國立政治大學應用數學系

碩士學位論文

完全圖上的 (a, d)-antimagic 圖標號

On (a, d)-antimagic labelings of complete graphs

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中 華 民 國 109 年 6 月

DOI:10.6814/NCCU202000786

致謝

First, I would like to express my gratitude to my advisor Professor Yi-Wu Chang, and I would also like to thank all the Professors in NCCU and my fellow classmates. Finally, I would like to express my gratitude to my family members.

DOI:10.6814/NCCU202000786

中文摘要

圖標號是將整數分配到一個圖的邊或點。圖標號的發展源起於 1967 年，由 Rosa 所提出，在 1990 年，Hartsfield 和 Ringel 引進了 antimagic graphs 的概念，而 (a, d)-antimagic graph 的觀念則是 Bodendiek 和 Walther 在 1993 年引入，詳細的資料可以在 [2] 中參考。在本篇論文我們探討一些 (a, d)-antimagic 圖標號的概念，特別是探討完全圖的 (a, d)-antimagic 圖標號。

DOI:10.6814/NCCU202000786

Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labeling methods was introduced by Rosa in 1967. Hartsfield and Ringel introduced the concept of an antimagic graph in 1990. The concept of an (a, d)-antimagic labelings was introduced by Bodendiek and Walther in 1993. See [2] for further introduction. In this paper, we investigate properties of an (a, d)-antimagic graph. In particular, we study the (a, d)-antimagic labelings of complete graphs.

DOI:10.6814/NCCU202000786

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Chapter 1

Introduction

In this paper, all graphs are finite, undirected and simple. An labeling of a graph with m edges and n vertices is a bijection from the set of edges to the integers $1, ..., m$, where the image of each edge in the function is the label of the edge. *Antimagic labeling* is a graph labeling such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with the same vertex. A graph is called antimagic if it has an antimagic labeling.

Definition 1.1. *A graph $G = (V, E)$ is said to be (a, d)-antimagic if there exist positive integers a, d and a bijection mapping $f : E \to \{1, 2, ..., |E|\}$ such that the induced mapping $g_f : V \to \mathbb{N}$ defined by $g_f(v) = \sum_{e \in I(v)} f(e)$, $v \in V$, where $I(v) = \{e \in E \mid e$ is incident to $v\}$, is injective and $g_f(v)$ form an arithmetic progression with initial value a and common differences d.*

Remark 1.2. *Every (a, d)-antimagic graph is antimagic, but the converse is not true. For example, P_4 is antimagic, which can be seen by the labeling shown in Figure 1.1, but it is not (a, d)-antimagic, which can be seen by simply checking all 6 labelings from its edge set into the set {1,2,3} (In fact, we only need to check 3 types of labeling due to symmetry), as shown in Figure 1.2, or by a theorem which will be given later.*

Figure 1.1: P_4 is antimagic

DOI:10.6814/NCCU202000786

Figure 1.2: P_4 is not (a, d)-antimagic

In [1], Bodendiek and Walther introduced the concept of an (a, d)-antimagic graph, and also give the following necessary condition for a graph to admit an (a, d)-antimagic labeling.

Theorem 1.3. *(Theorem 1. of [1])*

If $G = (V, E)$, $|V| = n \geq 3$, $|E| = m \geq 2$ *is an* (a, d)*-antimagic graph, where* $a, d \in \mathbb{N}$*, then* a, d *satisfy the following conditions:*

1. $a, d \in \mathbb{N}$ *are positive solutions of the equation* $2an + n(n-1)d = 2m(m+1)$.

2. $a \geq 1 + 2 + ... + \delta = \delta(\delta + 1)/2$*, where* δ *is the minimum degree of* G

Proof. Since $g_f(v)$ form an arithmetic progression with initial value a and common differences d, we have $\sum_{v \in V} g_f(v) = a + (a+d) + ... + [a + (n-1)d]$. Also, every edge label was used twice in the vertex sum. That is, $\sum_{v \in V} g_f(v) = 2 \sum_{e \in E} f(e)$. Therefore, $a + (a+d) + ... + [a + (n-1)d] = 2(1 + 2 + ... + m)$, and this is clearly equivalent to the first condition. The second condition follows from the fact that every vertex of G is incident to at least δ edges.

DOI:10.6814/NCCU202000786

Chapter 2

Examples

In this chapter, we examine a few types of graph to see if they are (a, d)-antimagic, most of the examples in Chapter 2 has been discussed in [1], some of them without going into detail, we fill up some of the proofs in this Chapter.

2.1 Cycles, Paths, and Stars

In this section, we examine whether cycles, paths, and stars are (a, d)-antimagic.

Example 2.1. *If $n \in \mathbb{N}$ is even, then C_n is not (a, d)-antimagic.*

Proof. It follows from the fact that C_n has n vertices and n edges, so Theorem 1.3 implies that if C_n is an (a, d)-antimagic graph, then we must have $2a+(n-1)d = 2(n+1)$ and $a \geq 3$. Because n is even, $2a + (n-1)d = 2(n+1)$ implies d is even, Therefore, $2(n+1) = 2a + (n-1)d \geq 6 + (n-1)2 = 2n + 4 = 2(n+2)$, which is impossible. Therefore, every even cycle is not (a, d)-antimagic.

Example 2.2. *Every odd cycles $C_{2k+1}, k \geq 1$ is $(k + 2, 1)$-antimagic.*

Proof. Since $n = m = 2k + 1$, Theorem 1.3 implies that $a + kd = 2k + 2$ and $a \geq 3$, $a + kd = 2k + 2$ is equivalent to $k(2 - d) = a - 2$. Since $a \geq 3$, we have $a - 2 \geq 1$, which implies that d can only be 1, so the only possible solution is $(a, d) = (k + 2, 1)$. See Figure 2.1 for the actual $(k + 2, 1)$-antimagic labeling of C_{2k+1} .

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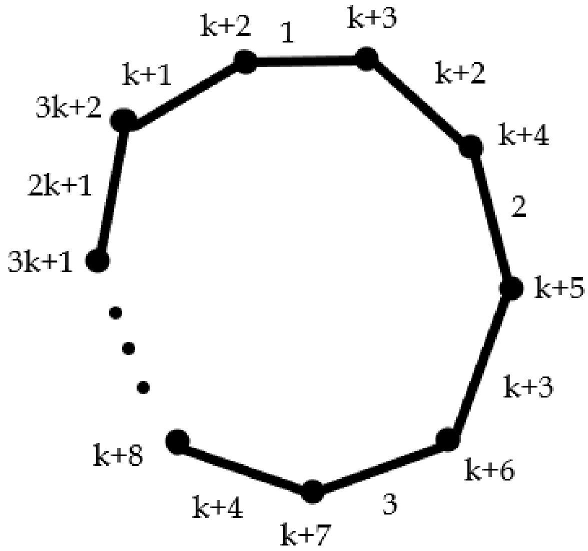

Figure 2.1: C_{2k+1} is $(k+2,1)$-antimagic

Example 2.3. *If* $n \in \mathbb{N}$ *is even, then the path* P_n *with* n *vertices is not* (a,d)*-antimagic.*

Proof. Since P_n has n vertices and $n-1$ edges, Theorem 1.3 implies that $2a+(n-1)d = 2(n-1)$ and $a \geq 1$, $2a+(n-1)d = 2(n-1)$ is equivalent to $2a = (n-1)(2-d)$, which implies that $d = 1$, and we get $2a = n-1$,which is impossible since $2a$ is even and $n-1$ is odd. Therefore, every even path is not (a,d)-antimagic.

Example 2.4. *Every path* $P_{2k+1}, k \geq 1$ *is* $(k,1)$*-antimagic.*

Proof. Similarly, since $n = 2k+1$, $m = 2k$, Theorems 1.3 implies that $a = k(2-d)$ and $a \geq 1$,which implies that $d = 1$, and the only solution is $(a,d) = (k,1)$. See Figure 2.2 for the actual $(k,1)$-antimagic labeling of P_{2k+1}.

Example 2.5. *If k is odd, then the stars* S_k *with* $k+1$ *vertices and k edges is not* (a,d)*-antimagic.*

Proof. The star S_k has $k+1$ vertices and k edges, so Theorem 1.3 implies that $2a = k(2-d)$ and $a \geq 1$, which implies that $d = 1$,but $d = 1$ implies $2a = k$, which is impossible since $2a$ is even and k is odd.

DOI:10.6814/NCCU202000786

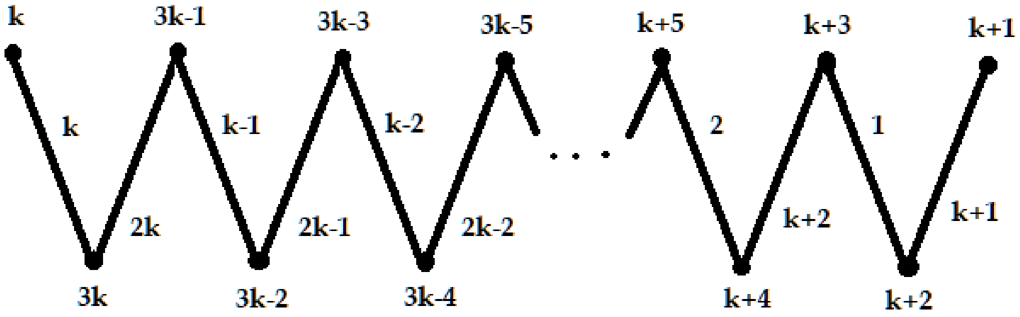

Figure 2.2: P_{2k+1} is $(k, 1)$-antimagic

Example 2.6. *If $k > 2$ is even, then the stars S_k with $k + 1$ vertices and k edges is not (a, d)-antimagic.*

Proof. Similarly, the only possible solution is $(a, d) = (k/2, 1)$, but $k > 2$ implies that $a > 1$, which is impossible since there must be one leaf v_0 of S_k such that $g_f(v_0) = 1$.

2.2 Complete graphs K_n

In this section, we introduce a theorem to help us examine whether a complete graph K_n is (a, d)-antimagic.

Notice that in a complete graph K_n, there are $m = (n - 1)n/2$ edges, apply Theorem 1.3 we get the following theorem:

Theorem 2.7. *Consider the complete graph K_n ,we have the following properties:*

1. *For $K_n, n = 4k, k \geq 1$, all possible $a, d \in \mathbb{N}$ such that K_n is (a, d)-antimagic are $(a, d) = (n(n-1)/2+(n-1)h, n(n-3)/2+1-2h)$, where $h = 0, 1, 2, ..., n(n-3)/4$.*

2. *For $K_n, n = 4k + 2, k \geq 1$, all possible $a, d \in \mathbb{N}$ such that K_n is (a, d)-antimagic are $(a, d) = (n(n-1)/2+(n-1)h, n(n-3)/2+1-2h)$, where $h = 0, 1, 2, ..., [n(n-3)-2]/4$.*

3. *For $K_n, n = 2k + 1, k \geq 1$, all possible $a, d \in \mathbb{N}$ such that K_n is (a, d)-antimagic are $(a, d) = (n(n-1)/2+(n-1)h/2, n(n-3)/2+1-h)$, where $h = 0, 1, 2, ..., n(n-3)/2$.*

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Proof. We only consider the first case, the other two cases are similar.

Consider $K_n, n = 4k, k \geq 1$, which has n vertices and $m = (n-1)n/2$ edges, Theorem 1.3 implies that $2an + n(n-1)d = n(n-1)[n(n-1)/2+1]$ and $a \geq (n-1)n/2$. For $a = (n-1)n/2$, we have $d = n(n-3)/2+1$, which is a positive integer since $n = 4k$ is even, therefore, $d \leq n(n-3)/2+1$, and every time d decrease by 1, a increase by $(n-1)/2$. Because n is even, $n-1$ is odd, that is, $(n-1)/2$ is not an integer, so we get a new set of solutions for (a, d) whenever d decrease by 2, and the smallest positive integer d can be is 1, so there are $|\{1, 3, ..., n(n-3)/2+1\}| = n(n-3)/4+1$ possible solutions for (a, d).

Remark 2.8. *With the help of Theorem 2.7, we will be able to identify all the candidates for* (a, d) *for a complete* K_n *to be* (a, d)*-antimagic without having to solve the equation in Theorem 1.3. However, these candidates may not be true for* K_n*. For example, by Theorem 2.7, all the possible candidate for* K_4 *are* $(6, 3)$ *and* $(9, 1)$*, but* K_4 *is neither* $(6, 3)$*-antimagic nor* $(9, 1)$*-antimagic, the proof will be given in the next chapter.*

DOI:10.6814/NCCU202000786

Chapter 3

Main results

In this chapter, we will show that K_4 is not (a, d)-antimagic, and we will present a theorem with stronger restrictions to the choice of (a, d) for a complete graph K_n.

3.1 K_4 is not (a, d)-antimagic

In this section, we prove in detail that the complete graph K_4 is not (a, d)-antimagic. The method being used in the proof will play an important role in the theorem we are going to present.

Proposition 3.1. *K_4 is not (a, d)-antimagic.*

Proof. By Theorem 2.7, the only possible solutions for (a, d) are $(6, 3)$ and $(9, 1)$. We discuss the two cases below:

1. If K_4 is $(6, 3)$-antimagic, then the set of all $g_f(v), v \in V$ is $\{6, 9, 12, 15\}$. Let $v_1 \in V$ be the vertex with $g_f(v_1) = 6$, then the only possible edges incident to v_1 are edges with label $1, 2, 3$. Without lose of generality, say $v_2, v_3, v_4 \in V$ such that $f(v_1v_2) = 1$, $f(v_1v_3) = 2$, and $f(v_1v_4) = 3$, see Figure 3.1 below. Now, consider $v_2, v_3, v_4 \in V$, because each $v_i, i = 2, 3, 4$ is incident to v_1 and two other vertices, the largest possible value for $g_f(v_2)$ is $1 + 5 + 6 = 12$, the largest possible value for $g_f(v_3)$ is $2 + 5 + 6 = 13$, $g_f(v_4)$ is $3 + 5 + 6 = 14$, contradict to the fact that in a $(6, 3)$-antimagic labeling, one of the vertex $v \in V$ must have $g_f(v) = 6 + 3 + 3 + 3 = 15$.

2. If K_4 is $(9, 1)$-antimagic, then the set of all $g_f(v), v \in V$ is $\{9, 10, 11, 12\}$. Let $v_1 \in V$

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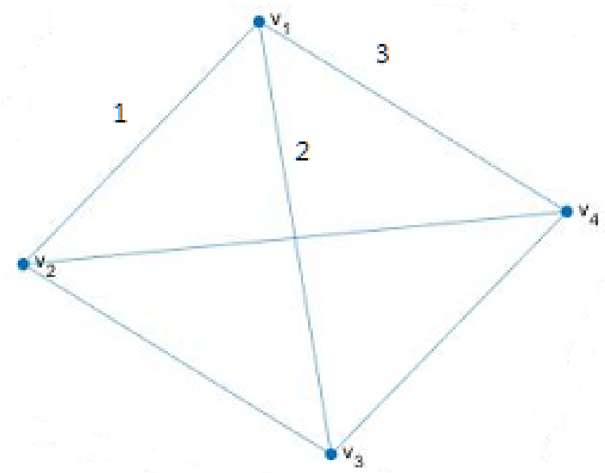

Figure 3.1

be the vertex with $g_f(v_1) = 9$, then the possible edges incident to v_1 are edges with label sets $\{1, 2, 6\}$, or $\{1, 3, 5\}$, or $\{2, 3, 4\}$. We have three subcases:

(a) If the edges incident to v_1 are edges with label sets $\{1, 2, 6\}$, Without lose of generality, say $v_2, v_3, v_4 \in V$ such that $f(v_1v_2) = 1$, $f(v_1v_3) = 2$, and $f(v_1v_4) = 6$. Consider v_4, all possible value for $g_f(v_4)$ are $6 + 3 + 4 = 13$, or $6 + 3 + 5 = 14$, or $6 + 4 + 5 = 15$, which is impossible for an $(9, 1)$-antimagic of K_4 since $13 \notin \{9, 10, 11, 12\}$, $14 \notin \{9, 10, 11, 12\}$, and $15 \notin \{9, 10, 11, 12\}$

(b) If the edges incident to v_1 are edges with label sets $\{1, 3, 5\}$, Without lose of generality, say $v_2, v_3, v_4 \in V$ such that $f(v_1v_2) = 1$, $f(v_1v_3) = 3$, and $f(v_1v_4) = 5$. First, consider v_2, all possible value for $g_f(v_2)$ are $1 + 2 + 4 = 7$, or $1 + 2 + 6 = 9$, or $1 + 4 + 6 = 11$, since $7 \notin \{9, 10, 11, 12\}$ and $g_f(v_1) = 9$, the only possible value for $g_f(v_2)$ is 11, next, consider v_3, all possible value for $g_f(v_3)$ are $3 + 2 + 4 = 9$, or $3 + 2 + 6 = 11$, or $3 + 4 + 6 = 13$, since $g_f(v_1) = 9$, and $13 \notin \{9, 10, 11, 12\}$, $g_f(v_3)$ must be 11. We get $g_f(v_2) = 11 = g_f(v_3)$, which leads to a contradiction.

(c) If the edges incident to v_1 are edges with label sets $\{2, 3, 4\}$, Without lose of generality, say $v_2, v_3, v_4 \in V$ such that $f(v_1v_2) = 2$, $f(v_1v_3) = 3$, and $f(v_1v_4) = 4$. Consider v_2, all possible value for $g_f(v_2)$ are $2 + 1 + 5 = 8$, or $2 + 1 + 6 = 9$, or $2 + 5 + 6 = 13$, since $8 \notin \{9, 10, 11, 12\}$, $13 \notin \{9, 10, 11, 12\}$, we must have $g_f(v_2) = 9 = g_f(v_1)$, which leads to a contradiction.

DOI:10.6814/NCCU202000786

Using the same method in the proof of Propsition 3.1, we can also show that K_5 is neither $(10, 6)$-antimagic nor $(12, 5)$-antimagic.

3.2 Main results

In this section, we will present a theorem with stronger restriction to the choice of (a, d) for a complete graph K_n to be (a, d)-antimagic.

Theorem 3.2. *$K_n, n \geq 4$ is not $(n(n-1)/2, n(n-3)/2+1)$-antimagic.*

Proof. For $n \geq 4$, $m = (n-1)n/2 > n$. Consider $(a, d) = (n(n-1)/2, n(n-3)/2+1)$. Suppose K_n is $(n(n-1)/2, n(n-3)/2+1)$-antimagic, then there is a vertex $v_1 \in V$ such that $g_f(v_1) = n(n-1)/2$, which can only be the sum of $1, 2, ..., (n-1)$, that is, v_1 must be incident to the edges with labeling $\{1, 2, ..., (n-1)\}$ Now, consider any vertex $v' \in V, v' \neq v_1$, since v' must incidient to v_1 and $(n-2)'s$ other vertices, we have $g_f(v') \geq a+d \geq 1+[n+(n+1)+...(2n-3)] = 1+(3n-3)(n-2)/2$, which implies $(n(n-1)/2+n(n-3)/2+1) \geq 1+(3n-3)(n-2)/2$, but this is equivalent to $2(n-1)^2 \geq 2+3(n-1)(n-2)$, which is equivalent to $(n-1)(n-4) \leq -2$, which is impossible for $n \geq 4$. ∎

Remark 3.3. *With Theorem 3.2, we can modify Theorem 1.3 into:*

If $K_n, n \geq 4$ is an (a, d)-antimagic graph, where $a, d \in \mathbb{N}$, then a, d satisfy the following conditions:

1. *$a, d \in \mathbb{N}$ are positive solutions of the equation $2an + n(n-1)d = 2m(m+1)$.*

2. *$a > 1+2+...+\delta = \delta(\delta+1)/2$, where δ is the minimum degree of G*

Corollary 3.4. *K_4 is not $(6, 3)$-antimagic, K_5 is not $(10, 6)$-antimagic, K_6 is not $(15, 10)$-antimagic, K_7 is not $(21, 15)$-antimagic, etc.*

Remark 3.5. *In fact, K_5 is $(14, 4)$-antimagic, $(16, 3)$-antimagic, $(18, 2)$-antimagic, and $(20, 1)$-antimagic; K_6 is $(25, 6)$-antimagic, $(30, 4)$-antimagic, and $(35, 2)$-antimagic; K_7 is $(33, 11)$-antimagic, $(36, 10)$-antimagic, $(39, 9)$-antimagic, $(42, 8)$-antimagic,..., and $(63, 1)$-antimagic, we demonstrate the cases for K_5 in Figure3.2 and 3.3, see Figure 6, 8 of [1] for the labeling of the other cases.*

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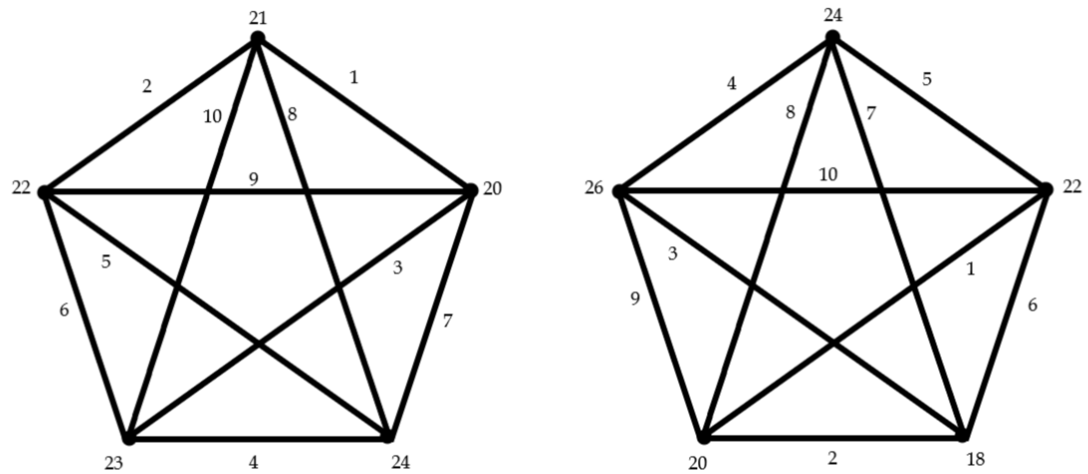

Figure 3.2: K_5 is $(20, 1)$-antimagic, $(18, 2)$-antimagic

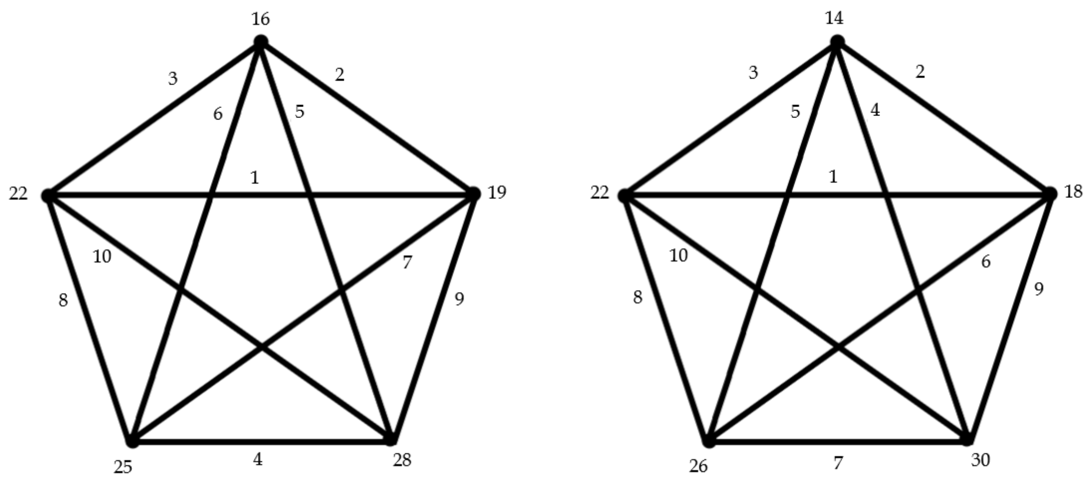

Figure 3.3: K_5 is $(16, 3)$-antimagic, $(14, 4)$-antimagic

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3.3 Conclusion

In this paper we discuss the concept of an (a, d)-antimagic graph, and give some examples to demonstrate how to verify whether or not a graph is (a, d)-antimagic, and in the end we give a stronger necessary conditions for a complete graph to admit an (a, d)-antimagic labeling.

However, from the observation such as K_4 is not $(9, 1)$-antimagic, K_5 is not $(12, 5)$-antimagic, and K_6 is not $(20, 8)$-antimagic, etc, we have a feeling that there are stricter necessary conditions for a complete graph K_n to be (a, d)-antimagic, and even possibly lead to a necessary and sufficient condition.

DOI:10.6814/NCCU202000786

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DOI:10.6814/NCCU202000786