}$$

i.e.,

(2.2)

m_2	$2m_2$		m_1m_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	m_2+1		$(m_1-1)m_2+1$

.

The ordering χ of (2.1) on $\mathbf{Z}_{m_1\times m_2}$ can now be passed to $\Sigma_{m_1\times m_2}$. Indeed, for each $U = (u_{\alpha_1,\alpha_2}) \in \Sigma_{m_1\times m_2}$, define

$$\begin{aligned} \chi(U) &\equiv \chi_{m_1\times m_2}(U) \\ &= 1 + \sum_{\alpha_1=1}^{m_1}\sum_{\alpha_2=1}^{m_2} u_{\alpha_1,\alpha_2} 2^{m_2(m_1-\alpha_1)+(m_2-\alpha_2)}. \end{aligned} \tag{2.3}$$

Obviously, there is an one-to-one correspondence between local patterns in $\Sigma_{m_1\times m_2}$ and positive integers in the set $\mathbf{N}_{2^{m_1m_2}} = \{k \in \mathbf{N} : 1 \le k \le 2^{m_1,m_2}\}$, where $\mathbf{N}$ denotes the set of positive integers. Therefore, U is referred to herein as the $\chi(U)$-th element in $\Sigma_{m_1\times m_2}$. Identifying the pictorial patterns by numbers $\chi(U)$ is a highly effictive means of proving theorems since computations can now be performed on $\chi(U)$. In a two dimensional case, we will keep the ordering (2.1)$\sim$ (2.3) χ on $\mathbf{Z}_{m_1\times m_2}$ and $\Sigma_{m_1\times m_2}$, respectively.

2.1. Ordering Matrices.

For $1\times n$ pattern $U = (u_k), 1 \le k \le n$ in $\Sigma_{1\times n}$, as in (2.3), U is assigned the number

$$i = \chi(U) = 1 + \sum_{k=1}^{n} u_k 2^{(n-k)}. \tag{2.4}$$

As denoted by the $1\times n$ column pattern $x_{n;i}$,

$$x_{n;i} = \begin{bmatrix} u_n \\ \vdots \\ u_1 \end{bmatrix} \quad or \quad \begin{array}{|c|} \hline u_n \\ \hline \vdots \\ \hline u_1 \\ \hline \end{array} \ . \tag{2.5}$$

In particular, when $n = 2$, as denoted by $x_i = x_{2;i}$,

$$i = 1 + 2u_1 + u_2$$

and

$$x_i = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} \quad or \quad \begin{array}{|c|}\hline u_2 \\ \hline u_1 \\ \hline \end{array} \ . \tag{2.6}$$

A 2×2 pattern $U = (u_{\alpha_1,\alpha_2})$ can now be obtained by a horizontal direct sum of two 1×2 patterns, i.e.,

$$\begin{aligned} x_{i_1,i_2} &\equiv x_{i_1} \oplus x_{i_2} \\ &\equiv \begin{bmatrix} u_{1,2} & u_{2,2} \\ u_{1,1} & u_{2,1} \end{bmatrix} \quad or \quad \begin{array}{|c|c|}\hline u_{1,2} & u_{2,2} \\ \hline u_{1,1} & u_{2,1} \\ \hline \end{array} \ , \end{aligned} \tag{2.7}$$

where

$$i_k = 1 + 2u_{k,1} + u_{k,2}, \quad 1 \le k \le 2. \tag{2.8}$$

Therefore, the complete set of all $16(= 2^{2\times 2})$ 2×2 patterns in $\Sigma_{2\times 2}$ can be listed by a 4×4 matrix $\mathbf{X}_2 = [x_{i_1,i_2}]$ with 2×2 pattern x_{i_1,i_2} as its entries in

$$\begin{array}{c|cccc|} & \begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array} & \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array} & \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array} \\ \begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 0&1\\\hline\end{array} \\ \begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 0&1\\\hline\end{array} \\ \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline\end{array} \\ \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&0\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array} \end{array} \tag{2.9}$$

It is easy to verify that

$$\chi(x_{i_1,i_2}) = 4(i_1 - 1) + i_2, \tag{2.10}$$

i.e, we are counting local patterns in $\Sigma_{2\times 2}$ by going through each row successively in Table (2.9). Correspondingly, $\mathbf{X}_2$ can be referred to as an ordering matrix for $\Sigma_{2\times 2}$. Similarly, a 2×2 pattern can also be viewed as a vertical direct sum of two 2×1 patterns, i.e,

$$y_{j_1,j_2} = y_{j_1} \oplus y_{j_2}, \tag{2.11}$$

where

$$y_{j_l} = \begin{bmatrix} u_{1,l} & u_{2,l} \end{bmatrix} \quad or \quad \begin{array}{|c|c|}\hline u_{1,l} & u_{2,l} \\ \hline \end{array} \ ,$$

and

$$j_l = 1 + 2u_{1,l} + u_{2,l}, \tag{2.12}$$

$1 \leq l \leq 2$. A 4×4 matrix $\mathbf{Y}_2 = [y_{j_1,j_2}]$ can also be obtained for $\Sigma_{2\times 2}$. i.e., we have

(2.13)
$$\begin{array}{c|cccc}
 & \begin{matrix}0&0\end{matrix} & \begin{matrix}0&1\end{matrix} & \begin{matrix}1&0\end{matrix} & \begin{matrix}1&1\end{matrix} \\
\hline
\begin{matrix}0&0\end{matrix} & \begin{matrix}0&0\\0&0\end{matrix} & \begin{matrix}0&1\\0&0\end{matrix} & \begin{matrix}1&0\\0&0\end{matrix} & \begin{matrix}1&1\\0&0\end{matrix} \\
\begin{matrix}0&1\end{matrix} & \begin{matrix}0&0\\0&1\end{matrix} & \begin{matrix}0&1\\0&1\end{matrix} & \begin{matrix}1&0\\0&1\end{matrix} & \begin{matrix}1&1\\0&1\end{matrix} \\
\begin{matrix}1&0\end{matrix} & \begin{matrix}0&0\\1&0\end{matrix} & \begin{matrix}0&1\\1&0\end{matrix} & \begin{matrix}1&0\\1&0\end{matrix} & \begin{matrix}1&1\\1&0\end{matrix} \\
\begin{matrix}1&1\end{matrix} & \begin{matrix}0&0\\1&1\end{matrix} & \begin{matrix}0&1\\1&1\end{matrix} & \begin{matrix}1&0\\1&1\end{matrix} & \begin{matrix}1&1\\1&1\end{matrix}
\end{array}$$

The relation between $\mathbf{X}_2$ and $\mathbf{Y}_2$ must be explored. Indeed, from (2.12), $u_{k,l}$ can be solved in terms of j_l, i.e., we have

$$u_{1,l} = [\frac{j_l - 1}{2}] \tag{2.14}$$

and

$$u_{2,l} = j_l - 1 - 2[\frac{j_l - 1}{2}], \tag{2.15}$$

where [ ] is the Gauss symbol, i.e., $[r]$ is the largest integer which is equal to or less than r. From (2.8), (2.12), (2.14) and (2.15), we have the following relations between indices i_1, i_2 and j_1, j_2.

$$j_1 = 1 + \sum_{k=1}^{2} [\frac{i_k - 1}{2}]\ 2^{2-k}, \tag{2.16}$$

$$j_2 = 1 + \sum_{k=1}^{2} \{\ i_k - 1 - 2\ [\frac{i_k - 1}{2}]\ \}\ 2^{2-k}, \tag{2.17}$$

and

$$i_1 = 1 + \sum_{l=1}^{2} [\frac{j_l - 1}{2}]\ 2^{2-l}, \tag{2.18}$$

$$i_2 = 1 + \sum_{l=1}^{2} \{\ j_l - 1 - 2\ [\frac{j_l - 1}{2}]\ \}\ 2^{2-l}. \tag{2.19}$$

From (2.16) and (2.17), (2.9) or $\mathbf{X}_2$ can also be represented by y_{j_1,j_2} as

$$\mathbf{X}_2 = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{2,1} & y_{2,2} \\ y_{1,3} & y_{1,4} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{4,1} & y_{4,2} \\ y_{3,3} & y_{3,4} & y_{4,3} & y_{4,4} \end{bmatrix}. \tag{2.20}$$

In (2.20), the indices j_1, j_2 are arranged by two Z-maps successively as

$$\begin{bmatrix} 1 & \longrightarrow & 2 \\ & \swarrow & \\ 3 & \longrightarrow & 4 \end{bmatrix} \tag{2.21}$$

i.e., the path from 1 to 4 in (2.21) is Z shaped and is then called a Z-map. More precisely, $\mathbf{X}_2$ can be decomposed by

$$\mathbf{X}_2 = \begin{bmatrix} Y_{2;1} & Y_{2;2} \\ Y_{2;3} & Y_{2;4} \end{bmatrix} \tag{2.22}$$

and

$$Y_{2;k} = \begin{bmatrix} y_{k,1} & y_{k,2} \\ y_{k,3} & y_{k,4} \end{bmatrix}. \tag{2.23}$$

Where $\mathbf{X}_2$ is arranged by a Z-map ($Y_{2;k}$) in (2.22) and each $Y_{2;k}$ is also arranged by a Z-map ($y_{k,l}$) in (2.23). Therefore, the indices of y in (2.20) consist of two Z-maps.

The expression (2.20) of all local patterns in $\Sigma_{2\times 2}$ by y can be extended to all patterns in $\Sigma_{2\times n}$ for any $n \geq 3$. Indeed, a local pattern U in $\Sigma_{2\times n}$ can be viewed as the horizontal direct sum of two $1 \times n$ local patterns, i.e., U_1 and U_2, and also the vertical direct sums of n many 2×1 local patterns. As in (2.9), all patterns in $\Sigma_{2\times n}$ can be arranged by the ordering matrix

$$\mathbf{X}_n = \left[\; x_{n;i_1,i_2} \; \right], \tag{2.24}$$

a $2^n \times 2^n$ matrix with entry $x_{n;i_1,i_2} = x_{n;i_1} \oplus x_{n;i_2}$, where $\chi(U_1) = i_1$ and $\chi(U_2) = i_2$ as in (2.4) and (2.5), $1 \leq i_1, i_2 \leq 2^n$. On the other hand, the two 2×2 patterns y_{j_1,j_2} and y_{j_2,j_3} can be combined to become a 2×3 pattern y_{j_1,j_2,j_3}, since the second row in y_{j_1,j_2} and the first row of y_{j_2,j_3} are identical, i.e.,

$$\begin{aligned} y_{j_1,j_2,j_3} &\equiv y_{j_1,j_2} \hat{\oplus} y_{j_2,j_3} \\ &\equiv y_{j_1} \oplus y_{j_2} \oplus y_{j_3}, \end{aligned} \tag{2.25}$$

Herein, a wedge direct sum $\hat{\oplus}$ is used for 2×2 patterns whenever they can be combined. In this way, a $2 \times n$ pattern $y_{j_1,\cdots,j_n}$ is obtained from $n-1$ many 2×2 patterns y_{j_1,j_2}, y_{j_2,j_3}, $\cdots$, y_{j_{n-1},j_n} by

$$\begin{aligned} y_{j_1,\cdots,j_n} &\equiv y_{j_1,j_2} \hat{\oplus} y_{j_2,j_3} \hat{\oplus} \cdots \hat{\oplus} y_{j_{n-1},j_n} \\ &\equiv y_{j_1} \oplus y_{j_2} \oplus \cdots \oplus y_{j_n}, \end{aligned} \tag{2.26}$$

where $1 \leq j_k \leq 4$, and $1 \leq k \leq n$. Now, $\mathbf{X}_n$ in y expression can be obtained as follows.

Theorem 2.1. *For any $n \geq 2$, $\Sigma_{2\times n} = \{y_{j_1,\cdots,j_n}\}$, where $y_{j_1,\cdots,j_n}$ is given in (2.26). Furthermore, the ordering matrix $\mathbf{X}_n$ can be decomposed by n Z-maps successively as*

$$\mathbf{X}_n = \begin{bmatrix} Y_{n;1} & Y_{n;2} \\ Y_{n;3} & Y_{n;4} \end{bmatrix}, \tag{2.27}$$

$$Y_{n;j_1,\cdots,j_k} = \begin{bmatrix} Y_{n;j_1,\cdots,j_k,1} & Y_{n;j_1,\cdots,j_k,2} \\ Y_{n;j_1,\cdots,j_k,3} & Y_{n;j_1,\cdots,j_k,4} \end{bmatrix}, \tag{2.28}$$

for $1 \leq k \leq n-2$, and

$$Y_{n;j_1,\cdots,j_{n-1}} = \begin{bmatrix} y_{j_1,\cdots,j_{n-1},1} & y_{j_1,\cdots,j_{n-1},2} \\ y_{j_1,\cdots,j_{n-1},3} & y_{j_1,\cdots,j_{n-1},4} \end{bmatrix}. \tag{2.29}$$

Proof. From (2.12), (2.14) and (2.15), we have following table:

j_l	1	2	3	4
$u_{1,l}$	0	0	1	1
$u_{2,l}$	0	1	0	1

Table 2.1

For any $n \geq 2$, by (2.12),(2.14) and (2.15), it is easy to generalize (2.18) and (2.19) to

$$i_{n;1} = 1 + \sum_{l=1}^{n} [\frac{j_l - 1}{2}]2^{n-l}, \tag{2.30}$$

and

$$i_{n;2} = 1 + \sum_{l=1}^{n} \{j_l - 1 - 2[\frac{j_l - 1}{2}]\}2^{n-l}. \tag{2.31}$$

From (2.30) and (2.31), we have

$$i_{n+1;1} = 2i_{n;1} - 1 + [\frac{j_{n+1} - 1}{2}], \tag{2.32}$$

and

$$i_{n+1;2} = 2i_{n;2} - 1 + \{j_{n+1} - 1 - 2[\frac{j_{n+1} - 1}{2}]\}. \tag{2.33}$$

Next, by induction on n, the theorem follows from last two formulae and the Table 2.1. The proof is complete. ■

Remark 2.2. The ordering matrix on $\Sigma_{m\times n}$ can also be introduced accordingly. Section 4 provides further details. However, when spatial entropy $h(\mathcal{B})$ of $\Sigma(\mathcal{B})$ is computed, only λ_n, the largest eigenvalue of $\mathbf{T}_n(\mathcal{B})$, must be known. Section 3 provides further details.

2.2. **More Symbols on Larger Lattices.** Consider the number of symbols is larger than two on $\mathbf{Z}_{2\times 2}$, i.e., $\mathcal{S} = \{0, 1, 2, \cdots, p-1\}$, $p \geq 3$. Formulae in the last subsection can be modified from 2 to p appropriately and similar results can be obtained. Here, only some key formulae are mentioned. Additionally, (2.8) and (2.12) are replaced by

$$i_r = 1 + pu_{r,1} + u_{r,2}, \tag{2.34}$$

$$j_s = 1 + pu_{1,s} + u_{2,s}, \tag{2.35}$$

$1 \leq i_r$, $j_s \leq p^2$, $r = 1, 2$ and $s = 1, 2$.

Now, the ordering matrices $\mathbf{X}_2 = [x_{i_1,i_2}]$ and $\mathbf{Y}_2 = [y_{j_1,j_2}]$ are both $p^2 \times p^2$ matrices. If we express $\mathbf{X}_2$ by y, then we have

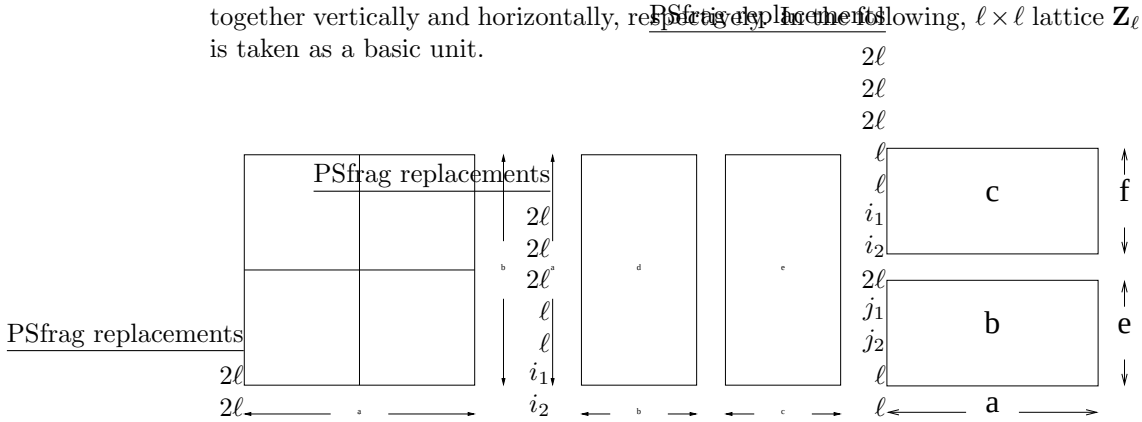
$$\mathbf{X}_2 = \begin{bmatrix} Y_1 & \cdots & Y_p \\ Y_{p+1} & \cdots & Y_{2p} \\ \vdots & \ddots & \vdots \\ Y_{(p-1)p+1} & \cdots & Y_{p^2} \end{bmatrix}_{p\times p}, \tag{2.36}$$

with

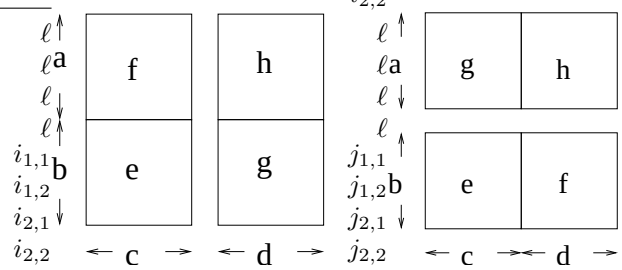
$$Y_{j_1} = \begin{bmatrix} y_{j_1,1} & \cdots & y_{j_1,p} \\ y_{j_1,(p+1)} & \cdots & y_{j_1,2p} \\ \vdots & \ddots & \vdots \\ y_{j_1,(p-1)p+1} & \cdots & y_{j_1,p^2} \end{bmatrix}_{p\times p}. \tag{2.37}$$

The higher ordering matrices $\mathbf{X}_n$ can also be expressed in y as in Theorem 2.1. Details are provided later.

Next, as we encounter in CNN, the system is often given by a 3×3 (or $m \times m$ with $m \geq 3$) template (e.g.[15, 16, 17, 18, 19, 29]). Here, local patterns $\Sigma_{m\times m}$ must be studied, where $m \geq 3$. The concept introduced in the last section can be generalized to cover this situation. Here, a case is first treated when m is even. Indeed, assume that $m = 2\ell$, $\ell \geq 2$ and $\mathcal{S}$ contains p elements. Then, $2\ell \times 2\ell$ lattice $\mathbf{Z}_{2\ell\times 2\ell}$ can be viewed as two $\ell \times 2\ell$ lattices $\mathbf{Z}_{\ell\times 2\ell}$ and two $2\ell \times \ell$ lattices $\mathbf{Z}_{2\ell\times \ell}$. Furthermore, $\mathbf{Z}_{\ell\times 2\ell}$ and $\mathbf{Z}_{2\ell\times \ell}$ can be viewed as two $\ell \times \ell$ lattices $\mathbf{Z}_{\ell\times \ell}$ glued together vertically and horizontally, respectively. In the following, $\ell \times \ell$ lattice $\mathbf{Z}_{\ell\times \ell}$ is taken as a basic unit.



PSfrag replacements



Now, the ordering matrices $\mathbf{X}_2 = [x_{i_1,i_2}]$ and $\mathbf{Y}_2 = [y_{j_1,j_2}]$ are introduced to $\Sigma_{2\ell\times 2\ell}$ as follows. Given a $U = (u_{\alpha_1,\alpha_2}) \in \Sigma_{2\ell\times 2\ell}$, define

$$i_{r_1,r_2} = \sum_{\alpha_1=1}^{\ell}\sum_{\alpha_2=1}^{\ell} u_{\widetilde{\alpha_1},\widetilde{\alpha_2}} p^{\ell(\ell-\alpha_1)+\ell-\alpha_2}, \tag{2.38}$$

$$\widetilde{\alpha_1} = (r_1-1)\ell+\alpha_1,\ \widetilde{\alpha_2} = (r_2-1)\ell+\alpha_2, \tag{2.39}$$

and

$$i_k = 1 + qi_{k,1} + i_{k,2}, \tag{2.40}$$

here, r_1, r_2, $k = 1, 2$, and

$$j_{s_1,s_2} = \sum_{\alpha_1=1}^{\ell}\sum_{\alpha_2=1}^{\ell} u_{\widehat{\alpha_1},\widehat{\alpha_2}} p^{\ell(\ell-\alpha_1)+\ell-\alpha_2}, \tag{2.41}$$

$$\widehat{\alpha_1} = (s_2-1)\ell+\alpha_1,\ \widehat{\alpha_2} = (s_1-1)\ell+\alpha_2, \tag{2.42}$$

and

$$j_l = 1 + qj_{l,1} + j_{l,2}, \tag{2.43}$$

here, s_1, s_2, $l = 1, 2$, and

$$q = p^{\ell^2}. \tag{2.44}$$

From (2.38)$\sim$ (2.44), (2.18) and (2.19) are replaced by

$$i_1 = 1 + [\frac{j_2-1}{q}] + q[\frac{j_1-1}{q}], \tag{2.45}$$

and

$$i_2 = 1 + \sum_{k=1}^{2}\{j_k - 1 - q[\frac{j_k-1}{q}]\}q^{2-k}. \tag{2.46}$$

From (2.45) and (2.46), $\mathbf{X}_2$ can be expressed by $y_{j_1j_2}$ as in (2.36) and (2.37) by replacing p with q, i.e.,

$$\mathbf{X}_2 = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_q \\ Y_{q+1} & Y_{q+2} & \cdots & Y_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{(q-1)q+1} & Y_{(q-1)q+2} & \cdots & Y_{q^2} \end{bmatrix}_{q\times q}, \tag{2.47}$$

with

$$Y_{j_1} = \begin{bmatrix} y_{j_1,1} & \cdots & y_{j_1,q} \\ y_{j_1,q+1} & \cdots & y_{j_1,2q} \\ \vdots & \ddots & \vdots \\ y_{j_1,(q-1)q+1} & \cdots & y_{j_1,q^2} \end{bmatrix}_{q\times q}. \tag{2.48}$$

Now, recursion formulae can be stated for a higher ordering matrix $\mathbf{X}_n = [x_{n;i_1,i_2}]_{q^n\times q^n}$ as follows.

Theorem 2.3. *Assume that we have p symbols, $p \geq 2$ and let $q = p^{\ell^2}$, $\ell \geq 2$. For any $n \geq 2$, $\Sigma_{2\ell\times n\ell} = \{y_{j_1,j_2,\cdots,j_n}\}$, where $y_{j_1,j_2,\cdots,j_n} \equiv y_{j_1,j_2}\hat{\oplus}y_{j_2,j_3}\hat{\oplus}\cdots\hat{\oplus}y_{j_{n-1},j_n}$, $1 \leq j_k \leq q^2$ and $1 \leq k \leq n$. Furthermore, the ordering matrix $\mathbf{X}_n$ can be decomposed by n Z-maps successively as*

$$\mathbf{X}_n = \begin{bmatrix} Y_{n;1} & Y_{n;2} & \cdots & Y_{n;q} \\ Y_{n;q+1} & Y_{n;q+2} & \cdots & Y_{n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n;(q-1)q+1} & Y_{n;(q-1)q+2} & \cdots & Y_{n;q^2} \end{bmatrix} \tag{2.49}$$

$$Y_{n;j_1,\cdots,j_k} = \begin{bmatrix} Y_{n;j_1,\cdots,j_k,1} & Y_{n;j_1,\cdots,j_k,2} & \cdots & Y_{n;j_1,\cdots,j_k,q} \\ Y_{n;j_1,\cdots,j_k,q+1} & Y_{n;j_1,\cdots,j_k,q+2} & \cdots & Y_{n;j_1,\cdots,j_k,2q} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n;j_1,\cdots,j_k,(q-1)q+1} & Y_{n;j_1,\cdots,j_k,(q-1)q+2} & \cdots & Y_{n;j_1,\cdots,j_k,q^2} \end{bmatrix} \tag{2.50}$$

for $1 \leq k \leq n-2$,

$$Y_{n;j_1,\cdots,j_{n-1}} = \begin{bmatrix} y_{j_1,\cdots,j_{n-1},1} & y_{j_1,\cdots,j_{n-1},2} & \cdots & y_{j_1,\cdots,j_{n-1},q} \\ y_{j_1,\cdots,j_{n-1},q+1} & y_{j_1,\cdots,j_{n-1},q+2} & \cdots & y_{j_1,\cdots,j_{n-1},2q} \\ \vdots & \vdots & \ddots & \vdots \\ y_{j_1,\cdots,j_{n-1},(q-1)q+1} & y_{j_1,\cdots,j_{n-1},(q-1)q+2} & \cdots & y_{j_1,\cdots,j_{n-1},q^2} \end{bmatrix} \tag{2.51}$$

Proof. By taking $\mathbf{Z}_{\ell\times\ell}$ as a basic unit, $2n$ $\mathbf{Z}_{\ell\times\ell}$ in $\mathbf{Z}_{2\ell\times n\ell}$ can be ordered by

n	2n
⋮	⋮
2	n+2
1	n+1

and

2n-1	2n
⋮	⋮
3	4
1	2

for x and y, respectively. Now, (2.30) and (2.31) are replaced by

$$i_{n;1} = 1 + \sum_{s=1}^{n} [\frac{j_s - 1}{q}] q^{n-s} \tag{2.52}$$

and

$$i_{n;2} = 1 + \sum_{s=1}^{n} \{j_s - 1 - q[\frac{j_s - 1}{q}]\} q^{n-s}. \tag{2.53}$$

(2.32) and (2.33) are replaced by

$$i_{n+1;1} = q i_{n;1} - (q-1) + [\frac{j_{n+1} - 1}{q}], \tag{2.54}$$

and

$$i_{n+1;2} = q i_{n;2} - (q-1) + \{j_{n+1} - 1 - q[\frac{j_{n+1} - 1}{q}]\}. \tag{2.55}$$

Table 2.1 is replaced by

j	1	2	$\cdots$	q	$q+1$	$q+2$	$\cdots$	$2q$	$\cdots$	q^2-q+1	$\cdots$	q^2
$[\frac{j-1}{q}]$	0	0	$\cdots$	0	1	1	$\cdots$	1	$\cdots$	$q-1$	$\cdots$	$q-1$
$j-1-q[\frac{j-1}{q}]$	0	1	$\cdots$	q-1	0	1	$\cdots$	q-1	$\cdots$	0	$\cdots$	$q-1$

Table 2.2

By induction on n as in proving Theorem 2.1, the results follow. ■

Next, the local $m_1 \times m_2$ patterns are first extended to even $2\ell \times 2\ell$ patterns, where 2ℓ is the smallest positive integer which is greater than m_1 and m_2. The study of $\Sigma_{2\ell\times 2\ell}$ is then proceeded with as in the previous paragraphs.

Obviously, the situation becomes more complex when a given larger lattice $\mathbf{Z}_{m\times m}$contains many symbols. However, the above theory can also be applied to derive recursion formulae for higher ordering matrices $\mathbf{X}_n$ from $\mathbf{X}_2,\ n \geq 3$.

3. Transition matrices

This section derives the transition matrices $\mathbf{T}_n$ for a given basic set $\mathcal{B}$. For simplicity, the study of two symbols $\mathcal{S} = \{0, 1\}$ on 2×2 lattice $\mathbf{Z}_{2\times 2}$ in two dimensional lattice space $\mathbf{Z}^2$ is of particular focus. The results can be extended to general cases.

3.1. 2×2 **systems.** Given a basic set $\mathcal{B} \subset \Sigma_{2\times 2}$, horizontal and vertical transition matrices H_2 and V_2 can be defined by

$$H_2 = [h_{i_1,i_2}] \ and \ V_2 = [v_{j_1,j_2}] \tag{3.1}$$

, two 4×4 matrices with entries either 0 or 1 , according to following rules:

$$\begin{cases} h_{i_1,i_2} & = 1 & if & x_{i_1,i_2} \in \mathcal{B}, \\ & = 0 & if & x_{i_1,i_2} \in \Sigma_{2\times 2} - \mathcal{B}, \end{cases} \tag{3.2}$$

and

$$\begin{cases} v_{j_1,j_2} & = 1 & if & y_{j_1,j_2} \in \mathcal{B}, \\ & = 0 & if & y_{j_1,j_2} \in \Sigma_{2\times 2} - \mathcal{B}. \end{cases} \tag{3.3}$$

Obviously, $h_{i_1,i_2} = v_{j_1,j_2}$, where (i_1, i_2) and (j_1, j_2) are related according to (2.16)∼(2.19). Now, the transition matrix $\mathbf{T}_2$ for $\mathcal{B}$ can be defined by

$$\begin{aligned} \mathbf{T}_2 &\equiv \mathbf{T}_2(\mathcal{B}) \\ &= \begin{bmatrix} v_{1,1} & v_{1,2} & v_{2,1} & v_{2,2} \\ v_{1,3} & v_{1,4} & v_{2,3} & v_{2,4} \\ v_{3,1} & v_{3,2} & v_{4,1} & v_{4,2} \\ v_{3,3} & v_{3,4} & v_{4,3} & v_{4,4} \end{bmatrix}. \end{aligned} \tag{3.4}$$

Define

$$v_{j_1,j_2,\cdots,j_n} = v_{j_1,j_2} \cdot v_{j_2,j_3} \cdots v_{j_{n-1},j_n}, \tag{3.5}$$

and

$$\mathbf{T}_n = [v_{j_1,j_2,\cdots,j_n}],$$

then the transition matrix $\mathbf{T}_n$ for $\mathcal{B}$ defined on $\mathbf{Z}_{2\times n}$ is a $2^n \times 2^n$ matrix with entries $v_{j_1,\cdots,j_n}$, which are either 1 or 0, by substituting $y_{j_1,\cdots,j_n}$ by $v_{j_1,\cdots,j_n}$ in $\mathbf{X}_n$, see (2.27)∼(2.29).

In the following, we give some interpretations for $\mathbf{T}_n$, one from an algebraic perspective and the other from Lindenmayer system (details can be found in Remark 3.2). For clarity, $\mathbf{T}_3$ can be written in a complete form as

(3.6)

$$\begin{bmatrix} v_{1,1}v_{1,1} & v_{1,1}v_{1,2} & v_{1,2}v_{2,1} & v_{1,2}v_{2,2} & v_{2,1}v_{1,1} & v_{2,1}v_{1,2} & v_{2,2}v_{2,1} & v_{2,2}v_{2,2} \\ v_{1,1}v_{1,3} & v_{1,1}v_{1,4} & v_{1,2}v_{2,3} & v_{1,2}v_{2,4} & v_{2,1}v_{1,3} & v_{2,1}v_{1,4} & v_{2,2}v_{2,3} & v_{2,2}v_{2,4} \\ v_{1,3}v_{3,1} & v_{1,3}v_{3,2} & v_{1,4}v_{4,1} & v_{1,4}v_{4,2} & v_{2,3}v_{3,1} & v_{2,3}v_{3,2} & v_{2,4}v_{4,1} & v_{2,4}v_{4,2} \\ v_{1,3}v_{3,3} & v_{1,3}v_{3,4} & v_{1,4}v_{4,3} & v_{1,4}v_{4,4} & v_{2,3}v_{3,3} & v_{2,3}v_{3,4} & v_{2,4}v_{4,3} & v_{2,4}v_{4,4} \\ v_{3,1}v_{1,1} & v_{3,1}v_{1,2} & v_{3,2}v_{2,1} & v_{3,2}v_{2,2} & v_{4,1}v_{1,1} & v_{4,1}v_{1,2} & v_{4,2}v_{2,1} & v_{4,2}v_{2,2} \\ v_{3,1}v_{1,3} & v_{3,1}v_{1,4} & v_{3,2}v_{2,3} & v_{3,2}v_{2,4} & v_{4,1}v_{1,3} & v_{4,1}v_{1,4} & v_{4,2}v_{2,3} & v_{4,2}v_{2,4} \\ v_{3,3}v_{3,1} & v_{3,3}v_{3,2} & v_{3,4}v_{4,1} & v_{3,4}v_{4,2} & v_{4,3}v_{3,1} & v_{4,3}v_{3,2} & v_{4,4}v_{4,1} & v_{4,4}v_{4,2} \\ v_{3,3}v_{3,3} & v_{3,3}v_{3,4} & v_{3,4}v_{4,3} & v_{3,4}v_{4,4} & v_{4,3}v_{3,3} & v_{4,3}v_{3,4} & v_{4,4}v_{4,3} & v_{4,4}v_{4,4} \end{bmatrix}$$

From an algebraic perspective, $\mathbf{T}_3$ can be defined through the classical Kronecker product (or tensor product) $\otimes$ and Hadamard product $\odot$, (e.g. [9]). Indeed, for any two matrices $A = (a_{i,j})$ and $B = (b_{k,l})$, the Kronecker product (tensor product) of $A \otimes B$ is defined by

$$A \otimes B = (a_{i,j}B). \tag{3.7}$$

On the other hand, for any two $n \times n$ matrices

$$C = (c_{i,j}) \text{ and } D = (d_{i,j}),$$

where $c_{i,j}$ and $d_{i,j}$ are numbers or matrices. Next, Hadamard product of $C \odot D$ is defined by

$$C \odot D = (c_{i,j} \cdot d_{i,j}), \tag{3.8}$$

where the product $c_{i,j} \cdot d_{i,j}$ of $c_{i,j}$ and $d_{i,j}$ may be a multiplication of numbers, numbers and matrices or matrices whenever it is well defined. For instance, $c_{i,j}$ is

a number and $d_{i,j}$ is a matrix.
Denoted by

$$\mathbf{T}_2 = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \tag{3.9}$$

where T_k is a 2×2 matrix with

$$T_k = \begin{bmatrix} v_{k,1} & v_{k,2} \\ v_{k,3} & v_{k,4} \end{bmatrix}. \tag{3.10}$$

Next, using Hadamard product, (3.6) can be written as

$$\mathbf{T}_3 = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{2,1} & v_{2,2} \\ v_{1,3} & v_{1,4} & v_{2,3} & v_{2,4} \\ v_{3,1} & v_{3,2} & v_{4,1} & v_{4,2} \\ v_{3,3} & v_{3,4} & v_{4,3} & v_{4,4} \end{bmatrix} \odot \begin{bmatrix} T_1 & T_2 & T_1 & T_2 \\ T_3 & T_4 & T_3 & T_4 \\ T_1 & T_2 & T_1 & T_2 \\ T_3 & T_4 & T_3 & T_4 \end{bmatrix}, \tag{3.11}$$

and can also be written by Kronecker product with Hadamard product as

$$\mathbf{T}_3 = \begin{pmatrix} \mathbf{T}_2 \end{pmatrix}_{4\times 4} \odot \left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right], \tag{3.12}$$

where $(\mathbf{T}_2)_{4\times 4}$ is interpreted as a 4×4 matrix given as in (3.4). Hereinafter, $(M)_{k\times k}$ is used as the $k \times k$ matrix; its entries may also be matrices.

Furthermore, by (3.9) and (3.12), $\mathbf{T}_3$ can also be written as

$$\mathbf{T}_3 = \begin{bmatrix} T_1 \odot \mathbf{T}_2 & T_2 \odot \mathbf{T}_2 \\ T_3 \odot \mathbf{T}_2 & T_4 \odot \mathbf{T}_2 \end{bmatrix}. \tag{3.13}$$

Now, from the perspective of Lindenmayer system, (3.13) can be interpreted as a rewriting rule as follows:

To construct $\mathbf{T}_3$ from $\mathbf{T}_2$, simply replace T_k in (3.9) by $T_k \odot \mathbf{T}_2$, i.e,

$$T_k \longmapsto T_k \odot \mathbf{T}_2 = \begin{bmatrix} v_{k,1}T_1 & v_{k,2}T_2 \\ v_{k,3}T_3 & v_{k,4}T_4 \end{bmatrix}. \tag{3.14}$$

Now, $\mathbf{T}_3$ can be written as

$$\mathbf{T}_3 = \begin{bmatrix} v_{1,1}T_1 & v_{1,2}T_2 & v_{2,1}T_1 & v_{2,2}T_2 \\ v_{1,3}T_3 & v_{1,4}T_4 & v_{2,3}T_3 & v_{2,4}T_4 \\ v_{3,1}T_1 & v_{3,2}T_2 & v_{4,1}T_1 & v_{4,2}T_2 \\ v_{3,3}T_3 & v_{3,4}T_4 & v_{4,3}T_3 & v_{4,4}T_4 \end{bmatrix}. \tag{3.15}$$

Since $v_{k,j}$ is either 0 or 1, the entries of $\mathbf{T}_3$ in (3.15) are T_k, i.e, T_k can be taken as the "basic element" in constructing $\mathbf{T}_n$, $n \geq 3$. As demonstrated later, (3.14) is an effective means of constructing $\mathbf{T}_{n+1}$ from $\mathbf{T}_n$ for any $n \geq 2$.

Now, by induction on n, the following properties of transition matrix $\mathbf{T}_n$ on $\mathbf{Z}_{2\times n}$ can be easily proven.

Theorem 3.1. *Let $\mathbf{T}_2$ be a transition matrix given by (3.4). Then, for higher order transition matrices $\mathbf{T}_n$, $n \geq 3$, we have the following three equivalent expressions*
(I) $\mathbf{T}_n$ can be decomposed into n successive 2×2matrices (or n-successive Z-maps) as follows:

$$\mathbf{T}_n = \begin{bmatrix} T_{n;1} & T_{n;2} \\ T_{n;3} & T_{n;4} \end{bmatrix},$$

$$T_{n;j_1,\cdots,j_k} = \begin{bmatrix} T_{n;j_1,\cdots,j_k,1} & T_{n;j_1,\cdots,j_k,2} \\ T_{n;j_1,\cdots,j_k,3} & T_{n;j_1,\cdots,j_k,4} \end{bmatrix},$$

for $1 \le k \le n-2$ *and*

$$T_{n;j_1,\cdots,j_{n-1}} = \begin{bmatrix} v_{j_1,\cdots,j_{n-1},1} & v_{j_1,\cdots,j_{n-1},2} \\ v_{j_1,\cdots,j_{n-1},3} & v_{j_1,\cdots,j_{n-1},4} \end{bmatrix}.$$

Furthermore,

$$T_{n;k} = \begin{bmatrix} v_{k,1}T_{n-1;1} & v_{k,2}T_{n-1;2} \\ v_{k,3}T_{n-1;3} & v_{k,4}T_{n-1;4} \end{bmatrix}. \tag{3.16}$$

(II) Starting from

$$\mathbf{T}_2 = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

with

$$T_k = \begin{pmatrix} v_{k,1} & v_{k,2} \\ v_{k,3} & v_{k,4} \end{pmatrix},$$

where $\mathbf{T}_n$ *can be obtained from* $\mathbf{T}_{n-1}$ *by replacing* T_k *by* $T_k \odot \mathbf{T}_2$ *according to (3.14).*

(III)

$$\mathbf{T}_n = (\mathbf{T}_{n-1})_{2^{n-1}\times 2^{n-1}} \odot \left(E_{2^{n-2}} \quad \otimes \quad \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \right),$$

where E_{2^k} *is the* $2^k \times 2^k$ *matrix with* 1 *as its entries.*

Proof.

(I)The proof is simply replaced $Y_{n;j_1,\cdots,j_k}$ and $y_{j_1,\cdots,j_n}$ by $T_{n;j_1,\cdots,j_k}$ and $v_{j_1,\cdots,j_n}$ in Theorem 2.1, respectively.

(II) follow from (I) directly.

To Prove (III), from (I) we have

$$\mathbf{T}_n = \begin{bmatrix} T_{n;1} & T_{n;2} \\ T_{n;3} & T_{n;4} \end{bmatrix}.$$

Additionally, base on (3.16), following formula is derived.

$$\begin{aligned} \mathbf{T}_n &= \begin{bmatrix} v_{1,1}T_{n;1} & v_{1,2}T_{n;2} & v_{2,1}T_{n;1} & v_{2,2}T_{n;2} \\ v_{1,3}T_{n;3} & v_{1,4}T_{n;4} & v_{2,3}T_{n;3} & v_{2,4}T_{n;4} \\ v_{3,1}T_{n;1} & v_{3,2}T_{n;2} & v_{4,1}T_{n;1} & v_{4,2}T_{n;2} \\ v_{3,3}T_{n;3} & v_{3,4}T_{n;4} & v_{4,3}T_{n;3} & v_{4,4}T_{n;4} \end{bmatrix} \\ &= (\mathbf{T}_{n-1})_{2^{n-1}\times 2^{n-1}} \odot \left(E_{2^{n-2}} \quad \otimes \quad \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \right). \end{aligned}$$

The proof is complete. ■

Remark 3.2. While studying the growth processes of plants, Lindenmayer, e.g.[38], derived a developmental algorithm, i.e., a set of rules which describes plant development in time. Hereinafter, a system with a set of rewriting rules is referred to as Lindenmayer system or L-system. From Theorem 3.1(III), the family of transition matrices $\{\mathbf{T}_n\}_{n\ge 2}$ is a two-dimensional L-system with a rewriting rule(3.16).

Similar to many L-systems, our system $\mathbf{T}_n$ also enjoys the simplicity of recursion formulae and self-similarity.

As for spatial entropy $h(\mathcal{B})$, we have the following theorem.

Theorem 3.3. *Given a basic set $\mathcal{B} \subset \Sigma_{2\times 2}$, let λ_n be the largest eigenvalue of the associated transition matrix $\mathbf{T}_n$ which is defined in Theorem 3.1. Then,*

$$h(\mathcal{B}) = \lim_{n\to\infty} \frac{\log \lambda_n}{n}. \tag{3.17}$$

Proof. By the same arguments as in [13], the limit (1.2) is well-defined and exists. From the construction of $\mathbf{T}_n$, we observe that for $m \geq 2$,

$$\begin{aligned} \Gamma_{m\times n}(\mathcal{B}) &= \sum_{1\leq i,j\leq 2^n} (\mathbf{T}_n^{m-1})_{i,j} \\ &\equiv \#(\mathbf{T}_n^{m-1}). \end{aligned} \tag{3.18}$$

As in a one dimensional case, we have

$$\lim_{m\to\infty} \frac{\log \#(\mathbf{T}_n^{m-1})}{m} = \log \lambda_n,$$

e.g. [42]. Therefore,

$$\begin{aligned} h(\mathcal{B}) &= \lim_{m,n\to\infty} \frac{\log \Gamma_{m\times n}(\mathcal{B})}{mn} \\ &= \lim_{n\to\infty} \frac{1}{n} (\lim_{m\to\infty} \frac{\log \Gamma_{m\times n}(\mathcal{B})}{m}) \\ &= \lim_{n\to\infty} \frac{\log \lambda_n}{n}. \end{aligned}$$

The proof is complete. ■

3.2. **Computation of Maximum Eigenvalues and Spatial Entropy.** Given a transition matrix $\mathbf{T}_2$, for any $n \geq 2$, the characteristic polynomials $|\mathbf{T}_n - \lambda|$ are of degree 2^n. In general, computing or estimating the largest eigenvalue $\lambda_n = \lambda_n(\mathbf{T}_2)$ of $|\mathbf{T}_n - \lambda|$ for a large n is relatively difficult. However, in this section, we present a class of $\mathbf{T}_2$ in which $\lambda_n(\mathbf{T}_2)$ can be computed explicitly. Indeed, assume that $\mathbf{T}_2$ has the form of $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ in (3.9), i.e.,

$$T_1 = T_4 = A = \begin{bmatrix} a & a_2 \\ a_3 & a \end{bmatrix}, \tag{3.19}$$

and

$$T_2 = T_3 = B = \begin{bmatrix} b & b_2 \\ b_3 & b \end{bmatrix}, \tag{3.20}$$

where a, a_2, a_3, b, b_2 and b_3 are either 0 or 1.

We need the following lemma.

Lemma 3.4. *Let A and B be non-negative and non-zero $m \times m$ matrices, respectively, and α and β are positive numbers. The maximum eigenvalue of* $\begin{bmatrix} A & \alpha B \\ \beta B & A \end{bmatrix}$ *is then the maximum eigenvalue of*

$$A + \sqrt{\alpha\beta}B.$$

Proof. Consider

$$\begin{vmatrix} A-\lambda & \alpha B \\ \beta B & A-\lambda \end{vmatrix} = 0.$$

For $|A-\lambda| \neq 0$, the last equation is equivalent to

$$\begin{vmatrix} A-\lambda & B \\ 0 & (A-\lambda) - \alpha\beta B(A-\lambda)^{-1}B \end{vmatrix} = 0,$$

or

$$|I - \alpha\beta((A-\lambda)^{-1}B)^2| = 0.$$

Then, we have

$$|A + \sqrt{\alpha\beta}B - \lambda| = 0 \quad or \quad |A - \sqrt{\alpha\beta}B - \lambda| = 0.$$

Since A and B are non-negative and α and β are positive, verifying that the maximum eigenvalue λ of $\begin{bmatrix} A & \alpha B \\ \beta B & A \end{bmatrix}$ and $A + \sqrt{\alpha\beta}B$ are equal is relatively easy. The proof is complete. ■

Now, we can state our computation results for $\lambda_n(\mathbf{T}_2)$ when $\mathbf{T}_2$ satisfies (3.19) and (3.20).

Theorem 3.5. *Assume that* $\mathbf{T}_2 = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$ *and* $A = \begin{bmatrix} a & a_2 \\ a_3 & a \end{bmatrix}$ *and* $B = \begin{bmatrix} b & b_2 \\ b_3 & b \end{bmatrix}$ *where* $a, b, a_2, a_3, b_2, b_3 \in \{0, 1\}$. *For $n \geq 2$, let λ_n be the largest eigenvalue of*

$$|\mathbf{T}_n - \lambda| = 0.$$

Then

$$\lambda_n = \alpha_{n-1} + \beta_{n-1}, \tag{3.21}$$

where α_k and β_k satisfy the following recursion relations:

$$\alpha_{k+1} = a\alpha_k + b\beta_k, \tag{3.22}$$

$$\beta_{k+1} = \sqrt{(a_2\alpha_k + b_2\beta_k)(a_3\alpha_k + b_3\beta_k)}, \tag{3.23}$$

for $k \geq 0$, and

$$\alpha_0 = \beta_0 = 1. \tag{3.24}$$

Furthermore, the spatial entropy $h(\mathbf{T}_2)$ is equal to $\log \xi_*$, where ξ_* is the maximum root of the following polynomials $Q(\xi)$:
(I) if $a_2 = a_3 = 1$,

$$\begin{aligned} Q(\xi) \equiv \quad & 4\xi^2(\xi - a)^2 + (\gamma^2 - 4\delta)(\xi - a)^2 \\ & -\gamma^2\xi^2 - 2\gamma(2b - a\gamma)\xi - (2b - a\gamma)^2, \end{aligned} \tag{3.25}$$

where

$$\gamma = b_2 + b_3 \text{ and } \delta = b_2 b_3. \tag{3.26}$$

(II) if $a_2 a_3 = 0$ *and* $a_2 b_3 + a_3 b_2 = 1$,

$$Q(\xi) \equiv \xi^3 - a\xi^2 - \delta\xi + a\delta - b. \tag{3.27}$$

Moreover, if $a_2 a_3 = 0$ *and* $a_2 b_3 + a_3 b_2 = 0$, *then* $h(\mathbf{T}_2) = 0$.

Proof. Owing to the special structure of $\mathbf{T}_2$, it is easy to verify that for any $k \geq 2$, we have

$$\mathbf{T}_k = \begin{bmatrix} A_k & B_k \\ B_k & A_k \end{bmatrix},$$

and

$$\mathbf{T}_{k+1} = \begin{bmatrix} A_{k+1} & B_{k+1} \\ B_{k+1} & A_{k+1} \end{bmatrix},$$

here

$$A_{k+1} = \mathbf{T}_k \odot A = \begin{bmatrix} aA_k & a_2 B_k \\ a_3 B_k & aA_k \end{bmatrix}, \tag{3.28}$$

and

$$B_{k+1} = \mathbf{T}_k \odot B = \begin{bmatrix} bA_k & b_2 B_k \\ b_3 B_k & bA_k \end{bmatrix}, \tag{3.29}$$

$A_2 = A$ and $B_2 = B$. Now by Lemma 3.4,

$$|\mathbf{T}_{n+1} - \lambda_{n+1}| = 0,$$

implies

$$|A_{n+1} + B_{n+1} - \lambda_{n+1}| = 0. \tag{3.30}$$

Let

$$\alpha_0 = 1 \text{ and } \beta_0 = 1.$$

By induction on k, $1 \leq k \leq n$, and using (3.28),(3.29),(3.30) and Lemma 3.4, it is straight forward to derive

$$|\alpha_k A_{n-k+1} + \beta_k B_{n-k+1} - \lambda_{n+1}| = 0, \tag{3.31}$$

where α_k and β_k satisfy (3.22) and (3.23). In particular,

$$\alpha_n = a\alpha_{n-1} + b\beta_{n-1}, \tag{3.32}$$

$$\beta_n = \{(a_2\alpha_{n-1} + b_2\beta_{n-1})(a_3\alpha_{n-1} + b_3\beta_{n-1})\}^{\frac{1}{2}}, \tag{3.33}$$

and

$$\lambda_{n+1} = \alpha_n + \beta_n.$$

This proves the first part of the theorem.
The rest of the proof demonstrates that $h(\mathbf{T}_2) = \log \lambda_*$ where λ_* is the maximum root of $Q(\lambda)$. From (3.33), we have

$$\begin{aligned} \beta_n^2 = a_2 a_3 \alpha_{n-1}^2 &+ (a_2 b_3 + a_3 b_2)\alpha_{n-1}\beta_{n-1} \\ &+ b_2 b_3 \beta_{n-1}^2. \end{aligned} \tag{3.34}$$

Now, in (3.34), α_{n-1} is solved in terms of β_{n-1} and β_n. Next, α_{n-1} and α_n are substituted into (3.32) to obtain difference equations involving β_{n+1}, β_n and β_{n-1}. There are two cases to be studied:

Case I. If $a_2 = a_3 = 1$, then we have

$$\alpha_{n-1} = \frac{1}{2}\{-\gamma\beta_{n-1} + (4\beta_n^2 + (\gamma^2 - 4\delta)\beta_{n-1}^2)^{\frac{1}{2}}\}. \tag{3.35}$$

Substituting (3.35) into (3.32) yields

$$\begin{aligned} \{4\beta_{n+1}^2 + (\gamma^2 - 4\delta)\beta_n^2\}^{\frac{1}{2}} &= \gamma\beta_n + (2b - a\gamma)\beta_{n-1} \\ &+ a\{4\beta_n^2 + (\gamma^2 - 4\delta)\beta_{n-1}^2\}^{\frac{1}{2}}. \end{aligned} \tag{3.36}$$

Now, let

$$\xi_n = \frac{\beta_n}{\beta_{n-1}}, \tag{3.37}$$

and after dividing (3.36) by β_{n-1}, we have

$$\xi_n\{4\xi_{n+1}^2 + (\gamma^2 - 4\delta)\}^{\frac{1}{2}} = \gamma\xi_n + (2b - a\gamma) + a\{4\xi_n^2 + (\gamma^2 - 4\delta)\}^{\frac{1}{2}}. \tag{3.38}$$

(3.38) can be written as the following iteration map:

$$\xi_{n+1} = G_1(\xi_n), \tag{3.39}$$

where

$$G_1(\xi) = \frac{1}{2}\{4\delta + 2\gamma g(\xi) + g^2(\xi)\}^{\frac{1}{2}}, \tag{3.40}$$

and

$$g(\xi) = (2b - a\gamma)\xi^{-1} + a\{4 + (\gamma^2 - 4\delta)\xi^{-2}\}^{\frac{1}{2}}. \tag{3.41}$$

According to our results, the fixed point ξ_* of $G_1(\xi)$, i.e., $\xi_* = G(\xi_*)$, is a root of $Q(\xi)$. Indeed, by letting $\xi_n = \xi_{n+1} = \xi_*$ in (3.38), we have

$$(\xi_* - a)(4\xi_*^2 + (\gamma^2 - 4\delta))^{\frac{1}{2}} = \gamma\xi_* + (2b - a\gamma),$$

which gives us $Q(\xi_*) = 0$.

It can be proven that the maximum fixed point of $G_1(\xi)$ or the maximum root ξ_* of $Q(\xi) = 0$ satisfies $1 \le \xi_* \le 2$ and

$$\xi_n \to \xi_* \quad as \quad n \to \infty. \tag{3.42}$$

Details are omitted here for brevity. By (3.21), (3.35) and (3.37), we can also prove that

$$\frac{\lambda_{n+1}}{\lambda_n} \to \xi_* \quad as \quad n \to \infty. \tag{3.43}$$

Hence, $h(\mathbf{T}_2) = \log \xi_*$.

Case II. If $a_2a_3 = 0$ and $a_2b_3 + a_3b_2 = 1$,
then, from (3.33), we have

$$\alpha_{n-1} = \beta_n^2\beta_{n-1}^{-1} - \delta\beta_{n-1}. \tag{3.44}$$

Again, substituting (3.44) into (3.32) and letting (3.37) lead to

$$\xi_{n+1}^2\xi_n - a\xi_n^2 - \delta\xi_n + a\delta - b = 0, \tag{3.45}$$

i.e.,

$$\xi_{n+1} = G_2(\xi_n),$$

where

$$G_2(\xi) = \{a\xi + \delta + (b - a\delta)\xi^{-1}\}^{\frac{1}{2}}. \tag{3.46}$$

The maximum fixed point ξ_* of (3.46) is the maximum root of $Q(\xi) = 0$ in (3.27). It can also be proven that (3.42) and (3.43) holds in this case.

Finally, if $a_2a_3 = 0$ and $a_2b_3 + a_3b_2 = 0$, then β_n are all equal for $n \geq 1$. Hence, α_n is at most linear growth in n, implying that $h(\mathbf{T}_2) = 0$. The proof is thus complete. ■

For completeness, we list all $\mathbf{T}_2$ which satisfy (3.19) and (3.20) and have positive entropy $h(\mathbf{T}_2)$. The table is arranged based on the magnitude of $h(T_2)$. The polynomial $Q(.)$ in either (3.25) or (3.27) has been simplified to its proper factor whenever possible.

	A	B	$Q(\lambda)$	λ_*
(1)	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda - 2$	2
(2)	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\lambda^3 - 2\lambda^2 + \lambda - 1$	(i)
(3)(α)	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda^2 - \lambda - 1$	g
(3)(β)	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\lambda^2 - \lambda - 1$	g
(3)(γ)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda^2 - \lambda - 1$	g
(4)	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\lambda^3 - \lambda^2 - 1$	(ii)
	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$		
(5)	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda^3 - \lambda - 1$	(iii)
(6)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\lambda^4 - \lambda - 1$	(iv)

(i) $\lambda_* \doteq 1.75488$
(ii) $\lambda_* \doteq 1.46557$
(iii) $\lambda_* \doteq 1.32472$
(iv) $\lambda_* \doteq 1.22074$
where, $g \doteq 1.61803$, is the golden mean, a root of $\lambda^2 - \lambda - 1 = 0$.

Table 3.1

The recursion formulae for λ_n are

(1)		$\lambda_n = 2^n,$
(2)		$\lambda_{n+1} = \lambda_n + (\lambda_n \lambda_{n-1})^{\frac{1}{2}},$
(3)	(α)	$\lambda_{n+1} = \lambda_n + (\lambda_n(\lambda_n - \lambda_{n-1}))^{\frac{1}{2}},$
	(β)	$\lambda_{n+1} = \lambda_n + \lambda_{n-1},$
	(γ)	$\lambda_{n+1} = \lambda_n + \lambda_{n-1},$
(4)		$\lambda_{n+1} = \lambda_n + (\lambda_{n-1}(\lambda_n - \lambda_{n-1}))^{\frac{1}{2}},$
(5)		$\lambda_{n+1} = (\lambda_n \beta_{n-1})^{\frac{1}{2}} + \beta_{n-1},$ where $\beta_{n-1} = \lambda_n - \lambda_{n-1} + \cdots + (-1)^n,$
(6)		$\lambda_{n+1} = \lambda_n + (\lambda_n \beta_{n-2})^{\frac{1}{2}} - \beta_{n-2}.$

Table 3.2

Remark 3.6.
(i) According to Table 3.2, for cases (1)$\sim$(4), λ_{n+1} depends only on two preceding terms, λ_n and λ_{n-1}. However, in (5) and (6), λ_{n+1} depends on all of their preceding terms $\lambda_1, \cdots, \lambda_n$.
(ii) From Lemma 3.4 and Theorem 3.5, in addition to the maximum eigenvalue, we can obtain a complete set of eigenvalues of $\mathbf{T}_n$ explicitly.
(iii) In Theorem 3.5, polynomial $Q(\xi)$ given in (3.25) or (3.27) is the limiting equation for $\lambda_n^{\frac{1}{n}}$. Identifying whether any limiting equation ia available for general $\mathbf{T}_n$ is a worthwhile task.

Remark 3.7. Similar to the concept in Theorem 3.5, if $\mathbf{T}_2$ does not satisfy (3.19) and (3.20), another special structure can allow us to obtain explicit recursion formulae of λ_n and compute its spatial entropy $h(\mathbf{T}_2)$ explicitly. Table 3.3 provides some examples.

		λ_n	λ_*
(1)	$\mathbf{T_2} = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$,		
	$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda_n = 2g^{n-1}$	g
(2)	$\mathbf{T}_2 = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$		
(i)	$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda_{n+1} = g\lambda_n$ $\lambda_2 = 2g$	g
(ii)	$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\lambda_n = g^n$	g
(3)	$\mathbf{T}_2 = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$		
	$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	$\lambda_{n+1} = \lambda_n + \sqrt{\lambda_{n-1} \cdot (\lambda_n - \lambda_{n-1})}$	g
	or $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$		
(4)	$\mathbf{T}_2 = \begin{bmatrix} A & A \\ B & A \end{bmatrix}$	$\lambda_{n+1} = \lambda_n + \sqrt{\lambda_n \lambda_{n-1}}$	g
	$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$Q(\lambda) = \lambda^3 - 2\lambda^2 + \lambda - 1$	(i) *in* Table 3.1

Table 3.3

3.3. $2\ell \times 2\ell$ **Systems.** Results in the last two subsections can be generalized to p-symbols on $\mathbf{Z}_{2\ell\times 2\ell}$. Given a basic set $\mathcal{B} \subset \Sigma_{2\ell\times 2\ell}$, horizontal and vertical transition matrices $H_2 = [h_{i_1,i_2}]_{q^2\times q^2}$ and $V_2 = [v_{j_1,j_2}]_{q^2\times q^2}$ can be defined according the rules (3.2) and (3.3) by replacing $\Sigma_{2\times 2}$ with $\Sigma_{2\ell\times 2\ell}$, respectively. Then, the transition matrix $\mathbf{T}_2(\mathcal{B})$ for $\mathcal{B}$ can be defined by

$$\mathbf{T}_2 = \mathbf{T}_2(\mathcal{B}) = \begin{bmatrix} V_1 & V_2 & \cdots & V_q \\ V_{q+1} & V_{q+2} & \cdots & V_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ V_{(q-1)q+1} & V_{(q-1)q+2} & \cdots & V_{q^2} \end{bmatrix} \tag{3.47}$$

where

$$V_m = \begin{bmatrix} v_{m,1} & v_{m,2} & \cdots & v_{m,q} \\ v_{m,(q+1)} & v_{m,q+2} & \cdots & v_{m,2q} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m,(q-1)q+1} & v_{m,(q-1)q+2} & \cdots & v_{m,q^2} \end{bmatrix}, \tag{3.48}$$

$1 \le m \le q^2$. The higher order transition matrix $\mathbf{T}_n = [v_{j_1,j_2,\cdots,j_n}]$ for $\mathcal{B}$ defined on $\mathbf{Z}_{2\ell\times n\ell}$ is a $q^n \times q^m$ matrix, where $v_{j_1,j_2,\cdots,j_n}$ is given by (3.5) which are either 1 or 0, by substituting $y_{j_1,\cdots,j_n}$ by $v_{j_1,\cdots,j_n}$ in $\mathbf{X}_n$, see (2.49)$\sim$(2.51). For completeness, we state the following theorem for $\mathbf{T}_n$ and omit the proof for brevity.

Theorem 3.8. *Let $\mathbf{T}_2$ be a transition matrix given by (3.47) and (3.48). Then, for higher order transition matrices $\mathbf{T}_n$, $n \ge 3$, we have the following three equivalent expressions*

(I) *$\mathbf{T}_n$ can be decomposed into n successive $q \times q$ matrices as follows:*

$$\mathbf{T}_n = \begin{bmatrix} T_{n;1} & \cdots & T_{n;q} \\ T_{n;q+1} & \cdots & T_{n;2q} \\ \vdots & \ddots & \vdots \\ T_{n;(q-1)q+1} & \cdots & T_{n;q^2} \end{bmatrix}$$

$$T_{n;j_1,\cdots,j_k} = \begin{bmatrix} T_{n;j_1,\cdots,j_k,1} & \cdots & T_{n;j_1,\cdots,j_k,q} \\ T_{n;j_1,\cdots,j_k,q+1} & \cdots & T_{n;j_1,\cdots,j_k,2q} \\ \vdots & \ddots & \vdots \\ T_{n;j_1,\cdots,j_k,(q-1)q+1} & \cdots & T_{n;j_1,\cdots,j_k,q^2} \end{bmatrix}$$

for $1 \le k \le n-2$ and

$$T_{n;j_1,\cdots,j_{n-1}} = \begin{bmatrix} v_{j_1,\cdots,j_{n-1},1} & \cdots & v_{j_1,\cdots,j_{n-1},q} \\ v_{j_1,\cdots,j_{n-1},q+1} & \cdots & v_{j_1,\cdots,j_{n-1},2q} \\ \vdots & \ddots & \vdots \\ v_{j_1,\cdots,j_{n-1},(q-1)q+1} & \cdots & v_{j_1,\cdots,j_{n-1},q^2} \end{bmatrix}.$$

Furthermore,

$$T_{n;k} = \begin{bmatrix} v_{k,1}T_{n-1;1} & \cdots & v_{k,q}T_{n-1;q} \\ v_{k,q+1}T_{n-1;q+1} & \cdots & v_{k,2q}T_{n-1;2q} \\ \vdots & \ddots & \vdots \\ v_{k,(q-1)q+1}T_{n-1;(q-1)q+1} & \cdots & v_{k,q^2}T_{n-1;q^2} \end{bmatrix}$$

(II) *Starting from*

$$\mathbf{T}_2 = \begin{bmatrix} T_1 & \cdots & T_q \\ T_{q+1} & \cdots & T_{2q} \\ \vdots & \ddots & \vdots \\ T_{(q-1)q+1} & \cdots & T_{q^2} \end{bmatrix},$$

with

$$T_k = \begin{bmatrix} v_{k,1} & \cdots & v_{k,q} \\ v_{k,q+1} & \cdots & v_{k,2q} \\ \vdots & \ddots & \vdots \\ v_{k,(q-1)q+1} & \cdots & v_{k,q^2} \end{bmatrix},$$

$\mathbf{T}_n$ *can be obtained from* $\mathbf{T}_{n-1}$ *by replacing* T_k *by* $T_k \odot \mathbf{T}_2$ *according to*

$$T_k \mapsto T_k \odot \mathbf{T}_2 = \begin{bmatrix} v_{k,1}T_1 & \cdots & v_{k,q}T_q \\ v_{k,q+1}T_{q+1} & \cdots & v_{k,2q}T_{2q} \\ \vdots & \ddots & \vdots \\ v_{k,(q-1)q+1}T_{(q-1)q+1} & \cdots & v_{k,q^2}T_{q^2} \end{bmatrix}$$

(III)

$$\mathbf{T}_n = (\mathbf{T}_{n-1})_{q^{n-1}\times q^{n-1}} \odot (E_{q^{n-2}} \otimes \mathbf{T}_2).$$

For the spatial entropy $h(\mathcal{B})$, we have a similar result as in Theorem 3.3.

Theorem 3.9. *Given a basic set* $\mathcal{B} \subset \Sigma_{m_1\times m_2}$*, let* ℓ *be the smallest integer such that* $2\ell \geq m_1$ *and* $2\ell \geq m_2$*, and let* $\widetilde{\mathcal{B}} = \Sigma_{2\ell\times 2\ell}(\mathcal{B})$*. Assume that* $\lambda_{n;\ell}$ *is the largest eigenvalue of the associated transition matrix* $\mathbf{T}_n$*, which is defined in Theorem 3.8. Then*

$$h(\mathcal{B}) \quad = \tfrac{1}{\ell^2} \lim_{n\to\infty} \tfrac{log\lambda_{n;\ell}}{n}$$

Proof.

As in Theorem 3.3,

$$\begin{aligned} h(\mathcal{B}) &= \lim_{m,n\to\infty} \frac{log\Gamma_{m\ell\times n\ell}(\widetilde{\mathcal{B}})}{m\ell\times n\ell} \\ &= \frac{1}{\ell^2}\lim_{n\to\infty}\frac{1}{n}(\lim_{m\to\infty}\frac{log\#(T_n^{m-1}(\widetilde{\mathcal{B}}))}{m}) \\ &= \frac{1}{\ell^2}\lim_{n\to\infty}\frac{1}{n}(\lim_{m\to\infty}\frac{log\lambda_{n;\ell}^{m-1}}{m}) \\ &= \frac{1}{\ell^2}\lim_{n\to\infty}\frac{log\lambda_{n;\ell}}{n}. \end{aligned}$$

The proof is complete. ■

3.4. **Relation with Matrix Shifts.** Under many circumstances, we are given a pair of horizontal transition matrix $H = (h_{i,j})_{p\times p}$ and vertical transition matrix $V = (v_{i,j})_{p\times p}$, where $h_{i,j}$ and $v_{i,j} \in \{0,1\}$, e.g. [13, 29, 30]. Now, the sets of all admissible patterns which can be generated by H and V on $\mathbf{Z}_{m_1\times m_2}$ and $\mathbf{Z}^2$ are denoted by $\Sigma_{m_1\times m_2}(H;V)$ and $\Sigma(H;V)$, respectively. Furthermore, $\Sigma_{m_1\times m_2}(H;V)$ and $\Sigma(H;V)$ can be characterized by

(3.49)

$$\begin{aligned}&\Sigma_{m_1\times m_2}(H;V)=\{U\in\Sigma_{m_1\times m_2,p}: h_{u_\alpha,u_{\alpha+e_1}}=1 \textit{ and } v_{u_\beta,u_{\beta+e_2}}=1,\\ &\textit{where } e_1=(1,0),\ e_2=(0,1),\ \alpha=(\alpha_1,\alpha_2),\ \beta=(\beta_1,\beta_2)\\ &\textit{with } 1\le\alpha_1\le m_1-1\ ,\ 1\le\alpha_2\le m_2 \textit{ and } 1\le\beta_1\le m_1\ ,1\le\beta_2\le m_2-1\}\end{aligned}$$

and

(3.50)

$$\Sigma(H;V)=\{U\in\Sigma_p^2: h_{u_\alpha,u_{\alpha+e_1}}=1 \textit{ and } v_{u_\beta,u_{\beta+e_2}}=1 \textit{for all } \alpha,\beta\in\mathbf{Z}^2\}.$$

Literature often refers to $\Sigma(H;V)$ as Matrix shift, Markov shift or subshift of finite types, e.g. [13, 30, 37]

As mentioned earlier, constructing $\Sigma_{m_1\times m_2}(H;V)$ is of priority concern. According to our results, the established theories can be applied to answer this question. Indeed, we introduce $\mathcal{S}=\{0,1,2,\cdots,p-1\}$. On $\mathbf{Z}_{2\times 2}$, consider local pattern $U=(u_{\alpha_1,\alpha_2})$ with $u_{\alpha_1,\alpha_2}\in\mathcal{S}$. Define the ordering matrices $\mathbf{X}_2=[x_{i_1,i_2}]_{p^2\times p^2}$ and $\mathbf{Y}_2=[y_{j_1,j_2}]_{p^2\times p^2}$ for $\Sigma_{2\times 2}$ as in (2.36) and (2.37). Now, the basic set $\mathcal{B}(H;V)$ determined by H and V can be expressed as

(3.51)

$$\mathcal{B}(H;V)=\{U=(u_{\alpha_1,\alpha_2})\in\Sigma_{2\times 2}: h_{u_{1,1},u_{2,1}}h_{u_{1,2},u_{2,2}}v_{u_{1,1},u_{1,2}}v_{u_{2,1},u_{2,2}}=1\}.$$

Therefore, the transition matrix $\mathbf{T}_2=\mathbf{T}_2(H;V)$ can be expressed as $\mathbf{T}_2=[t_{j_1,j_2}]_{p^2\times p^2}$ with $t_{j_1,j_2}=1$ if and only if $y_{j_1,j_2}\in\mathcal{B}(H;V)$, i.e., $t_{j_1,j_2}=1$ if and only if

$$h_{u_{1,1},u_{2,1}}h_{u_{1,2},u_{2,2}}v_{u_{1,1},u_{1,2}}v_{u_{2,1},u_{2,2}}=1, \tag{3.52}$$

where j_l is related to u_{α_1,α_2} according to (2.35)

Now, $\mathbf{T}_n=\mathbf{T}_n(H;V)$ can be constructed recursively from $\mathbf{T}_2(H;V)$ by Theorem 3.8. Then, λ_n and spatial entropy $h(H;V)$ can be studied by Theorem 3.9. Notably, verifying that $\mathbf{T}_n(H;V)=\overline{\mathbf{T}}^{(n)}_{H,V}$, the transition matrix obtained by Juang et al in [30],is relatively easy. Furthermore, $T^{(n)}_{H,V}$ in [30] can also be obtained by deleting the rows and columns formed by zeros in $\mathbf{T}_n(H;V)$.

On the other hand, given a basic set $\mathcal{B}\subset\Sigma_{2\times 2,p}$ (or $\Sigma_{2l\times 2l,p}$), in general there is no horizontal transition matrix $H=(h_{i,j})_{p\times p}$ and vertical transition matrix $V=(v_{i,j})_{p\times p}$ such that $\mathcal{B}=\mathcal{B}(H;V)$ given by (3.51). Indeed, the number of subsets of $\Sigma_{2\times 2,p}$ is 2^{p^4} and the number of $\mathcal{B}(H;V)$ is at most 2^{2p^2} and $2^{2p^2}<2^{p^4}$ for any $p\ge 2$. However, as mentioned in p.468[37], any shift of finite type can be recorded to a matrix subshift. For completeness, a recoding method is described as follows.

Here, patterns in $\mathcal{B}\subset\Sigma_{2l\times 2l,p}$ are taken as new symbols, i.e.,

$$\mathcal{S}_{\mathcal{B}}\equiv\mathcal{B}\equiv\{U_1,\cdots,U_m\}, \tag{3.53}$$

where m is the number of patterns in $\mathcal{B}$. Now, with $\mathcal{S}_{\mathcal{B}}$, the horizontal transition matrix $H=H(\mathcal{B})=(h_{i_1,i_2})_{m\times m}$ and the vertical transition matrix $V=V(\mathcal{B})=(v_{j_1,j_2})_{m\times m}$ for $\mathcal{B}$ can be defined as follows.

$$\begin{aligned}&h_{i_1,i_2}=1 \textit{ if and only if}\\ &U_{i_1}(\alpha_1,\alpha_2)=U_{i_2}(\alpha_1-1,\alpha_2)\\ &\textit{for all } 2\le\alpha_1\le 2\ell \textit{ and } 1\le\alpha_2\le 2\ell,\end{aligned} \tag{3.54}$$

$$\begin{aligned} & v_{j_1,j_2} = 1 \ if \ and \ only \ if \\ & U_{j_1}(\alpha_1,\alpha_2) = U_{j_2}(\alpha_1,\alpha_2-1) \\ & for \ all \ 1 \le \alpha_1 \le 2\ell \ and \ 2 \le \alpha_2 \le 2\ell, \end{aligned} \tag{3.55}$$

i.e., U_{i_1} and U_{i_2} can be glued together horizontally and become a $(2\ell+1)\times 2\ell$ admissible pattern if and only if $h_{i_1,i_2} = 1$. Similarly, U_{j_1} and U_{j_2}can be glued together and become a $2\ell \times (2\ell+1)$ admissible pattern if and only if $v_{j_1,j_2} = 1$. Therefore, the pattern generation problems of $\mathcal{B} \subset \Sigma_{2l\times 2l}$ are equivalent to the problem of a given pair of horizontal matrix $H(\mathcal{B})$ and vertical transition matrix $V(\mathcal{B})$ defined by (3.54) and (3.55) with m symbols $\mathcal{S}_{\mathcal{B}} = \mathcal{B}$.

Notably, the n-th order transition matrix $\mathbf{T}_n(\mathcal{B})$ is a $q^n \times q^n$ matrix with $q = p^{\ell^2}$ and the n-th order transition matrix $\mathbf{T}_n(H(\mathcal{B}); V(\mathcal{B})))$ generated by $\mathbf{T}_2(H(\mathcal{B}); V(\mathcal{B})))$ is a $m^n \times m^n$ matrix. Consequently, if $m = \#\mathcal{B}$ is smaller than $q = p^{l^2}$, the eigenvalue problems of $\mathbf{T}_n(H(\mathcal{B}); V(\mathcal{B}))$ can be studied. Clearly, a small m generates less admissible patterns and a subsequently smaller entropy. For $\mathcal{B}$ with positive entropy $h(\mathcal{B})$ as in Table 3.1 and Table 3.3 , $\#\mathcal{B}$ is much larger than $q = 2$. Therefore, working on $\mathbf{T}_n(\mathcal{B})$ is generally better than doing so on $\mathbf{T}_n(H(\mathcal{B}); V(\mathcal{B})))$.

4. Higher dimensional cases

This section extends the results of a two dimensional case to higher dimensions. For clarity, a three dimension case is first examined by studying $\Sigma_{2\times 2\times 2}$ with two symbols $\mathcal{S} = \{0, 1\}$.

On $\Sigma_{m_1\times m_2\times m_3}$, each pattern $U = (u_{\alpha_1,\alpha_2,\alpha_3})$ is again assigned by

$$\begin{aligned} \chi(U) &\equiv \chi_{m_1,m_2,m_3}(U) \\ &= 1 + \sum_{\alpha_1=1}^{m_1}\sum_{\alpha_2=1}^{m_2}\sum_{\alpha_3=1}^{m_3} u_{\alpha_1,\alpha_2,\alpha_3}\chi^{\alpha_1,\alpha_2,\alpha_3}_{m_1,m_2,m_3}, \end{aligned} \tag{4.1}$$

here

$$\chi^{\alpha_1,\alpha_2,\alpha_3}_{m_1,m_2,m_3} = 2^{m_3m_2(m_1-\alpha_1)+m_3(m_2-\alpha_2)+(m_3-\alpha_3)}. \tag{4.2}$$

In general, given one to one and onto mapping ψ from $\Sigma_{m_1\times m_2\times m_2}$ to $\mathbf{N}_{2^{m_1m_2m_3}} = \{n \in \mathbf{N} : 1 \le n \le 2^{m_1m_2m_3}\}$, U is referred to herein as the $\psi(U)$-th element in $\Sigma_{m_1\times m_2\times m_2}$ and the ordering matrix $\mathbf{X}_\psi$ is defined with respect to ψ. Obviously, both $\chi\circ\psi^{-1}$ and $\psi\circ\chi^{-1}$ are one to one and onto on $\mathbf{N}_{2^{m_1m_2m_3}}$. The ordering matrices and the associated transition matrices with respect to χ and ψ are similar for all ψ. In this section, χ given in (4.1) and (4.2) is used for $\Sigma_{2\times m\times 2}$ and $\hat{\chi}$ (which will be defined in (4.17))is used for $\Sigma_{2\times m\times n}$ for $n \ge 3$. The choices of χ and $\hat{\chi}$ allow us to derive simple recursion formulae for generating ordering matrices and then transition matrices.

4.1. Ordering Matrices.

With ordering rule (4.1), the ordering matrix $\mathbf{X}_{2\times 2} = [x_{i_1,i_2}]$ of $\Sigma_{2\times 2\times 2}$ can be expressed as

$$i_k = 1 + \sum_{\alpha_2=1}^{2}\sum_{\alpha_3=1}^{2} u_{k,\alpha_2,\alpha_3}2^{6-2\alpha_2-\alpha_3}, \tag{4.3}$$

$1 \le i_k \le 16$ and $k = 1, 2$.

Define

$$\beta = 2(\alpha_2 - 1) + \alpha_3, \tag{4.4}$$

i.e, in the y-z plane, $\mathbf{Z}_{2\times 2}$ is given an order

(4.5)

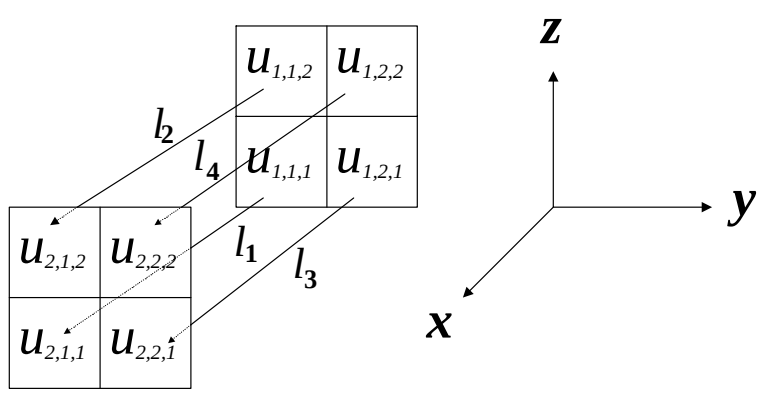
2	4
1	3

.

as in (2.1) and (2.2). Next, define

$$\ell_\beta = 1 + 2u_{1,\alpha_2,\alpha_3} + u_{2,\alpha_2,\alpha_3}, \tag{4.6}$$

$1 \le \ell_\beta \le 4$. From (4.4) and (4.6), we have the following diagram



.

From (4.4) and (4.6), it is easy to verify that $u_{\alpha_1,\alpha_2,\alpha_3}$ can be written in terms of ℓ_β by

$$u_{1,\alpha_2,\alpha_3} = [\frac{\ell_\beta - 1}{2}], \tag{4.7}$$

$$u_{2,\alpha_2,\alpha_3} = \ell_\beta - 1 - 2[\frac{\ell_\beta - 1}{2}]. \tag{4.8}$$

Furthermore, the relation between i_k and ℓ_β can also be derived as

$$i_1 = 1 + 2^3[\frac{\ell_1 - 1}{2}] + 2^2[\frac{\ell_2 - 1}{2}] + 2[\frac{\ell_3 - 1}{2}] + [\frac{\ell_4 - 1}{2}], \tag{4.9}$$

$$\begin{aligned} i_2 = {} & 1 + 2^3\{\ell_1 - 1 - 2[\tfrac{\ell_1-1}{2}]\} + 2^2\{\ell_2 - 1 - 2[\tfrac{\ell_2-1}{2}]\} + \\ & 2\{\ell_3 - 1 - 2[\tfrac{\ell_3-1}{2}]\} + \{\ell_4 - 1 - 2[\tfrac{\ell_4-1}{2}]\} \end{aligned} \tag{4.10}$$

and

$$\ell_1 = 1 + 2[\frac{i_1 - 1}{8}] + [\frac{i_2 - 1}{8}], \tag{4.11}$$

$$\ell_2 = 1 + 2[\frac{i_1 - 1}{4}] - 4[\frac{i_1 - 1}{8}] + [\frac{i_2 - 1}{4}] - 2[\frac{i_2 - 1}{8}], \tag{4.12}$$

$$\ell_3 = 1 + 2[\frac{i_1 - 1}{2}] - 4[\frac{i_1 - 1}{4}] + [\frac{i_2 - 1}{2}] - 2[\frac{i_2 - 1}{4}], \tag{4.13}$$

$$\ell_4 = 1 + 2(i_1 - 1) - 4[\frac{i_1 - 1}{2}] + (i_2 - 1) - 2[\frac{i_2 - 1}{2}]. \tag{4.14}$$

Denoted by

$$w_{\ell_1,\ell_2,\ell_3,\ell_4} = x_{i_1,i_2},$$

where i_k and ℓ_β are related to each other in (4.9)∼(4.14), leading to

$$\chi(w_{\ell_1,\ell_2,\ell_3,\ell_4}) = 1 + \sum_{\beta=1}^{4}\{2^4[\frac{\ell_\beta - 1}{2}] + \{\ell_\beta - 1 - 2[\frac{\ell_\beta - 1}{2}]\}\}2^{4-\beta} \tag{4.15}$$

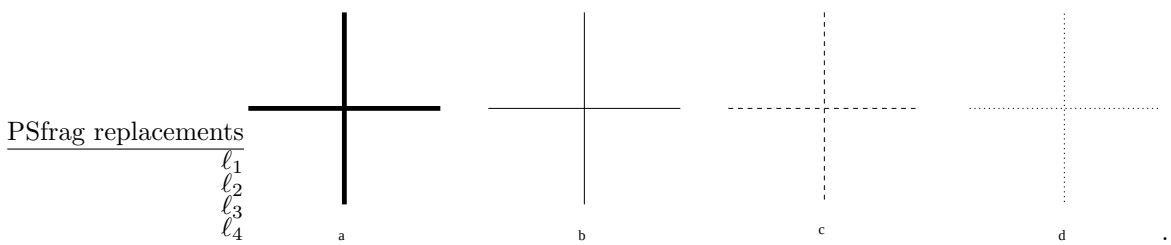
and the ordering matrix $\mathbf{X}_{2\times 2}$ can be represented in $w_{\ell_1,\ell_2,\ell_3,\ell_4}$ by 4 Z-maps successively as in Theorem 2.1, i.e ,

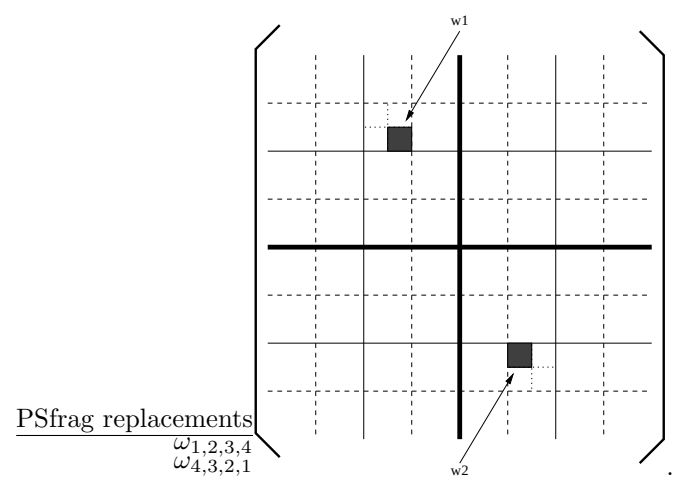
$$\begin{aligned}
\mathbf{X}_{2\times 2} &= \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}, \\
W_{\ell_1} &= \begin{pmatrix} W_{\ell_1,1} & W_{\ell_1,2} \\ W_{\ell_1,3} & W_{\ell_1,4} \end{pmatrix}, \\
W_{\ell_1,\ell_2} &= \begin{pmatrix} W_{\ell_1,\ell_2,1} & W_{\ell_1,\ell_2,2} \\ W_{\ell_1,\ell_2,3} & W_{\ell_1,\ell_2,4} \end{pmatrix},
\end{aligned}$$

and

$$W_{\ell_1,\ell_2,\ell_3} = \begin{pmatrix} w_{\ell_1,\ell_2,\ell_3,1} & w_{\ell_1,\ell_2,\ell_3,2} \\ w_{\ell_1,\ell_2,\ell_3,3} & w_{\ell_1,\ell_2,\ell_3,4} \end{pmatrix}.$$

For instance, $w_{1,2,3,4}$ and $w_{4,3,2,1}$ can be identified in the following diagram.





The rest of this subsection is devoted to constructing $\mathbf{X}_{m\times n}$ from $\mathbf{X}_{2\times 2}$ by the following three steps :

Step I : Define χ-ordering on $\mathbf{Z}_{m\times 2}$ by

(4.16)

2	4	•••	2k	•••	2m-2	2m
1	3	•••	2k-1	•••	2m-3	2m-1

and introduce ordering matrix $\mathbf{X}_{m\times 2}$ for $\Sigma_{2\times m\times 2}$.

Step II : Convert χ-ordering into $\hat{\chi}$-ordering on $\mathbf{Z}_{m\times 2}$ by

(4.17)

m+1	m+2	•••	m+k	•••	2m
1	2	•••	k	•••	m

and introduce ordering matrix $\hat{\mathbf{X}}_{m\times 2}$ for $\Sigma_{2\times m\times 2}$.

Step III : Define $\hat{\chi}$-ordering on $\mathbf{Z}_{m\times n}$ by

(4.18)

(n-1)m+1	(n-1)m+2	•••	nm-1	nm
• • •	• • •	•••	• • •	• • •
m+1	m+2	•••	2m-1	2m
1	2	•••	m-1	m

and introduce ordering matrix $\hat{\mathbf{X}}_{m\times n}$ for $\Sigma_{2\times m\times n}$.

To introduce $\mathbf{X}_{m\times 2}$, define

$$
\begin{aligned}
w_{\ell_1,\ell_2,\cdots,\ell_{2m}} = w_{\ell_1,\ell_2,\ell_3,\ell_4} \quad & \hat{\oplus}\ w_{\ell_3,\ell_4,\ell_5,\ell_6}\ \hat{\oplus}\ \cdots \\
& \hat{\oplus}\ w_{\ell_{2k+1},\ell_{2k+2},\ell_{2k+3},\ell_{2k+4}}\ \hat{\oplus}\ \cdots \\
& \hat{\oplus}\ w_{\ell_{2m-3},\ell_{2m-2},\ell_{2m-1},\ell_{2m}},
\end{aligned}
\tag{4.19}
$$

$0 \le k \le m-2$.

Then

(4.20)

$$\chi(w_{\ell_1,\ell_2,\cdots,\ell_{2m}}) = 1 + \sum_{\beta=1}^{2m}\{2^{2m}[\frac{\ell_\beta - 1}{2}] + \{\ell_\beta - 1 - 2[\frac{\ell_\beta - 1}{2}]\}\}2^{2m-\beta}.$$

From (4.16), (4.19) and (4.20), χ-ordering on $\mathbf{Z}_{m\times 2}$ is obviously one dimensional. It grows in m (y-direction). Therefore, by a similar argument as in proving Theorem 2.1, we have the following result for $\mathbf{X}_{m\times 2}$.

Theorem 4.1. *For any $m \geq 2$, $\Sigma_{2\times m\times 2} = \{w_{\ell_1,\ell_2,\cdots,\ell_{2m}}\}$, where $w_{\ell_1,\ell_2,\cdots,\ell_{2m}}$ is given in (4.17). Furthermore, the ordering matrix $\mathbf{X}_{m\times 2} = [w_{\ell_1,\ell_2,\cdots,\ell_{2m}}]$ which is a $2^{2m} \times 2^{2m}$ matrix can be decomposed into 2m Z-maps successively as*

$$\mathbf{X}_{m\times 2} = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix},$$

$$W_{\ell_1,\cdots,\ell_k} = \begin{pmatrix} W_{\ell_1,\cdots,\ell_k,1} & W_{\ell_1,\cdots,\ell_k,2} \\ W_{\ell_1,\cdots,\ell_k,3} & W_{\ell_1,\cdots,\ell_k,4} \end{pmatrix},$$

for $1 \leq k \leq 2m-2$,

$$W_{\ell_1,\cdots,\ell_{2m-1}} = \begin{pmatrix} w_{\ell_1,\cdots,\ell_{2m-1},1} & w_{\ell_1,\cdots,\ell_{2m-1},2} \\ w_{\ell_1,\cdots,\ell_{2m-1},3} & w_{\ell_1,\cdots,\ell_{2m-1},4} \end{pmatrix}.$$

Proof. From (4.7) and (4.8), we have following table.

l_β	1	2	3	4
u_{1,α_2,α_3}	0	0	1	1
u_{2,α_2,α_3}	0	1	0	1

Table 4.1

For any $m \geq 2$, by (4.6),(4.7) and (4.8), it is easy to generalize (4.9) and (4.10) to

$$i_{m;1} = 1 + \sum_{\beta=1}^{2m}[\frac{l_\beta - 1}{2}]2^{2m-\beta},$$

and

$$i_{m;2} = 1 + \sum_{\beta=1}^{2m}\{l_\beta - 1 - 2[\frac{l_\beta - 1}{2}]\}2^{2m-\beta}.$$

From above formulae, we have

$$i_{m+1;1} = 2^2 i_{m;1} + 2u_{1,m+1,1} + u_{1,m+1,2} + (1 - 2^2),$$

and

$$i_{m+1;2} = 2^2 i_{m;2} + 2u_{2,m+1,1} + u_{2,m+1,2} + (1 - 2^2).$$

Now, by induction on m, the theorem follows from last two formulae and Table 4.1. The proof is complete. ■

Next, χ-ordering is converted into $\hat{\chi}$-ordering for $\mathbf{Z}_{m\times 2}$.

Since $\mathbf{Z}_{m\times 2} = \{(\alpha_2, \alpha_3) : 1 \le \alpha_2 \le m, 1 \le \alpha_3 \le 2\}$, the position (α_2, α_3) is the β-th in (4.16), where $\beta = 2(\alpha_2 - 1) + \alpha_3$ given in (4.4). In (4.17), the position of (α_2, α_3) is the $\hat{\beta}$-th, where

$$\hat{\beta} = m(\alpha_3 - 1) + \alpha_2. \tag{4.21}$$

It is easy to verify

$$\hat{\beta} = m\beta + (1 - 2m)[\frac{\beta - 1}{2}] + (1 - m), \tag{4.22}$$

or

$$\hat{\beta} = k \qquad if \quad \beta = 2k - 1,$$

and

$$\hat{\beta} = m + k \qquad if \quad \beta = 2k,$$

$1 \le k \le m$.

Now, the ordering $\hat{\chi}$ in (4.17) on $\mathbf{Z}_{m\times 2}$ can be extended to $\mathbf{Z}_{m\times n}$ by (4.18).

For a fixed m, $\hat{\chi}$-ordering on $\mathbf{Z}_{m\times n}$ is obviously one dimensional; it grows in n (z-direction). With ordering (4.18) on $\mathbf{Z}_{m\times n}$, for $U = (u_{\alpha_1,\alpha_2,\alpha_3}) \in \Sigma_{2\times m\times n}$, denoted by

$$j_k = 1 + \sum_{\alpha_2=1}^{m} \sum_{\alpha_3=1}^{n} u_{k,\alpha_2,\alpha_3} 2^{m(n-\alpha_3)+(m-\alpha_2)}, \tag{4.23}$$

k=1,2. Then, we obtain

$$\hat{\chi}(U) = 2^{mn}(j_1 - 1) + j_2. \tag{4.24}$$

Now, let $\hat{x}_{j_1,j_2} = U = (u_{\alpha_1,\alpha_2,\alpha_3})$, then we have new ordering matrix $\hat{\mathbf{X}}_{m\times 2} = [\hat{x}_{j_1,j_2}]$ for $\Sigma_{2\times m\times 2}$. The relationship between $\mathbf{X}_{m\times 2}$ and $\hat{\mathbf{X}}_{m\times 2}$ is established before constructing $\hat{\mathbf{X}}_{m\times n}$ from $\hat{\mathbf{X}}_{m\times 2}$ for $n \ge 3$.

Here, a conversion sequence of orderings is first established from (4.16) to (4.17). Where P_k denotes the permutation of $\mathbf{N}_{2m} = \{1, 2, \cdots, 2m\}$ such that $P_k(k+1) = k$,$P_k(k) = k+1$ and the other numbers are fixed. Where P_k is denoted here as the permutation on $\mathbf{Z}_{m\times 2}$ such that it exchanges k and k+1 and maintains the other positions fixed, i.e,

(4.25)
$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & & k+1 & & \cdot & & \cdot \\ \hline \cdot & & \cdot & & k & & \cdot \\ \hline \end{array} \xrightarrow{P_k} \begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & & k & & \cdot & & \cdot \\ \hline \cdot & & \cdot & & k+1 & & \cdot \\ \hline \end{array}$$

Obviously,(4.16) can be converted into (4.17) in many ways by using a sequence of P_k. A systematic approach is presented as follows.

Lemma 4.2. *For $m \ge 2$, (4.16) can be converted into (4.17) by the following sequences of $\frac{m(m-1)}{2}$ permutations succesively*

$$\begin{array}{c}(P_2P_4\cdots P_{2m-2})(P_3P_5\cdots P_{2m-3})\cdots \\ (P_kP_{k+2}\cdots P_{2m-k})\cdots(P_{m-1}P_{m+1})P_m,\end{array} \tag{4.26}$$

$2 \le k \le m$.

Proof. When $m = 2$ and 3, verifying that (4.26) can convert (4.16) into (4.17) is relatively easy.

When $m \geq 4$, and for any $2 \leq k \leq m$, applying

$$(P_2P_4 \cdots P_{2m-2})(P_3P_5 \cdots P_{2m-3}) \cdots (P_kP_{k+2} \cdots P_{2m-k}) \tag{4.27}$$

to (4.16), then there are two intermediate cases:
(i) when $2 \leq k \leq [\frac{m}{2}]$, then we have

(4.28)

k+1	k+3	•••	3k-1	•••	•••	•••	3k-1+2l	•••	2m-k-1	2m-k+1	•••	2m-1	2m
1	2	•••	k	k+2	k+4	•••	k+2l	•••	•••	2m-3k+1	•••	2m-k-2	2m-k

where $0 \leq \ell \leq m - 2k$.
(ii) when $[\frac{m}{2}] + 1 \leq k \leq m - 1$, then we have

(4.29)

k+1	•••	2m-k-1	2m-k+1	2m-k+2	•••	•••	•••	2m-1	2m
1	2	•••	•••	•••	k-1	k	k+2	•••	2m-k

When k=m in (4.29), we have (4.17). We prove (4.28) and (4.29) by mathematical induction on k. When k=2, it is relatively easy to verify that (4.16) is converted into

3	5		•••	•••	2m-3	2m-1	2m
1	2	4	•••	•••		k	k+2

by $P_2P_4 \cdots P_{2m-2}$, i.e., (4.28) holds for k=2. Next, assume that (4.28) holds for $k \leq [\frac{m}{2}]$. Then, by applying $P_{k+1}P_{k+2} \cdots P_{2m-k-1}$ to (4.28), it can be verified that (4.28) holds for $k+1$ when $k+1 \leq [\frac{m}{2}]$ or becomes (4.29) when $k+1 \geq [\frac{m}{2}]$. When $k \geq [\frac{m}{2}] + 1$, we apply $P_{k+1}P_{k+2} \cdots P_{2m-k-1}$ to (4.29). It can also be verified that (4.29) holds for k+1. Finally, we conclude that (4.27) holds for $k = m$. The proof is thus complete. ■

By using Lemma 4.2, $\mathbf{X}_{m\times 2}$ can be converted into $\hat{\mathbf{X}}_{m\times 2}$ by the following construction. Let

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.30}$$

and for $2 \leq j \leq 2m - 2$, as denoted by

$$P_{2m;j} = I_{2^{j-1}} \otimes P \otimes I_{2^{2m-j-1}}, \tag{4.31}$$

where I_k is the $k \times k$ identity matrix. Furthermore, let

$$\mathbf{P}_{m\times 2} = (P_{2m;2}P_{2m;4}\cdots P_{2m;2m-2})\cdots(P_{2m;k}\cdots P_{2m;2m-k}) \cdots P_{2m;m}, \tag{4.32}$$

$2 \le k \le m$. Then, we have the following theorem.

Theorem 4.3. *For any $m \ge 2$,*

$$\hat{\mathbf{X}}_{m\times 2} = \mathbf{P}^t_{m\times 2}\mathbf{X}_{m\times 2}\mathbf{P}_{m\times 2}. \tag{4.33}$$

Proof. From Theorem 4.1, we have $\mathbf{X}_{m\times 2} = [w_{\ell_1,\ell_2,\cdots,\ell_{2m}}]_{2^{2m}\times 2^{2m}}$. It is easy to verify that for any $1 \le k \le 2m-1$,

$$P^t_{2m;k}[w_{\ell_1,\ell_2,\cdots,\ell_k,\ell_{k+1},\cdots,\ell_{2m}}]P_{2m;k} = [w_{\ell_1,\ell_2,\cdots,\ell_{k+1},\ell_k,\cdots,\ell_{2m}}] \tag{4.34}$$

,i.e., $P_{2m;k}$ exchanges ℓ_k and ℓ_{k+1} in $\mathbf{X}_{m\times 2}$. Therefore, from (4.32), (4.34) and Lemma 4.2, (4.33) follows. ■

Now, in Theorem 4.3, as denoted by

$$\hat{\mathbf{X}}_{m\times 2} = [\hat{w}_{r_1,r_2,\cdots,r_{2m}}], \tag{4.35}$$

$1 \le r_j \le 4$, by (4.16) and (4.19), we have

$$\hat{w}_{r_1,r_2,\cdots,r_{2m}} = w_{\ell_1,\ell_{m+1},\ell_2,\ell_{m+2},\cdots,\ell_k,\ell_{m+k},\cdots,\ell_m,\ell_{2m}}. \tag{4.36}$$

The $\hat{\chi}$-expression $\hat{\mathbf{X}}_{m\times 2} = [\hat{w}_{r_1,r_2,\cdots,r_{2m}}]$ for $\Sigma_{2\times m\times 2}$ enables us to construct $\hat{X}_{m\times n}$ for $\Sigma_{2\times m\times n}$. Indeed, from (4.22), for fixed $m \ge 2$ and $n \ge 2$, let

$$\begin{aligned}\hat{w}_{r_1,r_2,\cdots,r_{mn}} = \; & \hat{w}_{r_1,\cdots,r_m,r_{m+1},\cdots,r_{2m}} \hat{\oplus} \hat{w}_{r_{m+1},\cdots,r_{2m},r_{2m+1},\cdots,r_{3m}} \hat{\oplus} \cdots \\ & \hat{\oplus}\hat{w}_{r_{km+1},\cdots,r_{(k+1)m},r_{(k+1)m+1},\cdots,r_{(k+2)m}} \hat{\oplus} \cdots \\ & \hat{\oplus}\hat{w}_{r_{(n-2)m+1},\cdots,r_{(n-1)m},r_{(n-1)m+1},\cdots,r_{nm}},\end{aligned} \tag{4.37}$$

$0 \le k \le n-2$. Therefore, by a similar argument as in proving Theorem 2.1 and Theorem 4.1, we have the following theorem for $\hat{\mathbf{X}}_{m\times n}$. The detailed proof is omitted here for brevity.

Theorem 4.4. *By fixing $m \ge 2$ and for any $n \ge 2$, the ordering matrix $\hat{\mathbf{X}}_{m\times n}$ with respect to $\hat{\chi}$ ordering can be expressed as*

$$\hat{\mathbf{X}}_{m\times n} = [\hat{w}_{r_1,r_2,\cdots,r_{mn}}], \tag{4.38}$$

where $\hat{w}_{r_1,r_2,\cdots,r_{mn}}$ is given by (4.36) and (4.37).

4.2. **Transition matrices.** With the ordering matrices $\hat{\mathbf{X}}_{m\times n}$ for $\Sigma_{2\times m\times n}$ having been defined, higher order transition matrices $\hat{\mathbf{T}}_{m\times n}$ can now be derived from $\mathbf{T}_{2\times 2}$. As in the two dimensional case, assume that we have basic set $\mathcal{B} \subset \Sigma_{2\times 2\times 2}$. Define the transition matrix $\mathbf{T}_{2\times 2} = \mathbf{T}_{2\times 2}(\mathcal{B})$ by

$$\mathbf{T}_{2\times 2} = [t_{\ell_1,\ell_2,\ell_3,\ell_4}]_{2^4\times 2^4}, \tag{4.39}$$

where

$$\begin{aligned} t_{\ell_1,\ell_2,\ell_3,\ell_4} &= 1 \qquad if \;\; w_{\ell_1,\ell_2,\ell_3,\ell_4} \in \mathcal{B}, \\ &= 0 \qquad if \;\; w_{\ell_1,\ell_2,\ell_3,\ell_4} \in \Sigma_{2\times 2\times 2} - \mathcal{B}. \end{aligned} \tag{4.40}$$

Then, the transition matrix $\mathbf{T}_{m\times 2}$ for $\mathcal{B}$ defined on $\mathbf{Z}_{2\times m\times 2}$ is a $2^{2m} \times 2^{2m}$ matrix with entries $t_{\ell_1,\ell_2,\cdots,\ell_{2m}}$, where

$$t_{\ell_1,\ell_2,\cdots,\ell_{2m}} = \prod_{k=0}^{m-2} t_{\ell_{2k+1},\ell_{2k+2},\ell_{2k+3},\ell_{2k+4}}. \tag{4.41}$$

Based on Theorem 4.1, we can obtain results for $\mathbf{T}_{m\times 2}$ as $\mathbf{T}_n$ in Theorem 3.1 in the two dimensional case. Indeed, we have

Theorem 4.5. *Let $\mathbf{T}_{2\times 2}$ be a transition matrix given by (4.39) and (4.40). Then, for higher order transition matrices $\mathbf{T}_{m\times 2}$, $m \geq 3$, we have the following three equivalent expressions:*
(I) $\mathbf{T}_{m\times 2}$ can be decomposed into 2m successive 2×2 matrices

$$\begin{aligned} \mathbf{T}_{m\times 2} &= \begin{bmatrix} T_{2m;1} & T_{2m;2} \\ T_{2m;3} & T_{2m;4} \end{bmatrix}, \\ T_{2m;j_1,\cdots,j_k} &= \begin{bmatrix} T_{2m;j_1,\cdots,j_k,1} & T_{2m;j_1,\cdots,j_k,2} \\ T_{2m;j_1,\cdots,j_k,3} & T_{2m;j_1,\cdots,j_k,4} \end{bmatrix}, \end{aligned}$$

$1 \leq k \leq 2m-2$, *and*

$$T_{2m;j_1,\cdots,j_{2m-1}} = \begin{bmatrix} t_{j_1,\cdots,j_{2m-1},1} & t_{j_1,\cdots,j_{2m-1},2} \\ t_{j_1,\cdots,j_{2m-1},3} & t_{j_1,\cdots,j_{2m-1},4} \end{bmatrix}.$$

(II) Starting from

$$\mathbf{T}_{2\times 2} = [T_{\ell_1,\ell_2}]_{4\times 4}$$

and

$$T_{\ell_1,\ell_2} = [t_{\ell_1,\ell_2,\ell_3,\ell_4}]_{4\times 4},$$

for $m \geq 3$,$\mathbf{T}_{m\times 2}$ can be obtained from $\mathbf{T}_{(m-1)\times 2}$ by replacing T_{ℓ_1,ℓ_2} with

$$(T_{\ell_1,\ell_2})_{4\times 4} \odot (\mathbf{T}_{2\times 2})_{4\times 4}. \tag{4.42}$$

(III) For $m \geq 3$,

$$\mathbf{T}_{m\times 2} = (\mathbf{T}_{(m-1)\times 2})_{4^{m-1}\times 4^{m-1}} \odot (E_{4^{m-2}} \otimes \mathbf{T}_{2\times 2}). \tag{4.43}$$

Now, with respect to ordering matrix $\hat{\mathbf{X}}_{m\times 2}$, $\hat{\mathbf{T}}_{m\times 2}$ and $\mathbf{T}_{m\times 2}$ are similar. Additionally, by using Theorem 4.3, we have

Theorem 4.6.

$$\hat{\mathbf{T}}_{m\times 2} = \mathbf{P}^t_{m\times 2}\mathbf{T}_{m\times 2}\mathbf{P}_{m\times 2}. \tag{4.44}$$

By applying Theorem 4.4, transition matrix $\hat{\mathbf{T}}_{m\times n}$ can be obtained from $\hat{\mathbf{T}}_{m\times 2}$ as was done in Theorem 3.1 and Theorem 4.5. Indeed, we have

Theorem 4.7. *Let $\hat{\mathbf{T}}_{2\times 2} = [\hat{t}_{r_1,r_2,r_3,r_4}]_{2^4\times 2^4}$*
(I) $\hat{\mathbf{T}}_{m\times n}$ can be decomposed into mn Z-maps with successive 2×2 matrices:

$$\begin{aligned} \hat{\mathbf{T}}_{m\times n} &= \begin{bmatrix} \tilde{T}_{mn;1} & \tilde{T}_{mn;2} \\ \tilde{T}_{mn;3} & \tilde{T}_{mn;4} \end{bmatrix}_{2\times 2} \\ \tilde{T}_{mn;r_1,r_2,\cdots,r_k} &= \begin{bmatrix} \tilde{T}_{mn;r_1,r_2,\cdots,r_k,1} & \tilde{T}_{mn;r_1,r_2,\cdots,r_k,2} \\ \tilde{T}_{mn;r_1,r_2,\cdots,r_k,3} & \tilde{T}_{mn;r_1,r_2,\cdots,r_k,4} \end{bmatrix}_{2\times 2} \end{aligned}$$

$1 \leq k \leq mn-2$

$$\tilde{\mathbf{T}}_{mn;r_1,r_2,\cdots,r_{mn-1}} = \begin{bmatrix} \hat{t}_{mn;r_1,r_2,\cdots,r_{mn-1},1} & \hat{t}_{mn;r_1,r_2,\cdots,r_{mn-1},2} \\ \hat{t}_{mn;r_1,r_2,\cdots,r_{mn-1},3} & \hat{t}_{mn;r_1,r_2,\cdots,r_{mn-1},4} \end{bmatrix}_{2\times 2}.$$

(II) Let $\hat{\mathbf{T}}_{m\times 2} = [\hat{T}_{2m;r_1,r_2,r_3,\cdots,r_m}]_{2^m\times 2^m}$ *where* $\hat{T}_{2m;r_1,r_2,\cdots,r_m} = [\hat{t}_{r_1,r_2,\cdots,r_m,r_{m+1},\cdots,r_{2m}}]_{2^m\times 2^m}$. *Then, for any* $n \geq 3$, $\hat{\mathbf{T}}_{m\times n}$ *can be obtained from* $\hat{\mathbf{T}}_{m\times(n-1)}$ *by replacing* $\hat{T}_{2m;r_1,r_2,r_3,\cdots,r_m}$ *with*

$$(\hat{T}_{2m;r_1,r_2,r_3,\cdots,r_m})_{2^m\times 2^m} \odot (\hat{\mathbf{T}}_{m\times 2})_{2^m\times 2^m}. \tag{4.45}$$

(III) Furthermore, for $n \geq 3$ *we have*

$$\begin{aligned} &\hat{\mathbf{T}}_{m\times n} = \\ &(\hat{\mathbf{T}}_{m\times(n-1)})_{2^{m(n-1)}\times 2^{m(n-1)}} \odot (E_{2^{m(n-2)}} \otimes \hat{\mathbf{T}}_{m\times 2})_{2^{m(n-1)}\times 2^{m(n-1)}}. \end{aligned} \tag{4.46}$$

Details of the proof are omitted here for brevity.

Finally, the spatial entropy $h(\mathcal{B})$ can be computed through the maximum eigenvalue $\lambda_{m,n}$ of $\hat{\mathbf{T}}_{m\times n}$. Indeed, we have

Theorem 4.8. *Let* $\lambda_{m,n}$ *be the maximum eigenvalue of* $\hat{\mathbf{T}}_{m\times n}$, *then*

$$h(\mathcal{B}) = \lim_{m,n\to\infty} \frac{\log \lambda_{m,n}}{mn}. \tag{4.47}$$

The proof closely resembles that when proving Theorem 3.3. Details are omitted here for brevity.

4.3. **Computation of $\lambda_{m,n}$ and entropies.** From the last two subsections, we obtain a systematic means of writing down $\hat{\mathbf{T}}_{m\times n}$ from $\mathbf{T}_{2\times 2}$. As in a two dimensional case, recursion formulae for $\lambda_{m,n}$ can be obtained when $\mathbf{T}_{2\times 2}$ has a special structure. To demonstrate the methods developed in the last subsection, we provide an illustrative example in which $\hat{\mathbf{T}}_{m\times n}$ and $\lambda_{m,n}$ can be derived explicitly. More complete results will appear later.

Denoted by

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } E = E_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{4.48}$$

and let

$$\begin{aligned} \mathbf{T}_{2\times 2} &= \overset{2}{\otimes}(G\otimes E), \\ &= (G\otimes E)\otimes(G\otimes E). \end{aligned} \tag{4.49}$$

Proposition 4.9. *Let* $\mathbf{T}_{2\times 2}$ *be in (4.48) and (4.49). Then,*

$$(i) \quad \mathbf{T}_{m\times 2} = \overset{m}{\otimes}(G\otimes E), \tag{4.50}$$

$$(ii) \quad \hat{\mathbf{T}}_{m\times 2} = (\overset{m}{\otimes} G)\otimes(\overset{m}{\otimes} E), \tag{4.51}$$

$$(iii) \quad \hat{\mathbf{T}}_{m\times n} = (\overset{m(n-1)}{\otimes} G)\otimes(\overset{m}{\otimes} E). \tag{4.52}$$

Furthermore, for the maximum eigenvalue $\lambda_{m,n}$ of $\hat{\mathbf{T}}_{m\times n}$, we have the following recursion formulae:

$$\lambda_{m+1,n} = 2g^{n-1}\lambda_{m,n} \tag{4.53}$$

and

$$\lambda_{m,n+1} = g^m \lambda_{m,n} \tag{4.54}$$

for $m, n \geq 2$ with

$$\lambda_{2,2} = (2g)^2. \tag{4.55}$$

The topological entropy is

$$h(\mathbf{T}_{2\times 2}) = g, \tag{4.56}$$

where $g = \frac{1+\sqrt{5}}{2}$.

Proof. The proof is only described briefly, and the details are omitted for brevity.
(i) can be proved by Theorem 4.5 and induction on m. Indeed, by (4.43), we have

$$\begin{aligned}
\mathbf{T}_{3\times 2} &= (\mathbf{T}_{2\times 2})_{4\times 4} \odot (E_4 \otimes \mathbf{T}_{2\times 2})_{4\times 4} \\
&= (G \otimes E \otimes G \otimes E)_{4\times 4} \odot (E \otimes E \otimes (G \otimes E \otimes G \otimes E))_{4\times 4} \\
&= (G \odot E) \otimes (E \odot E) \otimes (G \odot G) \otimes (E_{2\times 2} \odot (E \otimes G \otimes E))_{2\times 2} \\
&= \overset{3}{\otimes} (G \otimes E).
\end{aligned}$$

Assume that $\mathbf{T}_{(m-1)\times 2} = \overset{m-1}{\otimes} (G \otimes E)$. Then by (4.43) again, we have

$$\begin{aligned}
\mathbf{T}_{m\times 2} &= (\mathbf{T}_{(m-1)\times 2}) \odot ((\overset{2(m-2)}{\otimes} E) \otimes \mathbf{T}_{2\times 2})) \\
&= (\overset{m-1}{\otimes} (G \otimes E))_{4^{m-2}\times 4^{m-2}} \odot ((\overset{m-2}{\otimes} E) \otimes (\overset{2}{\otimes} (G \otimes E)))_{4^{m-2}\times 4^{m-2}} \\
&= (\overset{m-2}{\otimes} (G \otimes E) \otimes (G \otimes E))_{4^{m-2}\times 4^{m-2}} \odot (\overset{m-2}{\otimes} (E \otimes E) \otimes (G \otimes E) \otimes (G \otimes E))_{4^{m-2}\times 4^{m-2}} \\
&= \overset{m-2}{\otimes} [(G \odot E) \otimes (E \odot E)] \otimes (G \odot G) \otimes (E \odot (E \otimes G \otimes E)) \\
&= \overset{m-2}{\otimes} (G \otimes E) \otimes (G \otimes E) \otimes (G \otimes E) \\
&= \overset{m}{\otimes} (G \otimes E).
\end{aligned}$$

(ii) The following property for matrices is needed and the detailed proof omitted: For any two 2×2 matrices A and B, we have

$$P(A \otimes B)P = B \otimes A, \tag{4.57}$$

where P is given in (4.30). We also prove in (4.51) by induction on m. When m=2, by Theorem 4.6,

$$\begin{aligned}
\hat{\mathbf{T}}_{2\times 2} &= \mathbf{P}^t_{2\times 2} \mathbf{T}_{2\times 2} \mathbf{P}_{2\times 2} \\
&= (P_{4;2})^t \mathbf{T}_{2\times 2} P_{4;2} \\
&= (I_2 \otimes P \otimes I_2)((G \otimes E) \otimes (G \otimes E))(I_2 \otimes P \otimes I_2) \\
&= G \otimes (P(E \otimes G)P) \otimes E \\
&= G \otimes G \otimes E \otimes E
\end{aligned}$$

by (4.57).

Now, assume that (4.51) holds for m-1, i.e.

$$\hat{\mathbf{T}}_{(m-1)\times 2} = (\overset{m-1}{\otimes} G) \otimes (\overset{m-1}{\otimes} E).$$

Then

$$\begin{aligned}
\hat{\mathbf{T}}_{m\times 2} &= \mathbf{P}^t_{m\times 2}\mathbf{T}_{m\times 2}\mathbf{P}_{m\times 2}\\
&= [(P_{2m;2}P_{2m;4}\cdots P_{2m;2m-2})(P_{2m;3}P_{2m;5}\cdots P_{2m;2m-3})\cdots(P_{2m;m})]^t\\
&\quad \mathbf{T}_{m\times 2}[(P_{2m;2}P_{2m;4}\cdots P_{2m;2m-2})(P_{2m;3}P_{2m;5}\cdots P_{2m;2m-3})\cdots(P_{2m;m})]\\
&= (P_{2m;m})\cdots(P_{2m;3}P_{2m;5}\cdots P_{2m;2m-3})\\
&\quad [(P_{2m;2}P_{2m;4}\cdots P_{2m;2m-2})(\overset{m}{\otimes}(G\otimes E))(P_{2m;2}P_{2m;4}\cdots P_{2m;2m-2})]\\
&\quad (P_{2m;3}P_{2m;5}\cdots P_{2m;2m-3})\cdots(P_{2m;m})\\
&= (P_{2m;m})\cdots(P_{2m;3}P_{2m;5}\cdots P_{2m;2m-3})\\
&\quad [G\otimes(\overset{m-1}{\otimes}(G\otimes E))\otimes E]\\
&\quad (P_{2m;3}P_{2m;5}\cdots P_{2m;2m-3})\cdots(P_{2m;m})\\
&= G\otimes\{(P_{2(m-1);m-1})\cdots(P_{2(m-1);2}P_{2(m-1);4}\cdots P_{2(m-1);2(m-1)-2})\\
&\quad [\overset{m-1}{\otimes}(G\otimes E)](P_{2(m-1);2}P_{2(m-1);4}\cdots P_{2(m-1);2(m-1)-2})\cdots(P_{2m-1;m-1})\\
&\quad \}\otimes E\\
&= G\otimes(\mathbf{P}^t_{(m-1)\times 2}\mathbf{T}_{(m-1)\times 2}\mathbf{P}_{(m-1)\times 2})\otimes E\\
&= G\otimes\hat{\mathbf{T}}_{(m-1)\times 2}\otimes E\\
&= G\otimes((\overset{m-1}{\otimes}G)\otimes(\overset{m-1}{\otimes}E))\otimes E\\
&= (\overset{m}{\otimes}G)\otimes(\overset{m}{\otimes}E).
\end{aligned}$$

(iii) For a fixed m, we prove the results by induction on $n\geq 2$. Assume that (4.52) holds for $n-1$, i.e.,

$$\hat{\mathbf{T}}_{m\times(n-1)} = (\overset{m(n-2)}{\otimes}G)\otimes(\overset{m}{\otimes}E).$$

Then, by (4.46), we have

$$\begin{aligned}
\hat{\mathbf{T}}_{m\times n} &= \hat{\mathbf{T}}_{m\times(n-1)}\odot((\overset{m(n-2)}{\otimes}E)\otimes\hat{\mathbf{T}}_{m\times 2})\\
&= ((\overset{m(n-2)}{\otimes}G)\otimes(\overset{m}{\otimes}E))\odot((\overset{m(n-2)}{\otimes}E)\otimes(\overset{m}{\otimes}G)\otimes(\overset{m}{\otimes}E))\\
&= (\overset{m(n-2)}{\otimes}G)\otimes(\overset{m}{\otimes}G)\otimes(\overset{m}{\otimes}E)\\
&= (\overset{m(n-1)}{\otimes}G)\otimes(\overset{m}{\otimes}E).
\end{aligned}$$

As for maximum eigenvalue $\lambda_{m,n}$, verifying (4.55) is easy. To show (4.53) for fixed n, by using (4.52), we have

$$\begin{aligned}
\hat{\mathbf{T}}_{(m+1)\times n} &= (\overset{(m+1)(n-1)}{\otimes}G)\otimes(\overset{m+1}{\otimes}E)\\
&= (\overset{n-1}{\otimes}G)\otimes(\overset{m(n-1)}{\otimes}G)\otimes(\overset{m}{\otimes}E)\otimes E\\
&= (\overset{n-1}{\otimes}G)\otimes\hat{\mathbf{T}}_{m\times n}\otimes E,
\end{aligned}$$

which implies

$$\lambda_{m+1,n} = 2g^{n-1}\lambda_{m,n},$$

see [8].

Similarly, for a fixed m, to prove (4.54), by using (4.52) again, we have

$$\begin{aligned}
\hat{\mathbf{T}}_{m\times(n+1)} &= (\overset{mn}{\otimes}G)\otimes(\overset{m}{\otimes}E)\\
&= (\overset{m}{\otimes}G)\otimes(\overset{m(n-1)}{\otimes}G)\otimes(\overset{m}{\otimes}E)\\
&= (\overset{m}{\otimes}G)\otimes\hat{\mathbf{T}}_{m\times n},
\end{aligned}$$

which implies

$$\lambda_{m,n+1} = g^m \lambda_{m,n}.$$

Finally, (4.56) follows from (4.53), (4.54) and Theorem 4.8. The proof is thus complete. ■

4.4. **Higher Dimensional Cases.** For completeness, this subsection, discusses how to generalize the theory developed previously to higher dimensions, i.e., dimension ≥ 4. We summarize the methods, outline the procedures and highlight key points, as well as omit the details which will appear in [4]. We begin by introducing orderings on sublattices $\mathbf{Z}_{m_1\times\cdots\times m_d} \subset \mathbf{Z}^d$. The associated ordering matrices $\mathbf{X}^{[d]}_{m_1\times\cdots\times m_d}$ on $\Sigma_{2\times m_1\times\cdots\times m_d}$ are then defined. Finally, the transition matrices for a given basic $\mathcal{B} \subset \Sigma_{2\times\cdots\times 2}$ are derived.

Given a finite lattice $\mathbf{Z}_{m_1\times\cdots\times m_d} \subset \mathbf{Z}^d$, denoted by

$$N_d = m_1 \times \cdots \times m_d. \tag{4.58}$$

We introduce d-many orderings on $\mathbf{Z}_{Nd}$. Indeed, define

$$M_0 = \prod_{i=1}^{d} m_i, \tag{4.59}$$

and for any j, $1 \leq j \leq d-1$, define

$$M_j \equiv M_j^{[1]} \equiv \prod_{i=j+1}^{d} m_i \quad \& \quad M_d^{[1]} = 1. \tag{4.60}$$

Next, for any $2 \leq k \leq d-1$ and $1 \leq j \leq d-1$, define

$$M_j^{[k]} = \prod_{i\neq k, i=j+1}^{d} m_i \quad \& \quad M_d^{[k]} = 1. \tag{4.61}$$

when $j \neq k$, and

$$M_k^{[k]} = \prod_{i=1, i\neq k}^{d} m_i. \tag{4.62}$$

Finally, for $1 \leq j \leq d-2$, define

$$M_j^{[d]} = \prod_{i=j+1}^{d-1} m_i \quad \& \quad M_{d-1}^{[d]} = 1. \tag{4.63}$$

and

$$M_d^{[d]} = \prod_{i=1}^{d-1} m_i. \tag{4.64}$$

The underlying notion behind these $M_j^{[k]}$ is that d-many orderings $[k]$, $1 \leq k \leq d$, are introduced to the set positive integers $\mathbf{N}_d = \{1, 2, \cdots, d\}$ by

$$
\begin{aligned}
&[1]: \qquad 1 \succ 2 \succ \cdots \succ d,\\
&\vdots\\
&[k]: \qquad k \succ 1 \succ 2 \succ \cdots \succ k-1 \succ k+1 \succ \cdots \succ d,\\
&\vdots\\
&[d]: \qquad d \succ 1 \succ 2 \succ \cdots \succ d-1.
\end{aligned}
$$

Therefore, for any $(\alpha_1, \cdots, \alpha_d) \in \mathbf{Z}_{N_d}$, we define

$$\chi^{[k]}((\alpha_1, \cdots, \alpha_d)) = 1 + \sum_{j=1}^{d} M_j^{[k]}(\alpha_j - 1), \tag{4.65}$$

i.e., we count the index $(\alpha_1, \cdots, \alpha_d)$ in $\mathbf{Z}_{N_d}$ as the $\chi^{[k]}((\alpha_1, \cdots, \alpha_d))$-th position in $\chi^{[k]}$ ordering. When d=2, $\chi^{[1]} = \chi$ and $\chi^{[2]} = \hat{\chi}$, see (4.16), (4.17) and (4.24).

As done previously, assume that $\mathcal{S} = \{0, 1\}$. Consider all patterns $U = (u_{\alpha_0, \alpha_1, \cdots, \alpha_d}) \in \Sigma_{2\times N_d}$ on $\mathbf{Z}_{2\times N_d} \subset \mathbf{Z}^{d+1}$. Then, we can define

$$
\begin{aligned}
&\chi^{[k]}((u_{\alpha_0, \alpha_1, \cdots, \alpha_d})) =\\
&1 + \sum_{\alpha_0=1}^{2} \sum_{\alpha_1=1}^{m_1} \cdots \sum_{\alpha_k=1}^{m_k} \cdots \sum_{\alpha_d=1}^{m_d} u_{\alpha_0, \alpha_1, \cdots, \alpha_d} \tilde{\chi}^{[k]}((\alpha_0, \alpha_1, \cdots, \alpha_d))
\end{aligned} \tag{4.66}
$$

where

$$
\begin{aligned}
&\log\{\tilde{\chi}^{[k]}(\alpha_0, \alpha_1, \cdots, \alpha_d)\} =\\
&\{M_0(2 - \alpha_0) + \sum_{j=1}^{d} M_j^{[k]}(m_j - \alpha_j)\} \log 2.
\end{aligned} \tag{4.67}
$$

Denoted by

$$N_0 = \underbrace{2 \times \cdots \times 2}_{d-times}, \tag{4.68}$$

and

$$N_k = m_1 \times \cdots \times m_k \times \underbrace{2 \times \cdots \times 2}_{d-k\ times}, \tag{4.69}$$

for $1 \le k \le d$, and d+1 numbers $n_0 = 2^d$, $n_k = m_1 \cdots m_k 2^{d-k}$, $1 \le k \le d$. Here ordering matrices $\mathbf{X}_{N_k}^{[k]}$ are introduced to $\Sigma_{2\times N_k}$. Indeed, in the first step, $\mathbf{X}_{N_0}^{[1]}$ is introduced to $\Sigma_{2\times N_0}$ with respect to $\chi^{[1]}$. Then, by proceeding as in Theorem 4.1, we obtain $\mathbf{X}_{N_1}^{[1]}$ on $\Sigma_{2\times N_1}$.

Next, a sequence of permutations can be obtained as in Lemma 4.2, which can convert $\chi^{[1]}$ ordering on $\mathbf{Z}_{N_1}$ into $\chi^{[2]}$ ordering on $\mathbf{Z}_{N_1}$. Therefore, after appropriately modifying Lemma 4.2 and Theorem 4.3, there is a $2^{n_1} \times 2^{n_1}$ matrix, $\mathbf{P}_{N1}^{[1]}$ such that

$$\mathbf{X}_{N_1}^{[2]} = (\mathbf{P}_{N_1}^{[1]})^t \mathbf{X}_{N_1}^{[1]} \mathbf{P}_{N_1}^{[1]} \tag{4.70}$$

as in Theorem 4.3 for d=2. Now, as in Theorem 4.4, we can construct $\mathbf{X}_{N_2}^{[2]}$ on $\Sigma_{2\times N_2}$ from $\mathbf{X}_{N_1}^{[2]}$. In this procedure, in k-step, we have $\mathbf{X}_{N_k}^{[k]}$ on $\Sigma_{2\times N_k}$. Next,

$\mathbf{X}_{N_k}^{[k]}$ is converted into $\mathbf{X}_{N_k}^{[k+1]}$ as in Theorem 4.3 and $\mathbf{X}_{N_k}^{[k+1]}$ is extended to $\mathbf{X}_{N_{k+1}}^{[k+1]}$ as in Theorem 4.4. Finally, we have $\mathbf{X}_{N_d}^{[d]}$ on the whole lattice $\mathbf{Z}_{2\times N_d}$. Notably, the extension of $\mathbf{X}_{N_k}^{[k+1]}$ to $\mathbf{X}_{N_{k+1}}^{[k+1]}$ is one dimensional, i.e., it grows in α_{k+1} direction only.

As for transition matrices, assume that we are given a basic set $\mathcal{B} \subset \Sigma_{2\times N_0}$, and $\mathbf{X}_{N_0}^{[1]}$ can be used to introduce the transition matrix $\mathbf{T}_{N_0}^{[1]}$, i.e.,

$$\mathbf{T}_{N_0}^{[1]} = [t_{\ell_1,\cdots,\ell_D}]_{2^{d+1}\times 2^{d+1}}$$

$D = 2^{d+1}$, $1 \le \ell_j \le 4$, where $t_{\ell_1,\cdots,\ell_D} = 1$ if and only if the associated pattern lies in $\mathcal{B}$. By proceeding as in Theorems 4.5 $\sim$ 4.7, we can obtain $\mathbf{T}_{N_d}^{[d]}$ on $\mathbf{Z}_{2\times Nd}$. After the maximum eigenvalue λ_{N_d} of $\mathbf{T}_{Nd}^{[d]}$ is computed, the entropy $h(\mathcal{B})$ can be obtained as

$$h(\mathcal{B}) = \lim_{Nd\to\infty} \frac{\log \lambda_{Nd}}{m_1 m_2 \cdots m_d}. \tag{4.71}$$

Remark 4.10. For $B \subset \Sigma_{2\ell\times\cdots\times 2\ell,p}$, the theories applied and detailed discussion will appear in [4].

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