國立政治大學 107 學年度 碩士班 招生考試試題

考 試 科 目	微 積 分	系 所 別	應用數學系	考 試 時 間	2 月 3 日(星期六)第 一節

1. Determine if each series converges or diverges.

(a) (6%) $\displaystyle\sum_{n=10}^{\infty}\frac{1}{n\log n\log\log n}$. (b) (6%) $\displaystyle\sum_{n=2}^{\infty}\ln(1-\frac{1}{n^2})$.

2. Evaluate the limits.

(a) (8%) $\displaystyle\lim_{x\to\infty}(\frac{e^x+1}{e^x-1})^{\ln x}$. (b) (8%) $\displaystyle\lim_{x\to 4}\frac{\sin^2(\pi x)}{e^{x-4}+3-x}$.

3. Evaluate the integrals.

(a) (8%) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{1-\cos x}{1+\cos x}\,dx$.

(b) (8%) $\displaystyle\int_1^2\int_{\frac{1}{y}}^{y}\sqrt{\frac{y}{x}}e^{\sqrt{xy}}\,dx\,dy$.

(c) (8%) $\displaystyle\int(\ln x)^{\ln x}(\frac{1}{x}+\frac{\ln\ln x}{x})\,dx$.

(d) (8%) $\displaystyle\oint_c(6y+x)\,dx+(y+2x)\,dy$, where $c: (x-2)^2+(y-3)^4$.

4. Evaluate the function $\varphi(t)$ defined by

(a) (10%) $\displaystyle\varphi(t)=\int_0^{\infty}e^{-\frac{x^2}{2}}\sin xt\,dx$.

(b) (10%) $\displaystyle\varphi(t)=\frac{d}{dt}\int_{\sin^3 t}^{2+\log_3 t^2}e^{-x^2}\,dx$.

5. (10%) Find r, s and t such that

$$\lim_{x\to 0}(\frac{\sin 3x\cos 2x}{x^5}+\frac{r}{x^4}+\frac{s}{x^2}+t)=0.$$

6. (10%) Let $\{a_n\}$ be any positive sequence. Suppose that $a_{n+1}\le a_n$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}a_n=0$. Show that $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent.

備 註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

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考試科目	線性代數	系別	應用數學系	考試時間	2月3日（星期六）第二節

注意事項：

- 作答時，請於答案卷上標明題號，並請勿任意更改題目符號，且請詳列過程，只有答案不給分。請盡量清楚完整回答你會的問題，不要只是每題回答一小部份。
- 本試題共有 5 個問題, 總計 100 分。

1. We say that a linear operator T on V is a **projection**, if there are two subspaces W_1, W_2 of V and $V = W_1 \oplus W_2$, such that for all $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ with $\mathbf{x}_1 \in W_1$, $\mathbf{x}_2 \in W_2$, we have $T(\mathbf{x}) = \mathbf{x}_1$. In this case, we called that T is the **projection on** W_1 **along** W_2.
   (a) (10 %) Let $T\colon V \to V$ be a projection. Show that $T^2 = T$.
   (b) (10 %) Let W be a subspace of a finite dimensional vector space of V. Show that $V = W \oplus W^{\perp}$. Define the projection T on W along $W^{\perp}$.
   (c) (10 %) Let W_1, W_2 be subspaces of $\mathbb{R}^3$, where $W_1 = \operatorname{span}\{(1,0,0),(1,0,1)\}$ and $W_2 = \operatorname{span}\{(1,1,1)\}$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the projection on W_1 along W_2. Let β be the standard ordered basis for $\mathbb{R}^3$. Find the matrix representation $[T]_\beta$ for T.

2. Let V be a vector space over $\mathbb{F}$. Let

$$V^* = \{f\colon V \to \mathbb{F} \mid f \text{ is a linear transformation.}\}$$

   Note that V^* is also a vector space. For a subset $S \subset V$, the **annihilator** S° of S is defined by

$$S^\circ = \{f \in V^* \mid f(\mathbf{x}) = \mathbf{o} \text{ for all } \mathbf{x} \in S\}.$$

   (a) (10 %) Show that S° is a subspace of V^*.
   (b) (10 %) Let W be a subspace of a finite dimensional vector space of V. Show that

$$\dim W + \dim W^\circ = \dim V.$$

3. (10 %) Let $T\colon V \to V$ be a linear operator on a finite dimensional vector space V. Let β, γ be two bases of V. Show that $\det([T]_\beta) = \det([T]_\gamma)$.

4. (20 %) Let A be a real $n \times n$ matrix. Show that A is invertible if and only if 0 is an eigenvalue of A.

5. (20 %) A real $n \times n$ matrix A is called **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for each nonzero vector $\mathbf{x} \in \mathbb{R}^n$. Let A be an $n \times n$ positive definite matrix. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T A\mathbf{x}$. Show that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。