國立政治大學應用數學系

碩士學位論文

在高維度下受波氏分配自我相斥隨機漫步的均場行為
Mean-field behavior for self-avoiding walks with Poisson interactions in high dimensions

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中 華 民 國 109 年 5 月

致謝

　　首先要感謝我的指導教授「陳隆奇」教授，在我碩一的時候就直接收我作為學生，並在很相信我可以做得動 self-avoiding walk 這個主題，雖然我做的只是佔了整個理論的一小部分，但還是令人覺得頭痛，學了很久，寫了很久，慢慢地終於也到了寫完的這一刻，再一次地感謝老師每一次的提點與信任，都讓我充滿了信心來寫這篇論文。

　　再來是要感謝推薦我去找陳隆奇教授的教官「游森棚」教授，在大學時期，因為教官的教學，讓我對數學產生了高度的興趣，當初要念研究所的時候，教官說了一句：「你一定會喜歡陳隆奇老師的研究的。」因為信任教官的說詞，不假思索地投注在機率的語言裡，就以結果而言，喜歡是喜歡，但是知識真的不簡單，雖說如此我還是努力地學會了一些東西，在這樣的過程當中，其實我成長許多，為此也非常感謝教官的推薦，今天沒有教官，就沒有成長過後的我了。

　　接著是感謝我的實變老師「陳天進」教授，一個喜歡學生叫老大的老師，在修習實變的第一堂課就說：「跟不上不是我的錯，是你們要跟上我，不是我要放慢速度。」難度很高加上速度很快，讓我繃緊神經每天讀書，時不時就去敲老大門，最終克服困難，高分過關。

　　最後是感謝這三年來一直陪伴著我成長的家人、朋友，因為有你們我才能走到現在，完成屬於我的第一篇著作。

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中文摘要

self-avoiding walk 是線性聚合物的模型。它是機率和統計力學中一個重要而有趣的模型。一些重要問題已經解決 (c.f. [5]). 然而，許多重要問題仍未解決，特別是涉及關鍵指數的問題，尤其是遠程模型的關鍵指數。在本文中，我們獲得了對於一個特殊的長域模型，其單步分佈是波松分佈的特殊敏感度模型，其敏感性指數滿足均值場行為，且其值大於上臨界值（$d_c = 4$）。參數 $\lambda \geq \lambda_d$ 的類型分佈，其中 λ_d 取決於維度。為此，我們選擇一組特殊的 bootstrapping functions，它們類似於 [4]，並使用 lace expansion 分析有關 bootstrapping functions 的複雜部分。此外，對於 $d > 4$，我們得到 λ_d 的確切值。

Abstract

Self-avoiding walk is a model for linear polymers. It is an important and interesting model in Probability and Statistical mechanics. Some of the important problems had been solved (c.f. [5]). However, many of the important problems remain unsolved, particularly those involving critical exponents, especially the critical exponents for long-range models. In this thesis, we see Lace expansion to obtain that the critical exponent of the susceptibility satisfies the mean-field behavior with the dimensions above the upper critical dimension ($d_c = 4$) for a special loge-range model in which each one-step distribution is the Poisson-type distribution with parameter $\lambda \geq \lambda_d$ where λ_d depends on the dimensions. To achieve this, we choose a particular set of bootstrapping functions which is similar as [4] and using a notoriously complicated part of the lace expansion analysis. Moreover we get the exactly value of λ_d for $d > 4$.

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Chapter 1

Introduction

A self-avoiding walk (SAW) is a model for linear polymers. In mathematics, a SAW is a sequence of moves on lattices (a lattice path) that does not visit the same point more than once. This is a special case of the theoretical graphic notion of a path. Very little is known rigorously about the SAW from a mathematical perspective, although physicists have provided numerous conjectures that are believed to be true and are strongly supported by numerical simulations.

In higher dimensions, the SAW is believed to behave much like the corresponding random walk. This behavior is called the mean-field behavior. The dimension d_c is called the upper critical dimension if the behavior is the mean-field behavior for dimensions $d > d_c$. It's well known that $d_c = 4$ for a finite-range SAW model (c.f. [4]) and $d_c = 2(\alpha \wedge 2)$ for a long-range model in which one-step distribution decays as $|x|^{-d-\alpha}$ for some $\alpha > 0$ (c.f. [7]) . The lace expansion is a powerful tool for analyzing the critical behavior for SAW on $\mathbb{Z}^d$ for $d > d_c$. The idea of the lace expansion was initiated by Brydges and Spencer for investigating weakly SAW for $d > 4$. Later, the lace-expansion was applied to various stochastic-geometrical models, such as SAW for $d > 4$, (c.f. [5]) , lattice trees/animals for $d > 8$, (c.f. [9]), percolation for $d > 6$, (c.f. [6]), oriented percolation for $d > 4$, (c.f. [13]), contact process for $d > 4$, (c.f. [11]), and Ising model for $d > 4$, (c.f. [10]). In this thesis, we consider the mean-field behavior for long-range SAW in which one-step distribution $D(x)$ is symmetric Poisson-type distribution that will be defined in the next chapter.

The rest of this thesis is organized as follows. In Chapter 2, we define the symmetry Poisson-type distribution and state the main results of this thesis and their proofs under some key propositions (Proposition 2.2.7-2.2.9). In Chapter 3, we follow the same argument of [3] to

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introduce the lace expansion that is the most important ingredient to prove the key propositions, and then we estimate the upper bound of the diagrams on lace expansion coefficients in Chapter 4. In Chapter 5, we evaluate the upper bounds of some random-walk quantities that are useful to estimate the upper bound of diagrams. Finally, in Chapter 6, we can use the previous analysis (Chapter 4-5) to prove the key propositions.

Chapter 2

Models and Main Results

First, we provide precise definitions of the long-range Poisson-type self-avoiding walk on $\mathbb{Z}^d$, whose translation-invariant 1-step distribution decays as the symmetric Poisson distribution. Then, we present the main results and explain their proofs according some key propositions.

2.1 Notations and Definitions

By Taylor's formula, we have the following equation

$$2^d = \prod_{j=1}^{d} \left[e^{-\lambda} \Big(\sum_{x_j=1}^{\infty} \frac{\lambda^{x_j-1}}{(x_j-1)!} \Big) + e^{-\lambda} \Big(\sum_{-x_j=1}^{\infty} \frac{\lambda^{-x_j-1}}{(-x_j-1)!} \Big) \right].$$

Then, we can define a symmetry Poisson distribution $D(x)$ on $\mathbb{Z}^d$ with parameter $\lambda > 0$ as follows. Since

$$
\begin{aligned}
1 \quad &= \quad \frac{e^{-\lambda d}}{2^d} \prod_{j=1}^{d} \Big(\sum_{x_j \neq 0} \frac{\lambda^{|x_j|-1}}{(|x_j|-1)!} \Big) \\
&= \quad \sum_{\substack{x=(x_1,x_2,\dots,x_d) \in \mathbb{Z}^d \\ x_k \neq 0, \forall k=1,2,\dots,d}} \Big(\prod_{j=1}^{d} \frac{e^{-\lambda}}{2} \frac{\lambda^{|x_j|-1}}{(|x_j|-1)!} \Big),
\end{aligned}
$$

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let

$$
D(x) = D(x_1, ..., x_d) = \begin{cases} \prod_{j=1}^{d} \frac{e^{-\lambda}}{2} \frac{\lambda^{|x_j|-1}}{(|x_j|-1)!}, & \text{if } x_1 x_2 \cdots x_d \neq 0, \\\\ 0, & \text{if } x_1 x_2 \cdots x_d = 0. \end{cases} \tag{2.1.1}
$$

In the rest of the thesis, we denote $\hat{f}(k) \equiv \sum_{x \in \mathbb{Z}^d} f(x) e^{ik \cdot x}$ where $k = (k_1, k_2, ..., k_d)$ and $(f * g)(x) \equiv \sum_{y \in \mathbb{Z}^d} f(y) g(x-y)$ the Fourier transform of function f on $\mathbb{Z}^d$ and the convolution of two function f and g on $\mathbb{Z}^d$, respectively.

By (2.1.1), we get

$$
\begin{aligned}
\hat{D}(k) &= \sum_{x \in \mathbb{Z}^d} D(x) e^{ik \cdot x} \\
&= \frac{e^{-\lambda d}}{2^d} \prod_{j=1}^{d} \left(\left(\sum_{x_j=1}^{\infty} \frac{\lambda^{x_j-1}}{(x_j-1)!} e^{ik_j x_j} \right) + \left(\sum_{-x_j=1}^{\infty} \frac{\lambda^{-x_j-1}}{(-x_j-1)!} e^{ik_j x_j} \right) \right) \\
&= \frac{e^{-\lambda d}}{2^d} \prod_{j=1}^{d} \left(\left(\sum_{u=0}^{\infty} \frac{\lambda^u}{u!} e^{ik_j(u+1)} \right) + \left(\sum_{u=0}^{\infty} \frac{\lambda^u}{u!} e^{ik_j(-u-1)} \right) \right) \\
&= \prod_{j=1}^{d} \frac{e^{-\lambda}}{2} \left(e^{\lambda e^{ik_j}} (\cos(k_j) + i\sin(k_j)) + e^{\lambda e^{-ik_j}} (\cos(k_j) - i\sin(k_j)) \right) \\
&= \prod_{j=1}^{d} e^{-\lambda + \lambda \cos(k_j)} \left(\cos(k_j) \cos(\lambda \sin(k_j)) - \sin(k_j) \sin(\lambda \sin(k_j)) \right) \\
&= \prod_{j=1}^{d} e^{-\lambda + \lambda \cos(k_j)} \cos\left(k_j + \lambda \sin(k_j) \right).
\end{aligned}
$$

Hence

$$
\hat{D}(k) = \prod_{j=1}^{d} e^{-\lambda + \lambda \cos(k_j)} \cos\left(k_j + \lambda \sin(k_j) \right).
$$

Let $\mathcal{W}_n(x, y)$ be the set of $\{w_0, w_1, ..., w_n\}$ with $w_0 = x$ and $w_n = y$, and δ be the Kronecker delta function. Then the two-point function of the random walk (RW) from the origin o to x is denoted by $\varphi_0^{RW}(x) \equiv \delta_{0,x}$ and, for $n \geq 1$,

$$
\varphi_n^{RW}(x) \equiv \sum_{w \in \mathcal{W}_n(o,x)} \prod_{i=1}^{n} D(w_i - w_{i-1}) = D^{*n}(x),
$$

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where D^{*n} is the n time convolution. Similarly, we define the two-point function of SAW from the origin o to x by $\varphi_0(x) \equiv \delta_{0,x}$ and, for $n \geq 1$,

$$\varphi_n(x) \equiv \sum_{w \in \mathcal{W}_n(o,x)} \prod_{i=1}^{n} D(w_i - w_{i-1}) \prod_{0 \leq i < j \leq n} (1 - \delta_{w_i, w_j}).$$

Since the green function of RW is defined as follows

$$S_p(x) \equiv \sum_{n=0}^{\infty} \varphi_n^{RW}(x) p^n = \delta_{0,x} + \sum_{n=1}^{\infty} D^{*n}(x) p^n, \quad p \in (0,1).$$

We can define the time-independent two-point function of SAW as follows

$$G_p(x) \equiv \sum_{n=0}^{\infty} \varphi_n(x) p^n, \quad p \in (0, p_c),$$

where p_c is the radius of convergence. Moreover, the susceptibility of SAW is defined as follows

$$\chi_p \equiv \sum_{x \in Z^d} G_p(x), \quad p \in (0, p_c). \tag{2.1.2}$$

In Physics, there is a conjecture about the susceptibility of SAW as follows,

$$\chi_p \asymp (p_c - p)^{-r},$$

where $f(p) \asymp g(p)$ means that $\frac{f(p)}{g(p)}$ is bounded away from zero and infinity, and r is a critical exponent that is not statistically distinguishable from a SAW instead on the integer lattice, on the hexagonal lattice, the triangular lattice, or indeed on any one of a wide variety of d-dimensional lattices. This feature is called universality. Given any dimension $d > 4$, $r = 1$ was proved by Slade (c.f. [6]) for spread-out model with sufficiently large L, and was proved by Hara and Slade (c.f. [5]) for nearest-neighbor model. In a general finite-range model it is predict that $r = \frac{43}{32}$ for $d = 2$, $r = 1.162$ for $d = 3$, $r = 1$ with logarithmic term for $d = 4$, and $r = 1$ for $d \geq 5$. However, up to now, there is no rigorous mathematical proof. For the long-range model which the one-step distribution decays as $|x|^{-d-\alpha}$ for some $\alpha > 0$, given any dimension $d > 2(\alpha \wedge 2)$, $r = 1$ was proved by Heydenreich (c.f. [7]). In particular, for $\alpha = 2$ and $d = 4$, $r = 1$ with logarithmic term was proved by Chen and Sakai (c.f. [3]). For more background and related

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results, we refer to [8] and [1]. It is interesting to get r for other long-range models. In our thesis, we discuss a special long-range model.

2.2 Main results and their proofs

Now, we present the main theorems in this thesis.

Theorem 2.2.1. *(Infrared bound) For SAW on $\mathbb{Z}^d$ whose one step distribution $D(x)$ is defined in (2.1.1) for any $d > 4$, there is a λ_d,*

$$
\lambda_d = \begin{cases}
60, & \text{if } d = 5, 6, \\
35, & \text{if } d = 7, \\
24, & \text{if } d = 8, \\
18, & \text{if } d = 9, \\
\frac{2e}{3}(1 + \frac{3}{d-4})e^{\frac{1}{d-1}[ln(3125000)-1+ln(d(d-4)^2)]}, & \text{if } d \geq 10,
\end{cases}
$$

such that for all $\lambda \geq \lambda_d$,

$$
||(1 - \hat{D})\hat{G}_p||_\infty \equiv \sup_{k \in \mathbb{Z}^d}(1 - \hat{D}(k))|\hat{G}_p(k)| \leq 3,
$$

uniformly in $p \in [1, p_c)$.

Remark 2.2.2. *According Theorem 2.2.1, we obtain that $\lambda_d \to \frac{2e}{3}$ as d goes to infinity. However, for $0 < \lambda < \frac{2e}{3}$, we don't obtain what we want.*

Remark 2.2.3. *The λ_d of Theorem 2.2.1 is not the best estimate for our Poisson-type.*

Define a bubble function $B_p = ||(pD)^{*2} * G_p^{*2}||_\infty$. By Theorem 2.2.1, we can estimate χ_p that is defined in (2.1.2) as follows.

Proposition 2.2.4. *Under the same assumption of Theorem 2.2.1, we have*

$$
\chi_p^2 \geq \frac{d}{dp}(p\chi_p) \geq \frac{\chi_p^2}{1 + B_p} \text{ and } B_{p_c} < \infty. \tag{2.2.1}
$$

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We will use the same argument of [7] to prove this proposition in Chapter 5. By Proposition 2.2.4, we get the second main result of this thesis directly.

Theorem 2.2.5. *Under the same assumption of Theorem 2.2.1, we have*

$$\frac{p_c}{p_c - p} \leq \chi_p \leq \frac{p_c(1 + B_{p_c})}{p_c - p}. \tag{2.2.2}$$

By Theorem 2.2.5, we obtain the third main result in our thesis obviously.

Corollary 2.2.6. *Under the same assumption of Theorem 2.2.1, we have*

$$\chi_p \asymp (p_c - p)^{-1}.$$

In Corollary 2.2.6, we get the ciritical exponent $r = 1$ in our model. We use the Proposition 2.2.4 and Theorem 2.2.1 to prove Theorem 2.2.5 as follow.

Proof of Theorem 2.2.5 according to the Proposition 2.2.4 and Theorem 2.2.1. Since B_p is increasing as p is increasing and from Proposition 2.2.4, we get $B_{p_c} < \infty$, by the first inequality of (2.2.1), we have

$$dp \geq \frac{1}{\chi_p^2}d(p\chi_p) \geq \frac{1}{1 + B_p}dp \geq \frac{1}{1 + B_{p_c}}dp, \text{ for } p \in (0, p_c].$$

Then,

$$\int_p^{p_c} \frac{1}{t^2}dt \geq \int_p^{p_c} \frac{1}{t^2\chi_t^2}d(t\chi_t) \geq \int_p^{p_c} \frac{1}{t^2(1 + B_{p_c})}dt.$$

This implies

$$\frac{p_c}{p_c - p} \leq \chi_p \leq \frac{p_c(1 + B_{p_c})}{p_c - p}.$$

$\square$

The proof of Theorem 2.2.1 is rather straightforward, assuming the following three propositions. To state those propositions, we first define

$$g_1(p) \equiv p, \text{ and } g_2(p) \equiv ||(1 - \hat{D})\hat{G}_p||_\infty, \tag{2.2.3}$$

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where the supremum near $k = 0$ should be interpreted as the supremum over the limit as $|k| \to 0$, and

$$g_3(p) \equiv \sup_{k,l} \frac{1}{\hat{U}(k,l)} \times |\hat{\Delta}_k \hat{G}_p(l)|, \tag{2.2.4}$$

where

$$\hat{U}(k,l) \equiv \big(1 - \hat{D}(k)\big)\bigg(\frac{\hat{S}_1(l+k) + \hat{S}_1(l-k)}{2}\hat{S}_1(l) + 4\hat{S}_1(l+k)\hat{S}_1(l-k)\bigg),$$

and the notation $\hat{\Delta}_k \hat{f}$ is defined as follows

$$\hat{\Delta}_k \hat{f}(l) = \frac{\hat{f}(l+k) + \hat{f}(l-k)}{2} - \hat{f}(l).$$

Now, we state the three aforementioned propositions and show that they indeed imply Theorem 2.2.1. To prove these three propositions, we need to use lace expansion and diagram estimates that will be explained in Chapter 3. Then we will show these three propositions in Chapter 6.

Proposition 2.2.7. *(Continuity). The functions $\{g_i(p)\}_{i=1}^3$ are continuous in $p \in [1, p_c)$.*

Let $K_1 = 1.01$, $K_2 = 1.02$, $K_3 = 2.1$ for $d = 5$, and $K_1 = 1.1$, $K_2 = 1.1$, $K_3 = 2.1$ for $d \geq 6$. Then, we have the following Propositions.

Proposition 2.2.8. *(Initial conditions). For $d > 4$, we have $g_i(1) < K_i$ for $i = 1, 2, 3$.*

Proposition 2.2.9. *(Bootstrapping argument). For $d > 4$, and $p \in (1, p_c)$, assume $g_i(p) \leq K_i$, $i = 1, 2, 3$, where $\{K_i\}_{i=1}^3$ are the same constants as in Proposition 2.2.8. Then, the stronger inequlities $g_i(p) < (1 - \epsilon)K_i$ for some $\epsilon > 0$, and $i = 1, 2, 3$, hold.*

We only use $g_1(p)$ and $g_2(p)$ in Proposition 2.2.7 - 2.2.9 to prove Theorem 2.2.1 as follow. As for $g_3(p)$, it is used to prove $g_2(p)$ in Proposition 2.2.9, and we will show that in later chapter. *Proof of Theorem 2.2.1 according to Proposition 2.2.7-2.2.9.* First, by Proposition 2.2.7, we have $g_1(p)$ is continuous in $p \in (1, p_c)$, and by Proposition 2.2.8 we have $g_1(1) < K_1$, and by Proposition 2.2.9 we have $g_1(p) < K_1$ in $p \in (1, p_c)$. Therefore, we can conclude that $g_1(p) < K_1$ in $p \in [1, p_c)$. In other words, $p_c \leq K_1$.

Then, we use same argument to obtain that $g_2(p) < K_2$ in $p \in [1, p_c)$. This complete prove Theorem 2.2.1.

$\square$

Chapter 3

The lace expansion for self-avoiding walk

The lace expansion was derived by Brydges and Spencer in [2]. It was later noted that the lace expansion can also be seen as a result from repeated application of the inclusion-exclusion relation. For a more combinatorial description of the lace expansion, see [14]. We use the inclusion-exclusion approach to the time-independent two-point function (2.1.2) as follows. By application of the inclusion-exclusion relation, we have

$$G_p(x) \equiv \sum_{n=0}^{\infty} \varphi_n(x) p^n = \delta_{0,x} + \sum_{n=1}^{\infty} (\varphi_1 * \varphi_{n-1})(x) p^n - R_p^{(1)}(x),$$

where

$$R_p^{(1)}(x) = \sum_{n=0}^{\infty} \sum_{y \in \mathbb{Z}^d} \sum_{\omega^{(1)} \in \mathcal{W}_{n-1}(y,x)} \varphi_1(y) I[o \in \omega^{(1)}] \prod_{i=1}^{n-1} D(\omega_i - \omega_{i-1}) \prod_{0 \le i < j \le n-1} (1 - \delta_{\omega_i, \omega_j}) p^n,$$

and I is an indicator function. Observe that

$$
\begin{aligned}
& \sum_{n=1}^{\infty} (\varphi_1 * \varphi_{n-1})(x) p^n \\
=\ & \sum_{n=1}^{\infty} \sum_{y \in \mathbb{Z}^d} \varphi_1(y) \varphi_{n-1}(x - y) p^n \\
=\ & \sum_{y \in \mathbb{Z}^d} \varphi_1(y) \sum_{n=0}^{\infty} \varphi_n(x - y) p^{n+1} \\
=\ & p(D * G_p)(x).
\end{aligned}
$$

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Then, we get

$$G_p(x) = \delta_{o,x} + p(D * G_p)(x) - R_p^{(1)}(x).$$

For convenience, we can use Feynman diagram to analyze $R_p^{(1)}(x)$. Here we first introduce some notations that will be needed. We define " $a \bullet\!\!-\!\!-\!\!-\!\!\bullet\, b$ " as a self-avoiding path from a to b, " $a \bullet\!\!-\!\!-^{b}\!\!\cdots\!\!\cdots\!\bullet\, c$ " as two independent self-avoiding paths and their intersection is only at b, and " $a \bullet \cdot \,\rule[-0.5ex]{0.6pt}{2ex}\,$ " to represent one step from a.

Using inclusion-exclusion, we can rewrite $R_p^{(1)}(x)$ as follows

$$= \sum_{y \in \mathbb{Z}} \underset{o \,=\, y}{\bigcirc} G_p(x - y) - R_p^{(2)}(x).$$

Then, we define

$$\pi_p^{(1)}(y) \equiv \underset{o \,=\, y}{\bigcirc} .$$

Continue the process, and we can rewrite $R_p^{(2)}(x)$ as follows

$$R_p^{(2)}(x) \quad = \quad \text{[diagram]}$$

$$= \quad \text{[diagram]} \; - \; \underbrace{\text{[diagram]}}_{:=R_p^{(3)}(x)}$$

$$= \quad \sum_{y \in \mathbb{Z}^d} \text{[diagram]}\, G_p(x-y) - R_p^{(3)}(x),$$

where y is a first hitting point of two independent self-avoiding paths. Then, we define

$$\pi_p^{(2)}(y) \equiv \text{[diagram]}.$$

By repeating the application of inclusion-exclusion relations, we obtain the lace expansion we redefine Feynman diagram of the lace-expansion coefficients depicted as (i.e. the solid line represents at least one step, and " $a \longrightarrow b$ " can be allowed to merge into one point, and all solid lines are independent. Their intersection points are only at the vertices in the following Feynman diagrams, and some vertices have no English code, e.g. " $\cdot$ ", which represents the sum of all possible points.)

$$\pi_p^{(3)}(x) \equiv \; {}_o\text{[diagram]}{}_x \, , \quad \pi_p^{(4)}(x) \equiv \; {}_o\text{[diagram]}{}^x , \quad \pi_p^{(5)}(x) \equiv \; {}_o\text{[diagram]}{}_x \, , \quad \ldots$$

In summary, we can get the equation as follow

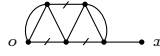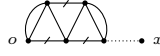$$G_p(x) = \delta_{0,x} + ((pD + \Pi_p^{(N)}) * G_p)(x) + (-1)^{N+1} R_p^{(N+1)}(x),$$

where

$$\Pi_p^{(N)}(x) = \sum_{n=1}^{N} (-1)^n \pi_p^{(n)}(x),$$

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and we can do a simple analytical method for $R_p^{(N)}(x)$ to get the following inequality

$$0 \leq R_p^{(N)}(x) \leq (\pi_p^{(N)} * G_p)(x). \tag{3.0.1}$$

For example, given $N = 5$, we have

$$
\begin{aligned}
R_p^{(5)}(x) \quad &= \quad \text{} \\
&\leq \quad \text{} \\
&= \quad (\pi_p^{(5)} * G_p)(x).
\end{aligned}
$$

Similarly, we can use the same argument to get the inequality (3.0.1) for $N \in \mathbb{N}$. Due to the construction explained above, we have the lace expansion as follow

Proposition 3.0.1. *(Lace expansion, c.f. [4]). For any $p < p_c$ and $N \in \mathbb{Z}_+ \equiv \{0\} \cup \mathbb{N}$, then we obtain the recursion equation*

$$G_p(x) = \delta_{0,x} + (J_p^{(N)} * G_p)(x) + (-1)^{N+1} R_p^{(N+1)}(x), \tag{3.0.2}$$

where

$$J_p^{(N)}(x) = pD(x) + \Pi_p^{(N)}(x), \tag{3.0.3}$$

and the remainder $R_p^{(N)}$ obeys the bound

$$0 \leq R_p^{(N)}(x) \leq (\pi_p^{(N)} * G_p)(x). \tag{3.0.4}$$

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Chapter 4

Diagrammatic bounds estimate

4.1 Diagrammatic bounds on the lace expansion coefficients

In this chapter, we need to evaluate the upper bound of $g_i(p)$ for $i = 1, 2, 3$ in (2.2.3) and (2.2.4). First, we need the following lemma to help us get the important result of our thesis.

Lemma 4.1.1. *For $p \in [1, p_c)$, we have*

$$\sum_{n=1}^{\infty} \hat{\pi}_p^{(n)}(0) < \infty, \; and \; \sup_k \sum_{n=1}^{\infty} \frac{-\hat{\Delta}_k \hat{\pi}_p^{(n)}}{1 - \hat{D}(k)} < 1.$$

The proof of Lemma 4.1.1 will be showed that in Chapter 6. Hence, by Lemma 4.1.1 we have

$$\lim_{N \to \infty} \hat{\pi}_p^{(N)}(0) = 0,$$

and by (3.0.4) we obtain

$$0 \leq \sum_{x \in \mathbb{Z}^d} R_p^{(N)}(x) \leq \hat{\pi}_p^{(N)}(0) \hat{G}_p(0) \longrightarrow 0 \; as \; N \to \infty.$$

Recall

$$\hat{\Pi}_p(k) = \sum_{n=1}^{\infty} (-1)^n \hat{\pi}_p^{(n)}(k), \; \text{and} \; \hat{J}_p(k) = p\hat{D}(k) + \hat{\Pi}_p(k).$$

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Then, (3.0.2) and (3.0.4) yield

$$\hat{G}_p(k) = 1 + \hat{J}_p(k)\hat{G}_p(k) + 0. \tag{4.1.1}$$

Through these simple calculations, we can get

$$\chi_p = \frac{1}{1 - \hat{J}_p(0)}.$$

Since $\chi_p \geq 0$, we can get $\hat{J}_p(0) \leq 1$. Using $\hat{D}(0) = 1$, we have

$$p \leq 1 - \hat{\Pi}_p(0). \tag{4.1.2}$$

By (3.0.3) and (4.1.1), we obtain

$$\hat{G}_p(k) = \frac{1}{1 - \hat{J}_p(k)} = \frac{1}{1 - p\hat{D}(k) - \hat{\Pi}_p(k)},$$

hence, we have

$$\chi_p = \hat{G}_p(0) = \frac{1}{1 - p - \hat{\Pi}_p(0)},$$

since $\hat{G}_p(0)$ is a positive continuous and increasing function on $p \in [0, p_c)$, and p_c is a radius of convergence. Then, we have

$$1 - p_c - \hat{\Pi}_{p_c}(0) = 0.$$

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As above,we have

$$\hat{G}_p(k)$$

$$= \frac{1}{(p_c + \hat{\Pi}_{p_c}(0)) - p\hat{D}(k) - \hat{\Pi}_p(k)}$$

$$= \frac{1}{(p - p_c) - 1 + 1 + (\hat{\Pi}_{p_c}(0) - \hat{\Pi}_p(0)) + p(1 - \hat{D}(k)) + \hat{\Pi}_p(0) - \hat{\Pi}_p(k)}$$

$$= \frac{1}{-(1 - p_c - \hat{\Pi}_{p_c}(0)) + (1 - p - \hat{\Pi}_p(0)) + p(-\hat{\Delta}_k \hat{D}(0)) + (-\hat{\Delta}_k \hat{\Pi}_p(0))}$$

$$= \frac{1}{-\frac{1}{\chi_{p_c}} + \frac{1}{\chi_p} + p(-\hat{\Delta}_k \hat{D}(0)) + (-\hat{\Delta}_k \hat{\Pi}_p(0))}$$

$$= \frac{1}{\frac{1}{\chi_p} + (-\hat{\Delta}_k \hat{J}_p(0))}.$$

Then, we have

$$0 \le \hat{G}_p(k) \le \frac{1}{-\hat{\Delta}_k \hat{J}_p(0)}. \tag{4.1.3}$$

In order to evaluate the upper bound of $g_{\hat{i}}(p)$ for $i = 1, 2, 3$, we must know the upper bound of $|\hat{\pi}_p^{(n)}(0)|$ and $|\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)|$. Since

$$\hat{\Pi}_p(0) = \sum_{m=1}^{\infty} \hat{\pi}_p^{(2m)}(0) - \sum_{m=0}^{\infty} \hat{\pi}_p^{(2m+1)}(0), \tag{4.1.4}$$

and we define $\hat{\Pi}_p^{even}(k) \equiv \sum_{m=1}^{\infty} \hat{\pi}_p^{(2m)}(k)$ and $\hat{\Pi}_p^{odd}(k) \equiv \sum_{m=0}^{\infty} \hat{\pi}_p^{(2m+1)}(k)$ where $k \in [-\pi, \pi]^d$. Hence, we can rewrite (4.1.4) as follow

$$\hat{\Pi}_p(0) = \hat{\Pi}_p^{even}(0) - \hat{\Pi}_p^{odd}(0),$$

and

$$-\hat{\Delta}_k \hat{\Pi}_p(0) = -\hat{\Delta}_k \hat{\Pi}_p^{even}(0) + \hat{\Delta}_k \hat{\Pi}_p^{odd}(0).$$

In the begining, we define

$$L_p = ||(pD)^{*2} * G_p||_\infty = \sup_x (\circ \!\!\mid\!\!\longrightarrow\!\!\mid x),$$

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and

$$B_p = ||(pD)^{*2} * G_p^{*2}||_\infty = \sup_x (\; \circ \!\longleftrightarrow\! \vdash x \;),$$

and

$$\gamma_p = p||D||_\infty + L_p + B_p = \sup_x (\; \circ \!\vdash\!\vdash x \;), + \sup_x (\; \circ \!\vdash\!\longrightarrow\!\vdash x \;) + \sup_x (\; \circ \!\longleftrightarrow\! \vdash x \;).$$

Then, we can obtain the bounds of $\hat{\pi}_p^{(n)}(0)$ and $|\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)|$ in Lemma 4.1.2 and Lemma 4.1.3.

Lemma 4.1.2. *(Diagrammatic bounds on the expansion coefficients.) The expansion coefficients* $\hat{\pi}_p^{(n)}(0) \equiv \sum_x \pi_p^{(n)}(x)$, *obey the following bound*

$$\hat{\pi}_p^{(n)}(0) \leq \begin{cases} L_p, & \text{if } n = 1, \\[2mm] B_p(p||D||_\infty + L_p)\gamma_p^{\,n-2}, & \text{if } n \geq 2. \end{cases}$$

Proof of Lemma 4.1.2. First, we have the following inequality

$$G_p(x)I[x \neq 0] \leq (pD * G_p)(x). \tag{4.1.5}$$

In the following, we will use the above inequality, and we can analyze $\hat{\pi}_p^{(n)}(0)$ for each $n \in \mathbb{N}$ to prove Lemma 4.1.2. For n = 1, we have

$$\hat{\pi}_p^{(1)}(0) \equiv \bigcirc_{o \; = \; y} \; \leq \; \bigcirc_{o \; = \; y} \; \leq \big((pD)^{*2} * G_p\big)(0) \leq L_p.$$

For $n \geq 2$, we first decompose $\hat{\pi}_p^{(n)}(0)$ by using subadditivity and then repeatedly apply

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(4.1.5) to obtain Lemma 4.1.2. For example,

$$
\begin{aligned}
\hat{\pi}_p^{(2)}(0) &= \sum_{x \neq o} \;\; \leq \; \Big(\sum_{x \neq o}\Big)\big(\sup_{x \neq o}\big) \\[2mm]
&\leq \Big(\sum_{x \neq 0} G_p(x)^2\Big)\big(\sup_{x \neq 0} G_p(x)\big) \\[2mm]
&\overset{(4.1.5)}{\leq} \underbrace{\big((pD)^{*2} * G_p^{*2}\big)(0)}_{\leq B_p} \big(\sup_{x \neq 0}(pD * G_p)(x)\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\pi}_p^{(3)}(0) &= \sum_{x \neq o} \;\; \leq \; \Big(\sum_{x \neq o}\Big)\big(\sup_{x \neq o}\sum_{y \neq x}\big)\big(\sup_{x \neq o}\big) \\[2mm]
&= \Big(\sum_{x \neq o} G_p(x)^2\Big)\big(\sup_{x \neq o}\sum_{y \neq x} G_p(y)G_p(x-y)\big)\big(\sup_{x \neq o} G_p(x)\big) \\[2mm]
&\overset{(4.1.5)}{\leq} B_p\Big(\sup_{x \neq o}\big((pD) * G_p^{*2}\big)(x)\Big)\Big(\sup_{x \neq o}\big((pD) * G_p\big)(x)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\hat{\pi}_p^{(4)}(0) &= \sum_{x \neq o} \;\; \leq \; \Big(\sum_{x \neq o}\Big)\big(\sup_{x \neq o}\sum_{y \neq x}\big)^2\big(\sup_{x \neq o}\big) \\[2mm]
&= \Big(\sum_{x \neq o} G_p(x)^2\Big)\big(\sup_{x \neq o}\sum_{y \neq x} G_p(y)G_p(x-y)\big)^2\big(\sup_{x \neq o} G_p(x)\big) \\[2mm]
&\overset{(4.1.5)}{\leq} B_p\Big(\sup_{x \neq o}\big((pD) * G_p^{*2}\big)(x)\Big)^2\Big(\sup_{x \neq o}\big((pD) * G_p\big)(x)\Big).
\end{aligned}
$$

By induction, we have

$$
\hat{\pi}_p^{(n)}(0) \leq B_p\Big(\sup_{x \neq o}\big((pD) * G_p^{*2}\big)(x)\Big)^{n-2}\Big(\sup_{x \neq o}\big((pD) * G_p\big)(x)\Big).
$$

In the following step, we will analyze the bound of $\sup_{x \neq o}\big((pD) * G_p\big)(x)$ and $\sup_{x \neq o}\big((pD) * G_p^{*2}\big)(x)$. Notice that, by omitting the spatial variables, we have

$$
pD * G_p = pD * \big(\delta G_p + (1-\delta)G_p\big) \overset{(4.1.5)}{\leq} pD + (pD)^{*2} * G_p,
$$

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where δ is the Kronecker delta. Therefore,

$$\sup_x (pD * G_p)(x) \le p||D||_\infty + ||(pD)^{*2} * G_p||_\infty = p||D||_\infty + L_p.$$

Similarly, we have

$$
\begin{aligned}
pD * G_p^{*2} \quad &= \quad pD * G_p * \big(\delta G_p + (1-\delta)G_p\big) \\
&\overset{(4.1.5)}{\le} \quad pD * G_p + (pD)^{*2} * G_p^{*2} \\
&= \quad pD * \big(\delta + (1-\delta)G_p\big) + (pD)^{*2} * G_p^{*2} \\
&\overset{(4.1.5)}{\le} \quad pD + (pD)^{*2} * G_p + (pD)^{*2} * G_p^{*2},
\end{aligned}
$$

hence

$$
\begin{aligned}
\sup_x (pD * G_p^{*2})(x) \quad &\le \quad p||D||_\infty + ||(pD)^{*2} * G_p||_\infty + ||(pD)^{*2} * G_p^{*2}||_\infty \\
&= \quad p||D||_\infty + L_p + B_p \equiv \gamma_p.
\end{aligned}
$$

$\square$

Then, we define the following equation to get the bound of $|\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)|$.

$$\hat{\mathcal{W}}_p(k) = \sup_x (1 - \cos k \cdot x) G_p(x).$$

Lemma 4.1.3. *(Diagrammatic bounds on the expansion coefficients.) The expansion coefficients* $|\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)| \equiv \sum_x (1 - \cos k \cdot x) \pi_p^{(n)}(x)$ *obey the following bound*

$$
|\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)| \le
\begin{cases}
B_p^2 \hat{\mathcal{W}}_p(k) m^2 \gamma_p^{2m-2}, & \text{if } n = 2m+1, \\[2ex]
B_p^2 \hat{\mathcal{W}}_p(k) m(m-1) \gamma_p^{2m-3} + B_p \hat{\mathcal{W}}_p(k) m \gamma_p^{2m-2}, & \text{if } n = 2m.
\end{cases}
$$

To prove this lemma, we need the following lemma which can be proved by following the argument in [4].

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Lemma 4.1.4. *(the telescopic inequality)*

$$1 - \cos\left(\sum_{j=1}^{d} t_j\right) \leq d \sum_{j=1}^{d}(1 - \cos t_j).$$

Proof of Lemma 4.1.4. First, take the real part of the telescopic identity

$$1 - e^{i\sum_{j=1}^{d} t_j} = \sum_{j=1}^{d}(1 - e^{it_j})e^{i\sum_{h=1}^{j-1} t_h}. \tag{4.1.6}$$

Then, using the inequalities respectively $|\sin\sum_{h=1}^{j-1} t_h| \leq \sum_{h=1}^{j-1}|\sin t_h|$, $|\sin t_j||\sin t_h| \leq (\sin^2 t_j + \sin^2 t_h)/2$ and $\sin^2 t_j \leq 2(1 - \cos t_j)$, and by (4.1.6), we have

$$1 - \cos\sum_{j=1}^{d} t_j = \sum_{j=1}^{d}(1 - \cos t_j)\cos\left(\sum_{h=1}^{j-1} t_h\right) + \sum_{j=1}^{d}\sin t_j \sin\left(\sum_{h=1}^{j-1} t_h\right).$$

Then

$$
1 - \cos\sum_{j=1}^{d} t_j - \sum_{j=1}^{d}(1 - \cos t_j)
$$

$$
= \quad -\sum_{j=1}^{d}(1 - \cos t_j)\left(1 - \cos\sum_{h=1}^{d-1} t_h\right) + \sum_{j=1}^{d}\sin t_j \sin\sum_{h=1}^{j-1} t_h
$$

$$
\leq \quad -\sum_{j=1}^{d}\frac{\sin^2 t_j}{2}\frac{\sin^2\sum_{h=1}^{j-1} t_h}{2} + \sum_{j=1}^{d}\sum_{h=1}^{j-1}|\sin t_j \sin t_h|
$$

$$
\leq \quad \sum_{j=1}^{d}\sum_{h=1}^{j-1}\sin t_j \sin|t_h| \leq \sum_{j=1}^{d}\sum_{h=1}^{j-1}\frac{(\sin^2 t_j + \sin^2 t_h)}{2}
$$

$$
= \quad \frac{1}{2}\left(\sin^2 t_2 + 2\sin^2 t_3 + ... + (d-1)\sin^2 t_d + \sum_{h=1}^{1}\sin^2 t_h + ... + \sum_{h=1}^{d-1}\sin^2 t_h\right)
$$

$$
= \quad \frac{d-1}{2}(\sin^2 t_1 + \sin^2 t_2 + ... + \sin^2 t_d)
$$

$$
= \quad \frac{d-1}{2}\sum_{j=1}^{d}\sin^2 t_j \leq (d-1)\sum_{j=1}^{d}(1 - \cos t_j).
$$

$\square$

Proof of Lemma 4.1.3 according to Lemma 4.1.4. First, we prove Lemma 4.1.3 for n = 2m+1. Since $\pi_p^{(1)}(x)$ is proportional to $\delta_{o,x}$ and therefore $\hat{\Delta}_k\hat{\pi}_p^{(1)}(0) \equiv 0$, we can assume $m \geq 1$.

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To bound $|\hat{\Delta}_k \hat{\pi}_p^{(2m+1)}(0)| \equiv \sum_x (1 - \cos k \cdot x)\pi_p^{(2m+1)}(x)$ for $m \geq 1$, we first identify the diagram vertices along the lowest diagram path from o to x, say $y_1, ..., y_{m-1}$, and then split x into $\{y_j - y_{j-1}\}_{j=1}^m$, where $y_0 = o$ and $y_m = x$.

For $m = 1$,

$$|\hat{\Delta}_k \hat{\pi}_p^{(3)}(0)| = \sum_{y_1 \in \mathbb{Z}^d} \left(1 - \cos k \cdot (y_1 - y_0)\right) {}_{y_0 = o}\!\!\triangle\!\!{}_{y_1}.$$

Then, by subadditivity, we obtain

$$
\begin{aligned}
|\hat{\Delta}_k \hat{\pi}_p^{(3)}(0)| &\leq \sum_{y_1 \in \mathbb{Z}^d} (1 - \cos k \cdot y_1) G_p(y_1)\, {}_o\!\!\triangle\!\!{}_{y_1} \\
&\leq \hat{\mathcal{W}}_p(k)\big(\sup_{y_1 \in \mathbb{Z}^d} {}_o\!\!\triangle\!\!{}_{y_1}\big) \leq \hat{\mathcal{W}}_p(k) B_p^2.
\end{aligned}
$$

For $m = 2$, we have

$$|\hat{\Delta}_k \hat{\pi}_p^{(5)}(0)| = \sum_{y_1, y_2 \in \mathbb{Z}^d} \left(1 - \cos \sum_{j=1,2} k \cdot (y_j - y_{j-1})\right) {}_{y_0 = o}\!\!\triangle\!\!\triangle\!\!{}_{y_2}.$$

Then, using Lemma 4.1.4 and subadditivity, we obtain

$$
\begin{aligned}
|\hat{\Delta}_k \hat{\pi}_p^{(5)}(0)| &\leq 2 \sum_{y_1, y_2 \in \mathbb{Z}^d} \big((1 - \cos k \cdot y_1) + (1 - \cos k \cdot (y_2 - y_1))\big) \\
&\quad \times \big(G_p(y_1)\, {}_o\!\!\triangle\!\!\triangle\!\!{}_{y_2} + G_p(y_2 - y_1)\, {}_o\!\!\triangle\!\!\triangle\!\!{}_{y_2}\big) \\
&\leq 2\hat{\mathcal{W}}_p(k)\big(\sup_{y_1 \in \mathbb{Z}^d} {}_o\!\!\triangle\!\!\triangle\!\!{}_{y_1} + \sup_{y_2 \in \mathbb{Z}^d} {}_o\!\!\triangle\!\!\triangle\!\!{}_{y_2}\big) \\
&\leq 4\hat{\mathcal{W}}_p(k) B_p^2 \gamma_p^2,
\end{aligned}
\tag{4.1.7}
$$

where the last inequality holds by the same argument of the proof in Lemma 4.1.2. By induction, we have

$$
\begin{aligned}
|\hat{\Delta}_k \hat{\pi}_p^{(2m+1)}(0)| &\leq m\hat{\mathcal{W}}_p(k) \times (m \text{ diagrams, each bounded by } B_p^2 \gamma_p^{2m-2}) \\
&\leq B_p^2 \hat{\mathcal{W}}_p(k) m^2 \gamma_p^{2m-2}.
\end{aligned}
$$

In the following, we prove Lemma 4.1.3 for $n = 2m$, and we follow the same lines as

above for $n = 2m + 1$. To bound $|\hat{\Delta}_k \hat{\pi}_p^{(2m)}(0)| \equiv \sum_x (1 - \cos k \cdot x) \pi_p^{(2m)}(x)$, we first identify the diagram vertices along the lowest diagram path from o to x, say, $y_1, ..., y_{m-1}$, and then split x into $\{y_j - y_{j-1}\}_{j=1}^m$, where $y_0 = o$ and $y_m = x$.

For $m = 1$,

$$|\hat{\Delta}_k \hat{\pi}_p^{(2)}| = \sum_{y_1 \in \mathbb{Z}^d} \left(1 - \cos k \cdot (y_1 - y_0)\right) \underset{y_0 = 0}{\overset{y_1}{\bigcirc}} .$$

Then, by subadditivity, we obtain

$$
\begin{aligned}
|\hat{\Delta}_k \hat{\pi}_p^{(2)}| &\leq \sum_{y_1 \in \mathbb{Z}^d} \left(1 - \cos k \cdot (y_1 - y_0)\right) G_p(y_1) \underset{y_0 = 0}{\overset{y_1}{\bigcirc}} \\
&\leq \hat{\mathcal{W}}_p(k) \left(\sup_{y_1 \in \mathbb{Z}^d} \underset{y_0 = 0}{\overset{y_1}{\bigcirc}} \right) \\
&\leq B_p \hat{\mathcal{W}}_p(k).
\end{aligned}
$$

For $m = 2$, we have

$$|\hat{\Delta}_k \hat{\pi}_p^{(4)}(0)| = \sum_{y_1, y_2 \in \mathbb{Z}^d} \left(1 - \cos \sum_{j=1,2} k \cdot (y_j - y_{j-1})\right) \underset{y_0 = o}{\overset{}{\diagup}}\!\!\diagdown^{y_2}_{y_1} .$$

Then, by using the same argument of (4.1.7), we obtain

$$
\begin{aligned}
|\hat{\Delta}_k \hat{\pi}_p^{(4)}(0)| &\leq 2 \sum_{y_1, y_2 \in \mathbb{Z}^d} \left((1 - \cos k \cdot y_1) + (1 - \cos k \cdot (y_2 - y_1))\right) \\
&\quad \times \left(G_p(y_1) \underset{o}{\overset{y_2}{\diagdown}}_{y_1} + G_p(y_2 - y_1) \underset{o}{\overset{y_2}{\diagdown}}_{y_1}\right) \\
&\leq 2\hat{\mathcal{W}}_p(k) \left(\underset{o}{\diagdown} + \underset{o}{\diagdown} \right) \\
&\leq 2\hat{\mathcal{W}}_p(k)(B_p^2 \gamma_p + B_p \gamma_p^2).
\end{aligned}
$$

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By induction, we have

$$|\hat{\Delta}_k \hat{\pi}_p^{(2m)}(0)| \quad \leq \quad m\hat{\mathcal{W}}_p(k) \times \Big(\big((m-1) \text{ diagrams, each bounded by } B_p^2 \gamma_p^{2m-3}\big)$$

$$+(1 \text{ diagram, bounded by } B_p \gamma_p^{2m-2})\Big)$$

$$\leq \quad B_p^2 \hat{\mathcal{W}}_p(k)m(m-1)\gamma_p^{2m-3} + B_p \hat{\mathcal{W}}_p(k)m\gamma_p^{2m-2}.$$

$\square$

From Lemma 4.1.2 and Lemma 4.1.3, we can simply get that if $\gamma_p < 1$ then $\hat{\pi}_p^{(n)}(0)$ and $|\hat{\Delta}_k \hat{\pi}_p^{(n)}|$ approaches zero as n goes to infinity. Hence, we need to suppose $\gamma_p < 1$ to achieve our purpose.

Lemma 4.1.5. *Suppose that* $\gamma_p \equiv p\|D\|_\infty + L_p + B_p < 1$. *We have*

$$0 \leq \hat{\Pi}_p^{odd}(0) \leq L_p + B_p(p\|D\|_\infty + L_p)\frac{\gamma_p}{1-\gamma_p^2}, \tag{4.1.8}$$

$$0 \leq \hat{\Pi}_p^{even}(0) \leq B_p(p\|D\|_\infty + L_p)\frac{1}{1-\gamma_p^2}, \tag{4.1.9}$$

$$\sup_k \frac{|\hat{\Delta}_k \hat{\Pi}_p^{odd}(0)|}{1-\hat{D}(k)} \leq \frac{B_p^2(1+\gamma_p^2)}{(1-\gamma_p^2)^3}\|\frac{\hat{\mathcal{W}}_p}{1-\hat{D}}\|_\infty, \tag{4.1.10}$$

$$\sup_k \frac{|\hat{\Delta}_k \hat{\Pi}_p^{even}(0)|}{1-\hat{D}(k)} \leq B_p^2\|\frac{\hat{\mathcal{W}}_p}{1-\hat{D}}\|_\infty \frac{2\gamma_p}{(1-\gamma_p^2)^3} + B_p\|\frac{\hat{\mathcal{W}}_p}{1-\hat{D}}\|_\infty \frac{1}{(1-\gamma_p^2)^2}. \tag{4.1.11}$$

Proof. Note that (4.1.8) and (4.1.9) are very easy to get from Lemma 4.1.2. Hence, we explain (4.1.10) and (4.1.11) as follows.

By lemma 4.1.3, we obtain

$$\sup_k \frac{|\hat{\Delta}_k \hat{\Pi}_p^{odd}(0)|}{1-\hat{D}(k)} \leq B_p^2\|\frac{\hat{\mathcal{W}}_p}{1-\hat{D}}\|_\infty \sum_{m=0}^{\infty} m^2 \gamma_p^{2m-2},$$

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where $\sum_{m=0}^{\infty} m^2 \gamma_p^{2m-2}$, and we can consider

$$\sum_{m=0}^{\infty} \gamma_p^{2m} = \frac{1}{1 - \gamma_p^2}.$$

Then, we have

$$\sum_{m=1}^{\infty} m^2 \gamma_p^{2m-2} = \frac{1 + \gamma_p^2}{(1 - \gamma_p^2)^3}.$$

Therefore, we get (4.1.10) as follows

$$\sup_k \frac{|\hat{\Delta}_k \hat{\Pi}_p^{odd}(0)|}{1 - \hat{D}(k)} \le \frac{B_p^2 (1 + \gamma_p^2)}{(1 - \gamma_p^2)^3} \| \frac{\hat{\mathcal{W}}_p}{1 - \hat{D}} \|_\infty.$$

Similarly, by lemma 4.1.3, we obtain

$$
\begin{aligned}
&\sup_k \frac{|\hat{\Delta}_k \hat{\Pi}_p^{even}(0)|}{1 - \hat{D}(k)} \\
&\le \quad B_p^2 \| \frac{\hat{\mathcal{W}}_p}{1 - \hat{D}} \|_\infty \sum_{m=1}^{\infty} m(m-1) \gamma_p^{2m-3} + B_p \| \frac{\hat{\mathcal{W}}_p}{1 - \hat{D}} \|_\infty \sum_{m=1}^{\infty} m \gamma_p^{2m-2}.
\end{aligned}
$$

Since $\sum_{m=1}^{\infty} \gamma_p^{2m} = \frac{\gamma_p^2}{1 - \gamma_p^2}$, we have

$$\sum_{m=1}^{\infty} m(m-1) \gamma_p^{2m-3} = \frac{2 \gamma_p}{(1 - \gamma_p^2)^3},$$

and

$$\sum_{m=1}^{\infty} m \gamma_p^{2m-2} = \frac{1}{(1 - \gamma_p^2)^2}.$$

Hence, we can obtain (4.1.11) as follows

$$\sup_k \frac{|\hat{\Delta}_k \hat{\Pi}_p^{even}(0)|}{1 - \hat{D}(k)} \le B_p^2 \| \frac{\hat{\mathcal{W}}_p}{1 - \hat{D}} \|_\infty \frac{2 \gamma_p}{(1 - \gamma_p^2)^3} + B_p \| \frac{\hat{\mathcal{W}}_p}{1 - \hat{D}} \|_\infty \frac{1}{(1 - \gamma_p^2)^2}.$$

$\square$

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4.2 Diagramatic bounds on the bootstrapping argument

In Section 4.1, we have already estimated the coefficients of lace expansion. In this section, we use them to estimate the upper bounds of $g_i(p)$ for $i = 1, 2, 3$. For $g_1(p)$, by (4.1.2), we can easily get

$$g_1(p) \leq 1 - \hat{\Pi}_p(0). \tag{4.2.1}$$

And then, by (4.1.3) and using the symmetry of $\hat{G}_p(k)$ (i.e. $\hat{J}_p(k) = \hat{J}_p(-k)$), we obtain $-\hat{\Delta}_k \hat{J}_p(0) = \hat{J}_p(0) - \hat{J}_p(k)$. Since $p \geq 1$, we have

$$
\begin{aligned}
\hat{G}_p(k) &\leq \frac{1}{-\hat{\Delta}_k \hat{J}_p(0)} = \frac{1}{\hat{J}_p(0) - \hat{J}_p(k)} \\
&= \frac{1}{p\hat{D}(0) + \hat{\Pi}_p(0) - p\hat{D}(k) - \hat{\Pi}_p(k)} \\
&= \frac{1}{p(1 - \hat{D}(k)) + (-\hat{\Delta}_k \hat{\Pi}_p(0))}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
g_2(p) &\leq \sup_{k \in [-\pi,\pi]^d} \left[\frac{1 - \hat{D}(k)}{p(1 - \hat{D}(k)) + (-\hat{\Delta}_k \hat{\Pi}_p(0))} \right] \\
&\overset{(p \geq 1)}{\leq} \sup_{k \in [-\pi,\pi]^d} \left[\frac{1 - \hat{D}(k)}{(1 - \hat{D}(k)) + (-\hat{\Delta}_k \hat{\Pi}_p(0))} \right].
\end{aligned}
$$

Then, we have

$$g_2(p) \leq \sup_{k \in [-\pi,\pi]^d} \left[1 + \frac{-\hat{\Delta}_k \hat{\Pi}_p(0)}{1 - \hat{D}(k)} \right]^{-1}. \tag{4.2.2}$$

Suppose $-\sum_{n=0}^{\infty} \hat{\Delta}_k \hat{\pi}_p^{(n)}(0) = \sum_{n=0}^{\infty} \sum_x (1 - \cos(k \cdot x)) \pi_p^{(n)}(x)$ is smaller than $1 - \hat{D}(k)$. Then, we can ensure the $g_2(p)$ to be uniformly bounded.

To evaluate $g_3(p)$, we need the following proposition which can be found in ([12], Lemma 5.7).

Proposition 4.2.1. *([12], Lemma 5.7) Suppose $\hat{A}(k) = (1 - \hat{a}(k))^{-1}$, where $\hat{a}$ is the Fourier transform of a symmetric $a(x) = a(-x)$ for all $x \in \mathbb{Z}^d$, then we have the following identity*

$$|\Delta_k \hat{A}(l)| \quad \leq \quad \frac{1}{2}[\hat{A}(l-k) + \hat{A}(l+k)]\hat{A}(l)[\hat{a}^{av}(0) - \hat{a}^{av}(k)]$$
$$+ 4\hat{A}(l-k)\hat{A}(l)\hat{A}(l+k)[\hat{a}^{av}(0) - \hat{a}^{av}(k)][\hat{a}^{av}(0) - \hat{a}^{av}(l)],$$

where $a^{av}(x) = |a(x)|$.

Then, we use Proposition 4.2.1, and substitute A to G_p, noting that $\hat{G}_p(k) = (1 - \hat{J}_p(k))^{-1}$,we obtain the upper bound of $g_3(p)$ as follows

$$g_3(p) \quad \equiv \quad \sup_{k,l} \frac{1}{\hat{U}(k,l)} \times |\hat{\Delta}_k \hat{G}_p(l)|, \quad \text{where} \quad \hat{G}_p(k) = \frac{1}{1 - \hat{J}_p(k)}$$
$$\leq \quad \sup_{k,l} \frac{1 - \hat{D}(k)}{\hat{U}(k,l)} \left\{ \left[\frac{\hat{G}_p(l+k) + \hat{G}_p(l-k)}{2} \right] \hat{G}_p(l) \frac{|\hat{\Delta}_k \hat{J}_p(l)|}{1 - \hat{D}(k)} \right.$$
$$\left. + 4\hat{G}_p(l+k)\hat{G}_p(l-k) \frac{-\hat{\Delta}_l |\hat{J}_p|(0)}{1 - \hat{J}_p(l)} \frac{-\hat{\Delta}_k |\hat{J}_p|(0)}{1 - \hat{D}(k)} \right\}, \tag{4.2.3}$$

where $|\hat{J}_p|(k) = \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} |\hat{J}_p(x)|$.

Applying these bounds in (4.2.1), (4.2.2), and (4.2.3), we obtain the following bounds on the bootstrapping functions $g_i(p)$ for $i = 1, 2, 3$.

Lemma 4.2.2. $p \in [1, p_c)$, *we have*

$$g_1(p) \quad \leq \quad 1 + L_p + \frac{B_p(p\|D\|_\infty + L_p)\gamma_p}{1 - \gamma_p^2},$$
$$g_2(p) \quad \leq \quad \left(1 - \frac{B_p^2(1 + \gamma_p^2)}{(1 - \gamma_p^2)^3}\|\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}\|_\infty\right)^{-1},$$
$$g_3(p) \quad \leq \quad \max\{g_2(p), 1\}^3 \left(p + B_p^2\|\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}\|_\infty \frac{2\gamma_p}{(1 - \gamma_p^2)^3} \right.$$
$$\left. + B_p\|\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}\|_\infty \frac{1}{(1 - \gamma_p^2)^2} + \frac{B_p^2(1 + \gamma_p^2)}{(1 - \gamma_p^2)^3}\|\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}\|_\infty \right)^2.$$

Proof. Since

$$\hat{\Pi}_p(0) = \hat{\Pi}_p^{even}(0) - \hat{\Pi}_p^{odd}(0),$$

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and

$$\hat{\Delta}_k \hat{\Pi}_p(0) = \hat{\Delta}_k \hat{\Pi}_p^{even}(0) - \hat{\Delta}_k \hat{\Pi}_p^{odd}(0).$$

We obtain

$$g_1(p) \overset{(4.2.1)}{\leq} 1 - \hat{\Pi}_p(0) \leq 1 + \hat{\Pi}_p^{odd}(0)$$

$$\overset{(4.1.8)}{\leq} 1 + L_p + B_p(p||D||_\infty + L_p)\frac{\gamma_p}{1 - \gamma_p^2},$$

and

$$g_2(p) \overset{(4.2.2)}{\leq} \sup_k (1 + \frac{-\hat{\Delta}_k \hat{\Pi}_p(0)}{1 - \hat{D}})^{-1} \leq \sup_k (1 - \frac{|\hat{\Delta}_k \hat{\Pi}_p^{odd}(0)|}{1 - \hat{D}})^{-1}$$

$$\overset{(4.1.10)}{\leq} (1 - \frac{B_p^2(1 + \gamma_p^2)}{(1 - \gamma_p^2)^3}||\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}||_\infty)^{-1}.$$

For $g_3(p)$,

$$g_3(p) \leq \sup_{k,l} \frac{1 - \hat{D}(k)}{\hat{U}(k,l)}\{[\frac{\hat{G}_p(l+k) + \hat{G}_p(l-k)}{2}]\hat{G}_p(l)\frac{|\hat{\Delta}_k \hat{J}_p(l)|}{1 - \hat{D}(k)}$$

$$+ 4\hat{G}_p(l+k)\hat{G}_p(l-k)\frac{-\hat{\Delta}_l|\hat{J}_p|(0)}{1 - \hat{J}_p(l)}\frac{-\hat{\Delta}_k|\hat{J}_p|(0)}{1 - \hat{D}(k)}\}$$

since $\hat{G}_p(k) \equiv 1/(1 - \hat{J}_p(k))$ and $|\hat{G}_p(k)| \leq g_2(p)\hat{S}_1(k) \equiv g_2(p)/(1 - \hat{D}(k))$, we obtain

$$g_3(p) \leq \sup_{k,l} \frac{1 - \hat{D}(k)}{\hat{U}(k,l)}\bigg\{(\frac{\hat{S}_1(l+k) + \hat{S}_1(l-k)}{2})\hat{S}_1(l)g_2(p)^2\frac{|\hat{\Delta}_k \hat{J}_p(0)|}{1 - \hat{D}(k)}$$

$$+ 4\hat{S}_1(l+k)\hat{S}_1(l-k)g_2(p)^3\frac{-\hat{\Delta}_l|\hat{J}_p|(0)}{1 - \hat{D}(l)}\frac{-\hat{\Delta}_k|\hat{J}_p|(0)}{1 - \hat{D}(k)})\bigg\}$$

$$\overset{(2.2.5)}{\leq} \max\{g_2(p), 1\}^3 \max\bigg\{\sup_{k,l} \frac{|\hat{\Delta}_k \hat{J}_p(l)|}{1 - \hat{D}(k)}, \big(\sup_k \frac{-\hat{\Delta}_k|\hat{J}_p|(0)}{1 - \hat{D}(k)}\big)^2\bigg\}.$$

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Since $J_p = pD + \Pi_p$, we have

$$
\begin{aligned}
\frac{|\hat{\Delta}_k \hat{J}_p(l)|}{1 - \hat{D}(k)} \quad &= \quad \frac{1}{1 - \hat{D}(k)} \Big| \sum_x (1 - \cos k \cdot x) e^{il \cdot x} (pD(x) + \Pi_p(x)) \Big| \\
&\leq \quad \frac{1}{1 - \hat{D}(k)} \sum_x (1 - \cos k \cdot x)(pD(x) + \Pi_p^{odd}(x) + \Pi_p^{even}(x)) \\
&\leq \quad p + \frac{|\hat{\Delta}_k \hat{\Pi}_p^{even}(0)|}{1 - \hat{D}(k)} + \frac{|\hat{\Delta}_k \hat{\Pi}_p^{odd}(0)|}{1 - \hat{D}(k)},
\end{aligned}
$$

which is larger than 1, since $p \geq 1$. It is easy to check that $-\hat{\Delta}_k |\hat{J}_p|(0)/(1 - \hat{D}(k))$ obeys the same bound. Therefore, by using (4.1.10) - (4.1.11), we obtain

$$
\begin{aligned}
g_3(p) \quad &\leq \quad \max\{g_2(p), 1\}^3 \Big(p + \frac{|\hat{\Delta}_k \hat{\Pi}_p^{even}(0)|}{1 - \hat{D}(k)} + \frac{|\hat{\Delta}_k \hat{\Pi}_p^{odd}(0)|}{1 - \hat{D}(k)}\Big)^2 \\
&\leq \quad \max\{g_2(p), 1\}^3 \cdot \Big(p + B_p^2 \|\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}\|_\infty \frac{2\gamma_p}{(1 - \gamma_p^2)^3} \\
&\qquad + B_p \|\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}\|_\infty \frac{1}{(1 - \gamma_p^2)^2} + \frac{B_p^2(1 + \gamma_p^2)}{(1 - \gamma_p^2)^3} \|\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}\|_\infty \Big)^2,
\end{aligned}
$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Therefore, in order to prove the main result of our thesis, we also need to evaluate the upper bound of L_p, B_p, $\|\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}\|_\infty$, and γ_p for $p \in [1, p_c)$ in the next chapter.

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Chapter 5

Random-walk estimate

In Section 5.1, we use the inequlity $G_1(x) \leq S_1(x)$ to evaluate the diagrams (i.e. $L_p, B_p, ||\frac{\hat{\mathcal{W}}_p}{1-\hat{D}}||_\infty$) at $p = 1$. In Section 5.2, we use bootstrapping argument to deal with the diamgrams for $p \in (1, p_c)$.

5.1   The diagrams bound of random-walk quantities for p = 1

To estimate the upper bound of $L_1, B_1, ||\frac{\hat{\mathcal{W}}_1}{1-\hat{D}}||_\infty$, we can use two RW quantities, such as the RW loop ε_1, the RW bubble ε_2, defined as

$$\varepsilon_1 \equiv (D^{*2} * S_1^{*1})(0) = \sum_{n=2}^\infty D^{*n}(0),$$

$$\varepsilon_2 \equiv (D^{*2} * S_1^{*2})(0) = \sum_{n=2}^\infty D^{*n}(0)(n-1). \tag{5.1.1}$$

Since the symmetry Poisson distribution is long-range, we can easy to get $(D^{*2} * S_1^{*1})(0) = \sum_{n=2}^\infty D^{*n}(0)$. Similarly, the analysis of $(D^{*2} * S_1^{*2})(0)$ can be written as follows

$$
\begin{aligned}
(D^{*2} * S_1^{*2})(0) &= \sum_{m=1}^\infty \sum_{n=1}^\infty D^{*n}(x) D^{*m}(-x) = \sum_{m=1}^\infty \sum_{n=1}^\infty D^{*(n+m)}(0) \\
&= \sum_{u=2}^\infty \sum_{m=1}^\infty \sum_{n=1}^\infty D^{*u}(0) I\{n+m=u\} \\
&= \sum_{u=2}^\infty (u-1) D^{*u}(0) = \sum_{n=2}^\infty (n-1) D^{*n}(0).
\end{aligned}
$$

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And then, we evaluate the diagrams for $p \in [1, p_c)$. First, by the Fourier transform of the RW Green function $\hat{S}_1(k) \equiv (1 - \hat{D}(k))^{-1}$, we have

$$\varepsilon_1 \equiv (D^{*2} * S_1)(0) = \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{1 - \hat{D}(k)} \frac{d^d k}{(2\pi)^d},$$

and

$$\varepsilon_2 \equiv (D^{*2} * S_1^{*2})(0) = \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{(1 - \hat{D}(k))^2} \frac{d^d k}{(2\pi)^d}.$$

Hence we can estimate the upper bound as follows

$$\varepsilon_1 = \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{1 - \hat{D}(k)} \frac{d^d k}{(2\pi)^d} \le \int_{[-\pi,\pi]^d} \frac{|\hat{D}(k)|^2}{1 - |\hat{D}(k)|} \frac{d^d k}{(2\pi)^d},$$

Since

$$|\hat{D}(k)| \le \prod_{j=1}^{d} e^{-\lambda + \lambda \cos(k_j)} \equiv \widetilde{D}(k).$$

We get

$$\varepsilon_1 \le \int_{[-\pi,\pi]^d} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} \frac{d^d k}{(2\pi)^d} = \int_{[0,\pi]^d} 2^d \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} \frac{d^d k}{(2\pi)^d}$$

$$= \frac{1}{\pi^d} \int_{[0,\pi]^d} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k.$$

As above, we have

$$\varepsilon_1 \le \frac{1}{\pi^d} \int_{[0,\pi]^d} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k. \tag{5.1.2}$$

Then, we do the analysis and calculation for this formula as follows. First, we divide the scope into three parts, namely I_1, I_2 and I_3,

$$I_1 = \{k : ||k||_2 \in [0, \frac{1}{\sqrt{\lambda}}]\}, \text{ and } I_2 = \{k : ||k||_2 \in [\frac{1}{\sqrt{\lambda}}, \lambda^{-\frac{1}{3}}]\},$$

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and

$$I_3 = \left\{ k : ||k||_2 \geq \lambda^{-\frac{1}{3}}, \text{ and } |k_j| \leq \pi \text{ for all } 1 \leq j \leq d \right\}.$$

So we can divide the (5.1.2) into three parts. i.e.

$$
\begin{aligned}
\varepsilon_1 &\leq \frac{1}{\pi^d} \int_{[0,\pi]^d} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k \\
&= \underbrace{\frac{1}{\pi^d} \int_{I_1} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k}_{(1)} + \underbrace{\frac{1}{\pi^d} \int_{I_2} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k}_{(2)} + \underbrace{\frac{1}{\pi^d} \int_{I_3} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k}_{(3)}.
\end{aligned}
$$

First, we evaluate (1) as follow. Since $I_1 = \{k : ||k||_2 \in [0, \frac{1}{\sqrt{\lambda}}]\}$ and $\cos(k_j) \leq 1 - \frac{k_j^2}{2} + \frac{k_j^4}{4!} \leq 1 - \frac{k_j^2}{2}(1 - \frac{1}{12\lambda})$, we have

$$\widetilde{D}(k) \leq \prod_{j=1}^{d} e^{(-\frac{1}{2} k_j^2 (\lambda - \frac{1}{12}))} = e^{(-\frac{1}{2} ||k||^2 (\lambda - \frac{1}{12}))}.$$

Then

$$1 - \widetilde{D}(k) \geq \frac{1}{2} ||k||_2^2 (\lambda - \frac{1}{12}) [\frac{3}{4} + \frac{1}{48\lambda}].$$

Therefore

$$
\begin{aligned}
\frac{1}{\pi^d} \int_{I_1} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k &= \frac{1}{\pi^d} \int_{I_1} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k = \frac{1}{\pi^d} \int_{||k||_2 \in [0, \frac{1}{\sqrt{\lambda}}]} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k \\
&\leq \frac{1}{\pi^d} \int_{||k||_2 \in [0, \frac{1}{\sqrt{\lambda}}]} \frac{1}{\frac{1}{2} ||k||_2^2 (\lambda - \frac{1}{12}) [\frac{3}{4} + \frac{1}{48\lambda}]} d^d k \\
&= \frac{1}{\pi^d} \frac{1}{\frac{1}{2} (\lambda - \frac{1}{12})(\frac{3}{4} + \frac{1}{48\lambda})} \int_{||k||_2 \in [0, \frac{1}{\sqrt{\lambda}}]} \frac{1}{||k||_2^2} d^d k \\
&\leq \frac{1}{\pi^d} \frac{\sigma_d}{\frac{1}{2} (\lambda - \frac{1}{12})(\frac{3}{4} + \frac{1}{48\lambda})} \int_0^{\frac{1}{\sqrt{\lambda}}} \frac{r^{d-1}}{r^2} dr \\
&= \frac{1}{\pi^d} \frac{\sigma_d}{\frac{1}{2} (\lambda - \frac{1}{12})(\frac{3}{4} + \frac{1}{48\lambda})} \frac{1}{d-2} (\frac{1}{\sqrt{\lambda}})^{d-2} \quad , d > 2,
\end{aligned}
$$

where σ_d is the surface area of the d-dimensional unit ball.

To evaluate (2) as follow. Since $I_2 = \{k : ||k||_2 \in [\frac{1}{\sqrt{\lambda}}, \lambda^{-\frac{1}{3}}]\}$ and $\cos(k_j) \leq 1 - \frac{k_j^2}{2} + \frac{k_j^4}{4!} = 1 - \frac{k_j^2}{2}(1 - \frac{k_j^2}{12}) \leq 1 - \frac{k_j^2}{2}(1 - \frac{1}{12\lambda^{2/3}})$, we have

$$\widetilde{D}(k) \leq e^{(-\frac{\lambda}{2}(1 - \frac{1}{12\lambda^{2/3}})||k||_2^2)} \leq e^{(-\frac{\lambda}{2}(1 - \frac{1}{12\lambda^{2/3}})\frac{1}{\lambda})} = e^{(-\frac{1}{2}(1 - \frac{1}{12\lambda^{2/3}}))}.$$

Therefore

$$\frac{1}{\pi^d} \int_{I_2} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k \quad = \quad \frac{1}{\pi^d} \int_{I_2} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k = \frac{1}{\pi^d} \int_{||k||_2 \in [\frac{1}{\sqrt{\lambda}}, \lambda^{-\frac{1}{3}}]} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k$$

$$\leq \quad \frac{1}{\pi^d} \frac{e^{-(1 - \frac{1}{12\lambda^{2/3}})}}{1 - e^{-\frac{1}{2}(1 - \frac{1}{12\lambda^{2/3}})}} \sigma_d \int_0^{\lambda^{-\frac{1}{3}}} r^{d-1} dr$$

$$= \quad \frac{e^{-(1 - \frac{1}{12\lambda^{2/3}})} \sigma_d}{\pi^d [1 - e^{(-\frac{1}{2}(1 - \frac{1}{12\lambda^{2/3}}))}]} \frac{1}{d} (\frac{1}{\lambda^{\frac{1}{3}}})^d \quad , d \in \mathbb{N}.$$

Finally, we evaluate (3) as follow. Since $I_3 = \{k : ||k||_2 \geq \lambda^{-\frac{1}{3}} \text{ and } |k_j| \leq \pi, \text{ for all } 1 \leq j \leq d\}$ and $\cos(k_j) \leq 1 - \frac{1}{2} k_j^2 (1 - \frac{\pi^2}{12})$, we have

$$\widetilde{D}(k) \leq e^{(-\frac{\lambda}{2}(1 - \frac{\pi^2}{12})||k||_2^2)} \leq e^{(-\frac{\lambda}{2}(1 - \frac{\pi^2}{12})\lambda^{-\frac{2}{3}})} = e^{(-\frac{1}{2}\lambda^{\frac{1}{3}}(1 - \frac{\pi^2}{12}))}.$$

Therefore

$$\frac{1}{\pi^d} \int_{I_3} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k$$

$$= \quad \frac{1}{\pi^d} \int_{I_3} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k = \frac{1}{\pi^d} \int_{I_3} \frac{\widetilde{D}(k)^2}{1 - \widetilde{D}(k)} d^d k$$

$$\leq \quad \frac{1}{\pi^d} \frac{1}{1 - e^{(-\frac{1}{2}(\lambda^{\frac{1}{3}}(1 - \frac{\pi^2}{12})))}} \int_{I_3} e^{(-\lambda(1 - \frac{\pi^2}{12})||k||_2^2)} d^d k$$

$$\leq \quad \frac{\sigma_d}{\pi^d (1 - e^{-\frac{1}{2}\lambda^{\frac{1}{3}}(1 - \frac{\pi^2}{12})})} \int_{\lambda^{-\frac{1}{3}}}^{\sqrt{d\pi^2}} e^{-\lambda(1 - \frac{\pi^2}{12})r^2} r^{d-1} dr$$

$$\leq \quad \frac{\sigma_d}{\pi^d (1 - e^{-\frac{1}{2}\lambda^{\frac{1}{3}}(1 - \frac{\pi^2}{12})})} e^{-\frac{d-1}{2}} (\frac{d-1}{2\lambda(1 - \frac{\pi^2}{12})})^{\frac{d-1}{2}} [\sqrt{d\pi^2} - \lambda^{-\frac{1}{3}}],$$

where the last inequality can be explained in the following inequaily.

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$$e^{-\lambda(1-\frac{\pi^2}{12})r^2}r^{d-1} \le e^{-\frac{d-1}{2}}\left(\frac{d-1}{2\lambda(1-\frac{\pi^2}{12})}\right)^{\frac{d-1}{2}}. \tag{5.1.3}$$

The proof of (5.1.3) is easy, if we let $f(r) = e^{-\lambda(1-\frac{\pi^2}{12})r^2}r^{d-1}$, using the first order derivative law, we can get, when $r = \sqrt{\frac{d-1}{2\lambda(1-\frac{\pi^2}{12})}}$, the maximum value of $f(r)$ is $e^{-\frac{d-1}{2}}\left(\frac{d-1}{2\lambda(1-\frac{\pi^2}{12})}\right)^{\frac{d-1}{2}}$.

As the above estimate, we obtain

$$\begin{aligned}
\varepsilon_1 \quad \le \quad & \frac{1}{\pi^d}\frac{\sigma_d}{\frac{1}{2}(\lambda-\frac{1}{12})(\frac{3}{4}+\frac{1}{48\lambda})}\frac{1}{d-2}\left(\frac{1}{\sqrt{\lambda}}\right)^{d-2} + \frac{e^{-(1-\frac{1}{12\lambda^{2/3}})}\sigma_d}{\pi^d[1-e^{(-\frac{1}{2}(1-\frac{1}{12\lambda^{2/3}}))}]}\frac{1}{d}\left(\frac{1}{\lambda^{\frac{1}{3}}}\right)^d \\
& + \frac{\sigma_d}{\pi^d(1-e^{-\frac{1}{2}\lambda^{\frac{1}{3}}(1-\frac{\pi^2}{12})})}e^{-\frac{d-1}{2}}\left(\frac{d-1}{2\lambda(1-\frac{\pi^2}{12})}\right)^{\frac{d-1}{2}}\left[\sqrt{d}\pi^2 - \lambda^{-\frac{1}{3}}\right], d > 2.
\end{aligned} \tag{5.1.4}$$

Similarly,

$$\varepsilon_2 \equiv (D^{*2} * S_1^{*2})(0) = \int_{[-\pi,\pi]^d}\frac{\hat{D}(k)^2}{(1-\hat{D}(k))^2}\frac{d^dk}{(2\pi)^d} \le \frac{1}{\pi^d}\int_{[0,\pi]^d}\frac{\widetilde{D}(k)^2}{(1-\widetilde{D}(k))^2}d^dk. \tag{5.1.5}$$

We can also divide the (5.1.5) into three parts. i.e.

$$\varepsilon_2 = \underbrace{\frac{1}{\pi^d}\int_{I_1}\frac{\widetilde{D}(k)^2}{(1-\widetilde{D}(k))^2}d^dk}_{(1')} + \underbrace{\frac{1}{\pi^d}\int_{I_2}\frac{\widetilde{D}(k)^2}{(1-\widetilde{D}(k))^2}d^dk}_{(2')} + \underbrace{\frac{1}{\pi^d}\int_{I_3}\frac{\widetilde{D}(k)^2}{(1-\widetilde{D}(k))^2}d^dk}_{(3')}.$$

And then, using the same arguments of estimating ϵ_1, we get

$$\begin{aligned}
\varepsilon_2 \quad \le \quad & \frac{4\sigma_d}{\pi^d(\lambda-\frac{1}{12})^2[\frac{3}{4}+\frac{1}{48\lambda}]^2}\frac{1}{d-4}\left(\frac{1}{\sqrt{\lambda}}\right)^{d-4} + \frac{e^{-(1-\frac{1}{12\lambda^{2/3}})}\sigma_d}{\pi^d[1-e^{-\frac{1}{2}(1-\frac{1}{12\lambda^{2/3}})}]^2}\frac{1}{d}\left(\frac{1}{\lambda^{1/3}}\right)^d \\
& + \frac{\sigma_d}{\pi^d(1-e^{-\frac{1}{2}\lambda^{\frac{1}{3}}(1-\frac{\pi^2}{12})})^2}e^{-\frac{d-1}{2}}\left(\frac{d-1}{2\lambda(1-\frac{\pi^2}{12})}\right)^{\frac{d-1}{2}}\left[\sqrt{d}\pi^2 - \lambda^{-\frac{1}{3}}\right], d > 4.
\end{aligned} \tag{5.1.6}$$

Note that $\sigma_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$, we can estimate the upper bound of ϵ_1 for $d > 2$ and the upper bound of ϵ_2 for $d > 4$, which are very important for estimating bootstrapping argument.

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Proposition 5.1.1. *Let $d \geq 5$ and $p = 1$. For all $\lambda > 0$, we have*

$$L_1 \leq \epsilon_1, B_1 \leq \epsilon_2, \text{ and } ||\frac{\hat{\mathcal{W}}_1}{1 - \hat{D}}||_\infty \leq 5(1 + 2\epsilon_1 + \epsilon_2).$$

Proof. For $p = 1$, using the inequality $G_1(x) \leq S_1(x), x \in \mathbb{Z}^d$ obtain the upper bound of L_1.

$$
\begin{aligned}
L_1 \leq ||D^{*2} * S_1||_\infty &= \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{1 - \hat{D}(k)} \frac{d^d k}{(2\pi)^d} \\
&= (D^{*2} * S_1)(0) \equiv \epsilon_1.
\end{aligned}
$$

Similarily, we have

$$
\begin{aligned}
B_1 \leq ||D^{*2} * S_1^{*2}||_\infty &= \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{(1 - \hat{D}(k))^2} \frac{d^d k}{(2\pi)^d} \\
&= (D^{*2} * S_1^{*2})(0) \equiv \epsilon_2.
\end{aligned}
$$

Finally, since

$$
\begin{aligned}
(1 - \cos k \cdot x)G_1(x) &\leq (1 - \cos k \cdot x)S_1(x) = \int_{[-\pi,\pi]^d} (-\hat{\Delta}_k \hat{S}_1(l))e^{il \cdot x} \frac{d^d l}{(2\pi)^d} \\
&\leq \int_{[-\pi,\pi]^d} \hat{U}(k,l) \frac{d^d l}{(2\pi)^d}.
\end{aligned}
$$

Then by Lemma 4.2.1 and using the Schwarz inequality, we have

$$
\begin{aligned}
\int_{[-\pi,\pi]^d} \hat{U}(k,l) \frac{d^d l}{(2\pi)^d} &\leq 5(1 - \hat{D}(k)) \int_{[-\pi,\pi]^d} \hat{S}_1(l)^2 \frac{d^d l}{(2\pi)^d} \\
&= 5(1 - \hat{D}(k))S_1^{*2}(0),
\end{aligned}
$$

by (5.1.1), we have

$$
\begin{aligned}
S_1^{*2}(0) &= \sum_{m=0}^\infty \sum_{n=0}^\infty D^{*n}(x)D^{*m}(-x) = \sum_{u=0}^\infty \sum_{m=0}^\infty \sum_{n=0}^\infty D^{*u}(0)I[n+m=u] \\
&= \sum_{u=0}^\infty (u+1)D^{*u}(0) \\
&= 1 + 2D(0) + 2\epsilon_1 + \epsilon_2 = 1 + 0 + 2\epsilon_1 + \epsilon_2.
\end{aligned}
$$

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Therefore

$$\int_{[-\pi,\pi]^d} \hat{U}(k,l) \frac{d^d l}{(2\pi)^d} \leq 5(1 - \hat{D}(k))(1 + 2\epsilon_1 + \epsilon_2).$$

Hence, we have

$$||\frac{\hat{\mathcal{W}}_1}{1 - \hat{D}}||_\infty \leq 5(1 + 2\epsilon_1 + \epsilon_2).$$

$\square$

5.2 The diagrams bound of random-walk quantities for p > 1

First, we estimate the upper bound of $L_p, B_p, ||\frac{\hat{\mathcal{W}}_p}{1-\hat{D}}||_\infty$ for $p \in [1, p_c)$ under the bootstrapping assumptions.

Proposition 5.2.1. *Let $d \geq 5$ and $p \in (1, p_c)$ and suppose that $g_i(p) \leq K_i$, $i = 1, 2, 3$ for some contants $\{K_i\}_{i=1}^3$. For all $\lambda > 0$, we have*

$$L_p \leq K_1^2 K_2 \epsilon_1, B_p \leq K_1^2 K_2^2 \epsilon_2, \text{ and } ||\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}||_\infty \leq 5K_3(1 + 2\epsilon_1 + \epsilon_2).$$

Proof.

$$
\begin{aligned}
L_p &= &&||(pD)^{*2} * G_p||_\infty \leq p^2 \int_{[-\pi,\pi]^d} \hat{D}(k)^2 |\hat{G}_p(k)| \frac{d^d k}{(2\pi)^d} \\
&\leq &&p^2 \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{1 - \hat{D}(k)} g_2(p) \frac{d^d k}{(2\pi)^d} \leq K_1^2 K_2 \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{1 - \hat{D}(k)} \frac{d^d k}{(2\pi)^d} \\
&= &&K_1^2 K_2 \epsilon_1,
\end{aligned}
$$

and

$$
\begin{aligned}
B_p &= &&||(pD)^{*2} * G_p^{*2}||_\infty \leq p^2 \int_{[-\pi,\pi]^d} \hat{D}(k)^2 \hat{G}_p(k)^2 \frac{d^d k}{(2\pi)^d} \\
&\leq &&p^2 \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{(1 - \hat{D}(k))^2} g_2(p)^2 \frac{d^d k}{(2\pi)^d} \leq K_1^2 K_2^2 \epsilon_2.
\end{aligned}
$$

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Finally, since $g_3(p) \leq K_3$, we obtain

$$
\begin{aligned}
0 \leq (1 - \cos(k \cdot x)) G_p(x) \quad &= \quad \int_{[-\pi,\pi]^d} (-\hat{\Delta}_k \hat{G}_p(l)) e^{il \cdot x} \frac{d^d l}{(2\pi)^d} \\
&\leq \quad K_3 \int_{[-\pi,\pi]^d} \hat{U}(k,l) \frac{d^d l}{(2\pi)^d},
\end{aligned}
$$

uniformly in x and k. Then, by the similar of analysis for $p = 1$, we have

$$
||\frac{\hat{\mathcal{W}}_p}{1 - \hat{D}}||_\infty \leq 5K_3(1 + 2\epsilon_1 + \epsilon_2).
$$

$\square$

Finally, we need to estimate the upper bounded of $||D(x)||_\infty$ as follows.

Lemma 5.2.2. *For any $\lambda \geq 1$,*

$$
||D(x)||_\infty \leq \frac{1}{(2\sqrt{2\pi})^d (\lfloor \lambda - 1 \rfloor)^{d/2}}.
$$

Proof. By (2.1.1)

$$
D(x) = \prod_{j=1}^d \frac{e^{-\lambda}}{2} \frac{\lambda^{|x_j|-1}}{(|x_j|-1)!}, \text{ if } x_1 x_2 \cdots x_d \neq 0.
$$

Then

$$
||D(x)||_\infty = \frac{e^{-d\lambda}}{2^d} \max_{\substack{x=(x_1,x_2,\ldots,x_d)\in\mathbb{Z}^d \\ x_1 x_2 \cdots x_d \neq 0}} \left(\prod_{j=1}^d \frac{\lambda^{|x_j|-1}}{(|x_j|-1)!} \right) = \frac{e^{-d\lambda}}{2^d} \prod_{j=1}^d \left(\max_{x_j \in \mathbb{Z}\backslash\{0\}} \frac{\lambda^{|x_j|}}{|x_j|!} \right).
$$

Using the Stirling's formula

$$
n! = \sqrt{2\pi n}(\frac{n}{e})^n e^a, \text{ where } \frac{1}{12n+1} < a < \frac{1}{12n},
$$

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we have the lower bound of $n!$ is $\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n+1}}$. Therefore

$$
\begin{aligned}
\frac{\lambda^{|x_j|}}{|x_j|!} &\leq \lambda^{|x_j|}\frac{1}{\sqrt{2\pi x_j}}e^{-x_j}x_j^{-x_j}e^{-\frac{1}{12x_j+1}} \\
&= \frac{1}{\sqrt{2\pi}}(\frac{\lambda e}{x_j})^{x_j}e^{-\frac{1}{12x_j+1}}x_j^{-\frac{1}{2}}, \quad x \geq 1 \\
&\leq \frac{1}{\sqrt{2\pi}}e^{x_j[\ln(\lambda e)-\ln(x_j)]}x_j^{-\frac{1}{2}} \\
&= \frac{1}{\sqrt{2\pi}}f(x_j),
\end{aligned}
$$

where

$$
f(t) = e^{t[ln(\lambda e)-ln(t)]}t^{-\frac{1}{2}}.
$$

Using the simple analysis of calculus, there is t_c such that $f'(t_c) = 0$, hence $f(t_c)$ is maximum value, and t_c satisfies the identity $\ln(\frac{\lambda}{t_c}) = \frac{1}{2t_c}$. However, it not easy to get t_c exactly, so we just evaluate the approximation of t_c. Let $h(t) = \ln(\frac{\lambda}{t}) - \frac{1}{2t}$. Clearly, $h(t_c) = 0$ and we want to find the interval of the root of $h(t)$ as follows. If $t = \lambda - \frac{1}{2}$, we have

$$
\ln(\frac{\lambda}{t}) = \ln(\frac{\lambda}{\lambda-\frac{1}{2}}) = -\ln(1-\frac{1}{2\lambda}) = \sum_{k=1}^{\infty}\frac{1}{k}(\frac{1}{2\lambda})^k = \frac{1}{2\lambda}+\frac{1}{8\lambda^2}+...
$$

and

$$
\frac{1}{2t} = \frac{1}{2\lambda-1} = \frac{1}{2\lambda}[\frac{1}{1-\frac{1}{2\lambda}}] = \frac{1}{2\lambda}\sum_{k=0}^{\infty}(\frac{1}{2\lambda})^k = \frac{1}{2\lambda}+\frac{1}{4\lambda^2}+...
$$

We obtain

$$
h(\lambda - \frac{1}{2}) < 0, \text{ and } h(t) \text{ is decreasing if } t > 1,
$$

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and

$$h(\lambda - 1)$$
$$= \quad \ln(\frac{\lambda}{\lambda - 1}) - \frac{1}{2(\lambda - 1)} = -\ln(1 - \frac{1}{\lambda}) - \frac{1}{2\lambda}[\frac{1}{1 - \frac{1}{\lambda}}]$$
$$= \quad \sum_{k=1}^{\infty}(\frac{1}{\lambda})^k\frac{1}{k} - \frac{1}{2\lambda}\sum_{k=0}^{\infty}(\frac{1}{\lambda})^k$$
$$= \quad \frac{1}{2\lambda} + \frac{1}{\lambda^3}\big((\frac{1}{3} + \frac{1}{4\lambda} + \frac{1}{5\lambda^2} + ...) - (\frac{1}{2} + \frac{1}{2\lambda} + \frac{1}{2\lambda^2} + ...)\big)$$
$$= \quad \frac{1}{2\lambda} + \frac{1}{\lambda^3}\sum_{n=0}^{\infty}(\frac{1}{n+3} - \frac{1}{2})(\frac{1}{\lambda})^n$$
$$> \quad \frac{1}{2\lambda} + \frac{1}{\lambda^3}(\frac{-1}{2}\frac{1}{1 - \frac{1}{\lambda}}) = \frac{1}{2}\frac{\lambda^2 - \lambda - 1}{\lambda^2(\lambda - 1)} > 0, \text{ if } \lambda > \frac{1 + \sqrt{5}}{2} \approx 1.6$$

By Intermediate value theorem, there is $t_c \in (\lambda - 1, \lambda - \frac{1}{2})$ such that $h(t_c) = 0$. Hence $f(t_c)$ is a maximum value of $f(t)$ for $t > 1$. Since $x_j \in \mathbb{Z}$,

$$\max_{x_j \in \mathbb{Z}} \frac{\lambda^{|x_j|}}{|x_j|!} \quad \leq \quad \max_{t \in \mathbb{Z}}\{\frac{1}{\sqrt{2\pi}}f(t)\}$$
$$\leq \quad \frac{1}{\sqrt{2\pi}}\max\{f(\lfloor\lambda - \frac{1}{2}\rfloor), f(\lceil\lambda - \frac{1}{2}\rceil), f(\lfloor\lambda - 1\rfloor)\},$$

for $j = 1, 2, ..., d$. Then, we evaluate $f(\lfloor\lambda - \frac{1}{2}\rfloor)$ and $f(\lfloor\lambda - 1\rfloor)$ and $f(\lceil\lambda - \frac{1}{2}\rceil)$ as follows

$$f(\lfloor\lambda - \frac{1}{2}\rfloor) = e^{(\lfloor\lambda - \frac{1}{2}\rfloor)[\ln(\lambda e) - \ln(\lambda - \frac{1}{2})]}\frac{1}{\sqrt{\lfloor\lambda - \frac{1}{2}\rfloor}}.$$

Since

$$\ln(\lambda e) - \ln(\lambda - \frac{1}{2}) \quad = \quad 1 + \ln(\frac{\lambda}{\lambda - \frac{1}{2}}) = 1 + \ln(\frac{1}{1 - \frac{1}{2\lambda}}) = 1 + \sum_{k=1}^{\infty}(\frac{1}{2\lambda})^k\frac{1}{k}$$
$$\leq \quad 1 + \frac{1}{2\lambda} + \frac{1}{2}\sum_{k=2}^{\infty}(\frac{1}{2\lambda})^k = 1 + \frac{1}{2\lambda} + \frac{1}{2}\frac{(\frac{1}{2\lambda})^2}{1 - \frac{1}{2\lambda}},$$

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so we have

$$f(\lfloor\lambda-\frac{1}{2}\rfloor) \quad\leq\quad e^{(\lfloor\lambda-\frac{1}{2}\rfloor)(1+\frac{1}{2\lambda}+\frac{1}{2-\frac{1}{\lambda}}\frac{1}{4\lambda^2})}\frac{1}{\sqrt{\lfloor\lambda-\frac{1}{2}\rfloor}}$$

$$= \quad\exp(\lambda-\frac{1}{4\lambda}[1-\frac{1}{2-\frac{1}{\lambda}}+\frac{1}{4\lambda-2}])\frac{1}{\sqrt{\lfloor\lambda-\frac{1}{2}\rfloor}}.$$

Similarily,

$$f(\lceil\lambda-\frac{1}{2}\rceil) \quad\leq\quad \exp(\lambda-\frac{1}{4\lambda}[1-\frac{1}{2-\frac{1}{\lambda}}+\frac{1}{4\lambda-2}])\frac{1}{\sqrt{\lceil\lambda-\frac{1}{2}\rceil}}.$$

and

$$f(\lfloor\lambda-1\rfloor) \quad\leq\quad \exp(\lambda-\frac{1}{\lambda}[1-\frac{1}{2-\frac{2}{\lambda}}+\frac{1}{2\lambda-2}])\frac{1}{\sqrt{\lfloor\lambda-1\rfloor}}.$$

As above, we obtain

$$e^{-\lambda}\max_{x_j\in\mathbb{Z}}(\frac{\lambda^{|x_j|}}{|x_j|!}) \quad\leq\quad \frac{1}{\sqrt{2\pi}}\max\{\frac{1}{\sqrt{\lfloor\lambda-\frac{1}{2}\rfloor}},\frac{1}{\sqrt{\lceil\lambda-\frac{1}{2}\rceil}},\frac{1}{\sqrt{\lfloor\lambda-1\rfloor}}\}$$

$$\leq\quad \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\lfloor\lambda-1\rfloor}}.$$

Therefore

$$||D(x)||_\infty \quad=\quad \prod_{j=1}^{d}(\frac{e^{-\lambda}}{2}\max(\frac{\lambda^{|x_j|}}{|x_j|!})) \leq \prod_{j=1}^{d}\frac{1}{2\sqrt{2\pi}}\frac{1}{\sqrt{\lfloor\lambda-1\rfloor}}$$

$$=\quad \frac{1}{(2\sqrt{2\pi})^d(\lfloor\lambda-1\rfloor)^{d/2}}.$$

$\square$

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Chapter 6

Proof of Proposition 2.2.7 - 2.2.9

In this chapter, we use the same argument of [4].

6.1   Proof of Proposition 2.2.7

First, we prove Proposition 2.2.7, and we have

$$g_1(p) = p, \quad g_2(p) = ||(1 - \hat{D})\hat{G}_p||_\infty, \text{ and } g_3(p) = \sup_{k,l} \frac{|\hat{\Delta}_k \hat{G}_p(l)|}{\hat{U}(k,l)},$$

where

$$\hat{U}(k,l) \equiv (1 - \hat{D}(k))(\frac{1}{2}(\hat{S}_1(l+k) + \hat{S}_1(l-k))\hat{S}_1(l) + 4\hat{S}_1(l+k)\hat{S}_1(l-k)).$$

Obviously, $g_1(p) = p$ is continuous. In order to prove the continuity of $g_2(p)$ and $g_3(p)$, we define

$$\tilde{g}_{2,k}(p) = (1 - \hat{D}(k)\hat{G}_p(k),$$

and

$$\tilde{g}_{3,k,l}(p) = \frac{1}{\hat{U}(k,l)} \times \hat{\Delta}_k \hat{G}_p(l),$$

and, clearly, they are continuous in $p \in [1, p_c)$ for every $k, l \in \mathbb{Z}^d$. However, since

$$g_2(p) = \sup_{k \in \mathbb{Z}^d} |\tilde{g}_{2,k}(p)|, \text{ and } g_3(p) = \sup_{k,l \in \mathbb{Z}^d} |\tilde{g}_{3,k,l}(p)|,$$

and the supremum of continuous functions is not necessarily continuous, we must use the following lemma to provide a sufficient condition for the supremum to be continuous.

Lemma 6.1.1. *(Lemma 5.13 of [12], in our language). Fix $p_0 \in [1, p_c)$ and let $\{\hat{f}_k(p)\}_{k \in \mathbb{Z}^d}$ be an equicontinuous family of functions in $p \in [1, p_0]$. Suppose that $\sup_{k \in \mathbb{Z}^d} \hat{f}_k(p) < \infty$ for every $p \in [1, p_0]$. Then, $\sup_{k \in \mathbb{Z}^d} \hat{f}_k(p) < \infty$ is continuous in $p \in [1, p_0]$.*

Therefore, in order to prove the continuity of $\{g_i(p)\}_{i=2,3}$. We want to show that $\{\tilde{g}_{2,k}(p)\}_{k \in [-\pi,\pi]^d}$ and $\{g_{3,k,l}\}_{k,l \in [-\pi,\pi]^d}$ are equicontinuous families of functions in $p \in [1, p_0]$ for each $p_0 \in [1, p_c)$. To prove this, it suffices to show that (i) and (ii) in the following lemma.

Lemma 6.1.2.

(i) $\tilde{g}_{2,k}(p)$ and $\partial_p \tilde{g}_{2,k}(p)$ are uniformly bounded in $k \in [-\pi, \pi]^d$ and $p \in [1, p_0]$.

and

(ii) $\tilde{g}_{3,k,l}(p)$ and $\partial_p \tilde{g}_{3,k,l}(p)$ are uniformly bounded in $k, l \in [-\pi, \pi]^d$ and $p \in [1, p_0]$.

Proof. First, we prove (i) as follows

By $0 \le 1 - \hat{D}(k) \le 2$, $|\hat{G}_p(k)| \le \hat{G}_p(0) \equiv \chi_p$ and the monotonicity of χ_p in p, we obtain $|\tilde{g}_{2,k}| = |(1 - \hat{D})\hat{G}_p| \le 2\chi_p \le 2\chi_{p_0} < \infty$, so $\tilde{g}_{2,k}(p)$ is uniformly bounded in $k \in [-\pi, \pi]^d$ and $p \in [1, p_0]$.

By subadditivity and translation-invariance, we have

$$\partial_p(pG_p(x)) = G_p(x) + p(\partial_p G_p), \tag{6.1.1}$$

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where

$$
\begin{aligned}
\partial_p G_p(x) \quad &= \quad \partial_p(\sum_{n=0}^{\infty} \varphi_n(x)p^n) = \sum_{n=0}^{\infty} n\varphi_n(x)p^{n-1} \\
&= \quad \sum_{n=1}^{\infty} n\varphi_n(x)p^{n-1} = \sum_{n=0}^{\infty} (n+1)\varphi_{n+1}(x)p^n \\
&\leq \quad \sum_{n=0}^{\infty} (n+1)D*\varphi_n(x)p^n = \sum_{n=0}^{\infty} D*\varphi_n(x)p^n \sum_{m=0}^{n} 1 \\
&= \quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D*\varphi_{n+m}(x)p^{n+m} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{y} D(y)\varphi_{n+m}(x-y)p^{n+m} \\
&= \quad \sum_{y} D(y) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varphi_{n+m}(x-y)p^{n+m} \\
&\leq \quad \sum_{y} D(y) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varphi_n*\varphi_m(x-y)p^{n+m} \\
&= \quad \sum_{y} D(y) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{z} \varphi_n(z)\varphi_m(x-y-z)p^{n+m} \\
&= \quad \sum_{y} D(y) \sum_{z}\sum_{n=0}^{\infty} \varphi_n(z)p^n \sum_{m=0}^{\infty} \varphi_m(x-y-z)p^m \\
&= \quad \sum_{y} D(y) \sum_{z} G_p(z)G_p(x-y-z) = \sum_{y} D(y)G_p^{*2}(x-y) \\
&= \quad D*G_p^{*2}(x).
\end{aligned}
$$

Then, by (6.1.1) we obtain

$$
\partial_p G_p(x) \leq D*G_p^{*2}(x),
$$

hence

$$
\begin{aligned}
|\partial_p \tilde{g}_{2,k}(p)| \quad &= \quad |(1-\hat{D})\partial_p \hat{G}_p(k)| \leq 2|\partial_p \hat{G}_p(k)| \\
&\leq \quad 2|\hat{D}(k)\hat{G}_p(k)^2| \leq 2|\hat{G}_p(k)|^2 \\
&\leq \quad 2\chi_p^2 \leq 2\chi_{p_0}^2.
\end{aligned}
$$

This is the complete proof of (i). Next, we prove (ii) as follows. Using Lemma 4.1.4, and

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we obtain

$$|\hat{\Delta}_k\hat{G}_p(l)| \equiv |\frac{1}{2}(\hat{G}_p(l+k) + \hat{G}_p(l-k)) - \hat{G}_p(l)|$$

$$= |\frac{1}{2}(\sum_x G_p(x)e^{i(l+k)\cdot x} + \sum_x G_p(x)e^{i(l-k)\cdot x} - \sum_x G_p(x)e^{il\cdot x}|$$

$$= |\frac{1}{2}(\sum_x G_p(x)\cos(l+k)\cdot x + \sum_x G_p(x)\cos(l-k)\cdot x - \sum_x G_p(x)\cos(l\cdot x)|$$

$$= |\sum_x G_p(x)\cos(l\cdot x)\cos(k\cdot x) - \sum_x G_p(x)\cos(l\cdot x)|$$

$$\leq |\sum_x G_p(x)\cos(l\cdot x)(\cos(k\cdot x) - 1)|$$

$$\leq \sum_x G_p(x)(1 - \cos(k\cdot x))$$

$$= \sum_x (1 - \cos(k\cdot x))\sum_{n=0}^{\infty}\sum_{\omega\in\mathcal{W}_n(x)}\prod_{i=1}^{n}D(\omega_i - \omega_{i-1})\prod_{0\leq i<j\leq n}(1 - \delta_{\omega_i,\omega_j})p^n$$

$$= \sum_x \sum_{n=0}^{\infty}\sum_{\omega\in\mathcal{W}_n(x)}(1 - \cos\sum_{i=1}^{n}k\cdot(\omega_i - \omega_{i-1}))$$

$$\times \prod_{i=1}^{n}D(\omega_i - \omega_{i-1})\prod_{0\leq i<j\leq n}(1 - \delta_{\omega_i,\omega_j})p^n$$

$$\leq \sum_{u,v,x}(1 - \cos k\cdot(v - u))\sum_{\omega\in\mathcal{W}_n(x)}n$$

$$\times \sum_{i=1}^{n}I[\omega_{i-1} = u \text{ and } \omega_i = v]p^n\prod_{i=1}^{n}D(\omega_i - \omega_{i-1})\prod_{0\leq i<j\leq n}(1 - \delta_{\omega_i,\omega_j}).$$

Ignoring the self-avoiding constraint between $\eta \equiv (\omega_0, \omega_1, ..., \omega_{i-1})$ and $\xi \equiv (\omega_i, \omega_{i+1}, ..., \omega_n)$ and using translation-invariance, we can further bound $|\hat{\Delta}_k\hat{G}_p(l)|$ as

$$|\hat{\Delta}_k\hat{G}_p(l)| \leq \sum_{u,v,x}(1 - \cos k\cdot(v - u))pD(v - u)\sum_{\substack{\eta\in\omega_{i-1}(0,u)\\\xi\in\omega_{n-i+1}(v,x)}}(|\eta| + |\xi| + 1)$$

$$\times p^{|\eta|}\prod_{i=1}^{|\eta|}D(\eta_i - \eta_{i-1})p^{|\xi|}\prod_{j=1}^{|\xi|}D(\xi_j - \xi_{j-1})\prod_{0\leq i<j\leq n}(1 - \delta_{\omega_i,\omega_j})$$

$$\leq 2p(1 - \hat{D}(k))\chi_p\sum_x\sum_{n=0}^{\infty}\sum_{\omega\in\mathcal{W}_n(x)}(n+1)p^n$$

$$\times \prod_{j=1}^{n}D(\omega_j - \omega_{j-1})\prod_{0\leq i<j\leq n}(1 - \delta_{\omega_i,\omega_j}).$$

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However, by the identity $|\omega| + 1 = \sum_y I[y \in \omega]$ for a self-avoiding path ω, subadditivity and translation-invariance, the sum in the last line is bounded as

$$\sum_x \sum_{n=0}^{\infty} \sum_{\omega \in \mathcal{W}_n(x)} (n+1)p^n \prod_{j=1}^{n} D(\omega_j - \omega_{j-1}) \prod_{0 \le i < j \le n} (1 - \delta_{\omega_i, \omega_j})$$

$$\le \sum_{x,y} G_p(y)G_p(x-y) = \chi_p^2.$$

As a result, we arrive at

$$|\hat{\Delta}_k \hat{G}_p(l)| \le 2p_0(1 - \hat{D}(k))\chi_{p_0}^3, \tag{6.1.2}$$

which implies that $\tilde{g}_{3,k,l}(p)$ is uniformly bounded in $k, l \in \mathbb{Z}^d$ and $p \in [1, p_0]$. For the derivative $\partial_p \tilde{g}_{3,k,l}(p) \equiv \hat{U}(k,l)^{-1} \hat{\Delta}_k \partial_p \hat{G}_p(k)$, we note that

$$\begin{aligned}
|\hat{\Delta}_k \partial_p \hat{G}_p(k)| \quad &= \quad \sum_x (1 - \cos k \cdot x) \partial_p G_p(x) \\
&\le \quad \sum_x (1 - \cos(k \cdot x))(D * G_p^{*2})(x) \\
&\overset{lemma\ 4.1.4}{\le} \quad 3\left((1 - \hat{D}(k))\chi_p^2 + 2\chi_p \underbrace{\sum_v (1 - \cos(k \cdot v))G_p(v)}_{\hat{\Delta}_k \hat{G}_p(0)}\right) \\
&\overset{(6.1.2)}{\le} \quad 3\left((1 - \hat{D}(k))\chi_{p_0}^2 + 4\chi_{p_0}^4 p_0(1 - \hat{D}(k))\right) \\
&= \quad 3(1 - \hat{D}(k))\chi_{p_0}^2(1 + 4\chi_{p_0}^2 p_0).
\end{aligned}$$

Therefore, $\partial_p \tilde{g}_{3,k,l}(p)$ is uniformly bounded in $k, l \in [-\pi, \pi]^d$ and $p \in [1, p_0]$.

$\square$

6.2 Proof of Proposition 2.2.8 - 2.2.9 and Lemma 4.1.1

First, we prove Proposition 2.2.8. In (5.1.4) and (5.1.6), it is obvious that ϵ_1 and ϵ_2 are decreasing when d and λ are increasing. To estimate the upper bounds of them, we only need to fix $d = 5$ and $\lambda = 60$.

$$\gamma_1 \equiv ||D||_\infty + L_1 + B_1 \le \frac{1}{(2\sqrt{2\pi})^d(\lfloor \lambda - 1 \rfloor)^{d/2}} + \epsilon_1 + \epsilon_2 \le 0.0417827.$$

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Remark 6.2.1. *For $d \geq 6$ and $\lambda = 60$, we get $\gamma_1 = 0.00789534870075646$.*

In addition, by (4.1.8) - (4.1.11) and Proposition 5.1.1, we have

$$\sum_{n=1}^{\infty} \hat{\pi}_1^{(n)}(0) \leq 0.0094953729577, \text{ and } \sup_k \sum_{n=1}^{\infty} \frac{-\hat{\Delta}_k \hat{\pi}_1^{(n)}(0)}{1 - \hat{D}(k)} \leq 0.176357145340,$$

which imply that the inequalities in Lemma 4.1.1 hold for $p = 1$ and $d > 4$. Moreover, by Lemma 4.2.2, we have

$$
\begin{aligned}
g_1(1) &\leq 1.009495372, \\
g_2(1) &\leq 1.005553012, \\
g_3(1) &\leq 1.406997428.
\end{aligned}
$$

Then Proposition 2.2.8 holds.

Next, we prove Proposition 2.2.9 holds for $p \in (1, p_c)$. First, we fix $d = 5$ and $\lambda = \lambda_5 \equiv 60$, according the assumptions $g_i(p) \leq K_i$ for $i = 1, 2, 3$, we have

$$\gamma_p \leq 0.044147.$$

In addition, by (4.1.8) - (4.1.11) and Proposition 5.2.1, we have

$$\sum_{n=1}^{\infty} \hat{\pi}_p^{(n)}(0) \leq 0.00988157659877, \text{ and } \sup_k \sum_{n=1}^{\infty} \frac{-\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)}{1 - \hat{D}(k)} \leq 0.3753353454618. \tag{6.2.1}$$

Then, we have

$$
\begin{aligned}
g_1(p) &\leq 1.0098815763136844 < K_1, \\
g_2(p) &\leq 1.0132463877593048 < K_2, \\
g_3(p) &\leq 2.0508909504570774 < K_3.
\end{aligned}
$$

Hence Proposition 2.2.9 holds for $p \in (1, p_c)$ and $\lambda \geq \lambda_5$. Moreover, this also implies that the inequalities in Lemma 4.1.1 hold for $p \in (1, p_c)$ and $d = 5$.

Next, we have to prove that Proposition 2.2.9 holds for $d > 5$ and $\lambda > \lambda_d$. To do this, we should estimate ϵ_1 and ϵ_2 in (5.1.4) and (5.1.6) for different dimensions d, respectively. First,

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substitute $d = 6$ and $\lambda = 2d$ in some small terms of right-hand sides of (5.1.4) and (5.1.6), it yields

$$\epsilon_1 \leq 5.10 \times 10^{-5} + 10.88A, \qquad (6.2.2)$$

and

$$\epsilon_2 \leq 0.0002 + 59.14A, \qquad (6.2.3)$$

where $A = \frac{\sqrt{d}\pi}{\Gamma(\frac{d}{2})} \left(\frac{d-1}{3\lambda} \right)^{\frac{d-1}{2}}$.

Using Lemma 4.2.2, we obtain

$$g_1(p) \leq 1 + L_p + \frac{B_p(p||D||_\infty + L_p)\gamma_p}{1 - \gamma_p^2} \leq 1 + L_p + \frac{B_p\gamma_p^2}{1 - \gamma_p^2}.$$

Since $g_1(1) < K_1$, $g_2(1) < K_2$, suppose $\gamma_p = 0.03 < 1$, Proposition 5.2.1, and Lemma 5.2.2 hold, we have

$$g_1(p) \leq 1 + K_1^2 K_2 \epsilon_1 + 0.00091(K_1^2 K_2^2 \epsilon_2) \leq 1.0001 + 14.6A.$$

Let $g_1(p) < (1 - \epsilon)K_1$ for all $\lambda \geq \lambda_d$ and some $\epsilon > 0$. Then, we obtain λ_d as follows

$$\lambda_d = \begin{cases} 60, & \text{if } d = 5, 6, \\ 35, & \text{if } d = 7, \\ 24, & \text{if } d = 8, \\ 18, & \text{if } d = 9, \end{cases}$$

We use them to obtain the upper bound of A is shown in Table 6.1 as follow,

Table 6.1: Upper bounds A for $6 \leq d \leq 9$

	$d = 6, \lambda_6 = 60$	$d = 7, \lambda_7 = 35$	$d = 8, \lambda_8 = 24$	$d = 9, \lambda_d = 18$
A	0.00028	0.00027	0.00024	0.00023

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For $d \geq 10$, let $A < 0.0002$ and using $\Gamma(\frac{d}{2}) \geq (\lfloor \frac{d}{2} \rfloor - 1)!$. Then, we obtain $\lambda \geq \frac{2e}{3}(1 + \frac{3}{d-4})e^{\frac{1}{d-1}[ln(3125000)-1+ln(d(d-4)^2)]}$. Hence, when $d \geq 10$, $\lambda_d = \frac{2e}{3}(1 + \frac{3}{d-4})e^{\frac{1}{d-1}[ln(3125000)-1+ln(d(d-4)^2)]}$.

We use them to evidence that $\gamma_p < 0.03$ holds as follows.

$$\gamma_p = p||D||_\infty + L_p + B_p \leq \frac{K_1}{2^d(\lambda-1)^{d-2}} + K_1^2 K_2 \epsilon_1 + K_1^2 K_2^2 \epsilon_2 < 0.03.$$

Next, we need to prove $g_2(p) < (1-\epsilon)K_2$ and $g_3(p) < (1-\epsilon)K_3$ for some $\epsilon > 0$. Suppose $g_1(1) < K_1$, $g_2(1) < K_2$, $g_3(p) < K_3$, and $\gamma_p = 0.03 < 1$, Table 6.1, Lemma 5.2.1, and Lemma 5.2.2, we have

$$
\begin{aligned}
g_2(p) &\leq & (1 - 1.004(K_1^2 K_2^2 \epsilon_2)^2 5K_3(1 + 2\epsilon_1 + \epsilon_2))^{-1} \\
&<& 1.016 < (1-\epsilon)K_2, \\
g_3(p) &\leq & 1.1\Big(K_1 + (K_1^2 K_2^2 \epsilon_2)^2 5K_3(1 + 2\epsilon_1 + \epsilon_2)0.061 \\
&& +K_1^2 K_2^2 \epsilon_2 5K_3(1 + 2\epsilon_1 + \epsilon_2)1.002 \\
&& +(K_1^2 K_2^2 \epsilon_2)^2 5K_3(1 + 2\epsilon_1 + \epsilon_2)1.004\Big)^2 \\
&<& 1.66 < (1-\epsilon)K_3.
\end{aligned}
$$

Therefore, This proves Proposition 2.2.9 and Lemma 4.1.1 for $d > 5$ and $p \in (1, p_c)$. This completes the proof of Proposition 2.2.9 and Lemma 4.1.1.

$\square$

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Appendix A

Proof of Proposition 2.2.4. First, we prove $\frac{d}{dp}(p\chi_p) \geq \frac{\chi_p^2}{1+B_p}$. Using Fubini's theorem, we have

$$
\begin{aligned}
\frac{d}{dp}(p\chi_p) &= \frac{d}{dp}(p\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}\varphi_n(x)p^n) = \frac{d}{dp}[p + \sum_{x\in\mathbb{Z}^d}\sum_{n=1}^{\infty}\varphi_n(x)p^{n+1}] \\
&= 1 + \sum_{x\in\mathbb{Z}^d}\sum_{n=1}^{\infty}(n+1)\varphi_n(x)p^n = \sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}(n+1)\varphi_n(x)p^n \\
&= \sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}\varphi_n(x)p^n(\sum_{m=0}^{n}1) \\
&= \sum_{x\in\mathbb{Z}^d}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\varphi_n(x)p^n \equiv Q.
\end{aligned}
\tag{A.0.1}
$$

Since B_p at least one step for our definition, we have

$$
\begin{aligned}
Q = \sum_{x\in\mathbb{Z}^d}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\varphi_n(x)p^n &\leq \sum_{n,m} \overset{m}{\underset{[1]}{\vdash}}\overset{n-m}{\underset{[2]}{\dashv}} - \sum_{n,m} \\
&\geq \chi_p^2 - QB_p.
\end{aligned}
\tag{A.0.2}
$$

Hence, we have

$$
Q \geq \frac{\chi_p^2}{1+B_p}.
\tag{A.0.3}
$$

Next, we claim $B_{p_c} < \infty$. By Theorem 2.2.1 we obtain

$$
g_2(p) \leq 3 \text{ and } \hat{G}_p(k) \leq \frac{3}{1-\hat{D}(k)},
\tag{A.0.4}
$$

then by monotone convergence theorem we have

$$\lim_{p \uparrow p_c} B_p \leq \int_{[-\pi,\pi]^d} \frac{9}{(1 - \hat{D}(k))^2} \frac{d^d k}{(2\pi)^d} < \infty, d > 4. \qquad (A.0.5)$$

$\square$

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