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Complexity of shift spaces on semigroups

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Abstract
Let $G = \langle S|R_A \rangle$ be a semigroup with generating set S and equivalences R_A among S determined by a matrix A. This paper investigates the complexity of G-shift spaces by yielding the Petersen–Salama entropies [defined in Petersen and Salama (Theoret Comput Sci 743:64–71, 2018)]. After revealing the existence of Petersen–Salama entropy of G-shift of finite type (G-SFT), the calculation of Petersen–Salama entropy of G-SFT is equivalent to solving a system of nonlinear recurrence equations. The complete characterization of Petersen–Salama entropies of G-SFTs on two symbols is addressed, which extends (Ban and Chang in On the topological entropy of subshifts of finite type on free semigroups, 2018. arXiv:1702.04394) in which G is a free semigroup.

Keywords Complexity · Entropy · Nonlinear recurrence equation · Non-free semigroups

Mathematics Subject Classification Primary 37E05 · 11A63

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1 Introduction

Let $\mathcal{A}$ be a finite alphabet and G be a group. A *configuration* is a function $f : G \to \mathcal{A}$ and a *pattern* is a function from a finite subset of G to $\mathcal{A}$. A subset $X \subseteq \mathcal{A}^G$ is called a *shift space* if $X = X_{\mathcal{F}}$ which is a set of configurations which avoid patterns from some set $\mathcal{F}$ of patterns. If $\mathcal{F}$ is finite, we call such a shift space *shift of finite type* (SFT). Let $n \in \mathbb{N}$ and denote by E_n the set of elements in G whose length is less than or equal to n. We define $\Gamma_n(X)$ the set of all possible patterns of X in E_n and set $\gamma_n = |\Gamma_n|$, i.e., the number of Γ_n. The *Petersen–Salama entropy* (*entropy* for short) of X is defined as

$$h(X) = \limsup_{n\to\infty} \frac{\ln \gamma_n}{|E_n|}. \tag{1}$$

It is known that the value (1) measures the exponential growth rate of the number of admissible patterns. From the dynamics viewpoint, it is a measure of the randomness or complexity of a given physical or dynamical system (cf. [12,18,24,25,27]). From the information theory viewpoint, it measures how much information can be stored in the set of allowed sequences.

The value $h(X)$ (we write it simply h) exists for $G = \mathbb{Z}^1$ under the classical subadditive argument, and its entropy formula and algebraic characterization are given by Lind [22,23] ; that is, the nonzero entropies of $\mathbb{Z}^1$-SFTs are exactly the nonnegative rational multiples of the logarithm of Perron numbers[1]. Generally, if G is *amenable* ($\mathbb{Z}^d$ is amenable), h exists due to the fact that G satisfies the Følner condition [13,17,21]. The characterization of the entropies for $G = \mathbb{Z}^d$, $d \geq 2$ is given by Hochman–Meyerovitch [19]; namely, the entropies of such G-SFTs are the set of *right recursively enumerable*[2] numbers. In [10], the authors found the recursive formula of h in $\mathbb{Z}^2$-SFTs, and some explicit values of h can be computed therein. Finally, the quantity (1) can also be used to characterize the chaotic spatial behavior for $\mathbb{Z}^2$ lattice dynamical systems (cf. [8,9,14–16]).

For $G = FS_d$, the free semigroup with generators $S = \{s_1, \ldots, s_d\}$ (it is not amenable), the existence of the limit (1) is due to the recent result of Petersen–Salama [25]. Its entropy formula for $d = 2$ and $|\mathcal{A}| = 2$ is presented by Ban–Chang [6]. While a significant phenomenon of the Petersen–Salama entropy is that it is conjugacy invariant for F_1-shifts (i.e., $\mathbb{Z}$-shifts), the invariance does not hold for F_d-shifts when $d \geq 2$. Notably, Petersen–Salama entropy coincides with classical topological entropy for F_1-shifts and one-sided ($\mathbb{Z}^+$)-shifts, but not in other cases. A general entropy known as the *sofic entropy*, which is conjugacy invariant, is introduced by Bowen (cf. [11,20]). Despite the Petersen–Salama entropy is not conjugacy invariant, it still reflects the complexity of a F_d-shift. For $G = F_2$, the free group with two generators, Piantadosi [26] finds an approximation of $h \approx 0.909155$ for the F_2-*golden mean shift* , i.e., the forbidden set $\mathcal{F} \subset \mathcal{A} \times S \times \mathcal{A}$ is defined by $\mathcal{F} = \{(2, s_i, 2) : i = 1, 2\}$[3] for $|\mathcal{A}| = 2$.

[1] A *Perron number* is a real algebraic integer greater than 1 and greater than the modulus of its algebraic conjugates.

[2] The infimum of a monotonic recursive sequence of rational numbers.

[3] This means that the consecutive 22 is a forbidden pattern along the generators s_1 and s_2.

Either the existence of the limit of (1) or to find the exact value of h for a F_d-SFT is an open problem. The specification properties or the chaotic behavior of FS_d-SFTs can be found in [3,4]

A *semigroup* is a set $G = \langle S|R\rangle$ together with a binary operation which is closed and associative, where R is a set of equivalences which describe the relations among S. A *monoid* is a semigroup with an identity element e. Suppose $A \in \{0,1\}^{d\times d}$ is a binary matrix. A semigroup/monoid G of the form $G = \langle S|R_A\rangle$ means that $s_i s_j = s_i$ if and only if $A(i,j) = 0$. We note that $FS_d = \langle S|R_A\rangle$, where $S = \{s_1, \ldots, s_d\}$ and $A = \mathbf{E}_d$, the $d \times d$ matrix with all entries being $1'$s. The aim of this paper is to find the entropy formula of G-SFTs. Although the discussion works for general cases, we focus on the case where $d = |\mathcal{A}| = 2$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ for the clarity and compactness of the idea.

In Sect. 2, we demonstrate that γ_n solves some nonlinear recurrence equations with respect to the lattice G and the rules $T = (T_1, T_2)$ (Theorem 2.2). Various types of recurrence equations, namely the *zero entropy type* (type **Z**, Proposition 3.1), *equal growth type* (type **E**, Theorem 3.2), *dominating type* (type **D**, Theorem 3.4 and Theorem 3.5) and *oscillating types* (type **O**, Proposition 3.8), are introduced, and the algorithms of the entropy computation for these types are presented in Sect. 3. We give the complete characterization of h of G-SFTs with $|\mathcal{A}| = 2$ (Theorem 4.1) in Sect. 4. That is, all the nonlinear recurrence equations of G-SFTs with $|\mathcal{A}| = 2$ are equal to $\mathbf{Z} \cup \mathbf{E} \cup \mathbf{D} \cup \mathbf{O}$. Some open problems related to this topic are listed in Sect. 5.

2 Preliminaries

In this section, we give the notations and some known results of G-SFTs. Let $\mathcal{A}$ be a finite set and $t \in \mathcal{A}^G$, for $g \in G$, $t_g = t(g)$ denotes the label attached to the vertex g of the right *Cayley graph*[4] of G. The *full shift* $\mathcal{A}^G$ collects all configurations, and the *shift map* $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is defined as $(\sigma_g t)_{g'} = t_{gg'}$ for $g, g' \in G$. For $n \geq 0$, let $E_n = \{g \in G : |g| \leq n\}$ denote the initial n-subgraph of the Cayley graph. An n-*block* is a function $\tau : E_n \to \mathcal{A}$. A configuration t *accepts* an n-block τ if there exists $g \in G$ such that $t_{gg'} = \tau_{g'}$ for all $g' \in E_n$; otherwise, we call τ a *forbidden block* of t (or t *avoids* τ). A G-shift space is a set $X \subseteq \mathcal{A}^G$ of all configurations which avoid a set of forbidden blocks. For $n \in \mathbb{N}$ and $g \in G$, let $\Gamma_n^{[g]}(X)$ be the set of n-blocks of X rooted at g, i.e., the support of each block of $\Gamma_n^{[g]}(X)$ is gE_n. Let $\gamma_n^{[g]} = \left|\Gamma_n^{[g]}(X)\right|$, the cardinality of $\Gamma_n^{[g]}(X)$, and denote $\gamma_n(X) = \gamma_n^{[e]}(X)$. The *topological degree* of X is defined as

$$\deg(X) = \limsup_{n\to\infty} \frac{\ln \ln \gamma_n(X)}{n}. \tag{2}$$

[4] The *right Cayley graph* of G with respect to S is the directed graph whose vertex set is G and its set of arcs is given by $E = \{(g, gs) : g \in G, s \in S\}$.

In [2], the authors show that the limit of (2) exists for G-SFTs, where $G = \langle S|R_A\rangle$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The relation of $\deg(X)$ and h is described as follows. For a G-SFT, γ_n behaves approximately like $\lambda_1\lambda_2^{\kappa^n}$ for some $\kappa, \lambda_1, \lambda_2 \in \mathbb{R}$ while $\deg(X) = \ln \kappa$. The formulation $\lambda_1\lambda_2^{\kappa^n}$ reveals that we could use λ_2 colors (in average) to *fill up* the elements of E_n in G. The $\deg(X)$ represents the logarithm of the degree κ (in average) of G in the viewpoint of the graph theory. For examples, let $\mathcal{A} = \{1, 2\}$ and X be a $\mathbb{Z}^1$- full shift over $\mathcal{A}$, then $\gamma_n = 2^n$ while $|E_n| = n$ and $\deg(X) = 0$. If X is a FS_3-full shift with $\mathcal{A}$, then $\gamma_n = 2^{3^n}$ while $|E_n| = 3^n$ and $\deg(X) = \ln 3$. Thus, the formulation $\lambda_1\lambda_2^{\kappa^n}$ can also be symbolized as $\gamma_n \approx |\mathcal{A}|^{|E_n|}$. It can be easily checked that if $\deg(X) = \ln \rho_A$, where ρ_A denotes the spectral radius of the matrix A, then $h = \ln \lambda_2$. However, it is not always the case we prove that $\{\kappa(X_\mathcal{F}) : X_\mathcal{F}$ is a FS_d-SFT$\} = \{\xi^{\frac{1}{p}} : \xi \in \mathcal{P}, p \geq 1\}$ [5] for FS_d-SFTs, where $\mathcal{P}$ is the set of Perron numbers. Suppose $G = \langle S|R_A\rangle$ has at least one *right free generator*[5], and Ban–Chang–Huang [7] provide the necessary and sufficient conditions for $\kappa = \rho_A$. Roughly speaking, the more information of the tuple $(\kappa, \lambda_1, \lambda_2) \in \mathbb{R}^3$ we know, the more explicit value of γ_n we obtain. A natural question arises: *How to compute the value* λ_2*?* If $G = FS_2$, i.e., $A = \mathbf{E}_2$, the authors give the complete characterization of the entropies for $|\mathcal{A}| = 2$ [1]. This study is to extend the previous work to $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

2.1 The existence of the entropy

From now on, we assume that $G = \langle S|R_A\rangle$, where $S = \{s_1, s_2\}$, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{A} = \{1, 2\}$. Theorem 2.1 shows that the limit (1) exists for G-shifts. The proof is based on the concept of the proof for FS_2-shifts [25]. Since the proof for G-shifts is not straightforward, we include the proof for the convenience of the reader.

Theorem 2.1 *Let X be a G-shift. The entropy* (1) *exists.*

Proof Let $L_n = \{g \in G : |g| = n\}$ and $l_n = |L_n|$ for $n \geq 0$. Set

$$\underline{h} = \liminf_{n\to\infty} \frac{\ln \gamma_n}{|E_n|} = \liminf_{n\to\infty} \frac{\ln \gamma_n}{l_0 + l_1 + \cdots + l_n}.$$

Since $l_0 = 1$, for $\varepsilon > 0$, there exists a large $m \in \mathbb{N}$ such that

$$\frac{\ln \gamma_m}{l_1 + \cdots + l_m} < \underline{h} + \varepsilon.$$

Write $n = pm + q$ where $0 \leq q \leq m - 1$, then we have

$$\gamma_{pm+q} = \gamma_{[(p-1)m+q]+m} \leq \gamma_{(p-1)m+q}\gamma_m^{l_{(p-1)m+q}}$$

[5] s_i is a right (resp. left) free generator if and only if $A(i, j) = 1$ (resp. $A(j, i) = 1$) for $1 \leq j \leq d$.

$$
\begin{aligned}
&\leq (\gamma_{(p-2)m+q}\gamma_m^{l_{(p-2)m+q}})\gamma_m^{l_{(p-1)m+q}} \\
&\ \ \vdots \\
&\leq \gamma_q \gamma_m^{l_q+l_{m+q}+\cdots+l_{(p-1)m+q}}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{pmatrix} l_{n+1} \\ l_n \end{pmatrix} = A \begin{pmatrix} l_n \\ l_{n-1} \end{pmatrix}.
$$

It can be easily checked that $l_{m+n} = l_m l_{n+1} + l_{m-1} l_n$. Thus, we have $l_{m+n} > l_m l_{n+1} > l_m l_n$ and

$$
\begin{aligned}
l_{q+1} + l_{q+2} + \cdots + l_{q+pm} &> l_q(l_1 + \cdots + l_m) + l_{q+m}(l_1 + \cdots + l_m) \\
&\quad + \cdots + l_{q+(p-1)m}(l_1 + \cdots + l_m) \\
&= (l_1 + \cdots + l_m)(l_q + l_{q+m} + \cdots + l_{q+(p-1)m}).
\end{aligned}
$$

Therefore, for n large enough we have

$$
\begin{aligned}
\underline{h} \leq \frac{\ln \gamma_n}{|E_n|} &= \frac{\ln \gamma_{pm+q}}{\left|E_{pm+q}\right|} \\
&\leq \frac{\ln \gamma_q}{\left|E_{pm+q}\right|} + \frac{l_q + l_{q+m} + \cdots + l_{q+(p-1)m}}{l_1 + l_2 + \cdots + l_{q+pm}} \ln \gamma_m \\
&\leq \frac{\ln \gamma_q}{\left|E_{pm+q}\right|} + \frac{l_q + l_{q+m} + \cdots + l_{q+(p-1)m}}{l_{q+1} + \cdots + l_{q+pm}} \ln \gamma_m \\
&\leq \frac{\ln \gamma_q}{\left|E_{pm+q}\right|} + \frac{\ln \gamma_m}{l_1 + \cdots + l_m} < \underline{h} + 2\varepsilon.
\end{aligned}
$$

Thus, we conclude that (1) exists and equals $\underline{h}$. This completes the proof. □

2.2 Nonlinear recurrence equations

Let $\mathcal{F} \subseteq \mathcal{A} \times S \times \mathcal{A}$ be a set of forbidden blocks and $X = X_{\mathcal{F}}$ be the corresponding G-SFT. The associated *adjacency matrices* $T_1 = (t^1_{ij})^2_{i,j=1}$ and $T_2 = (t^2_{ij})^2_{i,j=1}$ of $\mathcal{F}$ are defined as follows. For $i = 1, 2$, $T_i(a, b) = 0$ if $(a, s_i, b) \in \mathcal{F}$ and $T_i(a, b) = 1$, otherwise. Let $T = (T_1, T_2)$, the *G-vertex shift* X_T is defined.

$$
X_T = \{t \in \mathcal{A}^G : T_i(t_g, t_{gs_i}) = 1 \ \forall g, gs_i \in G\}. \tag{3}
$$

It is obvious that X_T is equal to X and thus $h(X) = h(X_T)$. Throughout the paper, we assume that T_i has no zero rows for $i = 1, 2$.

Fix $i \in \mathcal{A}$, we set $\Gamma^{[g]}_{i,n}$ the set which consists of all n-blocks τ in $\Gamma^{[g]}_n$ such that $\tau_g = i$ and $\gamma^{[g]}_{i,n} = \left|\Gamma^{[g]}_{i,n}\right|$, the cardinality of $\Gamma^{[g]}_{i,n}$. For $g \in G$, we define $\mathbf{F}_g = \{g' \in$

$G : gg' \in G$ and $\left|gg'\right| = |g| + \left|g'\right|\}$. If $g = g_1 g_2 \dots g_n \in G$, it is easily seen that $\mathbf{F}_g = G$ if $g_n = s_1$ and $\mathbf{F}_g = \mathbf{F}_{s_2}$ if $g_n = s_2$, note that $\mathbf{F}_{s_2} \neq G$. Thus,

$$\gamma_{i,n}^{[s_1]} = \sum_{j_1, j_2=1}^{2} t_{ij_1}^1 t_{ij_2}^2 \gamma_{j_1,n-1}^{[s_1 s_1]} \gamma_{j_2,n-1}^{[s_1 s_2]} = \sum_{j_1, j_2=1}^{2} t_{ij_1}^1 t_{ij_2}^2 \gamma_{j_1,n-1}^{[s_1]} \gamma_{j_2,n-1}^{[s_2]}, \tag{4}$$

$$\gamma_{i,n}^{[s_2]} = \sum_{j=1}^{2} t_{ij}^1 \gamma_{j,n-1}^{[s_2 s_1]} = \sum_{j=1}^{2} t_{ij}^1 \gamma_{j,n-1}^{[s_1]}, \ 1 \le i \le 2 \text{ and } n \ge 3. \tag{5}$$

The first equality in (4) means that the $\Gamma_{i,n}^{[s_1]}$ is generated by $\Gamma_{j_1,n-1}^{[s_1 s_1]}$ (resp. $\Gamma_{j_2,n-1}^{[s_1 s_2]}$) according to the rule of $t_{ij_1}^1$ (resp. $t_{ij_2}^2$) of T_1 (resp. T_2). Since $\Gamma_{j_1,n-1}^{[s_1 s_1]}$ and $\Gamma_{j_2,n-1}^{[s_1 s_2]}$ are in different branches, the summation $\sum_{j_1, j_2=1}^{2} t_{ij_1}^1 t_{ij_2}^2 \gamma_{j_1,n-1}^{[s_1 s_1]} \gamma_{j_2,n-1}^{[s_1 s_2]}$ demonstrates the cardinality of $\Gamma_{i,n}^{[s_1]}$. The second equality in (4) comes from the fact that $\mathbf{F}_{s_1 s_1} = \mathbf{F}_{s_1}$ and $\mathbf{F}_{s_1 s_2} = \mathbf{F}_{s_2}$. Similar argument derives (5), and the only difference is that we do not have the item $\gamma_{j,n-1}^{[s_2 s_2]}$ in the summation since $A(2, 2) = 0$, i.e., $s_2 s_2 = s_2$. Since equations (4) and (5) are the nonlinear recurrence equations which describe the numbers $\gamma_{i,n}^{[s_1]}$ and $\gamma_{i,n}^{[s_2]}$ for $i = 1, 2$, we continue to write $\{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^2$ to represent nonlinear recurrence equations (4) and (5) by abuse of notation.

The nonlinear recurrence equation $\{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^2$ can be described in an efficient way. Let $\mathcal{M}_m$ be the collection of $m \times m$ binary matrices. Let $v = (v_i)_{i=1}^n$ and $w = (w_i)_{i=1}^n$ be two vectors over $\mathbb{R}$. Denote by $\otimes$ the *dyadic product* of v and w; that is,

$$v \otimes w = (v_1 w_1, v_1 w_2, \dots, v_1 w_n, \dots, v_n w_1, v_n w_2, \dots, v_n w_n).$$

Let $M \in \mathcal{M}_m$ and $M^{(i)}$ denote the ith row of M. Define

$$\alpha_i = T_1^{(i)} \otimes T_2^{(i)} \text{ for } i = 1, 2 \tag{6}$$

and

$$\Theta_n^{[s_j]} = \begin{cases} (\gamma_{1,n}^{[s_1]}, \gamma_{2,n}^{[s_1]}) \otimes (\gamma_{1,n}^{[s_2]}, \gamma_{2,n}^{[s_2]}), & \text{if } j = 1; \\ (\gamma_{1,n}^{[s_1]}, \gamma_{2,n}^{[s_1]}) \otimes (1, 1), & \text{if } j = 2. \end{cases} \tag{7}$$

Let "$\cdot$" be the usual inner product, we define

$$\widehat{\gamma}_{i,n}^{[s_j]} = \alpha_i \cdot \Theta_{n-1}^{[s_j]} \text{ for } 1 \le i, j \le 2. \tag{8}$$

Theorem 2.2 *Let $T = (T_1, T_2) \in \mathcal{M}_2 \times \mathcal{M}_2$, formulas (4) and (5) can be reformulated as follows. For $i = 1, 2$,*

(1) $\gamma_{i,n}^{[s_1]} = \widehat{\gamma}_{i,n}^{[s_1]}$,

(2) *$\gamma_{i,n}^{[s_2]}$ is derived from $\widehat{\gamma}_{i,n}^{[s_2]}$ by letting all coefficients of the items $\gamma_{1,n-1}^{[s_1]}$ and $\gamma_{2,n-1}^{[s_1]}$ in $\widehat{\gamma}_{i,n}^{[s_2]}$ to be* 1.

Proof It follows from (4)

$$\begin{aligned}\gamma_{i,n}^{[s_1]} &= \sum_{j_1,j_2=1}^{2} t_{ij_1}^1 t_{ij_2}^2 \gamma_{j_1,n-1}^{[s_1s_1]} \gamma_{j_2,n-1}^{[s_1s_2]} \\ &= \left(\sum_{j_1=1}^{2} t_{ij_1}^1 \gamma_{j_1,n-1}^{[s_1s_1]}\right)\left(\sum_{j_2=1}^{2} t_{ij_2}^2 \gamma_{j_2,n-1}^{[s_1s_2]}\right).\end{aligned}$$

Since $\gamma_{j_1,n-1}^{[s_1s_1]} = \gamma_{j_1,n-1}^{[s_1]}$ and $\gamma_{j_1,n-1}^{[s_1s_2]} = \gamma_{j_1,n-1}^{[s_2]}$, we conclude that

$$\begin{aligned}\gamma_{i,n}^{[s_1]} &= \left[\left(t_{i1}^1, t_{i2}^1\right) \otimes \left(t_{i1}^2, t_{i2}^2\right)\right] \cdot \left[\left(\gamma_{1,n-1}^{[s_1]}, \gamma_{2,n-1}^{[s_1]}\right) \otimes \left(\gamma_{1,n-1}^{[s_2]}, \gamma_{2,n-1}^{[s_2]}\right)\right] \\ &= \alpha_i \cdot \Theta_{n-1}^{[s_j]} = \widehat{\gamma}_{i,n}^{[s_1]}.\end{aligned}$$

Thus, $\gamma_{i,n}^{[s_1]} = \widehat{\gamma}_{i,n}^{[s_1]}$. On the other hand, it follows from (8) that we have

$$\begin{aligned}\widehat{\gamma}_{i,n}^{[s_2]} &= \alpha_i \cdot \Theta_{n-1}^{[s_2]} \\ &= \left(t_{i1}^1 t_{i1}^2, t_{i1}^1 t_{i2}^2, t_{i2}^1 t_{i1}^2, t_{i2}^1 t_{i2}^2\right) \cdot \left[(\gamma_{1,n-1}^{[s_1]}, \gamma_{2,n-1}^{[s_1]}) \otimes (1,1)\right] \\ &= \left(t_{i1}^1 t_{i1}^2, t_{i1}^1 t_{i2}^2, t_{i2}^1 t_{i1}^2, t_{i2}^1 t_{i2}^2\right) \cdot \left(\gamma_{1,n-1}^{[s_1]}, \gamma_{1,n-1}^{[s_1]}, \gamma_{2,n-1}^{[s_1]}, \gamma_{2,n-1}^{[s_1]}\right) \\ &= t_{i1}^1 t_{i1}^2 \gamma_{1,n-1}^{[s_1]} + t_{i1}^1 t_{i2}^2 \gamma_{1,n-1}^{[s_1]} + t_{i2}^1 t_{i1}^2 \gamma_{2,n-1}^{[s_1]} + t_{i2}^1 t_{i2}^2 \gamma_{2,n-1}^{[s_1]}.\end{aligned} \tag{9}$$

Suppose that there is no restriction on the node s_2 in G, and the same reasoning as $\gamma_{i,n}^{[s_1]}$ applied to $\gamma_{i,n}^{[s_2]}$ implies

$$\begin{aligned}\gamma_{i,n}^{[s_2]} = {} & t_{i1}^1 t_{i1}^2 \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]} + t_{i1}^1 t_{i2}^2 \gamma_{1,n-1}^{[s_1]} \gamma_{2,n-1}^{[s_2]} \\ & + t_{i2}^1 t_{i1}^2 \gamma_{2,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]} + t_{i2}^1 t_{i2}^2 \gamma_{2,n-1}^{[s_1]} \gamma_{2,n-1}^{[s_2]}.\end{aligned} \tag{10}$$

Since $s_2 s_2 = s_2$, formula (9) is derived from (10) by letting $\gamma_{i,n-1}^{[s_2]} = 1$ for $i = 1, 2$. However, compared to formula (5) $\gamma_{i,n}^{[s_2]} = \sum_{j=1}^{2} t_{ij}^1 \gamma_{j,n-1}^{[s_1]}$; formula (9) counts $\gamma_{i,n}^{[s_2]}$ repeatedly, e.g., if $t_{i1}^1 t_{i1}^2 = t_{i1}^1 t_{i2}^2 = 1$, then (9) counts the item $\gamma_{1,n-1}^{[s_1]}$ twice. Thus, $\gamma_{i,n}^{[s_2]}$ can be derived from $\widehat{\gamma}_{i,n}^{[s_2]}$ by letting all coefficients of the terms $\gamma_{1,n-1}^{[s_1]}$ and $\gamma_{2,n-1}^{[s_1]}$ in $\widehat{\gamma}_{i,n}^{[s_j]}$ to be 1. This completes the proof. □

It is worth noting that the intrinsic meaning of (8) is that the effect of T (*rules*) comes from the factor α_i (since $\alpha_i = \alpha_i(T)$) and the effect of G (*lattice*) comes from the factor $\Theta_{n-1}^{[s_j]}$.

Example 2.3 Let $T = (T_1, T_2)$, where $T_i = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ for $i = 1, 2$. Then, we have $\alpha_1 = (1, 1, 1, 1)$ and $\alpha_2 = (1, 0, 0, 0)$. Applying Theorem 2.2, we have $\gamma_{1,n}^{[s_1]} = \sum_{i,j=1}^{2} \gamma_{i,n-1}^{[s_1]} \gamma_{j,n-1}^{[s_2]}$ and $\gamma_{2,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]}$. On the other hand, it follows from $\widehat{\gamma}_{1,n}^{[s_2]} = 2\gamma_{1,n-1}^{[s_1]} + 2\gamma_{2,n-1}^{[s_1]}$ and $\widehat{\gamma}_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}$ that we have $\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]}$ and $\gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}$.

Since $T = (T_1, T_2) \in \mathcal{M}_2 \times \mathcal{M}_2$, there are only finite possibilities of $T_i^{(1)}$ and $T_i^{(2)}$, namely

$$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\} = \{(1, 1), (1, 0), (0, 1), (0, 0)\}.$$

The above vectors correspond to 2×2 matrices which represent the possible directed graphs having two vertices. Hence, we have only finite choices of α_i [recall (6)] for $i = 1, 2$ as follows:

$$\begin{aligned} &v_1 = (1, 1, 1, 1), v_2 = (1, 0, 1, 0), v_3 = (0, 1, 0, 1), v_4 = (1, 1, 0, 0), \\ &v_5 = (0, 0, 1, 1), v_6 = (1, 0, 0, 0), v_7 = (0, 1, 0, 0), v_8 = (0, 0, 1, 0), \\ &v_9 = (0, 0, 0, 1). \end{aligned}$$

The collection of vectors $\{v_i\}_{i=1}^{9}$ comes from the Kronecker product $\varpi_i \otimes \varpi_j$ for $1 \le i, j \le 3$. We reorganize the vectors in the above manner so that we can process a systematic analysis for the different types of shifts represented by these vectors (see Sect. 3). For the convenience of the discussion, we define $F_{kl} = \{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^{2}$ the nonlinear recurrence equation of Theorem 2.2 by choosing $(\alpha_1, \alpha_2) = (v_k, v_l)$, and $h = h_{kl}$ if the corresponding recurrence equation is F_{kl}.

Remark 2.4 (1) Given $(\alpha_1, \alpha_2) = (v_k, v_l)$, the pair (T_1, T_2) is also uniquely determined. For instance, if $(\alpha_1, \alpha_2) = (v_2, v_3)$, then $T_1^{(1)} = (1, 1)$, $T_2^{(1)} = (1, 0)$, $T_1^{(2)} = (1, 1)$ and $T_1^{(2)} = (0, 1)$. Thus, one can reconstruct $T = (T_1, T_2)$ as

$$T_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

(2) Note that $\mathbf{F}_g = G$ if $g_n = s_1$. The entropy (1) can also be represented as

$$h = \lim_{n\to\infty} \frac{\ln \gamma_n}{|E_n|} = \lim_{n\to\infty} \frac{\ln\left(\gamma_{1,n}^{[s_1]} + \gamma_{2,n}^{[s_1]}\right)}{|E_n|}, \tag{12}$$

where the existence of the limit is due to Theorem 2.1.

2.3 Equivalence of the recurrence equations

Given two nonlinear recurrence equations F_{kl} and F_{pq}, we say that F_{kl} is *equivalent* to F_{pq} (write $F_{kl} \simeq F_{pq}$) if F_{kl} is equal to F_{pq} by interchanging items $\gamma_{1,n}^{[s_1]}$ with $\gamma_{2,n}^{[s_1]}$ and $\gamma_{1,n}^{[s_2]}$ with $\gamma_{2,n}^{[s_2]}$. It follows from (12) that the entropies of two G-SFTs are equal if their corresponding nonlinear recurrence equations are equivalent.

Example 2.5 $F_{48} \simeq F_{75}$.

Proof It follows from Theorem 2.2 that we obtain

$$F_{48} = \begin{cases} \gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]} + \gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}, \\ \gamma_{2,n}^{[s_1]} = \gamma_{2,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}, \\ \gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\ \gamma_{2,n}^{[s_2]} = \gamma_{2,n-1}^{[s_1]}, \\ \gamma_{1,1}^{[s_1]} = 2, \gamma_{2,1}^{[s_1]} = 1, \gamma_{1,1}^{[s_2]} = 1, \gamma_{2,1}^{[s_2]} = 1. \end{cases}$$

If we interchange $\gamma_{1,n}^{[s_1]}$ (resp. $\gamma_{1,n}^{[s_2]}$) with $\gamma_{2,n}^{[s_1]}$ (resp. $\gamma_{2,n}^{[s_2]}$), we have

$$\begin{cases} \gamma_{2,n}^{[s_1]} = \gamma_{2,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}, \\ \gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}, \\ \gamma_{2,n}^{[s_2]} = \gamma_{2,n-1}^{[s_1]}, \\ \gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\ \gamma_{1,1}^{[s_1]} = 1, \gamma_{2,1}^{[s_1]} = 2, \gamma_{1,1}^{[s_2]} = 1, \gamma_{2,1}^{[s_2]} = 1. \end{cases}$$

One can check that it is indeed F_{75}. This completes the proof. □

3 Formula and estimate of entropy

In what follows, λ stands for the *spectral radius* of A, i.e., $\lambda = \frac{1+\sqrt{5}}{2}$ and $\bar{\lambda}$ is its conjugate. We provide various types of nonlinear recurrence equations in which the formula (or estimate) of h is presented in this section. By abuse of notation, we also denote by $|v| = \sum_{i=1}^{n} \left|v^{(i)}\right|$ the *norm* of $v \in \mathbb{R}^n$ and $v^{(i)}$ the ith coordinate of v. It can be easily checked that $|E_n| = \left(\sum_{i=0}^{n} A^i \mathbf{1}\right)^{(1)}$, where $\mathbf{1} = (1, 1)'$ and v' denotes the *transpose* of v.

3.1 Zero entropy type

Proposition 3.1 indicates that $h_{kl} = 0$ if F_{kl} satisfies $|v_k| = |v_l| = 1$, e.g., $k, l = 6, 7, 8, 9$. We call such F_{kl} *zero entropy type* (write type $\mathbf{Z}$).

Proposition 3.1 *Let $T = (T_1, T_2) \in \mathcal{M}_2 \times \mathcal{M}_2$, and $\alpha_i = \alpha_i(T)$ be defined as above for $i = 1, 2$. If $|\alpha_1| = |\alpha_2| = 1$, then $h = 0$.*

Proof For simplicity, we only prove the case F_{67}, and the other cases can be treated similarly. Indeed, F_{67} is of the following form.

$$\begin{cases} \gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}, \\ \gamma_{2,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}, \\ \gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\ \gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\ \gamma_{1,1}^{[s_1]} = \gamma_{2,1}^{[s_1]} = \gamma_{1,1}^{[s_2]} = \gamma_{2,1}^{[s_2]} = 1. \end{cases} \tag{13}$$

Note that $\gamma_{i,1}^{[s_j]} = 1$, and if we assume that $\gamma_{i,k-1}^{[s_j]} = 1$ for $1 \leq i, j \leq 2$, (13) infers that $\gamma_{i,k}^{[s_j]} = 1$. Thus, $\gamma_{i,n}^{[s_j]} = 1$ for all $1 \leq n$ and $1 \leq i, j \leq 2$ by induction. This shows that $h_{67} = 0$. This completes the proof. □

3.2 Equal growth type

Let $T = (T_1, T_2)$ and $F = \{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^{2}$ be its nonlinear recurrence equations, we say that F is of the *equal growth type* ($F \in \mathbf{E}$) if $|\alpha_1| = |\alpha_2|$. Denote by $k_{i,j}$ the number of different items of $\gamma_{i,n}^{[s_j]}$ for $1 \leq i, j \leq 2$. If $|\alpha_1| = |\alpha_2|$, it can be checked that $k_{1,1} = k_{2,1} = \alpha$, but $k_{1,2}$ may not equal to $k_{2,2}$ in general.

Theorem 3.2 *Let $T = (T_1, T_2)$ and the corresponding α_1, α_2 satisfy $|\alpha_1| = |\alpha_2| = \alpha \in \mathbb{N}$. If $k_{1,2} = k_{2,2} =: \beta$, then*

$$h = \left(\frac{1 - \bar{\lambda} \frac{\ln \beta}{\ln \alpha}}{\lambda^2} \right) \ln \alpha.$$

Furthermore, if $k_{1,2} = k_{2,2} = \alpha$, then $h = \frac{\ln \alpha}{\lambda}$.

Proof 1 First, we claim that $\gamma_{1,n}^{[s_j]} = \gamma_{2,n}^{[s_j]}$ for $1 \leq j \leq 2$ and we prove it by induction. Note that $\gamma_{1,0}^{[s_j]} = \gamma_{2,0}^{[s_j]} = 1$ and assume that $\gamma_{1,n}^{[s_j]} = \gamma_{2,n}^{[s_j]}$ for $1 \leq j \leq 2$. Theorem 2.2 is applied to show that

$$\gamma_{i,n+1}^{[s_1]} = \widehat{\gamma}_{i,n+1}^{[s_1]} = \alpha_i \cdot \Theta_n^{[s_1]} = \alpha_i \cdot [(\gamma_{1,n}^{[s_1]}, \gamma_{2,n}^{[s_1]}) \otimes (\gamma_{1,n}^{[s_2]}, \gamma_{2,n}^{[s_2]})], \tag{14}$$

and $\gamma_{i,n+1}^{[s_2]}$ is constructed by letting all the coefficients of

$$\widehat{\gamma}_{i,n+1}^{[s_2]} = \alpha_i \cdot \Theta_n^{[s_2]} = \alpha_i \cdot [(\gamma_{1,n}^{[s_1]}, \gamma_{2,n}^{[s_1]}) \otimes (1, 1)]$$

to be 1 (Theorem 2.2). Since $k_{1,2} = k_{2,2}$, we conclude that $\gamma_{1,n+1}^{[s_2]} = \gamma_{2,n+1}^{[s_2]}$. Combining (14) with $\gamma_{1,n+1}^{[s_2]} = \gamma_{2,n+1}^{[s_2]}$ we can assert that $\gamma_{1,n+1}^{[s_1]} = \gamma_{2,n+1}^{[s_1]}$, and this proves the claim.

2 Since $|\alpha_1| = |\alpha_2|$ and $\gamma_{1,n}^{[s_j]} = \gamma_{2,n}^{[s_j]}$ for $1 \le j \le 2$. We have $\gamma_{i,n}^{[s_1]} = |\alpha_i|\,\gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]} = \alpha\gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}$ and $\gamma_{i,n}^{[s_2]} = \beta\gamma_{1,n-1}^{[s_1]}$. Thus, the nonlinear equation $F = \{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^{2}$ can be reduced to the simplified form

$$\begin{cases} \gamma_{1,n}^{[s_1]} = \alpha\gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}, \\ \gamma_{1,n}^{[s_2]} = \beta\gamma_{1,n-1}^{[s_1]}. \end{cases} \tag{15}$$

Let $w_n = (\ln\gamma_{1,n}^{[s_1]}, \ln\gamma_{1,n}^{[s_2]})'$. We have $w_n = Aw_{n-1} + b$, where $b = (\ln\alpha, \ln\beta)'$. Iterate w_n we have $w_n = A^{n-1}w_1 + \sum_{i=0}^{n-2} A^i b$. Observe that $w_1 = b$, thus

$$w_n = \sum_{i=0}^{n-1} A^i b = \ln\alpha\left(\sum_{i=0}^{n-1} A^i\widetilde{b}\right), \tag{16}$$

where $\widetilde{b} = (1, \frac{\ln\beta}{\ln\alpha})'$. Combining (12) with the fact that $\gamma_{1,n}^{[s_1]} = \gamma_{2,n}^{[s_1]}$, we assert that

$$\begin{aligned} h &= \lim_{n\to\infty}\frac{\ln\sum_{i=1}^{2}\gamma_{i,n}^{[s_1]}}{|E_n|} = \lim_{n\to\infty}\frac{\ln\gamma_{1,n}^{[s_1]}}{|E_n|} = \lim_{n\to\infty}\frac{w_n^{(1)}}{|E_n|} \\ &= \lim_{n\to\infty}\frac{(\ln\alpha)\left(\sum_{i=0}^{n-1}A^i\widetilde{b}\right)^{(1)}}{|E_n|}. \end{aligned} \tag{17}$$

Substituting $|E_n| = \left(\sum_{i=0}^{n} A^i\mathbf{1}\right)^{(1)}$ into (17) yields

$$h = (\ln\alpha)\lim_{n\to\infty}\frac{\left(\sum_{i=0}^{n-1}A^i\widetilde{b}\right)^{(1)}}{\left(\sum_{i=0}^{n}A^i\mathbf{1}\right)^{(1)}}.$$

Setting $A = PDP^{-1}$, $P = (p_{ij})$, $P^{-1} = (q_{ij})$, and $D = diag(\lambda, \bar{\lambda})$, we have

$$\begin{aligned} \left(\sum_{i=0}^{n-1}A^i\widetilde{b}\right)^{(1)} &= \left[P\left(\sum_{i=0}^{n-1}D^i\right)P^{-1}\widetilde{b}\right]^{(1)} \\ &= \left[P\left(\sum_{i=0}^{n-1}D^i\right)\left(q_{11} + q_{12}\frac{\ln\beta}{\ln\alpha}, q_{21} + q_{22}\frac{\ln\beta}{\ln\alpha}\right)'\right]^{(1)} \\ &= \sum_{i=0}^{n-1}\left[\lambda^i p_{11}\left(q_{11} + q_{12}\frac{\ln\beta}{\ln\alpha}\right) + \bar{\lambda}^i p_{12}\left(q_{21} + q_{22}\frac{\ln\beta}{\ln\alpha}\right)\right] \\ &= p_{11}\left(q_{11} + q_{12}\frac{\ln\beta}{\ln\alpha}\right)\frac{\lambda^n - 1}{\lambda - 1} + p_{12}\left(q_{21} + q_{22}\frac{\ln\beta}{\ln\alpha}\right)\frac{\bar{\lambda}^n - 1}{\bar{\lambda} - 1}. \end{aligned}$$

It follows the same computation that we have

$$\left(\sum_{i=0}^{n-1} A^i \mathbf{1}\right)^{(1)} = p_{11}\left(q_{11}+q_{12}\right)\frac{\lambda^n-1}{\lambda-1} + p_{12}\left(q_{21}+q_{22}\right)\frac{\bar{\lambda}^n-1}{\bar{\lambda}-1}. \tag{18}$$

Thus,

$$\lim_{n\to\infty}\frac{\left(\sum_{i=0}^{n-1} A^i\widetilde{b}\right)^{(1)}}{\left(\sum_{i=0}^{n} A^i\mathbf{1}\right)^{(1)}} = \frac{q_{11}+q_{12}\frac{\ln\beta}{\ln\alpha}}{\left(q_{11}+q_{12}\right)\lambda}.$$

Direct computation shows that

$$P = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \text{ and } P^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix}.$$

Thus, $\lim_{n\to\infty}\frac{\left(\sum_{i=0}^{n-1} A^i\widehat{b}\right)^{(1)}}{\left(\sum_{i=0}^{n} A^i\mathbf{1}\right)^{(1)}} = \frac{1-\bar{\lambda}\frac{\ln\beta}{\ln\alpha}}{\lambda^2}$. If $\alpha=\beta$, we have $\frac{1-\bar{\lambda}\frac{\ln\beta}{\ln\alpha}}{\lambda^2} = \frac{1}{\lambda}$. This completes the proof. □

Example 3.3 (1) $h_{42} = \frac{1-\bar{\lambda}\frac{1}{2}}{\lambda^2}\ln 4 = \frac{1}{2}\ln 4 = \ln 2$.

(2) $h_{23} = \frac{1-\bar{\lambda}}{\lambda^2}\ln 2 = \frac{1}{\lambda}\ln 2 \approx 0.428\,39$. Since $\deg(X) = \ln\lambda$ (cf. [2]), we have $\gamma_n \approx \left(2^{\frac{1}{\lambda}}\right)^{\lambda^n} = 2^{\lambda^{n-1}}$, e.g., $\gamma_9 \approx (2)^{\lambda^8} \approx 1.386\,8\times 10^{14}$.

(3) $h_{45} = \frac{1}{\lambda^2}\ln 2 \approx 0.264\,76$.

3.3 Dominating type

Let $\gamma_{i,n}^{[s_j]} = \sum_{l=1}^{n_{ij}} f_{l,n-1}^{ij}$, where $f_{l,n-1}^{ij}$ denotes the lth item of $\gamma_{i,n}^{[s_j]}$. We say that $\gamma_{i,n}^{[s_j]}$ has a *dominate item* if there exists integer $1\le r\le n_{ij}$ such that $f_{r,n}^{ij} \ge f_{l,n}^{ij}$ for all $r\ne l$ and $n\ge 1$. We say $F = \{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^{2}$ is of the *dominating type* ($F\in\mathbf{D}$) if each $\gamma_{i,n}^{[s_j]}$ has a dominate item for all $1\le i, j\le 2$. If $F\in\mathbf{D}$, we assume that $f_{1,n-1}^{ij}$ is the corresponding dominate item for all $1\le i, j\le 2$. Thus,

$$\gamma_{i,n}^{[s_j]} = \sum_{l=1}^{n_{ij}} f_{l,n-1}^{ij} = f_{1,n-1}^{ij}\left(1+\sum_{l=2}^{n_{ij}}\frac{f_{l,n-1}^{ij}}{f_{1,n-1}^{ij}}\right)$$

and $1 \le 1+\sum_{l=2}^{n_{ij}}\frac{f_{l,n-1}^{ij}}{f_{1,n-1}^{ij}} \le 4$, where the number 4 comes from the extreme case where $n_{ij}=4$ and $\frac{f_{l,n-1}^{ij}}{f_{1,n-1}^{ij}} \le 1$ for $l=2,\ldots,n_{ij}$. Let $w_n = (\ln\gamma_{1,n}^{[s_1]}, \ln\gamma_{2,n}^{[s_1]}, \ln\gamma_{1,n}^{[s_2]}, \ln\gamma_{2,n}^{[s_2]})'$, it follows immediately from (4) and (5) that

$$w_n = K w_{n-1} + b_{n-1}, \tag{19}$$

for some $K \in \mathcal{M}_4$ and

$$b_{n-1} = \begin{pmatrix} \ln\left(1 + \frac{\sum_{l=2}^{n11} f_{l,n-1}^{11}}{f_{1,n-1}^{ij}}\right) \\ \ln\left(1 + \frac{\sum_{l=2}^{n21} f_{l,n-1}^{21}}{f_{1,n-1}^{ij}}\right) \\ \ln\left(1 + \frac{\sum_{l=2}^{n12} f_{l,n-1}^{12}}{f_{1,n-1}^{ij}}\right) \\ \ln\left(1 + \frac{\sum_{l=2}^{n22} f_{l,n-1}^{22}}{f_{1,n-1}^{ij}}\right) \end{pmatrix}. \tag{20}$$

Ban–Chang [7] prove that if the symbol $\gamma_i^{[s_j]}$ is *essential*[6] for $1 \leq i, j \leq 2$, then $\rho_K = \lambda$, where ρ_B is the spectral radius of the matrix B. Let $v, w \in \mathbb{R}^n$, we say $v \geq w$ if $v_i \geq w_i$ for $1 \leq i \leq n$.

Proposition 3.4 *Suppose $\alpha_1 > \alpha_2$ (or $\alpha_2 > \alpha_1$), and then $F \in \mathbf{D}$.*

Proof We only prove the case where $\alpha_1 > \alpha_2$ and the other case is similar. The proof is divided into small cases.

1 $\alpha_1 = v_1$. In this case, there are eight possibilities of α_2, namely $\alpha_2 = v_i$ for $i = 2, \ldots, 9$. If $\alpha_2 = v_2$, the nonlinear recurrence equation $F_{12} = \{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^2$ is as follows:

$$\begin{cases} \gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]} + \gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}, \\ \gamma_{2,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}, \\ \gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]}, \\ \gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]}, \\ \gamma_{1,1}^{[s_1]} = 4, \gamma_{2,1}^{[s_1]} = 2, \gamma_{1,1}^{[s_2]} = 1, \gamma_{2,1}^{[s_2]} = 1. \end{cases}$$

Since $\gamma_{1,n}^{[s_2]} \geq \gamma_{2,n}^{[s_2]}$ and $\gamma_{1,n}^{[s_1]} \geq \gamma_{2,n}^{[s_1]}$, we deduce that $\gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}$ (resp. $\gamma_{1,n-1}^{[s_1]}$) is the dominate item for $\gamma_{1,n}^{[s_1]}$ and $\gamma_{2,n}^{[s_1]}$ (resp. $\gamma_{1,n}^{[s_2]}$ and $\gamma_{2,n}^{[s_2]}$). Thus, $F_{12} \in \mathbf{D}$. $F_{13}, F_{14}, F_{15} \in \mathbf{D}$ can be treated in the same manner. For $\alpha_2 = v_6, v_7, v_8, v_9$, let $\alpha_2 = v_6$, F_{16} is of the following form

$$\begin{cases} \gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]} + \gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}, \\ \gamma_{2,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}, \\ \gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]}, \\ \gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\ \gamma_{1,1}^{[s_1]} = 2, \gamma_{2,1}^{[s_1]} = 1, \gamma_{1,1}^{[s_2]} = 2, \gamma_{2,1}^{[s_2]} = 1. \end{cases}$$

6 We call the symbol $\gamma_i^{[s_j]}$ *essential* if there exists $n \in \mathbb{N}$ such that $\gamma_{i,n}^{[s_j]} > 1$, and *inessential* otherwise. In [6], the authors find a finite checkable conditions to characterize whether $\gamma_i^{[s_j]}$ is essential or inessential.

Since $\gamma_{2,n}^{[s_1]}$ (resp. $\gamma_{2,n}^{[s_2]}$) have only one item, it is the dominate item. It follows from the fact that $\gamma_{1,n}^{[s_1]} \geq \gamma_{2,n}^{[s_1]}$ and $\gamma_{1,n}^{[s_2]} \geq \gamma_{2,n}^{[s_2]}$, and it is concluded that $\gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}$ (resp. $\gamma_{1,n-1}^{[s_1]}$) is still the dominate item of $\gamma_{1,n}^{[s_1]}$ (resp. $\gamma_{1,n}^{[s_2]}$). Thus, $F \in \mathbf{D}$. $F_{17}, F_{18}, F_{19} \in \mathbf{D}$ can be treated similarly.

2 $\alpha_1 = v_2, v_3, v_4, v_5$. Assuming $\alpha_1 = v_2$, since $\alpha_1 > \alpha_2$, it suffices to check $\alpha_2 = v_6$ and v_8. For $\alpha_1 = v_2$, under the same argument as above we conclude that $\gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}$ (resp. $\gamma_{1,n-1}^{[s_1]}$) is the dominate item for $\gamma_{1,n}^{[s_1]}$ (resp. $\gamma_{2,n}^{[s_1]}$) and thus $F_{26} \in \mathbf{D}$. The same reasoning applies to other cases. This completes the proof. □

The entropy can be computed for $F \in \mathbf{D}$. Let us denote by $\mathcal{E}$ (resp. $\mathcal{I}$) the set of all essential (resp. inessential) symbols of $\{\gamma_i^{[s_j]}\}_{i,j=1}^2$, we note that $\{\gamma_i^{[s_j]}\}_{i,j=1}^2 = \mathcal{E} \cup \mathcal{I}$. The computation methods of h are divided into two subcases, namely $\mathcal{I} = \emptyset$ and $\mathcal{I} \neq \emptyset$. First, we take F_{16} as an example to illustrate how to compute h for this type and $\mathcal{I} = \emptyset$ in this case.

Theorem 3.5 *Let X be a G-SFT in which the corresponding nonlinear recurrence equation is F_{16}; that is, $T_1 = T_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Let b_n be constructed as (20),*

$$K = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} \lambda & \overline{\lambda} & 0 & 0 \\ \lambda & \overline{\lambda} & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{21}$$

be such that $QDQ^{-1} = K$ and $D = diag(\lambda, \bar{\lambda}, 0, 0)$. Then

$$h_{16} = \frac{(\lambda - 1)\, A_\infty}{\lambda^2} \approx 0.236\,07 A_\infty,$$

where

$$A_\infty = \lim_{n\to\infty} \left(\widehat{w}_1^{(1)} + \lambda^{-1}\widehat{b}_1^{(1)} + \cdots + \lambda^{-n+1}\widehat{b}_n^{(1)} \right) \tag{22}$$

and $\widehat{b_n} = Q^{-1}b_n$. Moreover, the limit (22) exists.

Proof Note that

$$F_{16} = \begin{cases} \gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]} + \gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}, \\ \gamma_{2,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}, \\ \gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]}, \\ \gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\ \gamma_{1,1}^{[s_1]} = 4, \gamma_{2,1}^{[s_1]} = 1, \gamma_{1,1}^{[s_2]} = 2, \gamma_{2,1}^{[s_2]} = 1. \end{cases}$$

It can be easily checked that $\gamma_{i,2}^{[s_j]} \geq 4$ for $1 \leq i, j \leq 2$. Thus, there is no inessential symbol and $\mathcal{I} = \emptyset$. Since $\gamma_{1,n}^{[s_1]} \geq \gamma_{2,n}^{[s_1]}$ and $\gamma_{1,n}^{[s_2]} \geq \gamma_{2,n}^{[s_2]}$, $\gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}$ (resp. $\gamma_{1,n-1}^{[s_1]}$)

is the dominate item of $\gamma_{1,n}^{[s_1]}$ (resp. $\gamma_{1,n}^{[s_2]}$), $F \in \mathbf{D}$. The above argument indicates that $w_n = K w_{n-1} + b_{n-1}$, and K is indeed (21). Along the identical line of the proof in Theorem 3.2, we have $w_n = K^{n-1} w_1 + \sum_{i=0}^{n-2} K^i b_i$,

$$h = \lim_{n\to\infty} \frac{\ln \sum_{i=1}^{2} \gamma_{i,n}^{[s_1]}}{|E_n|} = \lim_{n\to\infty} \frac{\ln \gamma_{1,n}^{[s_1]}}{|E_n|} = \lim_{n\to\infty} \frac{w_n^{(1)}}{|E_n|}, \tag{23}$$

and

$$\begin{aligned} w_n^{(1)} &= \left(K^{n-1} w_1 + K^{n-2} b_1 + \cdots + b_n\right)^{(1)} \\ &= \left(Q D^{n-1} Q^{-1} w_1 + Q D^{n-2} Q^{-1} b_1 + \cdots + Q Q^{-1} b_n\right)^{(1)}. \end{aligned} \tag{24}$$

Combining (24) with direct computation yields

$$\begin{aligned} w_n^{(1)} &= \left(Q D^{n-1} \widehat{w}_1 + Q D^{n-2} \widehat{b}_1 + \cdots + Q \widehat{b}_n\right)^{(1)} \\ &= \lambda^{n-1} Q_{11} \left(\widehat{w}_1^{(1)} + \lambda^{-1} \widehat{b}_1^{(1)} + \cdots + \lambda^{-n+1} \widehat{b}_n^{(1)}\right) + O(\bar{\lambda}^n) \\ &= \lambda^n \left(\widehat{w}_1^{(1)} + \lambda^{-1} \widehat{b}_1^{(1)} + \cdots + \lambda^{-n+1} \widehat{b}_n^{(1)}\right) + O(\bar{\lambda}^n). \end{aligned}$$

Combining (23), we obtain that

$$h = \lim_{n\to\infty} \frac{\lambda^n \left(\widehat{w}_1^{(1)} + \lambda^{-1} \widehat{b}_1^{(1)} + \cdots + \lambda^{-n+1} \widehat{b}_n^{(1)}\right)}{\left(\sum_{i=0}^{n} A^i \mathbf{1}\right)^{(1)}}. \tag{25}$$

Let $A_\infty = \lim_{n\to\infty} \left(\widehat{w}_1^{(1)} + \lambda^{-1} \widehat{b}_1^{(1)} + \cdots + \lambda^{-n+1} \widehat{b}_n^{(1)}\right)$, such limit exists due to the fact that $\widehat{b}_n^{(1)}$ is bounded for all n. Combining (25) with (18) yields

$$\begin{aligned} h &= \lim_{n\to\infty} \frac{\lambda^n A_\infty}{\left(\sum_{i=0}^{n} A^i \mathbf{1}\right)^{(1)}} \\ &= A_\infty \lim_{n\to\infty} \frac{\lambda^n}{p_{11}(q_{11}+q_{12}) \frac{\lambda^n - 1}{\lambda - 1} + p_{12}(q_{21}+q_{22}) \frac{\bar{\lambda}^n - 1}{\bar{\lambda} - 1}} \\ &= \frac{A_\infty}{\frac{1}{\lambda-1} p_{11}(q_{11}+q_{12})} \\ &= \frac{(\lambda-1) A_\infty}{\lambda^2} \\ &\approx 0.236\,07 A_\infty \approx 0.5011681177. \end{aligned}$$

This completes the proof. □

Next, we use F_{39} to illustrate the computation of h for the case where $\mathcal{I} \neq \emptyset$.

Proposition 3.6 $h_{39} = 0$.

Proof Note that

$$F_{39} = \begin{cases} \gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}, \\ \gamma_{2,n}^{[s_1]} = \gamma_{2,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}, \\ \gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]}, \\ \gamma_{2,n}^{[s_2]} = \gamma_{2,n-1}^{[s_1]}, \\ \gamma_{1,1}^{[s_1]} = 2, \gamma_{2,1}^{[s_1]} = 1, \gamma_{1,1}^{[s_2]} = 2, \gamma_{2,1}^{[s_2]} = 1. \end{cases} \tag{26}$$

It can be checked that $\gamma_{2,n}^{[s_1]} = \gamma_{2,n}^{[s_2]} = 1$ for all $n \in \mathbb{N}$, and $F_{39} \in \mathbf{D}$. Thus, $\mathcal{I} = \{\gamma_2^{[s_1]}, \gamma_2^{[s_1]}\} \neq \emptyset$. Under the same argument in the beginning of this section, we construct

$$K = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Since $w_n^{(2)} = w_n^{(4)} = 0$, formula (19) can be reduced to the following form.

$$\widetilde{w}_n = \widetilde{K}\widetilde{w}_{n-1} + \widetilde{b}_{n-1}, \tag{27}$$

where $\widetilde{w}_n = (w_n^{(1)}, w_n^{(3)})'$, $\widetilde{b}_n = (\ln(1 + \frac{\gamma_{2,n}^{[s_1]}}{\gamma_{1,n}^{[s_1]}}), \ln(1 + \frac{\gamma_{2,n}^{[s_1]}}{\gamma_{1,n}^{[s_1]}}))$ and $\widetilde{K} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ which is derived by deleting the second and fourth columns and rows of K. Note that if $\rho_{\widetilde{K}} > 1$, the method used in Theorem 3.5 still works. The induction formula (27) in fact provides us the formula of $w_n^{(1)}$ and $w_n^{(3)}$. Indeed, since $\gamma_{2,n}^{[s_2]} = 1$, $w_n^{(1)} = \ln\gamma_{1,n}^{[s_1]} = \ln(\gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]}) = \ln\gamma_{1,n}^{[s_2]} = w_n^{(3)}$, the recurrence equation (27) is reduced to the simple form

$$\begin{aligned} w_n^{(1)} &= w_{n-1}^{(1)} + \ln\left(1 + \frac{\gamma_{2,n-1}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}}\right) \\ &= w_1^{(1)} + \ln\left(1 + \frac{\gamma_{2,1}^{[s_1]}}{\gamma_{1,1}^{[s_1]}}\right) + \cdots + \ln\left(1 + \frac{\gamma_{2,n-1}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}}\right) \\ &= \ln\gamma_{1,1}^{[s_1]} + \ln\left(1 + \frac{\gamma_{2,1}^{[s_1]}}{\gamma_{1,1}^{[s_1]}}\right) + \cdots + \ln\left(1 + \frac{\gamma_{2,n-1}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}}\right). \end{aligned} \tag{28}$$

Equality (28) actually demonstrates the inductive formula $w_n^{(1)} = \ln\gamma_{1,n}^{[s_1]}$. Indeed,

$$\ln\gamma_{1,1}^{[s_1]} + \ln\left(1 + \frac{\gamma_{2,1}^{[s_1]}}{\gamma_{1,1}^{[s_1]}}\right) + \cdots + \ln\left(1 + \frac{\gamma_{2,n-1}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}}\right)$$

$$
\begin{aligned}
&= \ln \gamma_{1,1}^{[s_1]} \left(1 + \frac{\gamma_{2,1}^{[s_1]}}{\gamma_{1,1}^{[s_1]}}\right) \cdots \left(1 + \frac{\gamma_{2,n-1}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}}\right) \\
&= \ln \gamma_{1,1}^{[s_1]} \left(\frac{\gamma_{1,1}^{[s_1]} + \gamma_{2,1}^{[s_1]}}{\gamma_{1,1}^{[s_1]}}\right) \cdots \left(\frac{\gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}}\right). \\
&= \ln \gamma_{1,1}^{[s_1]} \left(\frac{\gamma_{1,2}^{[s_1]}}{\gamma_{1,1}^{[s_1]}}\right) \left(\frac{\gamma_{1,3}^{[s_1]}}{\gamma_{1,2}^{[s_1]}}\right) \cdots \left(\frac{\gamma_{1,n}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}}\right) \\
&= \ln \gamma_{1,n}^{[s_1]}.
\end{aligned}
$$

The second equality comes from the recurrence formula (26). For the computation of h, we know that $\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} + 1$. Thus, $\gamma_{1,n}^{[s_1]} = n$ and $h = 0$. □

The same reasoning applies to the cases $h_{62} = h_{64} = h_{72} = h_{58} = h_{59} = 0$.

Corollary 3.7 $h_{46} = h_{47} = h_{44} = h_{45} = \frac{1}{\lambda^2} \ln 2$.

Proof It follows from Theorem 3.2 that we have $h_{44} = h_{45}$. It suffices to prove that $h_{46} = h_{44}$, and the case where $h_{47} = h_{44}$ is similar. Since

$$
F_{46} = \begin{cases}
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]} + \gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}, \\
\gamma_{2,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,1}^{[s_1]} = 2, \gamma_{2,1}^{[s_1]} = 1, \gamma_{1,1}^{[s_2]} = 1, \gamma_{2,1}^{[s_2]} = 1,
\end{cases}
$$

we have $\gamma_{1,n}^{[s_2]} = \gamma_{2,n}^{[s_2]}$, and it follows from the fact that $F_{46} \in \mathbf{D}$, and we reduce F_{46} to (note $h_{46} = \lim_{n\to\infty} \frac{\ln\left(\gamma_{1,n}^{[s_1]}+\gamma_{2,n}^{[s_1]}\right)}{|E_n|} = \lim_{n\to\infty} \frac{\ln \gamma_{1,n}^{[s_1]}}{|E_n|}$)

$$
\begin{cases}
\gamma_{1,n}^{[s_1]} = 2\gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,1}^{[s_1]} = 2, \gamma_{1,1}^{[s_2]} = 1.
\end{cases}
$$

The same argument as in the proof of Theorem 3.2 infers that $h_{46} = h_{44} = \frac{1}{\lambda^2} \ln 2$. This completes the proof. □

3.4 Oscillating type

We call an $F = \{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^{2}$ the *oscillating type* ($F \in \mathbf{O}$) if there exist two sequences $\{m_n^1\}$, $\{m_n^2\}$ of $\mathbb{N}$ with $\{m_n^1\} \cap \{m_n^2\} = \emptyset$ and $\{m_n^1\} \cup \{m_n^2\} = \mathbb{N}$ such that $\gamma_{1,n}^{[s_1]} \geq \gamma_{2,n}^{[s_1]}$ for $n \in \{m_n^1\}$ and $\gamma_{1,n}^{[s_1]} < \gamma_{2,n}^{[s_1]}$ if $n \in \{m_n^2\}$. We say $F \in \mathbf{O}_2$ if the two sequences are odd and even numbers. For $F \in \mathbf{O}_2$, h can be computed along the same line of Theorem 3.5. The steps are listed as follows.

(1) Expand $F=\{\gamma_{i,n}^{[s_1]},\gamma_{i,n}^{[s_2]}\}_{i=1}^{2}$ to $(n-2)$-order, say $F^{(2)}$; that is, expand each item of $\gamma_{i,n}^{[s_j]}$ to next level according to the rule of F.
(2) Since $\gamma_{1,n}^{[s_1]}\geq\gamma_{2,n}^{[s_1]}$ for n being even and $\gamma_{1,n}^{[s_1]}<\gamma_{2,n}^{[s_1]}$ for n being odd, one assures that $F^{(2)}\in\mathbf{D}$.
(3) Construct $w_n=Kw_{n-2}+b_{n-2}$ as in the case of dominating type, and note that $K\in\mathcal{M}_{4\times 4}$ with $\rho_K=\lambda^2$ (cf. [7]).
(4) Iterate w_n and compute the growth rate of $\lim_{n\to\infty}\frac{w_{2n}}{|E_{2n}|}$. Since the limit h exists (Theorem 2.1), we have $h=\lim_{n\to\infty}\frac{w_{2n}}{|E_{2n}|}$.

The following proposition characterizes whether $F\in\mathbf{O}_2$.

Proposition 3.8 $F_{36},F_{56},F_{92},F_{94}\in\mathbf{O}_2$.

Proof Note that $F_{36}\simeq F_{92}$ and $F_{56}\simeq F_{94}$. Thus, we only need to prove F_{36} and F_{56}. Since the proofs of F_{36} and F_{56} are identical, it suffices to prove the case of F_{36}. F_{36} is of the following form

$$F_{36}=\begin{cases}\gamma_{1,n}^{[s_1]}=\gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}+\gamma_{2,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]},\\ \gamma_{2,n}^{[s_1]}=\gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]},\\ \gamma_{1,n}^{[s_2]}=\gamma_{1,n-1}^{[s_1]}+\gamma_{2,n-1}^{[s_1]},\\ \gamma_{2,n}^{[s_2]}=\gamma_{1,n-1}^{[s_1]},\\ \gamma_{1,1}^{[s_1]}=2,\gamma_{2,1}^{[s_1]}=1,\gamma_{1,1}^{[s_2]}=2,\gamma_{2,1}^{[s_2]}=1.\end{cases}\tag{29}$$

Let $\tau_n=\frac{\gamma_{1,n}^{[s_1]}}{\gamma_{2,n}^{[s_1]}}$ and $\chi_n=\frac{\gamma_{1,n}^{[s_2]}}{\gamma_{2,n}^{[s_2]}}$, we have

$$\begin{aligned}\tau_n&=\frac{\gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}+\gamma_{2,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}}{\gamma_{1,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}}=\frac{1}{\chi_{n-1}}+\frac{1}{\tau_{n-1}\chi_{n-1}},\\ \chi_n&=\frac{\gamma_{1,n-1}^{[s_1]}+\gamma_{2,n-1}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}}=1+\frac{1}{\tau_{n-1}}.\end{aligned}$$

The direct computation shows that $(\tau_1,\chi_1)=(\frac{1}{2},\frac{1}{2})$ and $(\tau_2,\chi_2)=(8,3)$. Note that if $\tau_n\leq\frac{1}{2}$ and $\chi_n\leq\frac{3}{2}$, then

$$\begin{aligned}\tau_{n+1}&=\frac{1}{\chi_n}+\frac{1}{\tau_n\chi_n}\geq\frac{2}{3}+\frac{4}{3}=2,\\ \chi_{n+1}&=1+\frac{1}{\tau_{n-1}}\geq 1+2=3.\end{aligned}$$

If $\tau_n\geq 2$, $\chi_n\geq 3$, we have

$$\tau_{n+1}=\frac{1}{\chi_n}+\frac{1}{\tau_n\chi_n}\leq\frac{1}{3}+\frac{1}{6}=\frac{1}{2},$$

$$\chi_{n+1} = 1 + \frac{1}{\tau_{n-1}} \leq 1 + \frac{1}{2} \leq \frac{3}{2}.$$

By induction, we have $\tau_n \leq \frac{1}{2}$ ($\gamma_{1,n}^{[s_1]} < \gamma_{2,n}^{[s_1]}$) and $\chi_n \leq \frac{3}{2}$ for n being an odd number and $\tau_n \geq 2$ ($\gamma_{1,n}^{[s_1]} > \gamma_{2,n}^{[s_1]}$), $\chi_n \geq 3$ for n being even, i.e., $F \in \mathbf{O}_2$. This completes the proof. ◻

4 Characterization

Theorem 4.1 *Let $\mathcal{A} = \{1, 2\}$. Suppose $F = \{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^{2}$ is the nonlinear recurrence equation of a G-SFT with $G = \langle S|R_A\rangle$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then, F is either one of the following four types.*

(1) *F is of the zero entropy type.*
(2) *F is of the equal growth type,*
(3) *F is of the dominating type,*
(4) *F is of the oscillating type.*

Proof Without loss of generality, we assume that $|\alpha_1| \geq |\alpha_2|$. The proof is divided into two subcases.

1 $|\alpha_1| = 4$. In this case, under the same proof of Theorem 3.4 we see that $F \in \mathbf{D}$.

2 $|\alpha_1| = 2$. Note that $(\alpha_1, \alpha_2) = (v_k, v_l)$ for $k, l = 2, 3$ and $(\alpha_1, \alpha_2) = (v_4, v_4)$ satisfy the assumptions of Theorem 3.2; they belong to type $\mathbf{E}$. For other cases, since most of them are type $\mathbf{D}$ under the routine check, we only pick those F which do not belong to type $\mathbf{D}$, namely F_{36}, F_{56}, F_{38}, F_{57}, F_{72}, F_{92}, F_{94} and F_{84}. Since $F_{36} \simeq F_{92}$, $F_{56} \simeq F_{94}$, $F_{38} \simeq F_{72}$ and $F_{57} \simeq F_{84}$, it suffices to check F_{36}, F_{56}, F_{38} and F_{57}. Proposition 3.8 indicates that $F_{36}, F_{56} \in \mathbf{O}_2$. Thus, we only need to discuss the cases of F_{38} and F_{57}. Actually, we prove $F_{38} \in \mathbf{O}\backslash\mathbf{O}_2$ (F_{57} is similar). F_{38} is of the form

$$\begin{cases} \gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}\gamma_{2,n-1}^{[s_2]}, \\ \gamma_{2,n}^{[s_1]} = \gamma_{2,n-1}^{[s_1]}\gamma_{1,n-1}^{[s_2]}, \\ \gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]}, \\ \gamma_{2,n}^{[s_2]} = \gamma_{2,n-1}^{[s_1]}, \\ \gamma_{1,1}^{[s_1]} = 2, \gamma_{2,1}^{[s_1]} = 1, \gamma_{1,1}^{[s_2]} = 2, \gamma_{2,1}^{[s_2]} = 1. \end{cases}$$

Let $\tau_n = \frac{\gamma_{1,n}^{[s_1]}}{\gamma_{2,n}^{[s_1]}}$ and $\chi_n = \frac{\gamma_{1,n}^{[s_2]}}{\gamma_{2,n}^{[s_2]}}$, then

$$\tau_n = \frac{\tau_{n-1+1}}{\chi_{n-1}} = \frac{\chi_n}{\chi_{n-1}} \text{ and } \chi_n = \tau_{n-1} + 1.$$

Direct examination shows that $(\tau_1, \chi_1) = (2, 2)$, $(\tau_2, \chi_2) = (\frac{3}{2}, 3)$, $(\tau_3, \chi_3) = (\frac{5}{6}, \frac{5}{2})$, $(\tau_4, \chi_4) = (\frac{11}{15}, \frac{11}{6})$, $(\tau_5, \chi_5) = (\frac{52}{55}, \frac{26}{15}) \ldots$; thus, $F_{38} \notin \mathbf{O}_2$.

3 $|\alpha_1| = 1$. Proposition 3.1 indicates that all these cases belong to type **Z**. This completes the proof. □

The following table indicates all types of F_{ij} for $1 \le i, j \le 9$.

$\alpha_2 \backslash \alpha_1$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	**E**	**D**	**D**	**D**	**D**	**D**	**D**	**D**	**D**
v_2	**D**	**E**	**E**	**D**	**D**	**D**	**O**	**D**	$\mathbf{O}_2$
v_3	**D**	**E**	**E**	**D**	**D**	**D**	**D**	**D**	**D**
v_4	**D**	**D**	**D**	**E**	**E**	**D**	**D**	**O**	$\mathbf{O}_2$
v_5	**D**	**D**	**D**	**E**	**E**	**D**	**D**	**D**	**D**
v_6	**D**	**D**	$\mathbf{O}_2$	**D**	$\mathbf{O}_2$	**Z**	**Z**	**Z**	**Z**
v_7	**D**	**D**	**D**	**D**	**O**	**Z**	**Z**	**Z**	**Z**
v_8	**D**	**D**	**O**	**D**	**D**	**Z**	**Z**	**Z**	**Z**
v_9	**D**	**D**	**D**	**D**	**D**	**Z**	**Z**	**Z**	**Z**

4.1 Numerical results

The numerical result of h is presented. We give some explanations as follows:

(1) We note that for each F_{kl}, there exists a unique F_{pq} such that $F_{kl} \simeq F_{pq}$, which gives $h_{kl} = h_{pq}$, e.g., $h_{48} = h_{75}$, $h_{14} = h_{51}$, etc.
(2) $h_{kl} = 0$ for $k, l \in \{6, 7, 8, 9\}$ (Proposition 3.1).
(3) $h_{11} = \ln 2$ and $h_{kl} = \frac{\ln 2}{\lambda}$ if $k, l \in \{2, 3\}$ and $h_{kl} = \frac{\ln 2}{\lambda^2}$ if $k, l \in \{4, 5\}$ (Theorem 3.2).
(4) $h_{44} = h_{45} = h_{46} = h_{47}$ (Corollary 3.7).
(5) $h_{39} = h_{62} = h_{64} = h_{72} = h_{58} = h_{59} = 0$ (Proposition 3.6).

$\alpha_2 \backslash \alpha_1$	v_1	v_2	v_3	v_4
v_1	0.6924441915	0.5827398718	0.5827398718	0.4446025684
v_2	0.5827398718	0.4282225063	0.4282225063	0.3480809809
v_3	0.5827398718	0.4282225063	0.4282225063	0.3785719508
v_4	0.5473654583	0.4046074815	0.2920492775	0.2648611178
v_5	0.4446025684	0.3785719508	0.3480809809	0.2648611178
v_6	0.5011681177	0.3384608728	0.2437451279	0.2647497426
v_7	0.4808946783	0.3062336239	0.2093165951	0.2647497426
v_8	0.3742043181	0.2747387680	0.1904155180	0.2009045358
v_9	0.3529894045	0.2396006045	0	0.1959187210

$\alpha_2\backslash\alpha_1$	v_5	v_6	v_7	v_8
v_1	0.5473654583	0.3529894045	0.3742043181	0.4808946783
v_2	0.2920492775	0	0.1904155180	0.2093165951
v_3	0.4046074815	0.2396006045	0.2747387680	0.3062336239
v_4	0.2648611178	0	0	0.1486957616
v_5	0.2648611178	0.1959187210	0.2009045358	0.2648611178
v_6	0.1312568310	0	0	0
v_7	0.1486957616	0	0	0
v_8	0	0	0	0
v_9	0	0	0	0

$\alpha_2\backslash\alpha_1$	v_9
v_1	0.5011681177
v_2	0.2437451279
v_3	0.3384608728
v_4	0.1312568310
v_5	0.2648611178
v_6	0
v_7	0
v_8	0
v_9	0

5 Conclusion and open problems

We list the results of this paper as follows:

(1) The existence of the entropy (1) for a G-shift is illustrated in Theorem 2.1.
(2) The nonlinear recurrence equation which describes the growth behavior of the admissible patterns $\gamma_n(X)$ in a G-SFT (or G-vertex shift) is established in Sect. 2.
(3) The $\mathbf{Z}$, $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{O}$ ($\mathbf{O}_2$) types of nonlinear recurrence equations are introduced. The algorithms of the entropy computations for these types are also presented (cf. Sect. 3).
(4) The characterization of the nonlinear recurrence equations of G-SFTs with two symbols is presented (Theorem 4.1).

We emphasize that the computation method of h can be easily extended to the case of more symbols. However, the general entropy formula for arbitrary G-SFTs is far from being solved. We list some problems in the further study.

Problem 5.1 Can we give the characterization for G-SFTs over symbol set $\mathcal{A}$ with $|\mathcal{A}| > 2$?

Problem 5.2 Let $H = \langle S|R_B\rangle$ with $S = \{s_1, \ldots, s_d\}$ and B be an arbitrary d-dimensional $\{0, 1\}$-matrix, can we develop the entropy theory for H-SFTs?

Problem 5.3 Can we extend the methods of G-SFTs to F_d-SFTs? More precisely, can we establish the entropy formula for F_d-SFTs?

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