The complexity of permutive cellular automata

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This paper studies one-dimensional permutive cellular automata in two aspects: Ergodic and topological behavior. Through investigating measure-theoretic entropy and topological pressure, we show taht Parry measure is the unique equilibrium measure whenever the potential function depends on one coordinate. In other words, permutive cellular automata exhibit no phase transition. Furthermore, the existence of snap-back repellers for a cellular automaton infers Li-Yorke chaos and bipermutive cellular automata guarantee the subsistence of snap-back repellers.

Key words: Cellular automata, permutive, equilibrium measure, phase transition, Parry measure, snap-back repeller, chaos

1 INTRODUCTION

Cellular automaton (CA), introduced by Ulam [18] and Neumann [19] as a model for self-production, is a particular class of dynamical systems which is defined by a local rule acting on a discrete space and is widely studied in a variety of contexts in physics, biology and computer science [3, 4, 6, 9, 8, 10, 15, 17, 21].

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The present paper devotes to study the complexity of permutive CA (defined later) in two aspects, say, the viewpoints of thermodynamics and topological behaviors.

Let (X, μ) be a measurable space and let $T : X \to X$ be a continuous transformation. Given a potential function $\phi : X \to \mathbb{R}$, the pressure function $P(T, \phi)$ of T with respect to ϕ indicates the energy of the system. Variational principle stands that $P(T, \phi) = \sup\{h(\mu) + \int \phi d\mu\}$, where $h(\mu)$ is the measure-theoretic entropy and the supremum is taken for ergodic measures. A measure that attains the supremum is called an equilibrium measure. The uniqueness of equilibrium measure asserts that there exists no phase transition in this system while the existence of two or more equilibrium measures implies phase transition may occur.

Investigating the formula of topological pressure and measure-theoretic entropy help for the determination of number of equilibrium measures. In other words, we want to answer whether permutive CA exhibit phase transition or not.

We demonstrate the formulae of measure-theoretic entropy (Theorem 3.1) and topological pressure (Theorem 3.4). The measure-theoretic entropy of permutive CA is also elucidated in [16]. This thesis gives it an alternative proof. Moreover, consider $\mu = (p_0, \ldots, p_{r-1})$ a Bernoulli measure and the potential function ϕ is given by $\phi(x) = \log p_{x_0}$ for $x \in \Omega$, where r is the number of alphabet S and $\Omega = S^{\mathbb{Z}}$ is the space of bi-infinite sequence. The permutivity of CA asserts that Parry measure is the unique equilibrium measure, thus there is no phase transition in such system (Corollary 3.6).

The second part studies the complexity of the topological behavior exhibited by permutive CA. Li and Yorke [12] discover that, if a first-order difference equation

$$x_{i+1} = f(x_i), \quad i \in \mathbb{Z}^+,$$

where $x_i \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous, admits a 3-cycle, then there exist many complex behavior: The lack of global stability and the existence of an uncountable set of orbits which do not approach any periodic path. Those systems assert such phenomena are called Li-Yorke chaos hereafter. Marotto demonstrates the existence of a snap-back repeller implements Li-Yorke chaos [13, 14]. García also shows the existence of snap-back repeller admits positive topological entropy, which is a sufficient condition for Li-Yorke chaos [7, 2].

This is a motivation that, in CA, does the existence of snap-back repeller also implements Li-Yorke chaos? The answer is affirmative (Theorem 4.1).

Notably, the dynamical system considered in [13] is differentiable. CA, however, is only a continuous system. Furthermore, each bipermutive CA exhibits a snap-back repeller (Proposition 4.3) and there is an example that permutive but not bipermutive CA is not Li-Yorke chaotic.

The rest of this elucidation is organized as follows. Section 2 gives some notations and definitions. Section 3 studies the ergodic properties of permutive CA while Section 4 investigates the existence of snap-back repeller implies Li-Yorke chaos and bipermutive CA is a collection which exhibits snap-back repeller.

2 NOTATION AND DEFINITION

Let $S = \{0, 1, 2, \dots, r-1\}$ be a finite alphabet and let $\Omega = S^{\mathbb{Z}}$ be the space of bi-infinite sequence $x = (x_n)_{-\infty}^{\infty}$. Hedlund examines CA in the viewpoint of symbolic dynamical systems [11]. He shows that $F : \Omega \to \Omega$ is a CA if and only if F can be represented as a sliding block code, i.e., there exists $k \in \mathbb{Z}^+$ and a block map $f : S^{2k+1} \to S$ such that $F(x)_i = f(x_{i-k}, \dots, x_{i+k})$ for $x \in \Omega$ and $i \in \mathbb{Z}$. Such f is called the local rule of F. The study of the local rule of a CA is essential for the understanding of this system.

A local rule $f : S^{2k+1} \to S$ is called leftmost (respectively rightmost) permutive if there exists an integer i, $-k \le i \le -1$ (respectively $1 \le i \le k$), such that

(i) f is a permutation at x_i whenever the other variables are fixed;

(ii) f does not depend on x_j for $j < i$ (respectively $j > i$).

f is called bipermutive provided f is both leftmost and rightmost permutive. The family of permutive cellular automata consists of the following three types of local rules.

1. f is leftmost permutive and does not depend on x_i for $i > 0$;
2. f is rightmost permutive and does not depend on x_i for $i < 0$;
3. f is bipermutive.

For reader's convenience, we recall definitions of measure-theoretic entropy, topological entropy, and topological pressures. Reader may refer to [20] for more details.

Let μ be an invariant probability measure on (Ω, F) and let α and β be two finite measurable partitions of Ω. Define $\alpha \bigvee \beta$ and $H_\mu(\alpha)$ by

$$\alpha \bigvee \beta = \{A \bigcap B : A \in \alpha, B \in \beta\}$$

and

$$H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A),$$

respectively. The measure-theoretic entropy of F is defined by

$$h_\mu(F) = \sup \left\{ \lim_{n \to \infty} \frac{1}{n} H_\mu(\bigvee_{m=0}^{n-1} F^{-m} \alpha) \right\}, \tag{1}$$

where the supremum is taken over all finite measurable partitions α.

Define $d : \Omega \times \Omega \to \mathbb{R}$ by

$$d(x, y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{r^{|i|}}, \quad x, y \in \Omega. \tag{2}$$

It is easy to verify that d is a metric and (Ω, d) is a compact metric space. Moreover, let ${}_a[s_a, \ldots, s_b]_b = \{x \in \Omega : x_a = s_a, \ldots, x_b = s_b\}$ be a cylinder in Ω, where $a \leq b, a, b \in \mathbb{Z}$. Then ${}_a[s_a, \ldots, s_b]_b$ is not only open but close in Ω.

Let $\mathcal{P}$ be an open cover of Ω, denote by

$$H(\mathcal{P}) = \inf\{\log \#\hat{\mathcal{P}}\},$$

where the infimum is taken over the set of finite subcovers $\hat{\mathcal{P}}$ of $\mathcal{P}$ and $\#A$ denotes the cardinality of A. The topological entropy of F is defined by

$$h_{top}(F) = \sup \left\{ \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{m=0}^{n-1} F^{-m} \mathcal{P}) \right\}, \tag{3}$$

where the supremum is taken over all open covers $\mathcal{P}$.

In addition, for α an open cover of Ω and $\phi \in C(\Omega, \mathbb{R})$ a continuous function from Ω to $\mathbb{R}$, denote by

$$p_n(F, \phi, \alpha) = \inf \left\{ \sum_{B \in \beta} \sup_{x \in B} e^{(S_n \phi)(x)} : \beta \text{ is a finite subcover of } \bigvee_{m=0}^{n-1} F^{-m} \alpha \right\},$$

where $n \in \mathbb{N}$ and $S_n \phi = \sum_{m=0}^{n-1} \phi \circ F^m$. Define

$$P(F, \phi) = \limsup_{\delta \to 0} \left\{ \lim_{n \to \infty} \frac{1}{n} \log p_n(F, \phi, \alpha) : diam(\alpha) \leq \delta \right\}. \tag{4}$$

The map $P(F, \cdot) : C(\Omega, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ is called the topological pressure of F. It comes immediately that $P(F, 0) = h_{top}(F)$.

Let X be a metric space and let $T : X \to X$ be continuous, a dynamical system associated with $x_n = T(x_{n-1})$ for $n \in \mathbb{Z}^+$ is said to be chaotic in the sense of Li-Yorke if and only if

1. there exists a positive integer N such that for each integer $p \geq N$, T has a point of period p;

2. there exists a scramble set $\mathcal{S}$, i.e., an uncountable set containing no periodic points such that

   (a) $T(\mathcal{S}) \subset \mathcal{S}$;

   (b) for every $x, y \in \mathcal{S}$ with $x \neq y$,

   $$\limsup_{m\to\infty} |T^m(x) - T^m(y)| > 0, \quad \liminf_{m\to\infty} |T^m(x) - T^m(y)| = 0;$$

   (c) for every $x \in \mathcal{S}$ and y a periodic point of T,

   $$\limsup_{m\to\infty} |T^m(x) - T^m(y)| > 0;$$

For F a cellular automaton, a point $z \in \Omega$ is called an expanding fixed point of F if

1. z is a fixed point of F;

2. there exists $\epsilon > 0$ such that for all $x \in B_\epsilon(z)$, $x \neq z$, $|F(x) - F(z)| > |x - z|$ and $F^{-m}(x) \to z$ as $m \to \infty$.

The radius ϵ such that each $x \neq z$ is expanding in $B_\epsilon(z)$ is called expanding radius.

Definition 2.1. *A point $z \in \Omega$ is called a snap-back repeller if*

1. *z is an expanding fixed point of F for some expanding radius ϵ;*

2. *there exists a point $x_0 \in B_\epsilon(z)$, $x_0 \neq z$, such that $F^M(x_0) = z$ for some positive integer M.*

3 EQUILIBRIUM MEASURES OF PERMUTIVE CELLULAR AUTOMATA

This section investigates equilibrium measures of permutive CA through studying the measure-theoretic entropy and topological pressure. Ban and Chang

[1] show that, if the potential function depends on one coordinate, uniform Bernouli measure is an equilibrium measure for linear CA with prime-state. We extends their result to permutive CA.

In the rest of this investigation, local rule f is permutive and depends on $x_i, \ldots, x_j$, where $i \leq j$ and $i, j \in \mathbb{Z}$.

3.1 Measure-theoretic entropy

Let $\mathcal{B}$ be a Borel σ-algebra on Ω, $\mu = (p_0, p_1, \ldots, p_{r-1})$ be an F-invariant Bernoulli measure, i.e.,

$$\mu({}_a[s_a, \ldots, s_b]_b) = p_{s_a} \cdots p_{s_b}, \quad \text{for } {}_a[s_a, \ldots, s_b]_b \subset \Omega.$$

Denote by $\widehat{i} = -\min\{i, 0\}$ and $\widehat{j} = \max\{j, 0\}$. The following theorem is also demonstrated in [16]. Here we give an alternative proof.

Theorem 3.1. *If f is permutive, then*

$$h_\mu(F) = -(\widehat{i} + \widehat{j}) \sum_{m=0}^{r-1} p_m \log p_m.$$

Before giving proof, we introduce a lemma. For $\ell \in \mathbb{N}$, denote by $\xi_\ell = \{{}_{-\ell}[s_{-\ell}, \ldots, s_\ell]_\ell : s_{-\ell}, \ldots, s_\ell \in S\}$ a measurable partition of Ω.

Lemma 3.2. *If f is permutive, then*

$$\bigvee_{m=0}^{n-1} F^{-m} \xi_\ell = \xi(-\ell - (n-1)\hat{i}, \ell + (n-1)\hat{j})$$

provided ℓ large enough, where $\xi(a, b) = \{{}_a[x_a, \ldots, x_b]_b : x_a, \ldots, x_b \in S\}$.

Proof. We discuss the case that f is permutive of type 1, i.e., $\hat{i} = -i > 0$ and $\hat{j} = 0$, the other cases can be done via analogous argument. First observe that for each $z = (z_a, \ldots, z_b) \in S^{b-a+1}$, $f_{b-a+i}^{-1} z \in S^{b-a-i+1}$, where $f_m : S^{-i+m+1} \to S^{m+1}$ is defined by

$$f_m(x_{i-m}, \ldots, x_0) = (f(x_{i-m}, \ldots, x_{-m}), \ldots, f(x_i, \ldots, x_0)), \quad \text{for } m \in \mathbb{N},$$

and $f_0 = f$. Denote f_{b-a+i} by f without ambiguity. For $s_{b+i+1}, \ldots, s_b \in S$, the leftmost permutivity of f at x_i implies there admits a unique $\tilde{z}_{b+i}$ such that $f(\tilde{z}_{b+i}, s_{b+i+1}, \ldots, s_{b+j}) = z_b$. Repeating this process there are uniquely determined $\tilde{z}_{b+i-1}, \ldots, \tilde{z}_{a+i} \in S$ such that

$$f(\tilde{z}) = z, \quad \text{where } \tilde{z} = (\tilde{z}_{a+i}, \ldots, \tilde{z}_{b+i}, s_{b+i+1}, \ldots, s_b) \in S^{b-a-i+1}.$$

Denote by $\xi_\ell = \{A_l\}_{l=1}^{r^{2\ell+1}}$, above discussion and induction asserts that

$$F^{-m}A_{n_1} \bigcap F^{-m}A_{n_2} = \varnothing, \quad \text{for } n_1 \neq n_2, m \in \mathbb{Z}^+.$$

Furthermore, if ℓ is chosen such that $\ell \geq [i/2]$, where $[x]$ is the greatest integer that is less than or equal to x. Then $\xi_\ell \bigvee F^{-1}\xi_\ell = \xi(-\ell+i,\ell)$. Inductively, $\bigvee_{m=0}^{n-1} F^{-m}\xi_\ell = \xi(-\ell+(n-1)i,\ell)$. This asserts the lemma. □

Proof of Theorem 3.1. The case that f is permutive of type 1 is proved, the other cases can be done similarly. Consider $\{\xi_\ell\}_{\ell=1}^\infty$ a sequence of finite partitions of Ω, it is easy to see that $\xi_1 \overset{\circ}{\subset} \xi_2 \overset{\circ}{\subset} \cdots$ and $\bigvee_{\ell=1}^\infty \xi_\ell \doteq \mathcal{B}$, where $A \overset{\circ}{\subset} B$ (respectively $A \doteq B$) means the σ-algebra generated by A is a subset of (respectively coincides with) that generated by B up to a measure zero set. For each $\ell \in \mathbb{N}$, observe that

$$\begin{aligned} H_\mu(\xi_\ell) &= -\sum_{A\in\xi_\ell} \mu(A)\log\mu(A) \\ &= -\sum_{s_{-\ell},\ldots,s_{\ell-1}} p_{s_{-\ell}}\cdots p_{s_{\ell-1}} \sum_{s_\ell} p_{s_\ell} \log p_{s_{-\ell}}\cdots p_{s_\ell} \\ &= -(2\ell+1)\sum_{m=0}^{r-1} p_m \log p_m. \end{aligned}$$

Applying Lemma 3.2 and mathematical induction,

$$H_\mu(\bigvee_{m=0}^{n-1} F^{-m}\xi_\ell) = -(2\ell-(n-1)i+1)\sum_{m=0}^{r-1} p_m \log p_m$$

whenever ℓ is large enough. Hence

$$h_\mu(F) = \lim_{\ell\to\infty} h_\mu(F,\xi_\ell) = i\sum_{m=0}^{r-1} p_m \log p_m.$$

This completes the proof. □

Example 3.3. *Let $S = \{0,1,2,3\}$ and let $f : S^5 \to S$ be defined by*

$$f(x_0,x_1,x_2,x_3,x_4) = 2x_0 + x_3x_0 + x_1^2 + 3x_4 \mod 4,$$

then f is permutive of type 2 and $\widehat{i} = 0, \widehat{j} = 4$. Theorem 3.1 shows that

$$h_\mu(F) = -4(p_0\log p_0 + p_1\log p_1 + p_2\log p_2 + p_3\log p_3).$$

3.2 Topological pressure

Let $\phi_k : S^k \to \mathbb{R}$ be given and let $\phi : \Omega \to \mathbb{R}$ be defined by $\phi(x) = \phi_k(x_0 \cdots x_{k-1})$. Set $\xi \equiv \xi(0, k-1)$ a measurable partition of Ω. Define transition matrix $\mathbf{T}_{\phi,F} = (t_{mn})_{1 \leq m,n \leq r^k}$ with respect to ϕ by

$$t_{mn} = d_{mn} \exp \phi(x), \quad x \in C_m, \tag{5}$$

where $C_m = [c_0, \ldots, c_{k-1}] \in \xi$, $m = 1 + \sum_{\ell=0}^{k-1} c_\ell \cdot r^{k-\ell-1}$ and

$$d_{mn} = \begin{cases} 1, & C_m \cap F^{-1}C_n \neq \varnothing; \\ 0, & \text{otherwise.} \end{cases}$$

We have the following theorem.

Theorem 3.4. *If f is permutive, then the topological pressure $P(F, \phi) = s \log r + \log \rho$, where ρ is the spectral radius of $\mathbf{T}_{\phi,F}$ and $s = \max\{0, \hat{\imath} + \hat{\jmath} - k\}$.*

Proof. The case f is permutive of type 1 is considered while the other cases can be proved similarly.

Without loss of generality we may assume $k = 2$. Let $a_{pq} \in \mathbb{R}$, $0 \leq p, q \leq r-1$, be given and let $\phi : \Omega \to \mathbb{R}$ be defined by $\phi(x) = a_{x_0 x_1}$. Set $\xi = \{{}_0[pq]_1 : 0 \leq p, q \leq r-1\} \equiv \{C_1, \ldots, C_{r^2}\}$. Define $\widetilde{\mathbf{T}}_{\phi,F} = (t_{mn})_{1 \leq m,n \leq r^2}$, where $t_{mn} = (s_{mn} \exp a_{pq})$, $m = pr + q + 1$ and

$$s_{mn} = \begin{cases} r^s, & C_m \cap F^{-1}C_n \neq \varnothing; \\ 0, & \text{otherwise.} \end{cases}$$

Let $D_p = \#\{q : C_p \cap F^{-1}C_q \neq \varnothing\}$. Then $D_p = D_q$ for $1 \leq p, q \leq r^2$ and thus can be denoted by a constant D. Observe that

$$p_1(F, \phi, \xi) = \sum_{i_1, i_2 \in S} \exp(a_{i_1 i_2}) = |\widetilde{\mathbf{T}}_{\phi,F}| / (D \cdot r^s),$$

where $|A| = \sum a_{mn}$ is the 1-norm for nonnegative matrix A. Moreover,

$$p_2(F, \phi, \xi) = \sum_{m_1, m_2 \in S^2} \sum_{A_{m_1;m_2} \neq \varnothing} \exp(\phi(x) + \phi(F(x))) = |\widetilde{\mathbf{T}}^2_{\phi,F}| / (D \cdot r^s),$$

where the summation is taken for all connected $A_{m_1;m_2} \in [m_1] \cap F^{-1}[m_2]$. It comes from induction that $p_n(F, \phi, \xi) = |\widetilde{\mathbf{T}}^n_{\phi,F}| / (D \cdot r^s)$ for $n \in \mathbb{N}$, Perron-Frobenius theorem demonstrates that $P(F, \phi, \xi) = \log \tilde{\rho} = s \log r + \log \rho$, where $\tilde{\rho}$ is the spectral radius of $\widetilde{\mathbf{T}}_{\phi,F}$ and ρ is the spectral radius of $\mathbf{T}_{\phi,F}$.

Fix $N \in \mathbb{N}$ and set $\xi_N = \xi(-N, N)$. It can be verified that $p_1(F, \phi, \xi_N) = r^{2(N-1)}(D \cdot r^s)^{-1}|\widetilde{\mathbf{T}}_{\phi,F}|$ and $p_n(F, \phi, \xi_N) = r^{2(N-1)}(D \cdot r^s)^{-1}|\widetilde{\mathbf{T}}^n_{\phi,F}|$ for $n \in \mathbb{N}$. This infers that $P(F, \phi, \xi_N) = \log \tilde{\rho} = s \log r + \log \rho$ for all $N \in \mathbb{N}$. The proof is done by letting N tend to infinity. □

If potential function ϕ depends on only one coordinate, that is, without loss of generality we may assume $\phi : \Omega \to \mathbb{R}$ is defined by $\phi(x) = a_{x_0}$, where $a_0, a_1, \ldots, a_{r-1} \in \mathbb{R}$ are given. Theorem 3.4 can be expressed in an explicit form.

Corollary 3.5. *Let $a_0, a_1, \ldots, a_{r-1} \in \mathbb{R}$ be given and let $\phi : \Omega \to \mathbb{R}$ be defined by $\phi(x) = a_{x_0}$. If f is permutive, then $P(F, \phi) = (\hat{\imath} + \hat{\jmath} - 1) \log r + \log(e^{a_0} + e^{a_1} + \cdots + e^{a_{r-1}})$.*

Proof. Let $\xi = \{[0], \ldots, [r-1]\}$ be the standard measurable partition of Ω. Permutivity of f asserts $\hat{\imath} + \hat{\jmath} \geq 1$ and thus the kth row of the transition matrix $\mathbf{T}_{\phi,F} \in \mathcal{M}_r(\mathbb{R})$ is $e^{a_{k-1}}(1\ 1 \cdots 1)$ for $1 \leq k \leq r$. Applying Theorem 3.4 derives the desired result. □

If X is a compact metric space and $T : X \to X$ is a continuous transformation, variational principle for topological pressure says that, for $\psi \in C(X, \mathbb{R})$,

$$P(T, \psi) = \sup\{h_\nu(T) + \int_X \psi \, d\nu : \nu \text{ is an ergodic measure}\}. \tag{6}$$

A measure ν is called an equilibrium measure provided $P(T, \psi) = h_\nu(T) + \int_X \psi \, d\nu$. Theorems 3.1 and 3.4 help for the determination of equilibrium measures of permutive CA.

Corollary 3.6. *If f is permutive and potential function ϕ is given as in Corollary 3.5, then Parry measure is the unique equilibrium measure. In other word, such cellular automaton possesses no phase transition.*

Proof. If f is permutive and $\hat{\imath} + \hat{\jmath} = 1$, then $P(F, \phi) = \log(e^{a_0} + e^{a_1} + \cdots + e^{a_{r-1}})$. Moreover,

$$h_\mu(F) + \int_\Omega \phi \, d\mu = -\sum_{m=0}^{r-1} p_m \log p_m + \sum_{m=0}^{r-1} a_m \cdot p_m = \sum_{m=0}^{r-1} p_m(a_m - \log p_m).$$

To determine whether μ is an equilibrium measure, define $\Phi : [0, \infty) \to \mathbb{R}$ by

$$\Phi(x) = \begin{cases} 0, & x = 0; \\ x \log x, & \text{otherwise.} \end{cases}$$

Then Φ is convex and $\Phi \in C^1((0,\infty),\mathbb{R})$. Moreover,

$$\Phi(\sum_{m=1}^{n}\alpha_m x_m) \le \sum_{m=1}^{n}\alpha_m\Phi(x_m), \quad \text{for } \sum_{m=1}^{n}\alpha_m = 1, \alpha_m \ge 0, x_m \in \mathbb{R}.$$

Let $\alpha_m = e^{a_m}/\lambda$ and $x_m = (p_m\lambda)/e^{a_m}$ for $0 \le m \le r-1$, where $\lambda = \sum_{m=0}^{r-1} e^{a_m}$.

$$\begin{aligned}
0 = \Phi(1) &= \Phi(\sum_{m=0}^{r-1}\alpha_m x_m) \\
&\le \sum_{m=0}^{r-1}\frac{e^{a_m}}{\lambda}\cdot\frac{p_m\lambda}{e^{a_m}}\log\frac{p_m\lambda}{e^{a_m}} = \sum_{m=0}^{r-1} p_m \log\frac{p_m\lambda}{e^{a_m}} \\
&= \log(e^{a_0}+e^{a_1}+\cdots+e^{a_{r-1}}) - \sum_{m=0}^{r-1} p_m(a_m - \log p_m).
\end{aligned}$$

The equality holds if and only if $(p_m\lambda)/e^{a_m} = 1$ for $0 \le m \le r-1$, i.e., μ is an equilibrium measure if and only if $p_k = e^{a_k}/\sum_{m=0}^{r-1} e^{a_m}$ for $0 \le k \le r-1$.

When $\hat{i}+\hat{j} \ge 2$, then $P(F,\phi) = (\hat{i}+\hat{j}-1)\log r + \log\sum e^{a_m}$ and $h_\mu(F) = -(\hat{i}+\hat{j})\sum p_m \log p_m$. The fact $\sum p_m \log p_m^{-1} \le \log r$ and the equality holds if and only if $p_k = p_\ell$ for $k \ne \ell$ implements that

$$-(\hat{i}+\hat{j})\sum p_m\log p_m + \sum p_m a_m \le (\hat{i}+\hat{j})\log r + \log\sum e^{a_m}.$$

Moreover, the equality holds if and only if

$$p_k = \frac{e^{a_k}}{\sum e^{a_m}} \quad \text{and} \quad p_k = p_\ell, \quad \text{for } 0 \le k,\ell \le r-1.$$

The proof is complete. □

4 TOPOLOGICAL PROPERTIES OF PERMUTIVE CELLULAR AUTOMATA

This section studies permutive cellular automata in the viewpoint of topological aspects. A dynamical system is said to be chaotic in the sense of Li-Yorke provided the existence of periodic points with period larger than some given integer and there exists an uncountable set such that any two distinct orbits of it would be arbitrary close but never merge together.

We show that, for a cellular automaton, the existence of snap-back repeller implies the exhibition of Li-Yorke chaos and bipermutive cellular automata possesses a snap-back repeller.

Theorem 4.1. *If F is a cellular automaton that possesses a snap-back repeller, then F is chaotic in the sense of Li-Yorke.*

Instead of giving a theoretical proof, an example is investigated to assert Theorem 4.1 since the proof is similar as the argument given in [13, 14].

Example 4.2. *Consider F Wolfram's rule 102 on $\Sigma_2^+ = \{x = (x_i)_{i \geq 0} : x_i \in \{0,1\}$ for all $i\}$, where the local rules $f : \{0,1\}^3 \to \{0,1\}$ is defined by $f(x_{-1}, x_0, x_1) = x_0 + x_1 \mod 2$ and the metric d on Σ_2^+ is defined by*

$$d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}. \tag{7}$$

Let $z = 0^\infty$ and $\epsilon = 1/2$, then z is a fixed point. For each $x \in B_\epsilon(z)$, $x_0 = x_1 = 0$. It is easily seen that

(1) $d(F_{102}(x), z) > d(x,z)$ for all $x \in B_\epsilon(z)$;

(2) $F_{102}^{-k}(x) \to z$ as $k \to \infty$ for all $x \in B_\epsilon(z)$;

(3) let $y = (0011001100110011\ldots)$, then $y \in B_\epsilon(z)$, $F_{102}(y) \notin B_\epsilon(z)$ and $F_{102}^3(y) = z$.

That is, z is a snap-back repeller.

To show that F is Li-Yorke chaotic, we need to find $N \in \mathbb{N}$ such that, for $n \geq N$, F exhibits an n-periodic orbit and show the existence of scramble set $\mathcal{S}$.

Let $\varsigma = 2^{-4}$. It is easily verified $F|_{B_\varsigma(z)}$ is well-defined and $B_\varsigma(z) = {}_0[00000]_4$. Set $G = F^{-3}|_{B_\varsigma(z)}$ and $Q = G(B_\varsigma(z))$, then Q, $F(Q)$ and $F^2(Q)$ are compact subsets of the complement of $B_\epsilon(z)$ and $F^{-3}(x) \in B_\varsigma(z)$ for all $x \in Q$. More than that, $F^{-k}Q \subset B_\varsigma(z)$ for all $k \geq 3$. Notably, $Q = G(B_\varsigma(z)) = F^{-3}(B_\varsigma(z))$ and

$$F^{-k} \circ G : B_\varsigma(z) \to B_\varsigma(z), \quad \text{for } k \geq 3. \tag{8}$$

Brouwer's fixed point theorem asserts that there exists $y_k \in B_\varsigma(z)$ such that $(F^{-k} \circ G)(y_k) = y_k$, i.e., $F^{k+3}(y_k) = y_k$, for all $k \geq 3$.

It remains to show that y_k is actually of period $k+3$.

Since $F^k(y_k) = G(y_k) \in Q$, F is expanding in $B_\varsigma(z)$ indicates that

$$F^n(y_k) \neq y_k, \quad \text{for } 1 \leq n \leq k.$$

Also, $F(Q), F^2(Q) \subset B_\epsilon(z)^c$ implies $F^n(y_k) \neq y_k$ for $n = k+1, k+2$. Hence, y_k is of period $k+3$.

As a conclusion, let $N = 6$. For all $n \geq N$, there exists $y_n \in B_\varsigma(z)$ such that $F^n(y_n) = y_n$ and $F^k(y_n) \neq y_n$ for $1 \leq k \leq n-1$.

The construction of the scramble set is similar as the method in [12], thus is skipped.

This completes the example.

Proposition 4.3. *Each bipermutive cellular automaton exhibits a snap-back repeller.*

Proof. Fagnani and Margara indicate that a bipermutive CA is topological conjugate to a one-sided shift [5]. It is easy to verify that it exhibits a snap-back repeller.

The proof is complete. □

Notably that bipermutivity is optimized for the exhibition of snap-back repellers when two-sided CA is considered. The following is a counterexample.

Example 4.4. *Wolfram's rule 102 exhibits no snap-back repellers on $\Sigma_2 = \{x = (x_i)_{i\in\mathbb{Z}} : x_i \in \{0,1\}$ for all $i\}$.*

Proof. For $n \in \mathbb{N}$ and $y \in \Sigma_2$ satisfies $F^{2^n}(y) = y$. It is easily seen that $y = 0^\infty$, which is a fixed point. This means F has no period 2^n points for all $n \in \mathbb{N}$. Hence F can never exhibit a snap-back repeller. □

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