國立政治大學應用數學系

碩士學位論文

一個卡特蘭等式的重新審視
A Catalan Identity revisited

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致謝

在本人的寫作過程中，首先最主要感謝的是我的指導老師，李陽明老師。在整個過程中他給了我很大的幫助，在論文題目制定時，他首先肯定了我的題目大方向，但是同時又協助確立目標，讓我在寫作時有了具體方向。在論文提綱制定時，我的思路不是很清晰，經過老師的幫忙，讓我具體寫作時思路頓時清晰。在完成初稿後，老師認真查看了我的內容，指出了我存在的很多問題。在此十分感謝李老師的細心指導，才能讓我順利完成畢業。感謝口試委員陳天進、蔡炎龍老師給予指正我論文中的錯誤，在此深表感謝！同時也感謝其他幫助和指導過我的老師和同學。最後要感謝在整個論文寫作過程中幫助過我的每一位人。

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中文摘要

本篇論文探討卡特蘭等式$(n+2)C_{n+1} = (4n+2)C_n$ 證明方式以往都以計算方式推導得出，當我參加劉映君的口試時，發現她使用組合方法來證明這個等式。當我在尋找論文的主題時，讀到李陽明老師的一篇論文"$The\ Chung-Feller\ theorem\ revisited$"，發現 Dyck 路徑也可以作為卡特蘭等式的組合證明，因此我們完成$(n+2)C_{n+1} = (4n+2)C_n$ 的組合證明。

通過 Dyck 路徑證明卡特蘭等式可以得到以下優勢：

1. 子路徑 C 在切換過程中不會改變。
2. 由於x_1 中的 P 的子路徑 B 為空，因此在交換 Ad 和 Bu 部分後，生成新的缺陷必連接在原始子路徑 C 之後。

   由於x_2 中的 Q 的子路徑 A 為空，因此在 Bu 交換和 Ad 部分後，生成新的提升必連接在原始子路徑 C 之後。
3. 在計算函數$g_1(g_2)$ 的反函數的過程中，缺陷（提升）恢復模式必遵循“後進先出”或“先進後出”規則。

關鍵字：卡特蘭等式、Dyck 路徑

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Abstract

When we first prove the Catalan identity, $(n + 2)C_{n+1} = (4n + 2)C_n$. We often prove it by calculation.When I participated in the oral examination of Ying-Jun Liu's essay, I found that she used a combinatorial proof to prove this identity. When I was looking for the subject of the thesis, I read a paper by professor Young-Ming Chen, $''The\ Chung - Feller\ theorem\ revisited''$, which found that Dyck paths could also be used as a combinatorial proof of the Catalan identity. Therefore, we completed the combinatorial proof of $(n + 2)C_{n+1} = (4n + 2)C_n$.

Proving the Catalan identity through the Dick paths can reveal the following advantages:

1.The subpath C does not change during the process of switching of the portions Ad and Bu.

2.Since the subpath B of P in x_1 is empty, a new flaw generated after switching of the portions Ad and Bu must be followed by the original subpath C. Since the subpath A of Q in x_2 is empty, a new lift generated after switching of the portions Bu and Ad must be followed by the original subpath C.

3.In the process of computing the preimage of a function g_1 (g_2), the flaws (lifts) recovery mode follows the $''Last{-}in{-}First{-}out''$ or $''First{-}in{-}Last{-}out''$.

Keywords: Catalan identity, Dyck path

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Chapter 1

Introduction

- When we first prove the Catalan identity, $(n+2)C_{n+1} = (4n+2)C_n$. We often prove it by calculation. The proof method is as follows:

$$(n+2)C_{n+1} = \frac{(n+2)C_{n+1}^{2n+2}}{n+2} = \frac{(2n+2)!}{(n+1)!(n+1)!} = \frac{(2n+1)(2n)!(2n+2)}{(n+1)n!n!(n+1)} = \frac{2(2n+1)(2n)!(2n+2)}{(n+1)n!n!(2n+2)} = \frac{(4n+2)(2n)!}{(n+1)n!n!} = \frac{(4n+2)C_n^{2n}}{n+1} = (4n+2)C_n$$

  When I participated in the oral examination of Ying-Jun Liu's essay, I found that she used a combinatorial of proofs to prove this identity. When I was looking for the subject of the thesis, I read a paper by teacher Young-Ming Chen, "The Chung-Feller theorem revisited", which found that Dyck paths could also be used as a combinatorial proof of the Catalan identity. Therefore, we complete the combinatorial proof of $(n+2)C_{n+1} = (4n+2)C_n$. [5] [7] [1]

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Definition 1.0.1. *An up-step is denoted by* $u = (1, 1)$.

A down-step is denoted by $d = (1, -1)$.

A step is either an up-step(u*) or a down-step(*d*).*

A path consists of consecutive steps.

A subpath is some of consecutive steps of a path.

Definition 1.0.2. *A path is that all up-steps and down-steps are above the x-axis.*

Definition 1.0.3. *A totally bad path is that all up-steps and down-steps are below the x-axis.*

Definition 1.0.4. *A flaw is a down-step below the x-axis.*

Definition 1.0.5. *A lift is a up-step above the x-axis.*

Definition 1.0.6. *In An* n*-Dyck path,* C_n *is the number of good paths from* $(0, 0)$ *to* $(2n, 0)$

Definition 1.0.7. *An* n*-Dyck path is a path from* $(0, 0)$ *to* $(2n, 0)$ *with* n *up-steps and* n *down-steps.*

An n*-Dyck path with* k *flaws if it has* k *down-steps below the x-axis.*

Definition 1.0.8. *Let* R *is a subpath of* n*-Dyck path. The number of up-steps and down-steps in* R *is denoted by* $|R| = r$*,*$0 \leq r \leq 2n$*.*

Definition 1.0.9. *The set* $\mathbb{D}_{n,k}$ *consists of all* n*-Dyck paths with* k *flaws,* $0 \leq k \leq n$*.*

For more details ,we refer to [4] [3] [2] [10] [9] [6] [8] [11]

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Chapter 2

Paths Start with Up-step

Definition 2.0.1. *Define a function f_1 from $\mathbb{D}_{n+1,k}$ into $\mathbb{D}_{n+1,k+1}$ by the following:*

1. *The set $\mathbb{D}_{n+1,k}$ consists of all $n+1$-Dyck paths with k flaws, $0 \leq k \leq n$. Each path in $\mathbb{D}_{n+1,k}$ can be factorized into $BuAdC$, where B is a subpath all bellow the x-xais, say with k_1 flaws, $0 \leq k_1 \leq k$, u is the first up step above the x-xais, A is a subpath all above the x-xais, d is the first step to contact the x-axis after A and above the x-axis, say uAd with 0 flaws, and C is the remaining path with $k-k_1$ flaws, $0 \leq k \leq n$.*

2. *The set $\mathbb{D}_{n+1,k+1}$ consists of all $n+1$-Dyck path with $k+1$ flaws, $0 \leq k \leq n$. Each path in $\mathbb{D}_{n+1,k+1}$ can be factorized into $AdBuC$, where A,dBu, and C have 0, k_1+1, and $k-k_1$ flaws.*

Note: *A、B、C may be empty, and they have the same number of up-steps and down-steps.*

i.e. $f_1(BuAdC) = AdBuC$

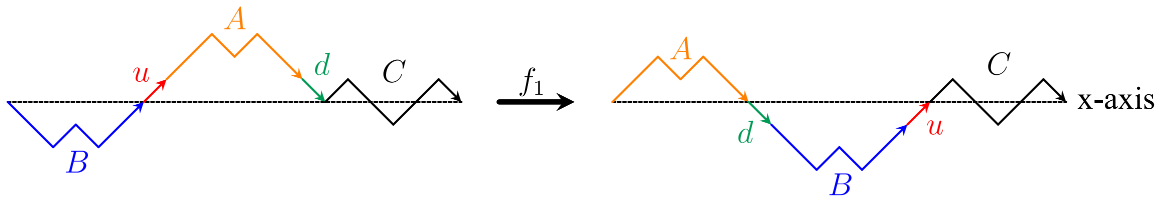

Figure 2.1: $f_1(BuAdC) = AdBuC$

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Theorem 2.0.1. *Define* f_1:$\mathbb{D}_{n+1,k} \rightarrow \mathbb{D}_{n+1,k+1}$ *by* $f_1(BuAdC) = AdBuC$.

The function f_1 *is one-to-one.*

Proof. Let $|B| = 2i, 0 \le i \le n$; $|u| = 1$; $|A| = 2j, 0 \le j \le n - i$; $|d| = 1$; $|C| = 2(n - i - j - 1)$.

Claim: f_1 one-to-one. (i.e.$f_1(BuAdC) = f_1(B'uA'dC') \Rightarrow AdBuC = A'dB'uC'$)

Suppose $|B'| = 2r, 0 \le r \le n$; $|u| = 1$; $|A'| = 2s, 0 \le s \le n-r$; $|d| = 1$; $|C'| = 2(n-r-s-1)$.

Claim 1: $|A| = |A'|$

case 1: $2j < 2s$

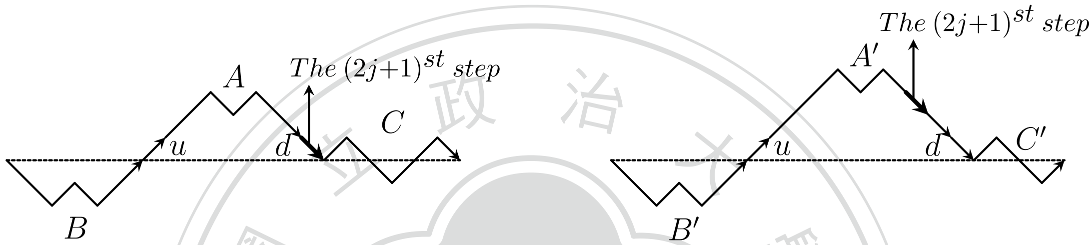

Figure 2.2: $BuAdC$ and $B'uA'dC'$

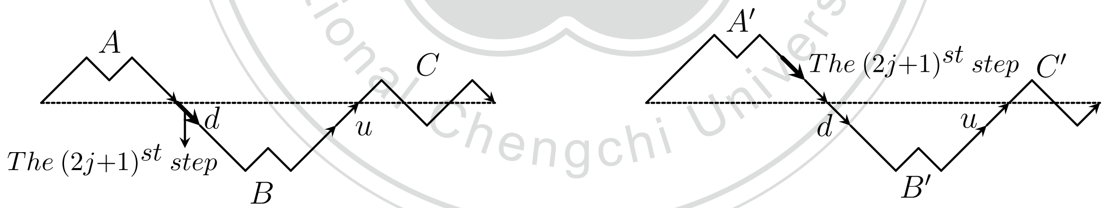

Figure 2.3: $f_1(BuAdC) = AdBuC$ and $f_1(B'uA'dC') = A'dB'uC'$

To see Figure 2.3 when we start on $(0, 0)$ to walk along the two path $AdBuC$ and $A'dB'uC'$. The $(2j + 1)^{st}$ step of $AdBuC$ is below the x-axis, but the $(2j + 1)^{st}$ step of $A'dB'uC'$ is still above the x-axis. This is a contradiction as two paths are the same. $\square$

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case 2: $2j > 2s$

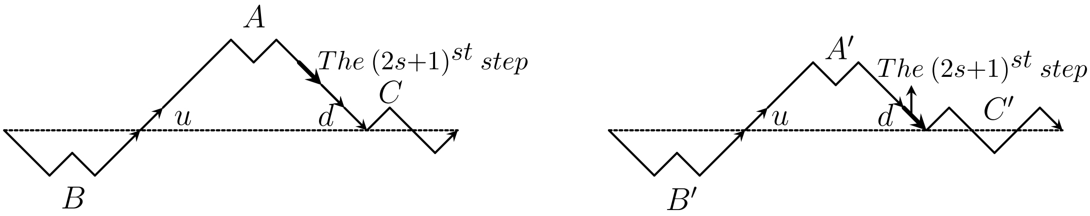

Figure 2.4: $BuAdC$ and $B'uA'dC'$

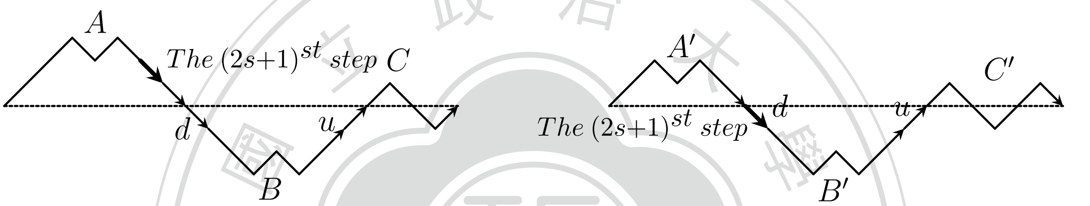

Figure 2.5: $f_1(BuAdC) = AdBuC$ and $f_1(B'uA'dC') = A'dB'uC'$

To see Figure 2.5 when we start on $(0, 0)$ to walk along the path the two path $AdBuC$ and $A'dB'uC'$. The $(2s+1)^{st}$ step of $AdBuC$ is still above the x-axis, but the $(2s+1)^{st}$ step of $A'dB'uC'$ is below the x-axis. This is a contradiction as two paths are the same. Thus, we have proof that $j = s$.

$\therefore |A| = |A'|$ and $A = A'$.

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Claim 2: $|B| = |B'|$

recall:Let $|B| = 2i$, $0 \leq i \leq n$; $|u| = 1$; $|A| = 2j$, $0 \leq j \leq n - i$; $|d| = 1$;

$|C| = 2(n - i - j - 1)$.

Suppose $|B'| = 2r$, $0 \leq r \leq n$; $|u| = 1$; $|A'| = 2s$, $0 \leq s \leq n - r$; $|d| = 1$;

$|C'| = 2(n - r - s - 1)$.

case 1: $2i < 2r$

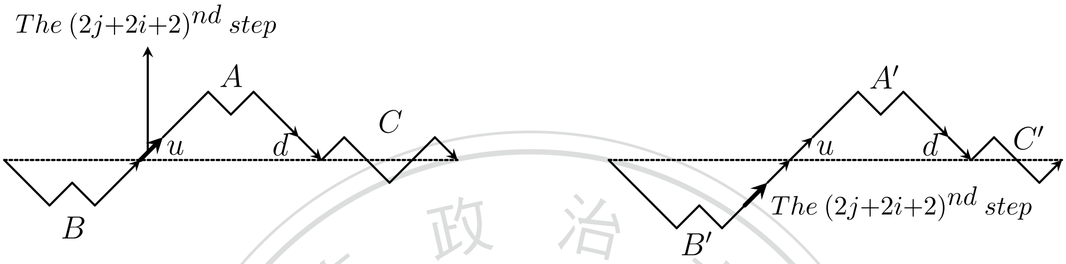

Figure 2.6: $BuAdC$ and $B'uA'dC'$

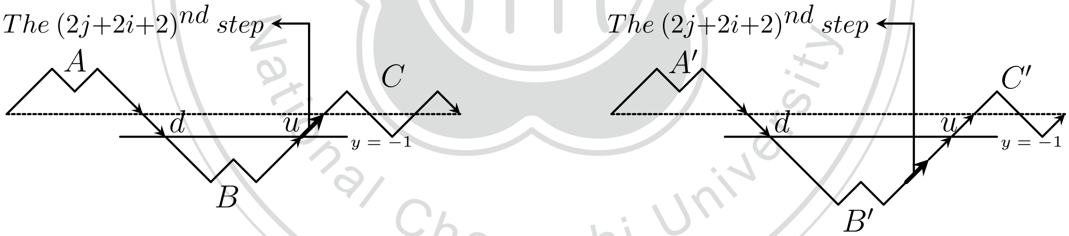

Figure 2.7: $f_1(BuAdC) = AdBuC$ and $f_1(B'uA'dC') = A'dB'uC'$

To see Figure 2.7 when we start on $(2j + 1, -1)$ to walk along the path B and B'. The $(2j + 2i + 2)^{nd}$ step of Bu is above $y = -1$, but the $(2j + 2i + 2)^{nd}$ step of B' is still below $y = -1$. This is a contradiction as two paths are the same.

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case 2: $2i > 2r$

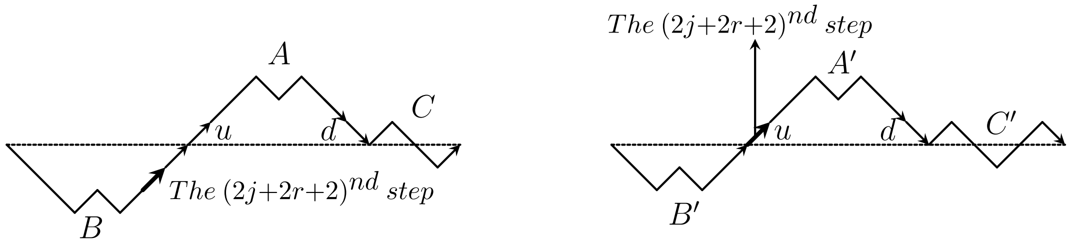

Figure 2.8: $BuAdC$ and $B'uA'dC'$

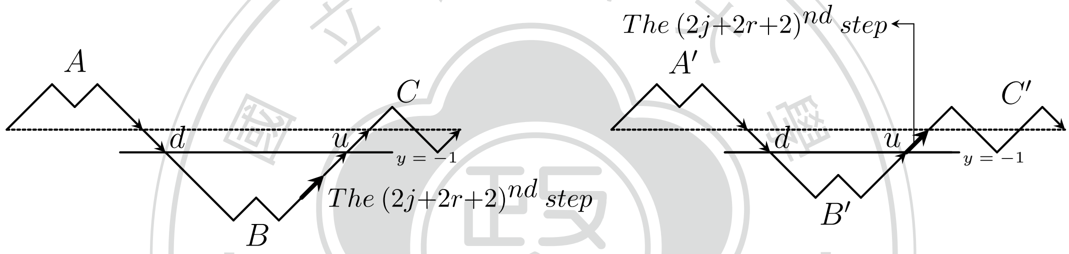

Figure 2.9: $f_1(BuAdC) = AdBuC$ and $f_1(B'uA'dC') = A'dB'uC'$

To see Figure 2.9 when we star on $(2j+1, -1)$ to walk along the path B and B'. The $(2j+2r+2)^{nd}$ step of $B'u$ is above $y=-1$, but the $(2j+2r+2)^{nd}$ step of B is still below $y=-1$. This is a contradiction as two paths are the same. Thus, we have proof that $j = s$.

$\therefore |B| = |B'|$ and $B = B'$.

Since $A = A'$ and $B = B'$,

$\because AdBuC = A'dB'uC' \Rightarrow C = C'$

$\therefore BuAdC = B'uA'dC'$

Therefore f_1 is one-to-one.

$\square$

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Lemma 2.0.2. *If P is a path in $\mathbb{D}_{n+1,k}$, then $f_1(P)$ has $k+1$ flaws.*

Moreover, the $(k+1)^{st}$ flaw will be connected to original flaws behind.

Note: A down step below the x-axis is called flaw. The flaws are counted from right to left and bottome-up.

Proof. The function f_1 from $\mathbb{D}_{n+1,k}$ into $\mathbb{D}_{n+1,k+1}$. Suppose P has k flaws in $\mathbb{D}_{n+1,k}$, and $P = BuAdC$, where B, uAd, C have $k-j$ flaws, 0 flaw, j flaws, respectively. The k^{th} flaw is in subpath B.Then $f_1(P)$ has $k+1$ flaws on $\mathbb{D}_{n+1,k+1}$, and $f_1(P) = AdBuC$, where A, dBu, C have 0 flaw, $k-j+1$ flaws, j flaws, respectively.

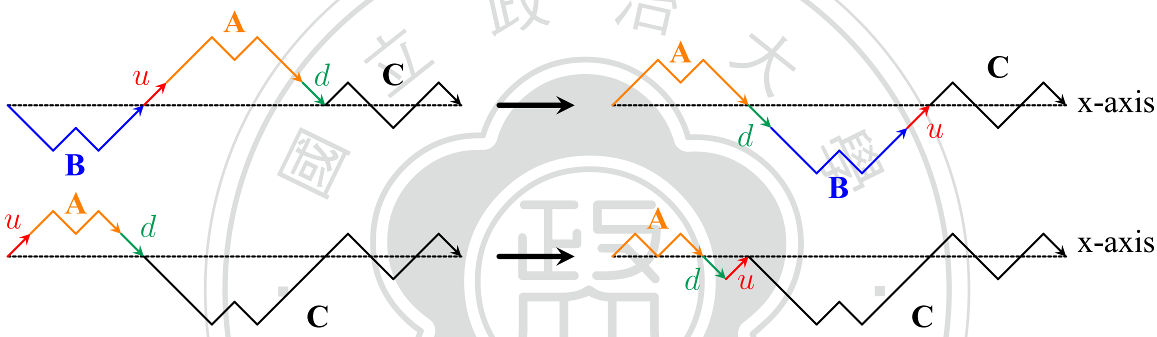

Figure 2.10: *the new flaw will be connected to original flaws behind*

Notice that the path $f_1(P)$, d is connected to the k^{th} flaw in subpath B behind. d becomes the $(k+1)^{st}$ flaw.

Therefore, no matter how many times we use f_1, d is conneted to the k^{th} flaw in subpath B behind. d is still the $(k+1)^{st}$ flaw.

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Note:

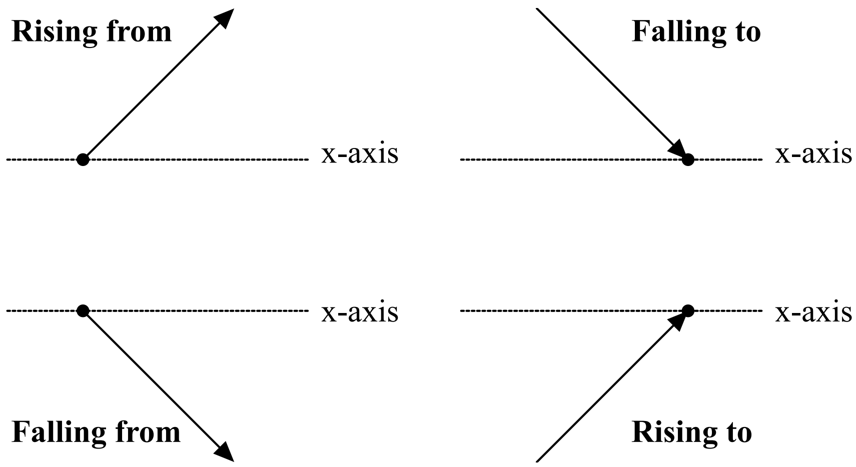

Figure 2.11: *Rising from*; *Falling to*; *Falling from*; *Rising to*

Lemma 2.0.3. *If Q is a path in $\mathbb{D}_{n+1,k+1}$, then $f_1^{-1}(Q)$ has k flaws.*

Moreover,the $(k+1)^{st}$ flaw in Q will be restored.

i.e. The $(k+1)^{st}$ flaw d and the first up-step u rising to the x-axis on the right side of d in $\mathbb{D}_{n+1,k+1}$ are restored to the first down-step d falling to the x-axis and the first up-step u rising from the x-axis in $\mathbb{D}_{n+1,k}$.

Show by formula:

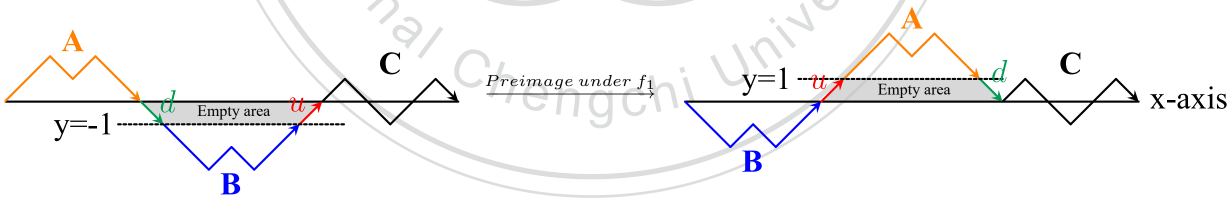

Figure 2.12: $AdBuC \xrightarrow{Preimage\ under\ f_1} BuAdC$

In Q, the $(k+1)^{st}$ flaw d and the first up-step u rising to the x-axis on the right side of d in $\mathbb{D}_{n+1,k+1}$, we can observe that there is an empty area enclosed by the u, d, the x-axis, and the horizontal line $y = -1$. After switching two portions Ad and Bu, another empty area is enclosed by the u is the first up-step rising from the x-axis, and d is the first down-step falling to the x-axis, and the horizontal line $y = 1$ in $f_1^{-1}(Q)$. And the remaining segments which are behind u is the fixed subpath C. □

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Theorem 2.0.2. *Define* f_1*:*$\mathbb{D}_{n+1,k}$$\rightarrow$$\mathbb{D}_{n+1,k+1}$ *by* $f_1(BuAdC) = AdBuC$*.*

The function f_1 *is onto.*

Proof. **Claim:** f_1 is onto. (f_1:$\mathbb{D}_{n+1,k} \to \mathbb{D}_{n+1,k+1}$)

For any path $Q = AdBuC$ in $\mathbb{D}_{n+1,k+1}$ which $1 \leq k+1 \leq n+1$. We choose the $(k+1)^{st}$ flaw d and choose the first up-step u rising to the x-axis on the right side of d. We switch the portions Ad and Bu then we can restore the $(k+1)^{st}$ flaw and get a new path $P = BuAdC$ in $\mathbb{D}_{n+1,k}$, by Lemma 2.2、Lemma 2.3.

Where Q has at least one flaw and P has at most n flaws. In fact, if Q has $k+1$ flaws then P has k flaws. So every Q in $\mathbb{D}_{n+1,k+1}$, we can find a path P in $\mathbb{D}_{n+1,k}$, such that $f_1(P) = Q$. Therefore f_1 is onto.

Hence, f_1 is one-to-one and onto by Theorem 2.1 and Theorem 2.2.

Note: Let f_1^{-1} be the inverse function of f_1. □

Definition 2.0.4. *The set* X_1 *consists of all paths in* $n+1$*-Dyck paths with* k *flaws.Each path in* X_1 *can be factorized into* $B\overrightarrow{u}A\overrightarrow{d}C$*. The set* Y_1 *consists of all* $n+1$*-Dyck paths which are totally bad paths.*

Define a function g_1 *from* X_1 *into* Y_1 *by the following:*

1. $g_1(P) = f_1^{n+1-k}(P)$*, where* P *in* X_1 *and* $f_1^{n+1-k} = \underbrace{f_1 \circ f_1 \circ \cdots \circ f_1}_{n+1-k\ times}$

2. *The first up-step rising from x-axis denote by* $\overrightarrow{u}$ *and the first down-step falling to x-axis denote by* $\overrightarrow{d}$*.*

Lemma 2.0.5. *Suppose that* P *in* X_1*. In* $f_1(P)$*, the first up-step* $\overrightarrow{u}$ *rising from the x-axis of* P *connectswith the first down-step* $\overrightarrow{d}$ *falling to the x-axis of P to be* $\overrightarrow{d}\overrightarrow{u}$ *and* $\overrightarrow{d}\overrightarrow{u}$ *is below the x-axis. Let* $P = \overrightarrow{u}A\overrightarrow{d}C$*, then* $f_1(P) = A\overrightarrow{d}\overrightarrow{u}C$*.*

Proof. We may assume that $P = B\overrightarrow{u}A\overrightarrow{d}C$ is any path of $\mathbb{D}_{n+1,k}$. Since the subpath B is empty, then $P = \overrightarrow{u}A\overrightarrow{d}C \in X_1$ and $f_1(P) = A\overrightarrow{d}\overrightarrow{u}C$.We know $\overrightarrow{u}A\overrightarrow{d}$ has 0 flaw, and C must have k flaws in P. Then A, $\overrightarrow{d}\overrightarrow{u}$, and C have 0 flaw, 1 flaws, and k flaws in $f_1(P)$, respectively. Therefore, $\overrightarrow{d}\overrightarrow{u}$ is below the x-axis.

□

Note: In $f_1^i(P)$, $1 \leq i \leq n+1-k$, the segments below the x-axis will be always below the x-axis.

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Theorem 2.0.3. *The function g_1:$X_1 \to Y_1$ is one-to-one.*

Proof. Suppose that $g_1(P) = g_1(Q)$, where P has k flaws and Q has h flaws in X_1.

To prove that $P = Q$.

By definition $g_1(P) = f_1^{n+1-k}(P)$, $g_1(Q) = f_1^{n+1-h}(Q)$

case 1: $k < h$

$f_1^{n+1-k}(P) = f_1^{n+1-h}(Q) \Rightarrow f_1(f_1^{n-k}(P)) = f_1(f_1^{n-h}(Q))$

$\because f_1$ is one-to-one $\quad \therefore f_1^{n-k}(P) = f_1^{n-h}(Q)$

For the same reason, the following can be obtained

$f_1(f_1^{n-k-1}(P)) = f_1(f_1^{n-h-1}(Q)) \Rightarrow f_1^{n-k-1}(P) = f_1^{n-h-1}(Q)$

Since f_1 is one-to-one, use this method $n + 1 - h$ times

$f_1(f_1^{h-k}(P)) = f_1(Q) \Rightarrow f_1^{h-k}(P) = Q$

The $\overrightarrow{d}\,\overrightarrow{u}$ of $f_1^{h-k}(P)$ are below the x-axis, but the $\overrightarrow{u}$ and $\overrightarrow{d}$ of Q are above the x-axis by Lemma 2.5.

This is a contradiction.

case 2: $k > h$

$f_1^{n+1-k}(P) = f_1^{n+1-h}(Q) \Rightarrow f_1(f_1^{n-k}(P)) = f_1(f_1^{n-h}(Q))$

$\because f_1$ is one-to-one $\quad \therefore f_1^{n-k}(P) = f_1^{n-h}(Q)$

For the same reason, the following can be obtained

$f_1(f_1^{n-k-1}(P)) = f_1(f_1^{n-h-1}(Q)) \Rightarrow f_1^{n-k-1}(P) = f_1^{n-h-1}(Q)$

Since f_1 is one-to-one, use this method $n + 1 - k$ times

$f_1(P) = f_1(f_1^{k-h}(Q)) \Rightarrow P = f_1^{k-h}(Q)$

The $\overrightarrow{u}$ and $\overrightarrow{d}$ of P are above the x-axis, but $\overrightarrow{d}\,\overrightarrow{u}$ of $f_1^{k-h}(Q)$ are below the x-axis by Lemma 2.5.

This is a contradiction.

case 3: $k = h$

$f_1^{n+1-k}(P) = f_1^{n+1-h}(Q) \Rightarrow f_1(f_1^{n-k}(P)) = f_1(f_1^{n-h}(Q))$

$\because f_1$ is one-to-one $\quad \therefore f_1^{n-k}(P) = f_1^{n-h}(Q)$

Use this method $n + 1 - h$ times, we have $f_1(P) = f_1(Q) \Rightarrow P = Q$

Therefore, g_1 is one-to-one.

□

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Theorem 2.0.4. *g_1:$X_1 \rightarrow Y_1$ is onto, where Y_1 is totally bad path of $n+1$-Dyck and $g_1(P) = f_1^{n+1-k}(P)$.*

Proof. Give $Q \in Y_1$, Q has $n+1$ flaw, and $\overrightarrow{d}\overrightarrow{u}$ is the $(k+1)^{st}$ flaw in Q.

Define: The preimage of the function $g_1^{-1} = f_1^{-(n+1-k)} = \underbrace{f_1^{-1} \circ f_1^{-1} \circ \cdots \circ f_1^{-1}}_{n+1-k\, times}$

Since $f_1^{-1}(Q)$ is the preimage of Q under f_1 and has n flaws.

Using this way for $n-k$ times, we get the path $f_1^{-(n-k)}(Q)$ which has $k+1$ flaws.

In $f_1^{-(n-k)}(Q)$, $\overrightarrow{d}\overrightarrow{u}$ is still under the x-axis and is the $(k+1)^{st}$ flaw.We use this way again, we get the path $f_1^{-(n+1-k)}(Q)$ which has k flaws and $\overrightarrow{u}A\overrightarrow{d}$ is above the x-axis, as f_1 is onto and by Lemma 2.5.

So we have $f_1^{-(n+1-k)}(Q) = f_1^{-(n+1-k)}(f_1^{n+1-k}(P)) = P$,

where P has k flaws,$P \in X_1$.

Let $P = f_1^{-(n+1-k)}(Q) \Rightarrow g_1(P) = f_1^{n+1-k}(P)$

$$= f_1^{n+1-k}((f_1^{-(n+1-k)})(Q))$$

$$= Q$$

Thus g_1 is onto.

□

Hence, g_1 is one-to-one and onto by Theorem 2.3 and Theorem 2.4.

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Definition 2.0.6. *The set Z_1 contains the totally bad path for all n-Dyck paths which replaces $\overrightarrow{d}\overrightarrow{u}$ in Y_1 with a dot mark, and all paths in Y_1 are $n+1$-Dyck paths which are totally bad path. Let h_1 be the function from Y_1 into Z_1.*

i.e. $Q = R\overrightarrow{d}\overrightarrow{u}S$ is $(2n+2, 0)$ path, where R, $\overrightarrow{d}\overrightarrow{u}$, and S are all totally bad paths. $h_1(Q) = h_1(R\overrightarrow{d}\overrightarrow{u}S) = R \bullet S$

Theorem 2.0.5. *h_1 is one-to-one and onto.*

Proof. It is clear that h_1 is one-to-one.

Given $Q' = R \bullet S \in Z_1$

We can change $\bullet$ into $\overrightarrow{d}\overrightarrow{u}$. Thus $R \bullet S \Rightarrow R\overrightarrow{d}\overrightarrow{u}S \in Y_1$.

Therefore, h_1 is one-to-one and onto.

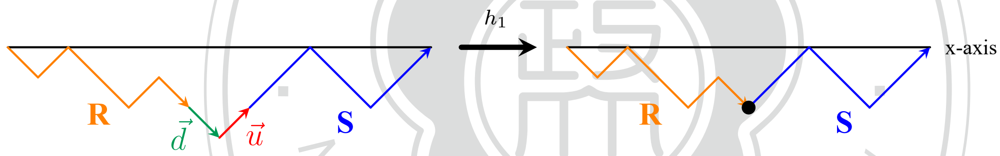

Figure 2.13: $h_1(R\vec{d}\vec{u}S) = R \bullet S$

□

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Chapter 3

Paths Start with Down-step

Definition 3.0.1. *Define a function* f_2 *from* $\mathbb{D}_{n+1,k+1}$ *into* $\mathbb{D}_{n+1,k}$ *by the following:*

1. *The set* $\mathbb{D}_{n+1,k+1}$ *consists of all* $n+1$*-Dyck paths with* $k+1$ *flaws,* $0 \leq k \leq n$*. Each path in* $\mathbb{D}_{n+1,k+1}$ *can be factorized into* $AdBuC$*, where* A *is a subpath all above the x-xais,* d *is the first down step below the x-xais,* B *is a subpath all bellow the x-xais, say with* k_2 *flaws,* $0 \leq k_2 \leq k$*,* u *is the first up step contact the x-axis after* B *and below the x-axis, say* uBd *with* k_2+1 *flaws, and* C *is the remaining path with* $k-k_2$ *flaws,* $0 \leq k \leq n$*.*

2. *The set* $\mathbb{D}_{n+1,k}$ *consists of all* $n+1$*-Dyck path with* k *flaws,* $0 \leq k \leq n$*. Each path in* $\mathbb{D}_{n+1,k}$ *can be factorized into* $BuAdC$*, where* B*,*dAu*, and* C *have* k_2*,* 0*, and* $k-k_2$ *flaws.*

Note: A 、B 、C *may be empty, and they have the same number of up-steps and down-steps.*

i.e. $f_2(AdBuC) = BuAdC$

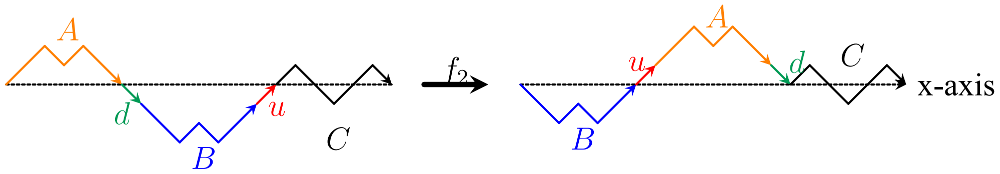

Figure 3.1: $f_2(AdBuC) = BuAdC$

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Theorem 3.0.1. *Define* f_2:$\mathbb{D}_{n+1,k+1} \rightarrow \mathbb{D}_{n+1,k}$ *by* $f_2(AdBuC) = BuAdC$.

The function f_2 *is one-to-one.*

Proof. Let $|B| = 2i$, $0 \le i \le n$; $|u| = 1$; $|A| = 2j$, $0 \le j \le n - i$; $|d| = 1$; $|C| = 2(n - i - j - 1)$.

Claim: f_2 one-to-one. (i.e.$f_2(AdBuC) = f_2(A'dB'uC') \Rightarrow BuAdC = B'uA'dC'$)

Suppose $|B'| = 2r$, $0 \le r \le n$; $|u| = 1$; $|A'| = 2s$, $0 \le s \le n-r$; $|d| = 1$; $|C'| = 2(n-r-s-1)$.

Claim 1: $|B| = |B'|$

case 1: $2i < 2r$

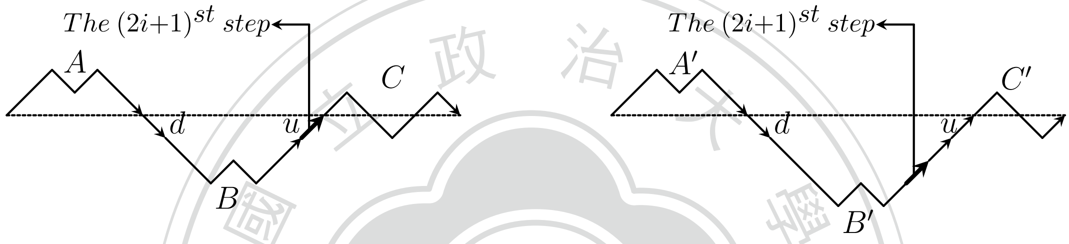

Figure 3.2: $AdBuC$ and $A'dB'uC'$

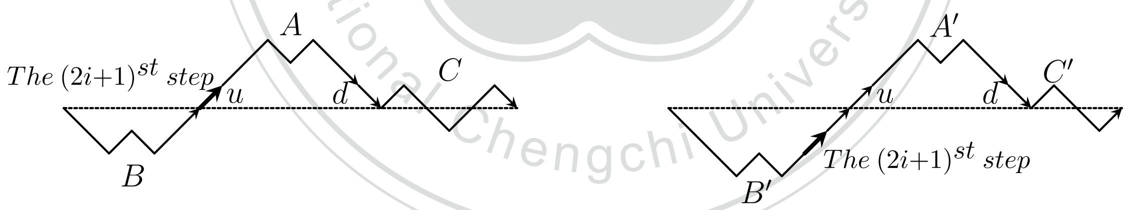

Figure 3.3: $f_2(AdBuC) = BuAdC$ and $f_2(A'dB'uC') = B'uA'dC'$

To see Figure 3.3 when we start on $(0, 0)$ to walk along the two path $BuAdC$ and $B'uA'dC'$. The $(2i + 1)^{st}$ step of $BuAdC$ is above the x-axis, but the $(2i + 1)^{st}$ step of $B'uA'dC'$ is still below the x-axis. This is a contradiction as two paths are the same.

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case 2: $2i > 2r$

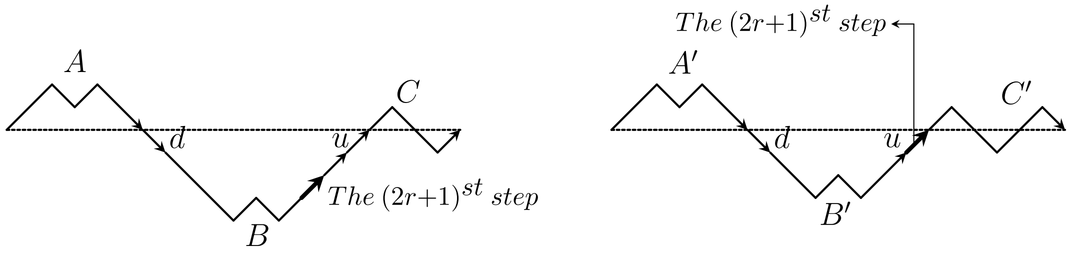

Figure 3.4: $AdBuC$ and $A'dB'uC'$

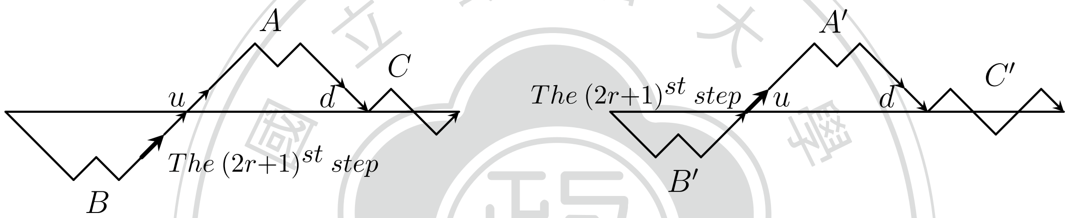

Figure 3.5: $f_2(AdBuC) = BuAdC$ and $f_2(A'dB'uC') = B'uA'dC'$

To see Figure 3.5 when we start on $(0, 0)$ to walk along the path the two path $BuAdC$ and $B'uA'dC'$. The $(2r+1)^{st}$ step of $BuAdC$ is below the x-axis, but the $(2r+1)^{st}$ step of $B'uA'dC'$ is still above the x-axis. This is a contradiction as two paths are the same. Thus, we have proof that $i = r$.

$\therefore |B| = |B'|$ and $B = B'$.

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Claim 2: $|A| = |A'|$

recall:Let $|B| = 2i$, $0 \leq i \leq n$; $|u| = 1$; $|A| = 2j$, $0 \leq j \leq n - i$; $|d| = 1$;

$|C| = 2(n - i - j - 1)$.

Suppose $|B'| = 2r$, $0 \leq r \leq n$; $|u| = 1$; $|A'| = 2s$, $0 \leq s \leq n - r$; $|d| = 1$;

$|C'| = 2(n - r - s - 1)$.

case 1: $2j < 2s$

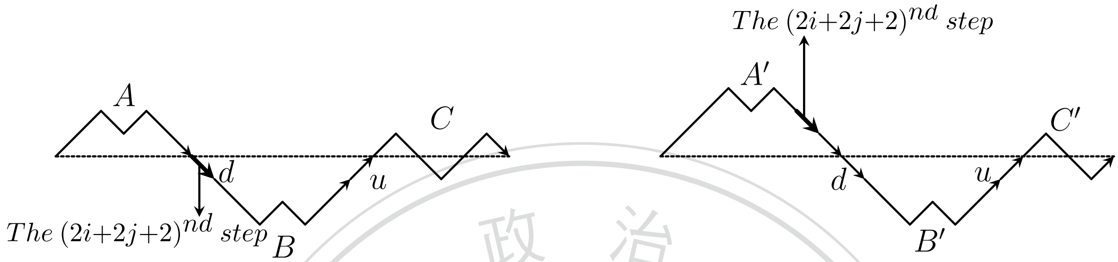

Figure 3.6: $AdBuC$ and $A'dB'uC'$

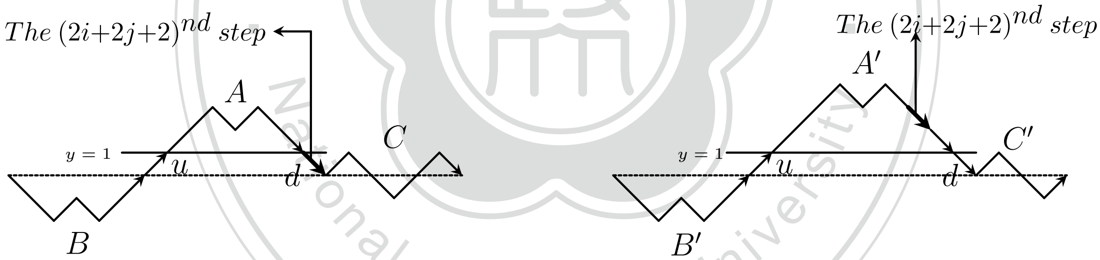

Figure 3.7: $f_2(AdBuC) = BuAdC$ and $f_2(A'dB'uC') = B'uA'dC'$

To see Figure 3.7 when we start on $(2j+1, 1)$ to walk along the path A and A'. The $(2i+2j+2)^{nd}$ step of Ad is below $y = 1$, but the $(2i+2j+2)^{nd}$ step of $A'd$ is still above $y = 1$. This is a contradiction as two paths are the same.

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case 2: $2j > 2s$

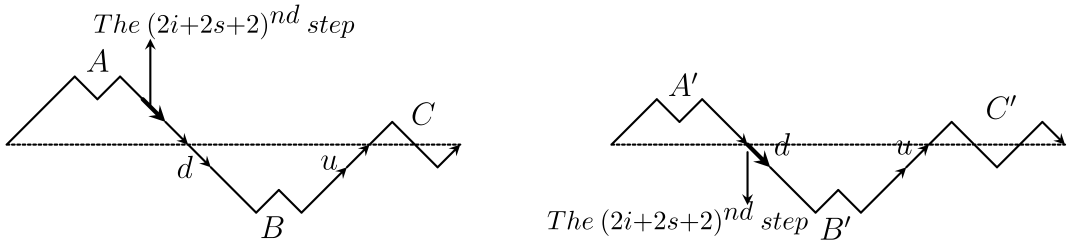

Figure 3.8: $AdBuC$ and $A'dB'uC'$

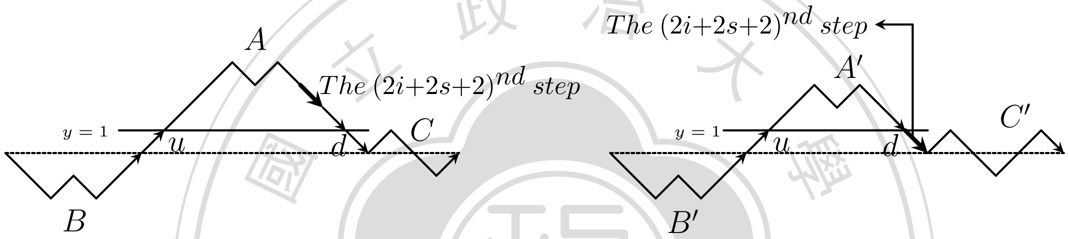

Figure 3.9: $f_2(AdBuC) = BuAdC$ and $f_2(A'dB'uC') = B'uA'dC'$

To see Figure 3.9 when we star on $(2i+1, 1)$ to walk along the path A and A'. The $(2i+2s+2)^{nd}$ step of Ad is above $y = 1$, but the $(2i+2s+2)^{nd}$ step of $A'd$ is still below $y = 1$. This is a contradiction as two paths are the same. Thus, we have proof that $j = s$.

$\therefore |A| = |A'|$ and $A = A'$.

Since $B = B'$ and $A = A'$,

$\because BuAdC = B'uA'dC' \Rightarrow C = C'$

$\therefore AdBuC = A'dB'uC'$

Therefore f_2 is one-to-one.

□

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Lemma 3.0.2. *If Q is a path in $\mathbb{D}_{n+1,k+1}$, then $f_2(Q)$ has k flaws. Moreover, the $k+1^{st}$ flaw will be restored and connected to original lift behind.*

Note: A up-step above the x-axis is called lift. The lifts are counted from right to left and top to bottom.

Proof. The function f_2 from $\mathbb{D}_{n+1,k+1}$ into $\mathbb{D}_{n+1,k}$. Suppose P has $k+1$ flaws in $\mathbb{D}_{n+1,k+1}$, and $Q = AdBuC$, where A, dBu, C have 0 flaw, $k+1-j$ flaws, j flaws, respectively. (i.e.A, dBu, C have $n-k-i$ lifts, 0 lift, i lifts, respectively.) The $(k+1)^{th}$ flaw is in subpath dBu.Then $f_2(Q)$ has k flaws on $\mathbb{D}_{n+1,k}$, and $f_2(Q) = BuAdC$, where B, uAb, C have $k-j$ flaws, 0 flaw, j flaws, respectively.

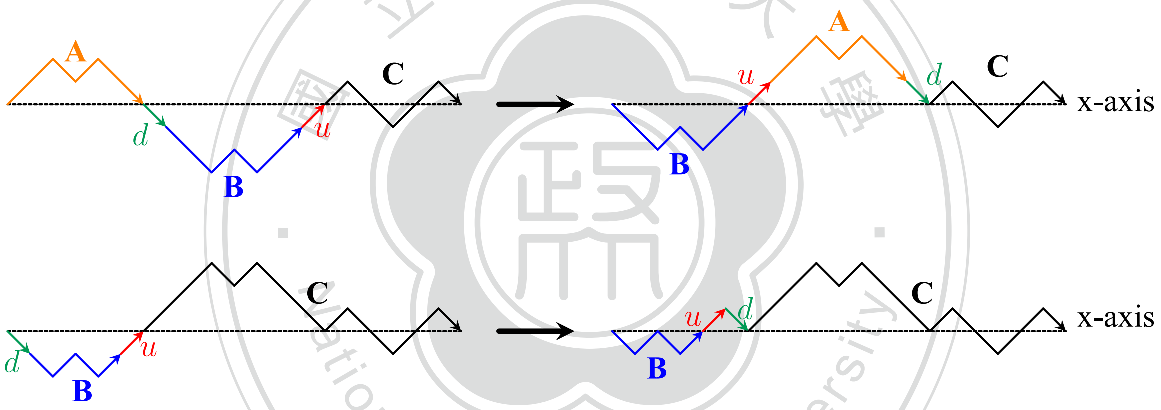

Figure 3.10: *the new lift will be connected to original lift behind*

Notice that the path $f_2(Q)$, u is connected to the $(n-k)^{th}$ flaw in subpath A behind. u becomes the $(n-k+1)^{st}$ lift.

Therefore, no matter how many times we use f_2, u is conneted to the $(n-k+1)^{st}$ lift in subpath A behind. u is still the $(n-k+1)^{st}$ lift.

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Note:

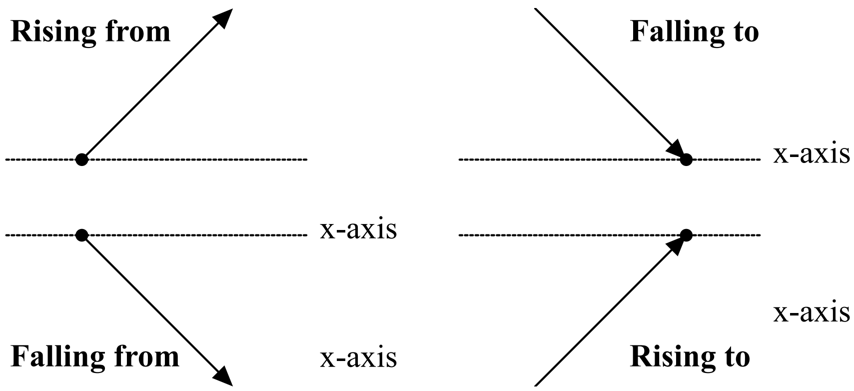

Figure 3.11: $Rising\,from$; $Falling\,to$; $Falling\,from$; $Rising\,to$

Lemma 3.0.3. *If P is a path in $\mathbb{D}_{n+1,k}$, then $f_2^{-1}(P)$ has $k+1$ flaws.Moreover, the $(n-k+1)^{st}$ lift in P will be dropped.*

i.e. The $(n-k)^{th}$ lift u and the first down-step d falling to the x-axis on the right side of u in $\mathbb{D}_{n+1,k}$ are dropped to the first up-step u rising to the x-axis and the first down-step d falling from the x-axis in $\mathbb{D}_{n+1,k+1}$.

Show by formula:

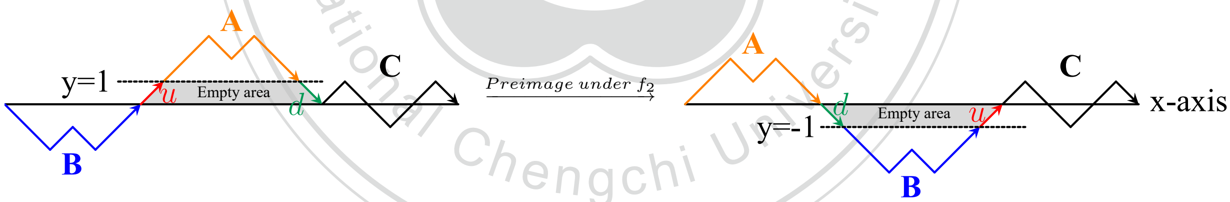

Figure 3.12: $BuAdC \xrightarrow{Preimage\ under\ f_2} AdBuC$

In P, the $(n-k+1)^{st}$ lift u and the first down-step d falling to the x-axis on the left side of u in $\mathbb{D}_{n+1,k}$, we can observe that there is an empty area enclosed by the u, d, the x-axis, and the horizontal line $y=1$. After switching two portions Bu and Ad, another empty area is enclosed by the d is the first down-step falling from the x-axis, and u is the first up-step rising to the x-axis, and the horizontal line $y=-1$ in $f_2^{-1}(P)$. And the remaining segments which are behind d is the fixed subpath C. □

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Theorem 3.0.2. *Define* f_2*:*$\mathbb{D}_{n+1,k+1}$*→*$\mathbb{D}_{n+1,k}$ *by* $f_2(AdBuC) = BuAdC$*.*

The function f_2 *is onto.*

Proof. **Claim:** f_2 is onto. (f_2:$\mathbb{D}_{n+1,k+1} \to \mathbb{D}_{n+1,k}$)

For any path $P = BuAdC$ in $\mathbb{D}_{n+1,k}$ which $0 \leq k \leq n$. We choose the $(n-k+1)^{st}$ lift u and choose the first down-step d falling to the x-axis on the right side of u. We switch the portions Bu and Ad then we can drop the $(n-k+1)^{st}$ lift (i.e. the $(k+1)^{st}$ flaw) and get a new path $Q = AdBuC$ in $\mathbb{D}_{n+1,k+1}$, by Lemma 3.2、Lemma 3.3. Where Q has at most $n+1$ flaws and P has at least one flaw. In fact, if P has k flaws then Q has $k+1$ flaws. So every P in $\mathbb{D}_{n+1,k}$, we can find a path Q in $\mathbb{D}_{n+1,k+1}$, such that $f_2(Q) = P$. Therefore f_2 is one-to one and onto.

Hence, f_2 is one-to-one and onto by Theorem 3.1 and Theorem 3.2.

Note: Let f_2^{-1} be the inverse function of f_2. □

Definition 3.0.4. *The set* X_2 *consists of all paths in* $n+1$*-Dyck paths with* $k+1$ *flaws.Each path in* X_2 *can be factorized into* $A\overrightarrow{d}B\overrightarrow{u}C$*. The set* Y_2 *consists of all* $n+1$*-Dyck paths which are good paths.*

Define a function g_2 *from* X_2 *into* Y_2 *by the following:*

1. $g_2(Q) = f_2^{k+1}(Q)$ *where* Q *in* X_2*, and* $f_2^{k+1} = \underbrace{f_2 \circ f_2 \circ \cdots \circ f_2}_{k+1\ times}$

2. *The first down-step falling from x-axis denote by* $\overrightarrow{d}$ *and the first up-step rising to x-axis denote by* $\overrightarrow{u}$*.*

Lemma 3.0.5. *Suppose that* Q *in* X_2*. In* $f_2(Q)$*, the first down-step* $\overrightarrow{d}$ *falling from the x-axis of* Q *connects with the first up-step* $\overrightarrow{u}$ *rising to the x-axis of* Q *to be* $\overrightarrow{u}\overrightarrow{d}$ *and* $\overrightarrow{u}\overrightarrow{d}$ *is above the x-axis. Let* $Q = \overrightarrow{d}B\overrightarrow{u}C$*, then* $f_2(Q) = B\overrightarrow{u}\overrightarrow{d}C$*.*

Proof. We may assume that $Q = A\overrightarrow{d}B\overrightarrow{u}C$ is any path of $\mathbb{D}_{n+1,k+1}$. Since the subpath A is empty, then $Q = \overrightarrow{d}B\overrightarrow{u}C \in X_2$ and $f_2(Q) = B\overrightarrow{u}\overrightarrow{d}C$.We know $\overrightarrow{d}B\overrightarrow{u}$ has $k+1-j$ flaws, and C must have j flaws in Q. Then B, $\overrightarrow{u}\overrightarrow{d}$, and C have $k-j$ flaws, 0 flaw, and j flaws in $f_2(Q)$, respectively. Therefore, $\overrightarrow{u}\overrightarrow{d}$ is above the x-axis.

□

Note: In $f_2^i(Q)$, $1 \leq i \leq K+1$, the segments above the x-axis will be always above the x-axis.

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Theorem 3.0.3. *The function* g_2*:*$X_2 \rightarrow Y_2$ *is one-to-one.*

Proof. Suppose that $g_2(P) = g_2(Q)$, where P has k flaws and Q has h flaws in X_2.

To prove that $P = Q$.

By definition $g_2(P) = f_2^{k+1}(P)$, $g_2(Q) = f_2^{h+1}(Q)$

case 1: $k < h$

$f_2^{k+1}(P) = f_2^{h+1}(Q) \Rightarrow f_2(f_2^{k}(P)) = f_2(f_2^{h}(Q))$

$\because f_2$ is one-to-one $\quad \therefore f_2^{k}(P) = f_2^{h}(Q)$

For the same reason, the following can be obtained

$f_2(f_2^{k-1}(P)) = f_2(f_2^{h-1}(Q)) \Rightarrow f_2^{k-1}(P) = f_2^{h-1}(Q)$

Since f_2 is one-to-one, use this method $k+1$ times

$f_2(P) = f_2(f_2^{h-k}(Q)) \Rightarrow P = f_2^{h-k}(Q)$

The $\overrightarrow{d}$ and $\overrightarrow{u}$ of P are below the x-axis, but the $\overrightarrow{u}\,\overrightarrow{d}$ of $f_2^{h-k}(Q)$ are above the x-axis by Lemma 3.5.

This is a contradiction.

case 2: $k > h$

$f_2^{k+1}(P) = f_2^{h+1}(Q) \Rightarrow f_2(f_2^{k}(P)) = f_2(f_2^{h}(Q))$

$\because f_1$ is one-to-one $\quad \therefore f_2^{k}(P) = f_2^{h}(Q)$

For the same reason, the following can be obtained

$f_2(f_2^{k-1}(P)) = f_2(f_2^{h-1}(Q)) \Rightarrow f_2^{k-1}(P) = f_2^{h-1}(Q)$

Since f_2 is one-to-one, use this method $h+1$ times

$f_2(f_2^{k-h}(P)) = f_2(Q) \Rightarrow f_2^{k-h}(P) = Q$

The $\overrightarrow{u}\,\overrightarrow{d}$ of $f_2^{k-h}(P)$ are above the x-axis, but $\overrightarrow{d}$ and $\overrightarrow{u}$ of Q are below the x-axis by Lemma 3.5.

This is a contradiction.

case3: $k = h$

$f_2^{k+1}(P) = f_2^{h+1}(Q) \Rightarrow f_2(f_2^{k}(P)) = f_2(f_2^{h}(Q))$

$\because f_2$ is one-to-one $\quad \therefore f_2^{k}(P) = f_2^{h}(Q)$

Use this method $h+1$ times, we have $f_2(P) = f_2(Q) \Rightarrow P = Q$

Therefore, g_2 is one-to-one.

□

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Theorem 3.0.4. *g_2:X_2→Y_2 is onto, where Y_2 is good path of $n+1$-Dyck and $g_2(Q) = f_2^{k+1}(Q)$.*

Proof. Give $P \in Y_2$, P has 0 flaw, and $\overrightarrow{u}\overrightarrow{d}$ is the $(n-k+1)^{st}$ lift in P.

Definition:The preimage of the function g_2= $f_2^{-(k+1)} = \underbrace{f_2^{-1} \circ f_2^{-1} \circ \cdots \circ f_2^{-1}}_{k+1\ times}$

Since $f_2^{-1}(P)$ is the preimage of P under f_2 and has 1 flaw.

Using this way for k times, we get the path $f_2^{-k}(P)$ which has k flaws.

In $f_2^{-k}(P)$, $\overrightarrow{u}\overrightarrow{d}$ is still above the x-axis and is the $(n-k+1)^{st}$ lift.We use this way again, we get the path $f_2^{-(k+1)}(p)$ which has $k+1$ flaws and $\overrightarrow{d}$B$\overrightarrow{u}$ is under the x-axis, as f_2 is onto and by Lemma 3.5.

So we have $f_2^{-(k+1)}(P) = f_2^{-(k+1)}(f_2^{k+1}(Q)) = Q$,

where Q has $k+1$ flaws,$Q \in X_2$.

Let $Q = f_2^{-(k+1)}(P) \Rightarrow g_2(Q) = f_2^{k+1}(Q)$

$$= f_2^{k+1}((f_2^{-(k+1)})(P))$$

$$= P$$

Thus g_2 is onto.

□

Hence, g_2 is one-to-one and onto by Theorem 3.3 and Theorem 3.4.

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Definition 3.0.6. *The set Z_2 contains the good path for all n-Dyck paths which replaces $\overrightarrow{u}\overrightarrow{d}$ in Y_2 with a dot mark, and all paths in Y_2 are $n+1$-Dyck paths which are good path. Let h_2 be the function from Y_2 into Z_2.*

i.e. $P = R\overrightarrow{u}\overrightarrow{d}S$ is $(2n+2, 0)$ path, where R, $\overrightarrow{u}\overrightarrow{d}$, and S are all good paths. $h_2(P) = h_2(R\overrightarrow{u}\overrightarrow{d}S) = R \bullet S$

Theorem 3.0.5. *h_2 is one-to-one and onto.*

Proof. It's clearly that h_2 is one-to-one.

Given $P' = R \bullet S \in Z_2$

We can change $\bullet$ into $\overrightarrow{u}\overrightarrow{d}$. Thus $R \bullet S \Rightarrow R\overrightarrow{u}\overrightarrow{d}S \in Y_2$.

Therefore, h_2 is one-to-one and onto.

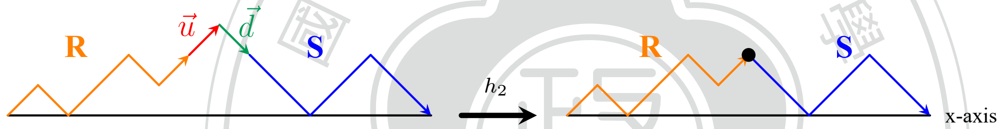

Figure 3.13: $h_2(R\vec{u}\vec{d}S) = R \bullet S$

□

We have completed the combinatorial proof of $(n+2)C_{n+1} = (4n+2)C_n$.

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Chapter 4

Summary

- In this thesis, we prove the Catalan identity in combinatorial way.

  In Chapter 2, we give a bijective proof between $``Paths\ Start\ with\ Up-step''$ and $``Dotted\,Totally\ Bad\ Paths''$. Then we construct the functions in $X_1 \xrightarrow{g_1} Y_1 \xrightarrow{h_1} Z_1$ that Paths Start with Up-step.

  In Chapter 3, we give a bijective proof between $``Paths\ Start\ with\ Down-step''$ and $``Dotted\,Good\ Paths''$.Then we construct the functions in $X_2 \xrightarrow{g_2} Y_2 \xrightarrow{h_2} Z_2$ that Paths Start with Down-step.

  Proving the Catalan identity through the Dyck paths can reveal the following advantages:

  1.The subpath C does not change during the process of switching of the portions Ad and Bu.

  2.Since the subpath B of P in x_1 is empty, a new flaw generated after switching of the portions Ad and Bu must be followed by the original subpath C. Since the subpath A of Q in x_2 is empty, a new lift generated after switching of the portions Bu and Ad must be followed by the original subpath C.

  3.In the process of computing the preimage of a function g_1 (g_2), the flaws (lifts) recovery mode follows the $``Last-in-First-out''$ or $``First-in-Last-out''$.

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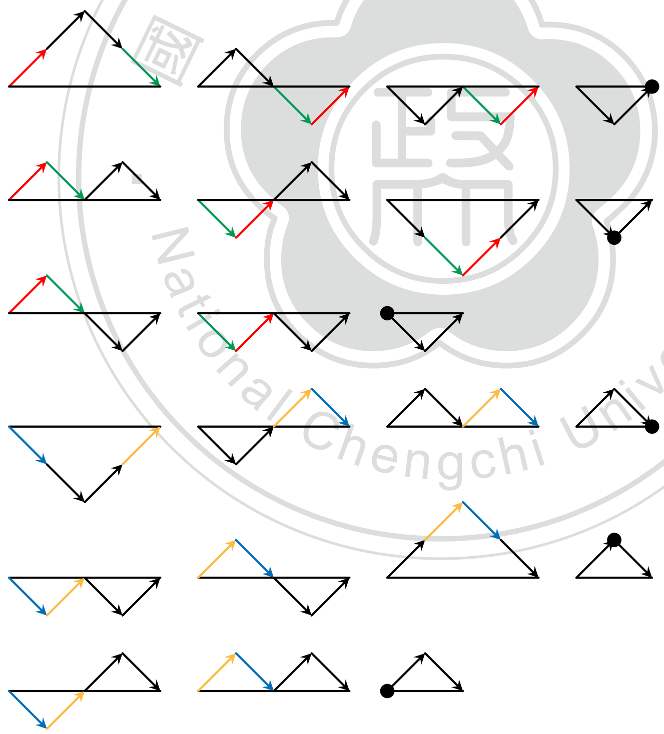
Appendix A

examples of Catalan identity

A.1 $(n+2)C_{n+1} = (4n+2)C_n$

- $n = 1$



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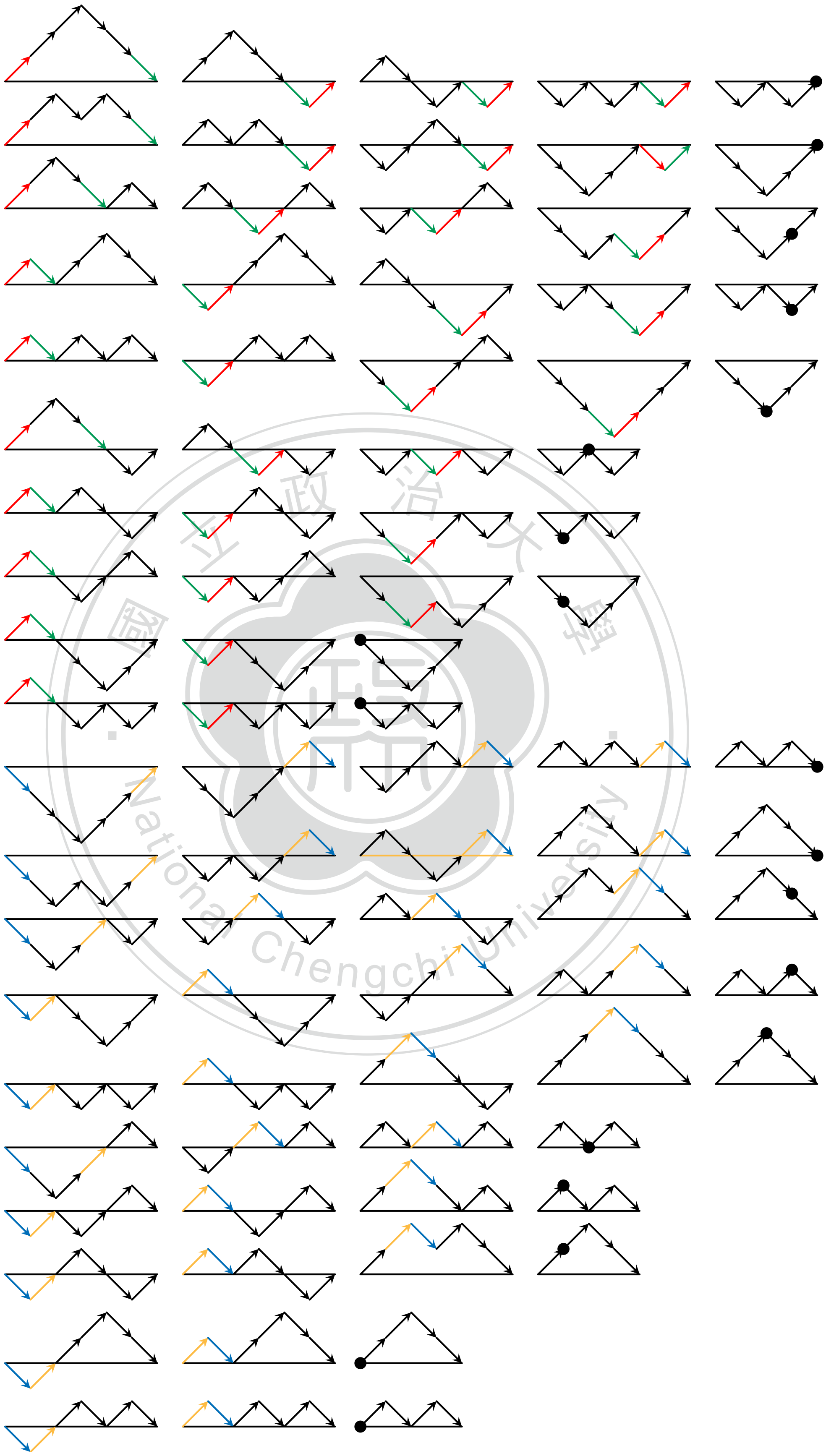
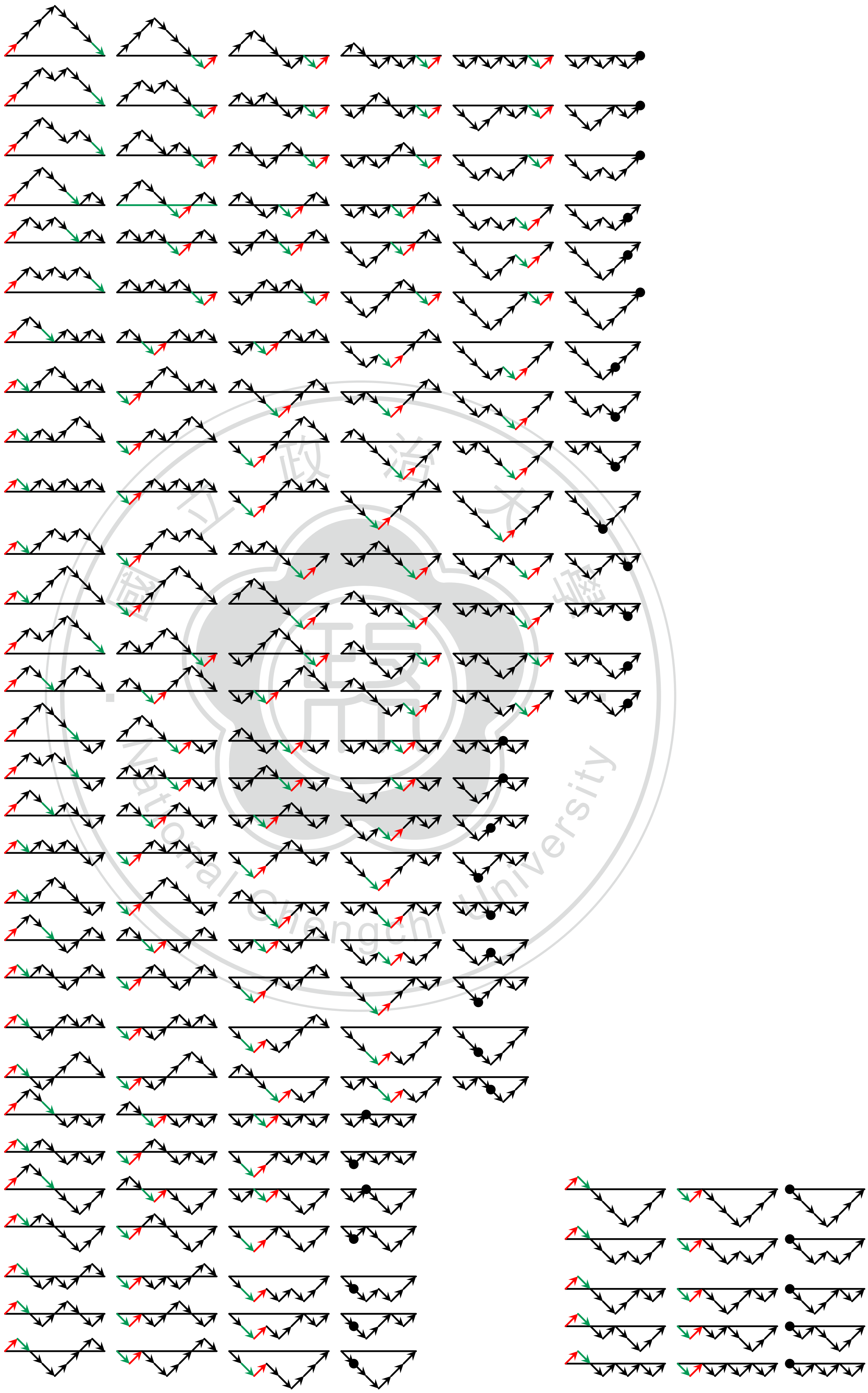
- $n = 2$

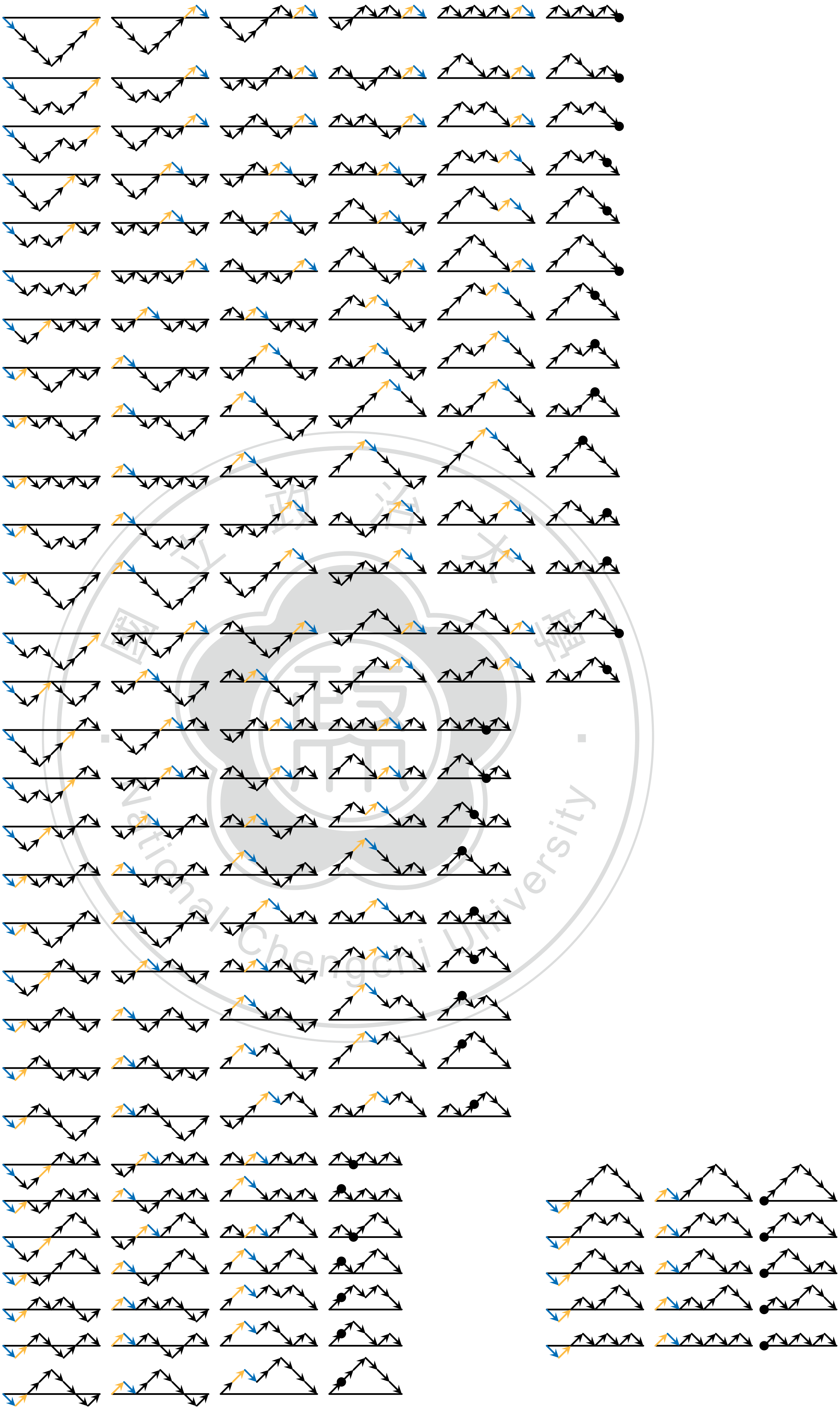


DOI:10.6814/NCCU202000719

- $n = 3$



DOI:10.6814/NCCU202000719



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