International Review of Economics and Finance 68 (2020) 131–149

Contents lists available at ScienceDirect

International Review of Economics and Finance

journal homepage: www.elsevier.com/locate/iref

A study of the differences among representative investment strategies

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ARTICLE INFO

JEL classification:
G11

Keywords:
Investment strategy
Anticipative model
Adaptive model
Static approach
Dynamic approach

ABSTRACT

This study compares the differences and efficiencies of investment strategies among anticipative and adaptive models using three representative decision approaches: the static approach (SA), semidynamic strategy (or re-assess by static approach, Re-SA), and dynamic programming (DP). We show that each approach has individual merits and weaknesses. A DP strategy may allow for relatively aggressive decisions because of opportunities to adapt the decisions later. However, that strategy may result in a serious downside risk. The suboptimal adaptive strategy, Re-SA, acts as a good proxy for the DP strategy. Therefore, both SA and Re-SA are important tools for addressing asset allocation problems.

1. Introduction

Investment strategy decisions can be classified into anticipative models or adaptive models. An anticipative model determines entire decisions at the valuation date and is prearranged, ignoring the feedback obtained from adaptive information. Since new information may become available before each decision date, a sensible strategy would be adaptable and take all new information into account, as is the case with an adaptive model. Both anticipative and adaptive models have been adopted to deal with the asset allocation problems of life insurance companies and pension funds.

Theoretical solutions for adaptive models rely on dynamic programming (or dynamic control; e.g., Merton, 1971; Vigna & Haberman, 2001; Haberman & Vigna, 2002; Devolder, Bosch, & Dominguez, 2003; Battocchio & Menoncin, 2004; Munk, Sørensen, & Vinther, 2004; Chiu & Li, 2006; Emms & Haberman, 2007; Hainaut & Devolder, 2007; Delong, Gerrard, & Haberman, 2008; Gao, 2008; Larsen, 2010; Han & Hung, 2012) and the Martingale method (Boulier, Huang, & Taillard, 2001; Cox & Huang, 1989; Deelstra, Grasselli, & Koehl, 2003; Wang, Xia, & Zhang, 2007). They reach the same goal by different means. Furthermore, the numerical approach for adaptive models has been well developed (Brennan, Schwartz, & Lagnado, 1998b, 1998a; Carino et al., 1994; Dempster, 1980; Inganger, 2006; Kusy & Ziemba, 1986; Musumeci & Musumeci, 1999; Ziemba, 2003). However, a perfect adaptive strategy might be impossible or computationally too heavy to find because of the complicated frameworks of reality. On the other hand, the anticipative model is more widely applied in the asset allocation issue of life insurance companies and pension funds in practice (Huang, 2010; Huang & Cairns, 2006; Huang & Lee, 2010; Hurlimann, 2002). By ignoring new information, an anticipative model solution is always achievable and can reduce the time cost associated with numerical approaches.

Despite the widespread application of these two decision models to asset allocation problems, no research has explored their

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https://doi.org/10.1016/j.iref.2020.03.007
Received 13 February 2018; Received in revised form 14 February 2020; Accepted 16 March 2020
Available online 30 March 2020
1059-0560/

differences in detail. This examination is interesting and necessary for both practitioners and researchers. Theoretically, an adaptive strategy (abbr. DP, dynamic programming) would be superior to an anticipative strategy (abbr. SA, static approach). However, questions remain: What are the differences among decisions of different models and what are their properties alone? Does the DP strategy always dominate the SA strategy? In other words, for a given asset return scenario, does the adaptive model certainly offer a superior performance? If not, what risks are associated with using an adaptive model? The purpose of this study is to explore these questions and fill this gap.

The asset allocation problem of real businesses is complicated and involves a large state space. Often, the DP strategy does not work in practice, both in the theoretical and numerical approaches. Alternatively, the SA strategy can be easily found by means of numerical methods. A compromised and suboptimal approach would be to re-assess the investment strategy by SA at every decision date. Specifically, decision makers can take all new information into account at each decision date and seek a new investment strategy in terms of SA. Thus, an investigation of the properties and the efficiencies of these strategies is important. We therefore compare the Re-SA against the SA and DP in this study. Moreover, we demonstrate that this "re-assess by static approach" (Re-SA) method can be a good proxy for the DP strategy.

Vigna and Haberman (2001) use an independent, two-asset return model and derive a closed-form solution for the dynamic programming problem for a defined contribution (DC) pension scheme in a discrete-time framework. They extend the two-asset model to arbitrary *n* assets and consider the correlation of asset returns in Haberman and Vigna (2002). In this study, we compare an anticipative method (SA) with an adaptive method (DP and Re-SA[1]) based on Vigna & Haberman, 2001 model. For the sake of simplicity, we compare the three decision approaches in the two-asset version and examine the efficiency by the distribution of objective function value.

The numerical analysis leads to four main findings. First, with the opportunities to adapt the decision, the initial strategy of DP, compared with SA, suggests a relatively aggressive decision. Second, the strategy proposed by DP optimizes the objective function better on average (a lower expected cost), and this confirms that DP is superior to SA. However, the byproduct of DP is higher uncertainty (variance). From the viewpoint of the mean variance criterion, investors select the approach that optimizes their objective functions better on average and minimizes the variance of the objective functions simultaneously. Thus, investors would not prefer the adaptive approach to the anticipative approach on the basis of the mean variance criterion. Third, in addition to a higher variance, the DP strategy also causes a more serious downside risk. Finally, the strategy and the efficiency of Re-SA are very similar to those of DP. The decision-making method Re-SA, as we propose herein, can be a satisfactory proxy for finding the solution of DP.

To the best of our knowledge, this study is the first investigation on the properties and the efficiencies of SA and DP for long-term asset allocation problems. Although DP is superior theoretically, we identify the merits and weaknesses of each approach. In particular, we propose the Re-SA strategy, which might be easier to compute in practice, as an alternative for DP. This study lays out the properties, efficiencies and risks associated with each strategy. Our investigation provides useful insights to investors, risk managers, DC participants and pension fund managers.

For simplicity throughout this article, we use the term "optimal dynamic investment strategy" to refer to the optimal investment decision proposed by DP, whereas "optimal static investment strategy" indicates the optimal investment decision proposed by SA. The structure of this article is as follows. In the next section, we introduce Vigna and Haberman's (2001) model and the corresponding optimal dynamic investment strategy, as well as the process for determining the optimal static strategy and the process for Re-SA. Sections 3 and 4 present the numerical results, such that we compare the strategies and the efficiencies between DP and SA in Section 3 and consider the numerical results for Re-SA in Section 4. Section 5 discusses the robustness of our inferences to the investment horizon. Finally, Section 6 is dedicated to the conclusions.

2. The asset allocation problem

This study employs the DC pension plan model of Vigna and Haberman (2001) to investigate the asset allocation issues associated with the SA, Re-SA and DP strategies. The upcoming subsection illustrates the problem of Vigna and Haberman (2001) briefly. We describe the problems of the SA strategy and the Re-SA strategy in the succeeding subsections. Since there is no closed-form solution for the optimal static investment strategy, we use a numerical approach to find it.

2.1. Vigna and Haberman's model[2]

Consider an employee entering into a DC pension scheme at time *0*. Contributions are to be paid yearly, in advance. The contribution rate (c) and the retirement date (n) are fixed, and no decrement other than retirement exists. The annual salary available is constant for all t and assumed to be 1, without loss of generality.

The fund can be invested in two assets: a low-risk asset and a high-risk asset. The logarithm return rates from time $t-1$ to t of the low-risk asset and the high-risk asset are denoted by $\mu(t)$ and $\lambda(t)$, respectively. Annual investment returns from the two assets are lognormally distributed, so $\mu(t)$ and $\lambda(t)$ are normally distributed, with

$$\mu(t)\tilde{}N\left(\mu,\sigma_1^2\right) \quad \text{and} \quad \lambda(t)\tilde{}N\left(\lambda,\sigma_2^2\right), \tag{1}$$

[1] Re-SA is an adaptive strategy since it is derived from the information available at each decision date.

[2] Please see Vigna and Haberman (2001) for more details.

where

$$\mu \leq \lambda \quad \text{and} \quad \sigma_1^2 \leq \sigma_2^2.$$

Furthermore, $\mu(t)$ and $\lambda(t)$ are independent for any t and are independent and identically distributed through time. The level of the fund at time t, denoted by $F(t)$, follows the recurrence relationship:

$$F(t+1) = [F(t) + c]\left[(1-\omega(t))e^{\mu(t+1)} + \omega(t)e^{\lambda(t+1)}\right], \tag{2}$$

where $\omega(t)$ is the proportion of the fund invested in the high-risk asset at the inception of the $(t+1)$th year (i.e., time t).

The employee intends to reach a sequence of yearly targets, $F^*(t)$, at each time point t, $t = 1, 2, \ldots, n$. The final target $F^*(n)$ is defined by

$$F^*(n) = F(0)e^{nr^*} + c\sum_{j=1}^{n} e^{(n+1-j)r^*}, \quad with \;\; r^* = \frac{1}{2}\left[\mu + \lambda + 0.5\left(\sigma_1^2 + \sigma_2^2\right)\right]. \tag{3}$$

The target at time 1, $F^*(1)$, is

$$F^*(1) = (F(0) + c)e^{r^*}, \tag{4}$$

and the interim targets $\{F^*(t)\}_{t=2,\ldots,n-1}$ are interpolated linearly from $F^*(1)$ to $F^*(n)$.

The employee's concern is how closely his portfolio follows the targets. The cost incurred at time t, $C(t)$, is defined as a quadratic function:

$$C(t) = [F(t) - F^*(t)]^2, \;\; \text{for t} = 1, 2, \ldots, \text{n} - 1, \;\; \text{and} \tag{5}$$

$$C(n) = \theta[F(n) - F^*(n)]^2, \;\; \theta > 1. \tag{6}$$

A large weight, θ, is imposed on the final target since the well-being after the employee's retirement is dependent on the final fund level. The total future cost at time t, $G(t)$, is defined as the sum of the discounted future costs:

$$G(t) = \sum_{s=t}^{n} \nu^{s-t} C(s), \tag{7}$$

where ν is a subjective discount rate. We define $\mathfrak{I}_t$ as the σ-field generated by all information available at time t,

$$\mathfrak{I}_t = \sigma\{F(0), F(1), \ldots, F(t), \omega(0), \ldots, \omega(t-1)\}, \;\; \text{for} \;\; \text{t} = 0, 1, \ldots, \text{n} - 1. \tag{8}$$

For any given t, the employee's task is to choose an optimal $\omega(t)$ that minimizes the expectation of the total future cost, $G(t)$, based on the information revealed by $\mathfrak{I}_t$. A formal description of the problem and the closed-form solution for the optimal dynamic investment strategy is presented in Appendix A.

2.2. Optimal static investment strategy

The static investment strategy means that the employee sets target allocations, weights of each asset, beforehand and then periodically rebalances the portfolio back to those targets. The weights of each asset are determined at the inception of the pension plan and will not change hereafter. In this model, the employee's objective is to determine a series of weights that minimize the expectation of the total future cost at time *0*:

$$\min_{\{\pi_0^S\}} E[G(0)|\mathfrak{I}_0] = = \min_{\{\pi_0^S\}} \sum_{s=0}^{n} \nu^s E[C(s)|\mathfrak{I}_0], \tag{9}$$

where $\{\pi_0^S\} = \{\omega_0(0), \omega_0(1), \ldots, \omega_0(n-1)\}$ and the optimal static weights are unchanged over time.[3] In the DC pension plan model, there is no closed-form solution for the optimal static investment strategy.

2.3. Re-assess by static approach (Re-SA)

In asset allocation practice, the weights of each asset may change over time as new information becomes available. The proposed Re-SA strategy uses the initial weight of the optimal static investment strategy and reassesses the static asset allocation problem at the next

[3] We use a subscript *0* to indicate that weights are determined at time *0*. The superscript S indicates a static investment strategy.

Table 1
Sets of parameters of the asset model.[51].

	μ	λ	σ_1	σ_2
Case 1	**4%**	**6%**	**5%**	**15%**
Case 2	4%	6%	2.5%	10%
Case 3	4%	6%	10%	20%
Case 4	4%	6%	2.5%	20%
Case 5	2%	4%	5%	15%
Case 6	2%	6%	5%	15%

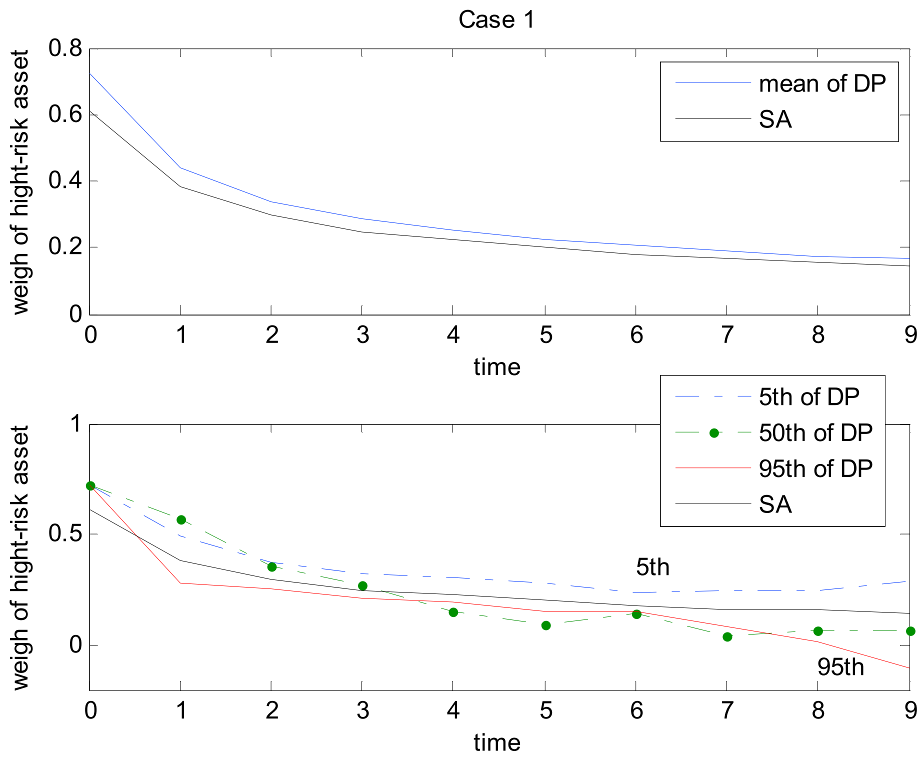

Fig. 1. Investment strategies: SA and DP.

decision date. At the valuation date ($t = 0$), the weights of each asset for Re-SA are the weights for SA. For any given t, the employee finds the optimal static investment strategy again:

$$\min_{\{\pi_t^S\}} E[G(t)|\mathfrak{I}_t] = = \min_{\{\pi_t^S\}} \sum_{s=t}^{n} \nu^{s-t} E[C(s)|\mathfrak{I}_t], \tag{10}$$

where $\{\pi_t^S\} = \{\omega_t(t), \omega_t(t+1), ..., \omega_t(n-1)\}$. That is, at time t, the employee finds the optimal static investment strategy, $\{\pi_t^{S*}\} = \{\omega_t^*(t), \omega_t^*(t+1), ..., \omega_t^*(n-1)\}$, that minimizes $E[G(t)|\mathfrak{I}_t]$ and takes $\omega_t^*(t)$ as the weight of the high-risk asset.

3. Numerical results: SA and DP

For our numerical analysis, in line with Vigna and Haberman (2001), we choose the following parameters: $c = 12\%$, $\nu = 0.95$, $\theta = 2$ and $F(0) = 0$. For simplicity, the length of future service n is equal to 10. Short selling is allowed. The optimal function “fminsearch” in MATLAB is employed to find the optimal static investment strategy.

We simulate future asset return scenarios using Equation (1). To obtain the optimal static investment strategy, we first simulate 100 groups, each containing 1000 paths for each asset, and find the optimal static investment strategy for each group numerically. We then use the average of the 100 strategies (weights) as the optimal static investment strategy[4]. In subsection 3.1, we use one specific group,

[4] Conceptually, the sample mean will tend toward the real optimal static investment strategy according to the weak law of large numbers.

Table 2
Simulated statistics (Case 1).

Case 1, $F^*(1) = 0.1269$ $F^*(10) = 1.6565$						
Pr(DP < SA), G(0)	0.6402					
	(0.0146)					
Pr(DP < SA), G(10)	0.6005					
	(0.0143)					
	DP			SA		
	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
Mean	**0.1282*****	**0.0626*****	**0.9524*****	0.1340	0.0673	0.9496
	(0.0041)	(0.0030)	(0.0031)	(0.0038)	(0.0029)	(0.0029)
St. dev.	0.1228	0.0906	**0.0956*****	**0.1131*****	**0.0843*****	0.0985
	(0.0068)	(0.0064)	(0.0024)	(0.0044)	(0.0044)	(0.0024)
1st	**0.0088*****	**1.1E-5****	0.7257	0.0095	1.4E-5	**0.7456*****
	(0.0010)	(6.9E-6)	(0.0131)	(0.0012)	(1.0E-5)	(0.0087)
5th	**0.0176*****	**2.5E-4*****	0.7972	0.0193	3.2E-4	**0.7977**
	(0.0012)	(7.1E-5)	(0.0063)	(0.0014)	(8.3E-5)	(0.0051)
25th	**0.0466*****	**0.0064*****	**0.8897*****	0.0520	0.0082	0.8803
	(0.0019)	(0.0008)	(0.0044)	(0.0021)	(0.0009)	(0.0042)
50th	**0.0892*****	**0.0284*****	**0.9514*****	0.0999	0.0353	0.9435
	(0.0036)	(0.0020)	(0.0036)	(0.0040)	(0.0025)	(0.0036)
75th	**0.1666*****	**0.0818*****	1.0130	0.1818	0.0950	**1.0117****
	(0.0069)	(0.0054)	(0.0039)	(0.0074)	(0.0055)	(0.0043)
95th	0.3696	0.2398	**1.1099*****	**0.3633*****	**0.2396**	1.1208
	(0.0175)	(0.0143)	(0.0074)	(0.0145)	(0.0120)	(0.0074)
99th	0.5850	0.4210	**1.1863*****	**0.5192*****	**0.3713*****	1.2029
	(0.0459)	(0.0386)	(0.0151)	(0.0312)	(0.0250)	(0.0168)

Note: For a given benchmark, the value of the superior strategy is marked in bold. In addition, we use Welch's *t*-test to examine if each pair of samples, observations from DP and from SA, comes from distributions with equal means. We use a two-tailed test and apply *, ** and *** as the 10%, 5% and 1% levels of significance, respectively (see also Appendix D, Case 1, DP vs. SA). All pairs of samples, except for the 95th percentile of G(10) and the 5th percentile of $F(10)/F^*(10)$, come from distributions with different means at the 95% significance level.

1000 paths of the 10-year asset return process, to compare the strategies of SA and DP. To assess efficiency and compare the effect of different investment strategies (the numerical results in subsection 3.2), we use all of the 100 groups.

In this study, the asset return parameters of Case 1 (see Table 1) serve as the basis for the analysis; we also conduct a sensitivity test for other cases.

3.1. *Investment strategies*

Fig. 1 compares the pattern of investment strategies between SA and DP. The optimal static investment strategy and the average of the optimal dynamic investment strategy (based on 1000 simulation paths) are displayed in the upper plot of Fig. 1. The strategy suggested by SA leads to a relatively conservative decision, probably because there is no opportunity to adjust the decision later. The strategy proposed by DP, instead, is information adapted, so it uses all information available before each decision moment, including the potential reaction in the future. With these opportunities to adapt the decision, the strategy suggested by DP allows for a relatively aggressive decision.

The lower plot of Fig. 1 illustrates some scenarios with optimal dynamic investment strategies over the 1000 paths. Three specific illustrated tracks refer to the scenarios corresponding to the 5th, 50th, and 95th percentiles of the final fund level, according to the optimal dynamic investment strategy. Specifically, we sort the final fund level in ascending order first and then consider the simulated asset return path and weights that bring in the p-th percentile of the final fund level. For the "5th of DP" scenario, the fund level in later years was generally lower, and the DP decision would propose holding more high-risk assets. For the "95th of DP" scenario, superior performance establishes a higher fund level a few years before retirement, so the proportion of high-risk assets decreases, coming close to 0 at time 8 and even negative at time 9, to achieve the predetermined target.

3.2. *Efficiency*

To assess efficiency and compare the effects of different investment strategies, we analyze the distributions of several critical variables[6]: total future cost at time *0* (G(0)), total future cost at retirement (G(10)), and final fund level. For the distribution of the final fund

[5] The first five cases are in line with Vigna and Haberman (2001). By adding the last case, we can investigate the effect of each parameter separately. For example, the effect of changing the mean of the low-risk asset can be studied by comparing Case 6 with Case 1.

[6] In the DC plan model of Vigna and Haberman (2001), the employee's initial target is to minimize the expectation of the total future cost, G(0), and his final target is to make the final fund level as close to the final target as possible, which is measured by G(10). Although we do not compare the total future cost for interim periods, i.e., G(1), G(2), …, G(9), the analyzed variables, G(0) and G(10), do include the information on future costs over the interim periods.

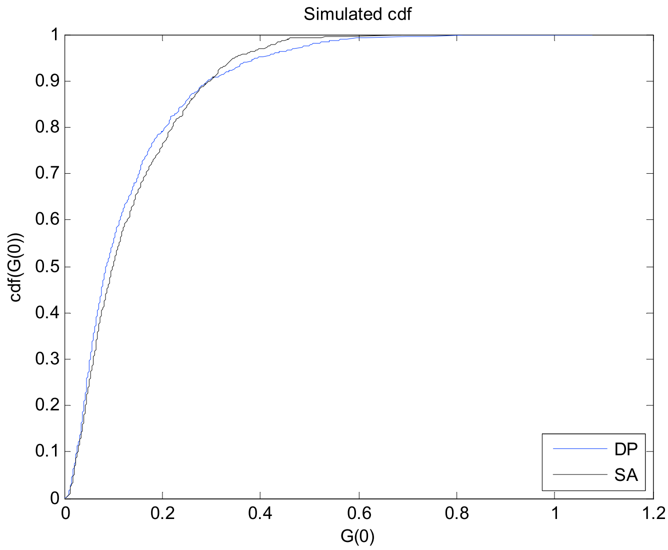

Fig. 2. Simulated cdf of G(0).[111]

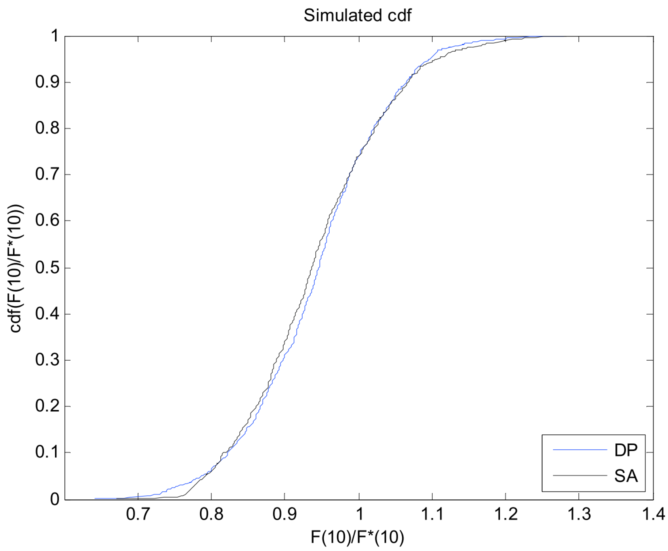

Fig. 3. Simulated cdf of F(10)/F*(10).[121]

level, we use the ratio $F(10)/F^*(10)$. This ratio indicates the percentage of the target achieved at retirement and enables the comparison among the different cases.

In this subsection, we first calculate the value of G(0), G(10) and F(10)/F*(10) over the 1000 paths for each group. The average and the sample standard deviation of a group serve as an observation for the mean and an observation for the standard deviation, and we have 100 observations from the 100 groups. In addition, by sorting the value of G(0), G(10) and/or F(10)/F*(10) in ascending order of a group, we have an observation for each percentile and we again have 100 observations from 100 groups. We report estimates for the mean, standard deviation, and percentiles (1st, 5th, 25th, 50th, 75th, 95th and 99th) over the simulations. Table 2 presents the simulated statistics of Case 1. We only present the numerical results for Case 1 in this subsection; the numerical results of other cases are in Appendix B (see columns of DP and of SA).[7]

The values in Table 2 are based on 100 groups, and each group provides an estimate, so we provide the average and the standard deviation over the 100 estimates. The value without brackets is the average, and the value in brackets presents the standard deviation. As Table 2 reveals, the means of G(0) and G(10) with DP (0.1282 and 0.0626) are smaller than those of SA (0.1340 and 0.0673).

[7] The properties that we discussed for Case 1 in this subsection still exist for other cases.

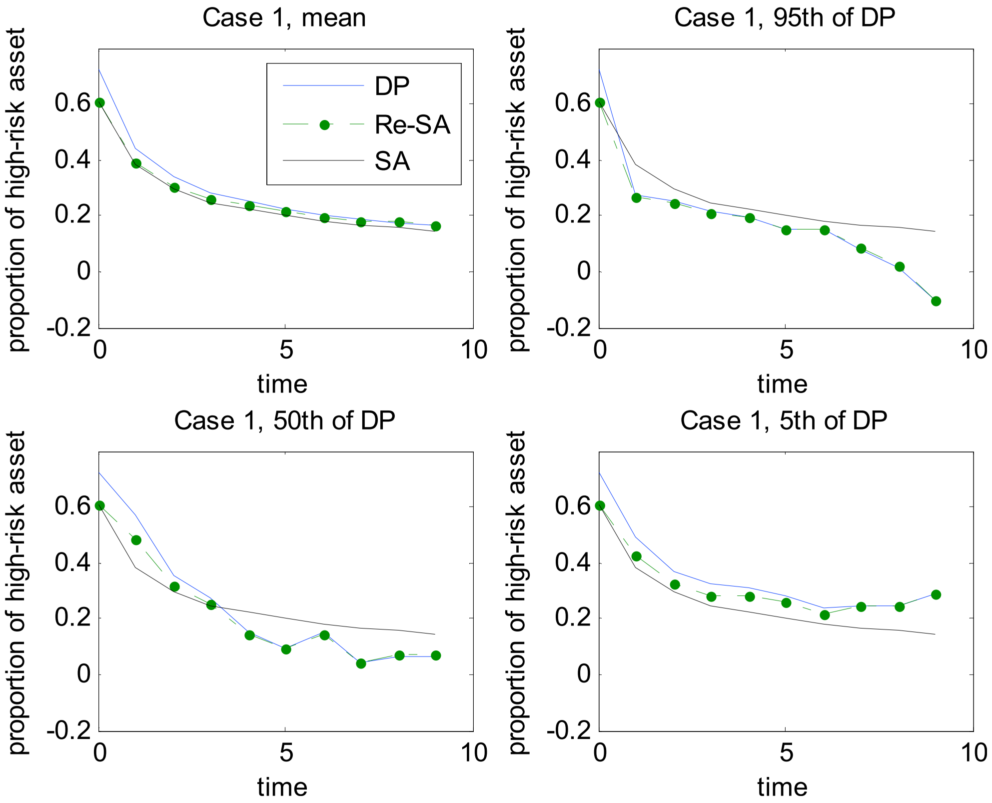

Fig. 4. The SA, Re-SA, and DP strategies.

However, the standard deviations indicate opposite relationships, such that they are smaller for SA (0.1131 and 0.0843) than for DP (0.1228 and 0.0906). According to the mean variance criterion, investors should select the approach that minimizes both the expected total future costs and variance simultaneously. In terms of the mean variance criterion, investors would not always prefer DP to SA.

For each benchmark, G(0), G(10), or $F(10)/F^*(10)$, a better strategy leads to a smaller value for G(0) and G(10) or a value closer to 1 for $F(10)/F^*(10)$. For example, the distribution of G(0) with DP leads to a smaller value at the 1st, 5th, 25th, 50th, and 75th percentiles, but it produces a greater cost in the 95th and 99th percentiles. The distribution of G(10) indicates similar properties. Fig. 2 depicts the numerical cumulative distribution function (cdf) of G(0) over one simulation group (i.e., 1000 simulation paths); it reveals that the strategy proposed by DP suffers a large right-tail (downside) risk. For example, the probability that G(0) is greater than 0.4, a higher total future cost at time *0*, is nearly 0.05 with DP but decreases to 0.03 for SA. Therefore, the strategy proposed by DP raises more outliers, leading to a higher standard deviation. Figure C-1 in Appendix C provides the corresponding data over 100,000 simulation paths.[8]

To appreciate a scenario with outliers, we also observe the distribution of $F(10)/F^*(10)$. In Table 2, the distribution with DP, compared to that with SA, leads to a value that is closer to 1 in most percentiles but farther from 1 in the 1st, 5th and 75th percentiles.[9] This result implies a greater downside risk of using DP, such that the final fund level may be much lower than the target. Fig. 3 confirms this finding by depicting the simulated cdf of $F(10)/F^*(10)$ for this simulation group. The left tail of DP is longer than that of SA when the value is less than 0.8. When investment performance is poor, the DP strategy would require a higher proportion of high-risk assets, and that would lead to a much lower final fund level once investment performance became poor again in subsequent periods.[10] Figure C-2 in Appendix C depicts the corresponding figure over 100,000 simulation paths.

The two probability values at the top of Table 2 reinforce the notion that DP is a superior tool on average (mean criterion). The investment target is to minimize the expectation of the total future cost at time *t*. Therefore, if one approach leads to a smaller total future cost at time *t* in a simulation path, that approach is better than the other approach in this path, conditional on time *t*. Accordingly, we

[12] For a detailed review of the figure, we draw the differences between the distance from the order statistics of F(10)/F*(10) to 1 with DP and that with SA, i.e., the difference in the horizontal distance from each line to 1, in Figure C-3 (see the blue line of the right plot).

[12] For a detailed review of the figure, we draw the differences between the distance from the order statistics of F(10)/F*(10) to 1 with DP and that with SA, i.e., the difference in the horizontal distance from each line to 1, in Figure C-3 (see the blue line of the right plot).

[8] We gather the 100 groups, each containing 1000 simulation paths, to produce the 100,000 paths.

[9] For the 75th percentile, the value of $F(10)/F^*(10)$ is close to 1 and the relative size is unsteady.

[10] Alternatively, larger values of $F(10)/F^*(10)$ are not risk. For example, though the 95th percentile of SA (1.1208) is greater than that of DP (1.1099), which results in greater costs, investors should prefer SA because it leads to a higher final fund level.

Table 3
Simulated statistics over three approaches (Case 1).

Case 1, $F^*(1) = 0.1269$ $F^*(10) = 1.6565$									
	DP			Re-SA			SA		
	Pr(DP < ReSA)	Pr(DP < SA)		Pr(ReSA < DP)	Pr(ReSA < SA)		Pr(SA < DP)	Pr(SA < ReSA)	
G(0)	0.6354	0.6402		0.3646	0.6133		0.3598	0.3867	
	(0.0137)	(0.0146)		(0.0137)	(0.0149)		(0.0146)	(0.0149)	
G(10)	0.5657	0.6005		0.4343	0.5926		0.3995	0.4074	
	(0.0144)	(0.0143)		(0.0144)	(0.0156)		(0.0143)	(0.0156)	
	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
Mean	**0.1282**	*0.0626*	**0.9524**	*0.1284*	**0.0624**	*0.9506*	0.1340	0.0673	0.9496
	(0.0041)	(0.0030)	(0.0031)	(0.0039)	(0.0029)	(0.0031)	(0.0038)	(0.0029)	(0.0029)
St. dev.	0.1228	0.0906	*0.0956*	*0.1185*	*0.0889*	**0.0945**	**0.1131**	**0.0843**	0.0985
	(0.0068)	(0.0064)	(0.0024)	(0.0060)	(0.0060)	(0.0023)	(0.0044)	(0.0044)	(0.0024)
1st	**0.0088**	**1.1E-05**	0.7257	*0.0093*	*1.2E-05*	*0.7283*	0.0095	1.4E-05	**0.7456**
	(0.0010)	(7.E−06)	(0.0131)	(0.0011)	(9.E−06)	(0.0126)	(0.0012)	(1.E−05)	(0.0087)
5th	**0.0176**	**2.5E-04**	0.7972	*0.0186*	*2.5E-04*	*0.7974*	0.0193	3.2E-04	**0.7977**
	(0.0012)	(7.E−05)	(0.0063)	(0.0013)	(7.E−05)	(0.0061)	(0.0014)	(8.E−05)	(0.0051)
25th	**0.0466**	**0.0064**	**0.8897**	*0.0484*	*0.0065*	*0.8884*	0.0520	0.0082	0.8803
	(0.0019)	(0.0008)	(0.0044)	(0.0019)	(0.0008)	(0.0043)	(0.0021)	(0.0009)	(0.0042)
50th	**0.0892**	**0.0284**	**0.9514**	*0.0913*	*0.0288*	*0.9494*	0.0999	0.0353	0.9435
	(0.0036)	(0.0020)	(0.0036)	(0.0036)	(0.0021)	(0.0035)	(0.0040)	(0.0025)	(0.0036)
75th	**0.1666**	**0.0818**	1.0130	*0.1677*	*0.0821*	**1.0106**	0.1818	0.0950	*1.0117*
	(0.0069)	(0.0054)	(0.0039	(0.0068)	(0.0053)	(0.0040	(0.0074)	(0.0055)	(0.0043)
95th	0.3696	0.2398	*1.1099*	*0.3635*	**0.2373**	**1.1067**	**0.3633**	*0.2396*	1.1208
	(0.0175)	(0.0143)	(0.0074)	(0.0172)	(0.0134)	(0.0073)	(0.0145)	(0.0120)	(0.0074)
99th	0.5850	0.4210	*1.1863*	*0.5646*	*0.4121*	**1.1823**	**0.5192**	**0.3713**	1.2029
	(0.0459)	(0.0386)	(0.0151)	(0.0463)	(0.0376)	(0.0154)	(0.0312)	(0.0250)	(0.0168)

Note: Values without brackets are the average, and values in brackets present the standard deviation. For a given benchmark, the value of the best strategy is marked in bold, and the value of the secondary strategy is marked in *italics*. Values of DP and SA are identical with values in Table 2. Appendix D (Case 1) presents the two-tailed Welch's *t*-test for each pair of samples.

calculate the probability that DP performs better than SA over the 100 groups of simulations. We use the estimated probability over one group as one observation. The average of the 100 observations is equal to the probability estimated directly from the 100,000 simulation paths.

For a specific simulation path, a better strategy would produce a smaller G(0), smaller G(10), and smaller $|F(10)/F^*(10) - 1|$. As Table 2 shows, the probability that DC performs better than SA is 64.02% according to G(0) and 60.05% by G(10); both are greater than one-half. The probability that DP performs better than SA, as determined by G(10) and $|F(10)/F^*(10) - 1|$, is equivalent. Hence, DP is superior to SA on average.

4. Re-assess by SA

In this section, we compare the strategies and the efficiencies among SA, Re-SA, and DP. For the Re-SA strategy, the fund level at time 1 is calculated according to the realized scenario and we thus have 1000 fund levels in 1000 paths of a group. We next need to determine the investment weight at time 1 for the fund level of each path. For scenarios of the future 9 years, we do not resimulate the rate of return. Alternatively, we use the existing random sample, the simulated rate of returns from time 1 to the end of the investment horizon for each group, as a return rate scenario. We again find the optimal static investment strategy for each group (scenario) numerically and use the average of the 100 strategies as the optimal static investment strategy for a specific path. This procedure is iterative until the last decision date.

The upper-left plot of Fig. 4 illustrates the mean strategy pattern of the three decision methods. We consider the specific group, the group used in subsection 3.1 and Figs. 2 and 3, as the realized scenario. The average of the strategies suggested by Re-SA is close to that of SA during the initial years, and then it gradually approximates the average of the strategies suggested by DP. In the remainder of Fig. 4, in line with the lower plot of Fig. 1, we illustrate potential investment strategy scenarios corresponding to the 5th, 50th, and 95th percentiles of the final fund level. It is obvious that, over the three specific scenarios, the strategic pattern proposed by Re-SA appears similar to what we find in the upper-left plot. The strategy suggested by Re-SA is close to that of SA initially and then approaches the DP strategy gradually. Thus, the efficiency of Re-SA falls in between that of DP and SA.

Table 3 reports the mean, standard deviation, and percentiles, as well as the probability that one approach performs better than the other for Case 1. Using rule G(0), the probability that DP performs better than Re-SA is 63.54% over the 100,000 paths; the probability that DP performs better than SA is 64.02%; and the probability that Re-SA performs better than SA is 61.33%. When performance is measured by G(10), these respective values are 56.57%, 60.05%, and 59.26%. Thus, the performance of DP is superior to Re-SA (and SA), and the performance of Re-SA is superior to SA in more than half of the paths. Table 3, an extension of Table 2, presents the numerical results for Case 1 and results for other cases are presented in Appendix B.

Theoretically, the mean of G(0) for DP should be the smallest among three strategies and the numerical results in Table 3, as well as tables in Appendix B, confirm this prediction. A further observation from Table 3 is that the means with Re-SA and with DP, as well as

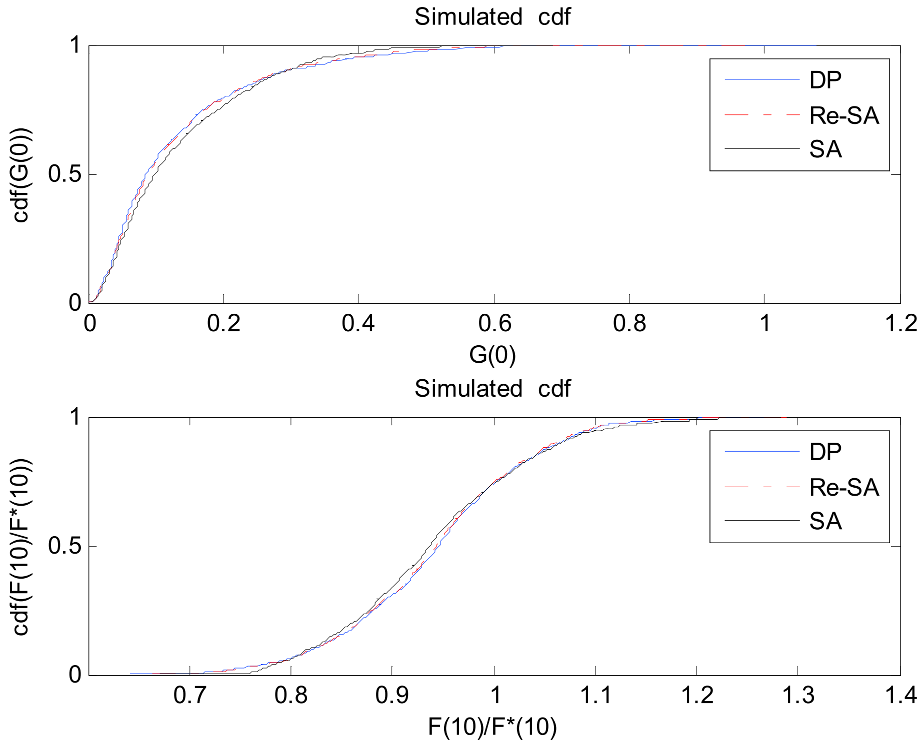

Fig. 5. Simulated cdf of G(0) and F(10)/F*(10)[131].

Table 4
Kolmogorov-Smirnov goodness-of-fit hypothesis test.

	DP vs. Re-SA	DP vs. SA	Re-SA vs. SA
Case 1			
G(0)	0.9519	0.0279	0.1048
G(10)	0.9999	0.0525	0.0748
$F(10)/F^{*}(10)$[a]	0.9672	0.1168	0.2350
Case 2			
G(0)	0.0214	0.0000	0.0050
G(10)	0.8228	0.0000	0.0000
$F(10)/F^{*}(10)$	0.3344	0.0001	0.0043
Case 3			
G(0)	1.0000	0.3072	0.6028
G(10)	1.0000	0.3935	0.5654
$F(10)/F^{*}(10)$	1.0000	0.5654	0.6785
Case 4			
G(0)	0.1298	0.0001	0.0068
G(10)	0.3632	0.0000	0.0006
$F(10)/F^{*}(10)$	0.3632	0.0002	0.0019
Case 5			
G(0)	0.9790	0.1441	0.2140
G(10)	1.0000	0.0464	0.0666
$F(10)/F^{*}(10)$	0.9995	0.2575	0.3935
Case 6			
G(0)	0.0525	0.0000	0.0006
G(10)	0.5654	0.0000	0.0000
$F(10)/F^{*}(10)$	0.2575	0.0000	0.0004

[a] This is equivalent to testing $F(10)$ with two approaches drawn from the same underlying population because $F^{*}(10)$ is a constant.

their standard deviations, are very close (an exception is the mean of $F(10)/F^{*}(10)$). Furthermore, most percentiles of Re-SA are close to DP, which suggests that the outcome reached from an investment strategy determined by Re-SA would be very close to that of DP.

Our final contribution is to demonstrate this inference. Again, we consider the group used in subsection 3.1, and Figs. 2 and 3, as the realized scenario. The upper plot of Fig. 5 depicts the cdf of G(0) and the lower plot depicts the cdf of $F(10)/F^{*}(10)$ among the three

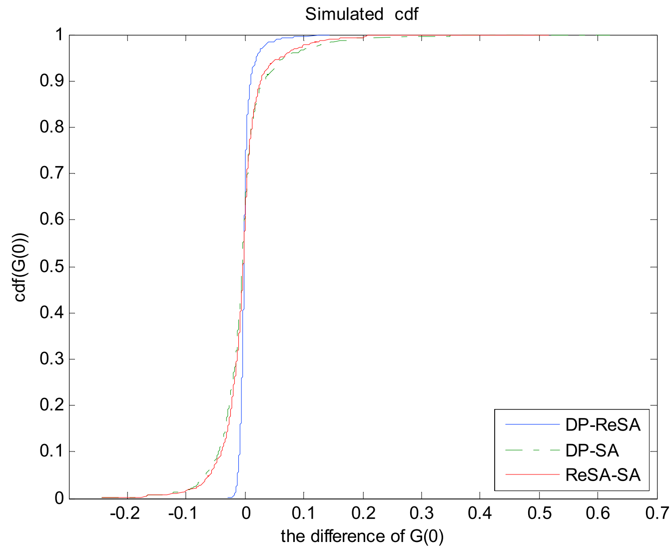

Fig. 6. Simulated cdfs of the differences in G(0).

Table 5
Maximum and minimum values of the differences in G(0).

	Minimum	Maximum
DP – Re-SA	−0.0285	0.1439
DP – SA	−0.2442	0.6203
Re-SA – SA	−0.2447	0.5161

strategies. The cdfs for Re-SA and for DP are nearly identical. Table 4 contains the p-value of the Kolmogorov-Smirnov goodness-of-fit hypothesis test for the 6 cases, with the null hypothesis being that the corresponding random samples are drawn from the same underlying continuous population. For example, in Case 1, the p-value is 0.9519 for the null hypothesis that G(0) with DP and G(0) with Re-SA are drawn from the same underlying population; it is 0.9999 and 0.9672 when considering the distribution of G(10) and $F(10)/F^*(10)$. When considering DP and Re-SA, except G(0) for Case 2, Table 4 suggests that we do not reject the null hypothesis at the 5% significance level for all cases. Conversely, we reject the null hypothesis at the 5% significance level for Cases 2, 4 and 6 when comparing DP (or Re-SA) and SA. Furthermore, a consistent conclusion is that the p-value when considering DP vs. Re-SA is always far greater than the p-value for the other two combinations. This indicates that among three potential combinations, DP and Re-SA share a distribution likeness.

To thoroughly confirm our inference, we still need to show that, for a specific simulation path, the G(0)s (G(10), and/or $F(10)/F^*(10)$) of Re-SA and of DP are close to each other. Therefore, we check the distribution of the differences using pairwise observations for each simulation path. Fig. 6 depicts the cdf of the differences for G(0), and Table 5 displays the maximum and minimum values of the differences. Differences between DP and Re-SA are nearly 0 (the blue line in Fig. 6); they are located within an extremely concentrated interval, with a maximum of 0.1439 and minimum of −0.0285 (Table 5) over the 1000 simulation paths. Thus, the numerical results of DP and Re-SA are very similar.

Fig. 6 and Table 5 also confirm that the downside risk would be greater if a strategy were decided by DP (or Re-SA). The distribution of the differences between DP (Re-SA) and SA is right-tailed, with a maximum of 0.6203 (0.5161) and minimum of −0.2442 (−0.2447). The values of the differences are negative when the performance of DP (Re-SA) is better than SA, and positive otherwise. Thus, there are extreme outliers when the performance of SA is better than DP (Re-SA), which implies a greater downside risk of using an adaptive investment strategy.

5. Robustness analysis for the investment horizon

The investment horizon for a life insurance company or a pension fund may be very long, such as 20 or 30 years. However, the duration of the aggregate liabilities is shorter. For example, the typical pension plan has a duration of approximately 15 (McCaulay, 2013). In this section, we examine the robustness of our inference for a large n. We consider only Case 1 and a 15-year investment

[13] In line with Figs. 2 and 3, we draw the horizontal distance of the lines in Figure C-3.

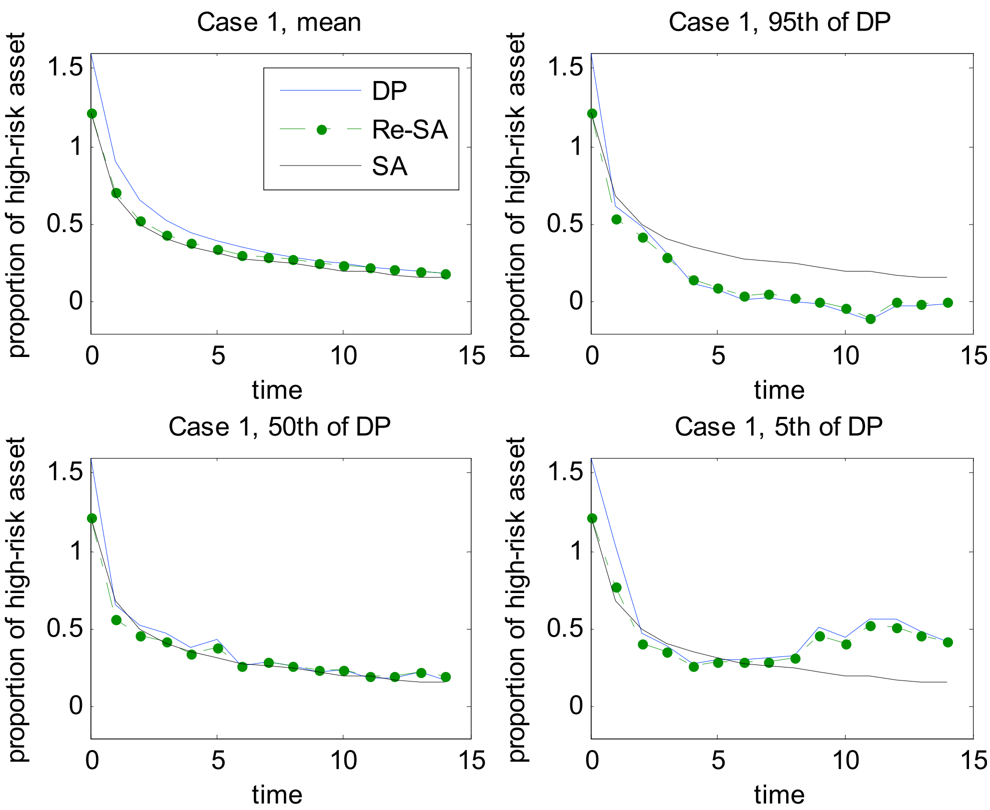

Fig. 7. The SA, Re-SA, and DP strategies for $n = 15$.

horizon for robustness analysis[14].

Fig. 7, similar to Fig. 4, depicts the investment strategies for a 15-year horizon. The upper-left plot of Fig. 7 shows the mean patterns of the three strategies over 1000 paths, and the other 3 plots illustrate potential investment strategy scenarios corresponding to the 5th, 50th, and 95th percentiles of the final fund level. Likewise, the strategy suggested by Re-SA is near that of SA initially and then approaches the DP strategy gradually.

Table 6, similar to Table 3, presents the mean, standard deviation and percentiles, as well as the probability that one approach performs better than the other for Case1.[15] Again, the performance of DP is superior to Re-SA (and SA), and the performance of Re-SA is superior to SA in more than half of the paths. In addition, DP (and Re-SA) has a better performance over most percentiles (1st, 5th, 25th, 50th and 75th) for G(0) and G(15). However, the SA strategy is better than DP, in the viewpoint of G(0) and G(15), when there is a poor investment performance (95th and 99th percentiles).

Table 7, similar to Table 4, presents the *p*-value of the Kolmogorov-Smirnov goodness-of-fit hypothesis test among three potential combinations. For combination DP vs. Re-SA, Table 4 suggests that we do not reject the null hypothesis at the 5% significance level when considering the distribution of G(15) and $F(15)/F^*(15)$ but reject it when considering G(0). However, the p-value for DP vs. Re-SA (0.0464) is much greater than that for DP vs. SA (0.0000) and Re-SA vs. SA (0.0011). DP and Re-SA still share much distribution likeness among the three combinations.

6. Conclusions

In this article, we examine the differences and the efficiencies of investment strategies for anticipative and adaptive models. Three investment strategies, SA, Re-SA, and DP, are subject to comparison, and we find that the investment strategy with SA is the most conservative, followed by the average investment scenarios with Re-SA and then DP. The strategy suggested by DP may allow for relatively aggressive decisions by posing subsequent adaptation opportunities.

The numerical results show that both models, anticipative and adaptive, have individual merits and weaknesses. The performance of DP is better than that of SA and Re-SA in more than half of the simulation paths; likewise, the performance of Re-SA is better than SA.

[14] The runtime cost is exponentially increasing by n and has been very large for $n = 15$.

[15] In Table 6, we consider only 10 groups (estimates) for the 15-year horizon because of the runtime cost. Furthermore, in line with Tables 2 and 3, the SA (and the Re-SA) strategy is determined by the average of the 100 optimal static investment strategies over 100 groups.

Table 6
Simulated statistics over three approaches (Case 1 and $n = 15$).

Case 1, $F^*(1) = 0.1269$ $F^*(15) = 2.9071$									
	DP			Re-SA			SA		
	Pr(DP < ReSA)	Pr(DP < SA)		Pr(ReSA < DP)	Pr(ReSA < SA)		Pr(SA < DP)	Pr(SA < ReSA)	
G(0)	0.7224	0.7295		0.2776	0.6753		0.2705	0.3247	
	(0.0152)	(0.0085)		(0.0152)	(0.0120)		(0.0085)	(0.0120)	
G(10)	0.5933	0.6511		0.4067	0.6399		0.3489	0.3601	
	(0.0152)	(0.0142)		(0.0152)	(0.0109)		(0.0142)	(0.0109)	
	G(0)	G(15)	F_{15}/F^*_{15}	G(0)	G(15)	F_{15}/F^*_{15}	G(0)	G(15)	F_{15}/F^*_{15}
Mean	**0.8263**	*0.2957*	**0.9457**	*0.8353*	**0.2917**	*0.9398*	0.8954	0.3385	0.9371
	(0.0307)	(0.0138)	(0.0042)	(0.0259)	(0.0130)	(0.0040)	(0.0257)	(0.0145)	(0.0040)
St. dev.	0.8161	0.4939	*0.1206*	*0.7199*	*0.4544*	**0.1167**	**0.6718**	**0.4232**	0.1267
	(0.0426)	(0.0205)	(0.0028)	(0.0324)	(0.0200)	(0.0026)	(0.0308)	(0.0262)	(0.0030)
1st	**0.0626**	*4.1E-05*	0.6294	*0.0791*	**3.6E-05**	*0.6419*	0.0798	7.0E-05	**0.6850**
	(0.0051)	(3.E−05)	(0.0134)	(0.0068)	(2.E−05)	(0.0138)	(0.0083)	(4.E−05)	(0.0103)
5th	**0.1218**	*0.0009*	0.7448	*0.1454*	**0.0008**	**0.7467**	0.1511	0.0016	*0.7457*
	(0.0057)	(0.0003)	(0.0085)	(0.0080)	(0.0003)	(0.0073)	(0.0103)	(0.0004)	(0.0059)
25th	**0.3148**	**0.0256**	**0.8722**	*0.3558*	*0.0269*	*0.8671*	0.3962	0.0412	0.8473
	(0.0110)	(0.0027)	(0.0043)	(0.0102)	(0.0030)	(0.0037)	(0.0166)	(0.0036)	(0.0040)
50th	**0.5737**	**0.1137**	**0.9478**	*0.6163*	*0.1188*	*0.9419*	0.7077	0.1774	0.9268
	(0.0215)	(0.0066)	(0.0039)	(0.0203)	(0.0064)	(0.0039)	(0.0199)	(0.0064)	(0.0047)
75th	**1.0362**	**0.3526**	1.0203	*1.0785*	*0.3669*	**1.0124**	1.2316	0.4857	*1.0148*
	(0.0424)	(0.0175)	(0.0049)	(0.0361)	(0.0196)	(0.0041)	(0.0430)	(0.0268)	(0.0051)
95th	2.3462	1.2091	*1.1378*	*2.2538*	**1.1512**	**1.1279**	**2.2446**	*1.1850*	1.1612
	(0.1937)	(0.0795)	(0.0087)	(0.1464)	(0.0627)	(0.0071)	(0.1265)	(0.0580)	(0.0082)
99th	3.9962	2.3652	*1.2405*	*3.5173*	*2.1848*	**1.2287**	**2.9892**	**1.8616**	1.2700
	(0.2367)	(0.1682)	(0.0269)	(0.1861)	(0.1798)	(0.0271)	(0.1642)	(0.1597)	(0.0235)

Note: For a given benchmark, the value of the best strategy is marked in bold, and the value of the secondary strategy is marked in *italics*. Appendix D (Case 1, $n = 15$) presents the two-tailed Welch's t-test for each pair of samples.

Table 7
Goodness-of-fit hypothesis test ($n = 15$).

	DP vs. Re-SA	DP vs. SA	Re-SA vs. SA
G(0)	0.0464	0.0000	0.0011
G(15)	0.9519	0.0000	0.0002
$F(15)/F^*(15)$	0.4253	0.0000	0.0031

The DP decision is better able to produce the smallest cost, which is the best decision method from the point of view of expectations. However, DP suffers greater variance and may result in serious downside risks. This property of the DP strategy is important to, but may be ignored by, investors and risk managers. Finally, the strategy, and thus the investment performance, of Re-SA is very similar to that of DP. As the DP strategy is often unreachable in practice, the Re-SA decision-making method that we propose herein can be a satisfactory proxy for finding the DP strategy.

Author statement

Hong-Chih Huang: Conceptualization, Funding acquisition, Software, Supervision, Validation, Writing - review & editing.

Yung-Tsung Lee: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Software, Writing - original draft, Writing - review & editing.

Declaration of competing interest

None.

CRediT authorship contribution statement

Hong-Chih Huang: Conceptualization, Funding acquisition, Software, Supervision, Validation, Writing - review & editing. **Yung-Tsung Lee:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Software, Writing - original draft, Writing - review & editing.

Acknowledgements

We appreciate the helpful comments and suggestions from two anonymous referees.

Appendix E. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.iref.2020.03.007.

Appendix A. Optimal dynamic investment strategy

In the model of Vigna and Haberman (2001), the value function at time t is defined as

$$J(\mathfrak{I}_t) = \min_{\{\pi_t\}} E[G(t)|\mathfrak{I}_t], \tag{A-1}$$

where $\{\pi_t\} = \{\{\omega(s)\}_{s=t,t+1,\dots,n-1}\}$ represents the set of the future possible investment strategies. The objective of the employee is to choose a feasible investment strategy that minimizes the expectation of the total future cost. Bellman's optimality principle gives

$$J(\mathfrak{I}_t) = \min_{\{\pi_t\}} E\left[\sum_{s=t}^{n} v^{s-t} C(s)|\mathfrak{I}_t\right] = \min_{\omega(t)}\{C(t) + vE[J(\mathfrak{I}_{t+1})|\mathfrak{I}_t]\Big\}, \tag{A-2}$$

and according to Vigna and Haberman (2001), the corresponding optimal dynamic investment strategy is

$$\omega^*(t) = \frac{Q_{t+1}V}{P_{t+1}(F(t)+c)D} - \frac{W}{D}, \quad \text{for } t = 0, 1, \dots, n-1; \tag{A-3}$$

where the sequences $\{P_t\}_{t=1,\dots,n}$ and $\{Q_t\}_{t=1,\dots,n}$ are provided recursively by

$$P_t = 1 + \nu H P_{t+1}, \text{ and } Q_t = F^*(t) - \nu c H P_{t+1} - \nu K Q_{t+1}, \tag{A-4}$$

with $P_n = \theta$ and $Q_n = \theta F^*(n)$. Furthermore, $D, H, K, V,$ and W are given respectively by

$$D = e^{2\mu+2\sigma_1^2} + e^{2\lambda+2\sigma_2^2} - 2e^{\mu+\lambda+0.5\left(\sigma_1^2+\sigma_2^2\right)}, \tag{A-5}$$

$$H = \frac{1}{D}\left[e^{2\mu+2\lambda+\sigma_1^2+\sigma_2^2}\left(e^{\sigma_1^2+\sigma_2^2} - 1\right)\right], \tag{A-6}$$

$$K = \frac{1}{D} e^{\mu+\lambda+0.5\left(\sigma_1^2+\sigma_2^2\right)}\left[e^{\mu+0.5\sigma_1^2} + e^{\lambda+0.5\sigma_2^2} - e^{\mu+1.5\sigma_1^2} - e^{\lambda+1.5\sigma_2^2}\right], \tag{A-7}$$

$$V = e^{\lambda+0.5\sigma_2^2} - e^{\mu+0.5\sigma_1^2}, \text{ and } \quad (A-8) W = e^{\mu+\lambda+0.5\left(\sigma_1^2+\sigma_2^2\right)} - e^{2\mu+2\sigma_1^2}. \tag{A-9}$$

For more details, please see Vigna and Haberman (2001).

Appendix B. Probability that DP Performs Better than SA and Simulated Statistics for Cases 2–6

Table B1
Simulated statistics over three approaches (Case 2)

Case 2, $F^*(1) = 0.1265$ $F^*(10) = 1.6215$

	DP		Re-SA		SA	
	Pr(DP < ReSA)	Pr(DP < SA)	Pr(ReSA < DP)	Pr(ReSA < SA)	Pr(SA < DP)	Pr(SA < ReSA)
G(0)	0.7405	0.7526	0.2595	0.6800	0.2475	0.3200
	(0.0137)	(0.0133)	(0.0137)	(0.0136)	(0.0133)	(0.0136)
G(10)	0.6090	0.6779	0.3910	0.6564	0.3221	0.3436
	(0.0151)	(0.0140)	(0.0151)	(0.0150)	(0.0140)	(0.0150)

	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
Mean	**0.0662**	*0.0228*	**0.9671**	*0.0668*	**0.0226**	*0.9640*	0.0708	0.0258	0.9633
	(0.0025)	(0.0015)	(0.0020)	(0.0021)	(0.0013)	(0.0019)	(0.0019)	(0.0011)	(0.0018)
St. dev.	0.0740	0.0421	*0.0570*	*0.0641*	*0.0383*	**0.0548**	**0.0570**	**0.0322**	0.0596
	(0.0069)	(0.0056)	(0.0018)	(0.0044)	(0.0039)	(0.0016)	(0.0020)	(0.0015)	(0.0014)
1st	**0.0041**	**2.6E-06**	0.8030	0.0051	*3.0E-06*	*0.8119*	*0.0048*	5.4E-06	**0.8354**
	(0.0006)	(2.E−06)	(0.0138)	(0.0007)	(2.E−06)	(0.0116)	(0.0006)	(3.E−06)	(0.0058)
5th	**0.0082**	**6.7E-05**	0.8675	*0.0101*	*7.6E-05*	*0.8687*	0.0103	1.2E-04	**0.8690**
	(0.0006)	(2.E−05)	(0.0055)	(0.0007)	(2.E−05)	(0.0053)	(0.0008)	(3.E−05)	(0.0033)
25th	**0.0219**	**0.0018**	**0.9347**	*0.0255*	*0.0019*	*0.9317*	0.0286	0.0031	0.9217

(continued on next page)

Table B1 (*continued*)

	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
	(0.0009)	(0.0002)	(0.0029)	(0.0010)	(0.0002)	(0.0028)	(0.0013)	(0.0004)	(0.0027)
50th	**0.0427**	**0.0082**	**0.9713**	*0.0472*	*0.0087*	*0.9675*	0.0544	0.0135	0.9606
	(0.0019)	(0.0006)	(0.0021)	(0.0018)	(0.0007)	(0.0021)	(0.0020)	(0.0010)	(0.0024)
75th	**0.0821**	**0.0256**	1.0040	*0.0855*	*0.0268*	**0.9999**	0.0971	0.0365	*1.0020*
	(0.0037)	(0.0020)	(0.0021)	(0.0035)	(0.0020)	(0.0021)	(0.0034)	(0.0021)	(0.0027)
95th	0.2032	0.0928	*1.0519*	*0.1899*	**0.0908**	**1.0468**	**0.1860**	*0.0926*	1.0654
	(0.0124)	(0.0073)	(0.0035)	(0.0097)	(0.0067)	(0.0035)	(0.0065)	(0.0045)	(0.0043)
99th	0.3579	0.1976	*1.0891*	*0.3088*	*0.1820*	**1.0827**	**0.2574**	**0.1432**	1.1109
	(0.0341)	(0.0279)	(0.0074)	(0.0251)	(0.0216)	(0.0075)	(0.0131)	(0.0098)	(0.0081)

Note: For a given benchmark, the value of the best strategy is marked in bold, and the value of the secondary strategy is marked in *italics*. Appendix D (Case 2) presents the two-tailed Welch's *t*-test for each pair of samples.

Table B2
Simulated statistics over three approaches (Case 3)

Case 3, $F^*(1) = 0.1277$ $F^*(10) = 1.7197$

	DP		Re-SA		SA	
	Pr(DP < ReSA)	Pr(DP < SA)	Pr(ReSA < DP)	Pr(ReSA < SA)	Pr(SA < DP)	Pr(SA < ReSA)
G(0)	0.5598	0.5698	0.4402	0.5643	0.4302	0.4357
	(0.0148)	(0.0141)	(0.0148)	(0.0151)	(0.0141)	(0.0151)
G(10)	0.5249	0.5581	0.4751	0.5579	0.4419	0.4421
	(0.0154)	(0.0167)	(0.0154)	(0.0169)	(0.0167)	(0.0169)

	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
Mean	**0.3058**	*0.1963*	**0.9504**	*0.3059*	**0.1959**	*0.9495*	0.3175	0.2064	0.9475
	(0.0101)	(0.0095)	(0.0055)	(0.0100)	(0.0094)	(0.0055)	(0.0107)	(0.0099)	(0.0052)
St. dev.	0.3020	0.2934	*0.1752*	*0.3000*	*0.2920*	**0.1748**	**0.2997**	**0.2877**	0.1792
	(0.0264)	(0.0364)	(0.0047)	(0.0263)	(0.0364)	(0.0047)	(0.0222)	(0.0288)	(0.0048)
1st	*0.0218*	**3.7E-05**	*0.5976*	0.0219	*3.8E-05*	0.5976	**0.0217**	4.1E-05	**0.6093**
	(0.0023)	(3.E−05)	(0.0162)	(0.0022)	(3.E−05)	(0.0159)	(0.0025)	(2.E−05)	(0.0135)
5th	**0.0422**	**8.2E-04**	**0.6896**	*0.0424*	*8.4E-04*	*0.6894*	0.0429	9.6E-04	0.6878
	(0.0026)	(0.0002)	(0.0086)	(0.0027)	(0.0002)	(0.0083)	(0.0031)	(0.0003)	(0.0081)
25th	**0.1093**	**0.0212**	**0.8291**	*0.1098*	*0.0212*	*0.8285*	0.1138	0.0244	0.8201
	(0.0046)	(0.0027)	(0.0072)	(0.0047)	(0.0027)	(0.0072)	(0.0048)	(0.0028)	(0.0067)
50th	**0.2113**	**0.0934**	**0.9357**	*0.2123*	*0.0934*	*0.9349*	0.2234	0.1050	0.9285
	(0.0079)	(0.0066)	(0.0068)	(0.0079)	(0.0065)	(0.0068)	(0.0087)	(0.0074)	(0.0067)
75th	**0.3992**	**0.2600**	1.0539	*0.4002*	*0.2603*	**1.0528**	0.4248	0.2829	*1.0531*
	(0.0173)	(0.0138)	(0.0078)	(0.0174)	(0.0138)	(0.0075)	(0.0184)	(0.0141)	(0.0080)
95th	*0.8784*	*0.7111*	*1.2585*	**0.8748**	**0.7090**	**1.2567**	0.8934	0.7228	1.2699
	(0.0453)	(0.0435)	(0.0160)	(0.0447)	(0.0434)	(0.0161)	(0.0401)	(0.0439)	(0.0162)
99th	1.3788	1.2677	*1.4333*	*1.3664*	*1.2561*	**1.4316**	**1.3372**	**1.2446**	1.4445
	(0.0993)	(0.1432)	(0.0364)	(0.0995)	(0.1398)	(0.0361)	(0.0877)	(0.1374)	(0.0331)

Note: For a given benchmark, the value of the best strategy is marked in bold, and the value of the secondary strategy is marked in *italics*. Appendix D (Case 3) presents the two-tailed Welch's *t*-test for each pair of samples.

Table B3
Simulated statistics over three approaches (Case 4)

Case 4, $F^*(1) = 0.1274$ $F^*(10) = 1.6956$

	DP		Re-SA		SA	
	Pr(DP < ReSA)	Pr(DP < SA)	Pr(ReSA < DP)	Pr(ReSA < SA)	Pr(SA < DP)	Pr(SA < ReSA)
G(0)	0.7358	0.7373	0.2642	0.6706	0.2627	0.3294
	(0.0135)	(0.0141)	(0.0135)	(0.0141)	(0.0141)	(0.0141)
G(10)	0.7022	0.7074	0.2978	0.6617	0.2927	0.3383
	(0.0146)	(0.0159)	(0.0146)	(0.0157)	(0.0159)	(0.0157)

	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
Mean	**0.1423**	**0.0583**	**0.9246**	*0.1429*	*0.0585*	*0.9218*	0.1476	0.0621	0.9211
	(0.0046)	(0.0029)	(0.0023)	(0.0041)	(0.0027)	(0.0022)	(0.0036)	(0.0022)	(0.0021)
St. dev.	0.1376	0.0839	*0.0667*	*0.1225*	*0.0779*	**0.0637**	**0.1080**	**0.0645**	0.0676
	(0.0084)	(0.0068)	(0.0020)	(0.0062)	(0.0053)	(0.0018)	(0.0031)	(0.0023)	(0.0017)
1st	**0.0073**	**1.3E-05**	0.7344	0.0090	*1.6E-05*	*0.7454*	*0.0083*	2.3E-05	**0.7805**
	(0.0011)	(1.E−05)	(0.0145)	(0.0013)	(1.E−05)	(0.0119)	(0.0014)	(2.E−05)	(0.0062)
5th	**0.0160**	**0.0003**	0.8051	0.0201	*0.0004*	*0.8084*	*0.0195*	0.0005	**0.8164**
	(0.0012)	(1.E−04)	(0.0065)	(0.0015)	(1.E−04)	(0.0056)	(0.0016)	(1.E−04)	(0.0033)
25th	**0.0496**	**0.0070**	**0.8858**	*0.0571*	*0.0082*	*0.8835*	0.0623	0.0117	0.8738
	(0.0023)	(0.0008)	(0.0037)	(0.0022)	(0.0009)	*(0.0035)*	(0.0031)	(0.0014)	(0.0030)

(*continued on next page*)

Table B3 (*continued*)

	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
50th	**0.1012**	**0.0281**	**0.9309**	*0.1088*	*0.0309*	*0.9271*	0.1235	0.0418	0.9170
	(0.0042)	(0.0022)	(0.0027)	(0.0040)	(0.0022)	(0.0026)	(0.0044)	(0.0027)	(0.0028)
75th	**0.1872**	**0.0750**	**0.9699**	*0.1899*	*0.0779*	*0.9654*	0.2080	0.0923	0.9642
	(0.0079)	(0.0049)	(0.0025)	(0.0075)	(0.0046)	(0.0024)	(0.0068)	(0.0042)	(0.0032)
95th	0.4103	0.2165	1.0219	*0.3829*	*0.2095*	1.0165	**0.3579**	**0.1934**	1.0383
	(0.0207)	(0.0140)	(0.0037)	(0.0165)	(0.0121)	(0.0036)	(0.0101)	(0.0069)	(0.0050)
99th	0.6530	0.3930	*1.0577*	*0.5796*	*0.3629*	**1.0516**	**0.4693**	**0.2737**	1.0942
	(0.0509)	(0.0419)	(0.0071)	(0.0391)	(0.0333)	(0.0071)	(0.0202)	(0.0155)	(0.0101)

Note: For a given benchmark, the value of the best strategy is marked in bold, and the value of the secondary strategy is marked in *italics*. Appendix D (Case 4) presents the two-tailed Welch's *t*-test for each pair of samples.

Table B4
Simulated statistics over three approaches (Case 5)

Case 5, $F^*(1) = 0.1244$ $F^*(10) = 1.4727$

	DP		Re-SA		SA	
	Pr(DP < ReSA)	Pr(DP < SA)	Pr(ReSA < DP)	Pr(ReSA < SA)	Pr(SA < DP)	Pr(SA < ReSA)
G(0)	0.6055	0.6199	0.3945	0.6031	0.3801	0.3970
	(0.0137)	(0.0134)	(0.0137)	(0.0150)	(0.0134)	(0.0150)
G(10)	0.5779	0.5926	0.4221	0.5851	0.4074	0.4149
	(0.0140)	(0.0154)	(0.0140)	(0.0168)	(0.0154)	(0.0168)

	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
Mean	**0.0807**	**0.0473**	**0.9505**	*0.0808*	*0.0473*	*0.9492*	0.0846	0.0506	0.9483
	(0.0027)	(0.0022)	(0.0030)	(0.0026)	(0.0022)	(0.0029)	(0.0026)	(0.0021)	(0.0028)
St. dev.	0.0824	0.0669	*0.0919*	0.0806	*0.0663*	**0.0912**	0.0777	**0.0630**	0.0948
	(0.0046)	(0.0045)	(0.0022)	(0.0042)	(0.0043)	(0.0022)	(0.0032)	(0.0031)	(0.0023)
1st	**0.0049**	**8.5E-06**	0.7345	*0.0050*	*8.8E-06*	*0.7360*	0.0052	1.0E-05	**0.7508**
	(0.0005)	(5.E−06)	(0.0120)	(0.0005)	(6.E−06)	(0.0117)	(0.0007)	(6.E−06)	(0.0089)
5th	**0.0096**	**2.0E-04**	*0.8013*	*0.0099*	*2.0E-04*	0.8011	0.0104	2.4E-04	**0.8017**
	(0.0006)	(5.E−05)	(0.0060)	(0.0007)	(6.E−05)	(0.0058)	(0.0007)	(5.E−05)	(0.0050)
25th	**0.0262**	**0.0050**	**0.8897**	*0.0268*	*0.0050*	*0.8888*	0.0288)	0.0062	0.8819
	(0.0011)	(0.0005)	(0.0042)	(0.0011)	(0.0005)	(0.0041)	(0.0012)	(0.0007)	(0.0039)
50th	**0.0532**	**0.0219**	**0.9492**	*0.0541*	*0.0221*	*0.9478*	0.0594	0.0268	0.9425
	(0.0023)	(0.0016)	(0.0034)	(0.0023)	(0.0016)	(0.0034)	(0.0025)	(0.0018)	(0.0035)
75th	**0.1050**	**0.0626**	1.0090	*0.1060*	*0.0629*	**1.0073**	0.1153	0.0719	*1.0082*
	(0.0047)	(0.0040)	(0.0038)	(0.0047)	(0.0041)	(0.0038)	(0.0050)	(0.0040)	(0.0043)
95th	0.2443	*0.1789*	*1.1025*	**0.2417**	**0.1783**	**1.1003**	*0.2436*	0.1804	1.1128
	(0.0122)	(0.0105)	(0.0071)	(0.0115)	(0.0101)	(0.0071)	(0.0108)	(0.0093)	(0.0071)
99th	0.3909	0.3098	*1.1758*	*0.3823*	*0.3056*	**1.1733**	**0.3555**	**0.2786**	1.1911
	(0.0327)	(0.0278)	(0.0151)	(0.0301)	(0.0277)	(0.0149)	(0.0224)	(0.0201)	(0.0151)

Note: For a given benchmark, the value of the best strategy is marked in bold, and the value of the secondary strategy is marked in *italics*. Appendix D (Case 5) presents the two-tailed Welch's *t*-test for each pair of samples.

Table B5
Simulated statistics over three approaches (Case 6)

Case 6, $F^*(1) = 0.1257$ $F^*(10) = 1.5613$

	DP		Re-SA		SA	
	Pr(DP < ReSA)	Pr(DP < SA)	Pr(ReSA < DP)	Pr(ReSA < SA)	Pr(SA < DP)	Pr(SA < ReSA)
G(0)	0.6921	0.7316	0.3079	0.6887	0.2684	0.3113
	(0.0137)	(0.0128)	(0.0137)	(0.0140)	(0.0128)	(0.0140)
G(10)	0.6584	0.6791	0.3416	0.6526	0.3209	0.3474
	(0.0142)	(0.0143)	(0.0142)	(0.0153)	(0.0143)	(0.0153)

	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
Mean	**0.1079**	**0.0584**	**0.9475**	*0.1102*	*0.0599*	*0.9403*	0.1253	0.0722	0.9377
	(0.0048)	(0.0035)	(0.0033)	(0.0041)	(0.0032)	(0.0032)	(0.0040)	(0.0031)	(0.0032)
St. dev.	0.1384	0.1038	*0.0959*	*0.1216*	*0.0969*	**0.0933**	**0.1142**	**0.0875**	0.1046
	(0.0152)	(0.0129)	(0.0029)	(0.0087)	(0.0088)	(0.0026)	(0.0046)	(0.0044)	(0.0026)
1st	**0.0060**	**8.0E-06**	0.6814	*0.0063*	*8.5E-06*	*0.6891*	0.0072	1.5E-05	**0.7253**
	(0.0006)	(4.E−06)	(0.0221)	(0.0007)	(6.E−06)	(0.0176)	(0.0008)	(9.E−06)	(0.0089)
5th	**0.0114**	**0.0002**	**0.7841**	*0.0123*	*0.0002*	*0.7812*	0.0144	0.0004	0.7784
	(0.0007)	(4.E−05)	(0.0088)	(0.0007)	(5.E−05)	(0.0079)	(0.0012)	(1.E−04)	(0.0052)
25th	**0.0301**	**0.0048**	**0.8914**	*0.0342*	0.0054	*0.8840*	0.0419	0.0093	0.8637
	(0.0013)	(0.0005)	(0.0044)	(0.0016)	(0.0005)	(0.0043)	(0.0022)	(0.0011)	(0.0042)
50th	**0.0626**	**0.0223**	**0.9515**	*0.0703*	*0.0244*	*0.9440*	0.0887	0.0398	0.9304

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Table B5 (*continued*)

	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
	(0.0028)	(0.0017)	(0.0038)	(0.0032)	(0.0019)	(0.0036)	(0.0043)	(0.0028)	(0.0040)
75th	**0.1301**	**0.0676**	1.0083	*0.1400*	*0.0729*	**1.0004**	0.1730	0.1039	*1.0033*
	(0.0067)	(0.0045)	(0.0037)	(0.0069)	(0.0049)	(0.0035)	(0.0076)	(0.0059)	(0.0053)
95th	*0.3548*	**0.2328**	*1.0943*	**0.3432**	*0.2367*	**1.0846**	0.3594	0.2518	1.1215
	(0.0246)	(0.0179)	(0.0066)	(0.0203)	(0.0170)	(0.0065)	(0.0155)	(0.0121)	(0.0080)
99th	0.6633	0.4846	*1.1663*	*0.5844*	*0.4631*	**1.1539**	**0.5127**	**0.3807**	1.2094
	(0.0669)	(0.0660)	(0.0147)	(0.0539)	(0.0506)	(0.0138)	(0.0276)	(0.0244)	(0.0174)

Note: For a given benchmark, the value of the best strategy is marked in bold, and the value of the secondary strategy is marked in *italics*. Appendix D (Case 6) presents the two-tailed Welch's *t*-test for each pair of samples.

Appendix C

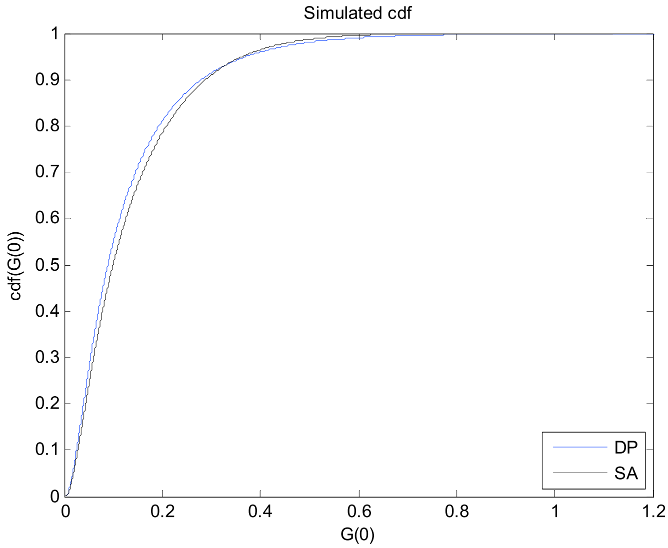

Fig. C1. Simulated cdf of G(0) over 100,000 paths (100 groups of 1000 simulation paths).

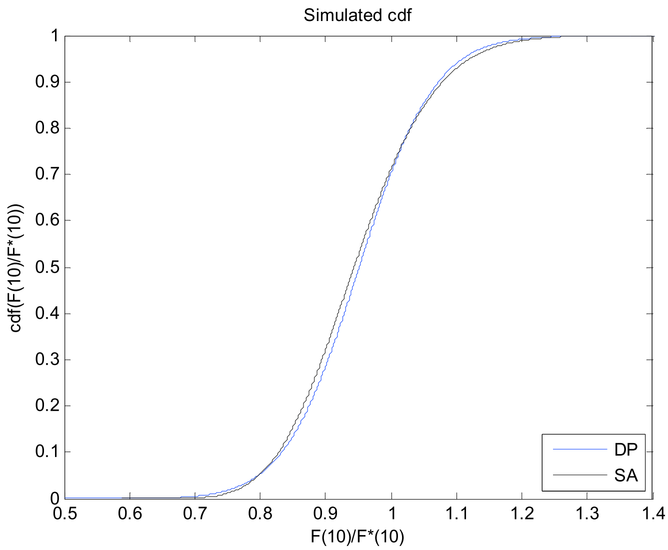

Fig. C2. Simulated cdf of F(10)/F*(10) over 100,000 paths (100 groups of 1000 simulation paths).

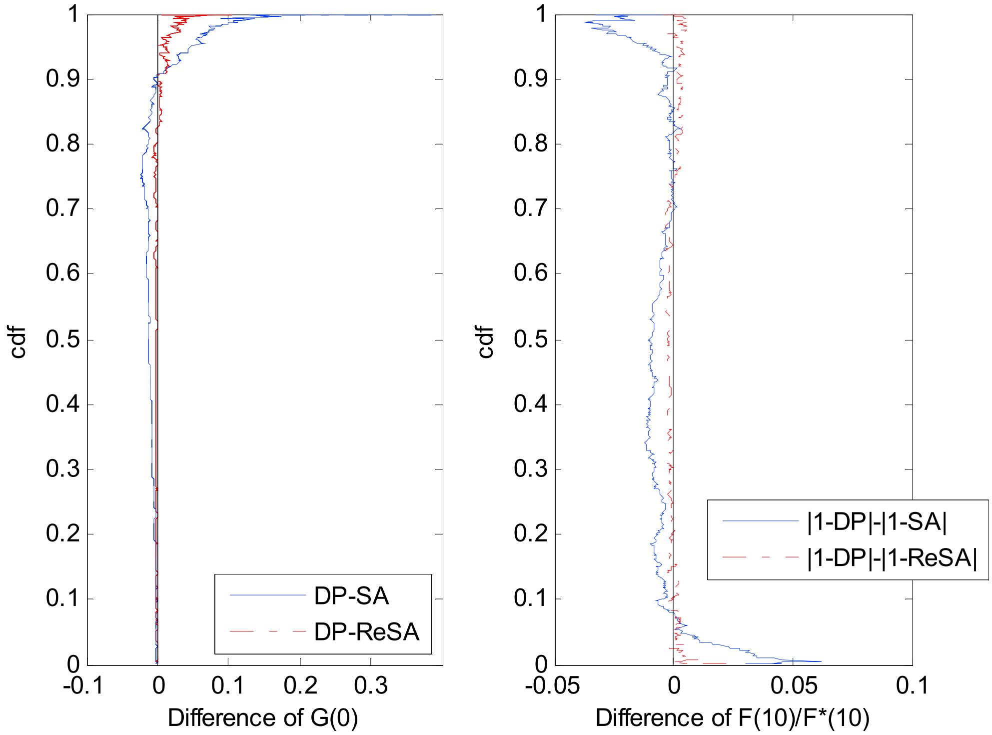

Fig. C3. The horizontal distance between the simulated lines.

The blue line on the left plot depicts the horizontal distance between the two simulated cdfs in Fig. 2; it also depicts the horizontal distance between the cdf with DP and that with SA in the upper plot of Fig. 5. The red line on the left plot depicts the horizontal distance between the cdf with DP and that with Re-SA in the upper plot of Fig. 5. A negative value indicates that DP has a smaller G(0) and a better performance than SA (Re-SA). SA is better than DP over large percentiles, which corresponds to a larger error (i.e., a larger total future cost at time 0).

The blue line on the right plot depicts the difference between the distance from the order statistics of F(10)/F*(10) to 1 with DP and that with SA, i.e., the difference between the horizontal distances from each line to 1, in Fig. 3 (and the lower plot of Fig. 5). The red line of the right plot depicts the difference between the distance with DP and that with Re-SA in the lower plot of Fig. 5. A negative value indicates that DP is closer to 1 and has a better performance than SA (Re-SA). SA is superior to DP when the final fund level is small, which implies that DP suffers a larger downside risk.

Furthermore, both red lines (the difference between DP and Re-SA) are very close to zero compared to the blue line. This indicates that Re-SA can be a good proxy for DP.

Appendix D. Welch's *t*-test for each pair of samples (Symbols *, ** and *** represent the 10%, 5% and 1% levels of significance; two-tailed test)

	DP vs. Re-SA			DP vs. SA			Re-SA vs. SA		
	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
Case 1									
Mean	–	–	***	***	***	***	***	***	**
St. dev.	***	*	***	***	***	***	***	***	***
1st	***	–	–	***	**	***	–	*	***
5th	***	–	–	***	***	–	***	***	–
25th	***	–	**	***	***	***	***	***	***
50th	***	–	***	***	***	***	***	***	***
75th	–	–	***	***	***	**	***	***	*
95th	**	–	***	***	–	***	–	–	***
99th	***	–	*	***	***	***	***	***	***
Case 2									
Mean	*	–	***	***	***	***	***	***	***
St. dev.	***	***	***	***	***	***	***	***	***
1st	***	*	***	***	***	***	***	***	***

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	DP vs. Re-SA			DP vs. SA			Re-SA vs. SA		
	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}	G(0)	G(10)	F_{10}/F^*_{10}
5th	***	***	–	***	***	**	*	***	–
25th	***	***	***	***	***	***	***	***	***
50th	***	***	***	***	***	***	***	***	***
75th	***	***	***	***	***	***	***	***	***
95th	***	*	***	***	–	***	***	**	***
99th	***	***	***	***	***	***	***	***	***
Case 3									
Mean	–	–	–	***	***	***	***	***	***
St. dev.	–	–	–	–	–	***	–	–	***
1st	–	–	–	–	–	***	–	–	***
5th	–	–	–	*	***	–	–	***	–
25th	–	–	–	***	***	***	***	***	***
50th	–	–	–	***	***	***	***	***	***
75th	–	–	–	***	***	–	***	***	–
95th	–	–	–	**	*	***	***	**	***
99th	–	–	–	***	–	**	**	–	***
Case 4									
Mean	–	–	***	***	***	***	***	***	**
St. dev.	***	***	***	***	***	***	***	***	***
1st	***	*	***	***	***	***	***	***	***
5th	***	***	***	***	***	***	***	***	***
25th	***	***	***	***	***	***	***	***	***
50th	***	***	***	***	***	***	***	***	***
75th	**	***	***	***	***	***	***	***	***
95th	***	***	***	***	***	***	***	***	***
99th	***	***	***	***	***	***	***	***	***
Case 5									
Mean	–	–	***	***	***	***	***	***	**
St. dev.	***	–	**	***	***	***	***	***	***
1st	–	–	–	***	**	***	***	*	***
5th	**	–	–	***	***	–	***	***	–
25th	***	–	–	***	***	***	***	***	***
50th	***	–	***	***	***	***	***	***	***
75th	–	–	***	***	***	–	***	***	–
95th	–	–	**	–	–	***	–	–	***
99th	*	–	–	***	***	***	***	***	***
Case 6									
Mean	***	***	***	***	***	***	***	***	***
St. dev.	***	***	***	***	***	***	***	***	***
1st	***	–	***	***	***	***	***	***	***
5th	***	***	**	***	***	***	***	***	***
25th	***	***	***	***	***	***	***	***	***
50th	***	***	***	***	***	***	***	***	***
75th	***	***	***	***	***	***	***	***	***
95th	***	–	***	–	***	***	***	***	***
99th	***	**	***	***	***	***	***	***	***
Case 1, $n = 15$									
Mean	–	–	***	***	***	***	***	***	–
St. dev.	***	***	***	***	***	***	***	***	***
1st	***	–	*	***	–	***	–	*	***
5th	***	–	–	***	***	–	–	***	–
25th	***	–	**	***	***	***	***	***	***
50th	***	*	***	***	***	***	***	***	***
75th	**	–	***	***	***	**	***	***	–
95th	–	*	**	–	–	***	–	–	***
99th	***	**	–	***	***	**	***	***	***

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