Entropy dimension of shift spaces on monoids

Article *in* Journal of Mathematical Physics · July 2020

DOI: 10.1063/1.5124073

3 authors, including:

Jung-Chao Ban
National Dong Hwa University

Chih-Hung Chang
National University of Kaohsiung

ENTROPY DIMENSION OF SHIFT SPACES ON MONOIDS

JUNG-CHAO BAN

Department of Mathematical Sciences, National Chengchi University, Taipei 11605, Taiwan, ROC.

Math. Division, National Center for Theoretical Science, National Taiwan University, Taipei 10617, Taiwan. ROC.

CHIH-HUNG CHANG*

Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 81148, Taiwan, ROC.

NAI-ZHU HUANG

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan, ROC.

Abstract. We consider the entropy dimension of G-shifts of finite type for the case where G is a nonabelian monoid. Entropy dimension tells whether a shift space has zero topological entropy. Suppose the Cayley graph C_G of G has a finite representation (that is, $\{C_{gG} : g \in G\}$ is a finite set up to graph isomorphism), and relations among generators of G are determined by a matrix A. We reveal an association between the characteristic polynomial of A and the finite representation of the Cayley graph. After introducing an algorithm for the computation of the entropy dimension, the set of entropy dimensions is related to a collection of matrices in which the sum of each row of every matrix is bounded by the number of leaves of the graph. Furthermore, the algorithm extends to G having finitely many free-followers.

1. Introduction

Let $\mathcal{A}$ be a finite alphabet. Given $d \in \mathbb{N}$, a (d-dimensional) *configuration* is a function from $\mathbb{Z}^d$ to $\mathcal{A}$, and a pattern is a function from a finite subset of $\mathbb{Z}^d$ to

2010 *Mathematics Subject Classification.* Primary 37A35, 37B10.

Key words and phrases. Entropy dimension; free generator; Cayley graph; finite representation; free-follower.

*Author to whom any correspondence should be addressed. Email: chchang@nuk.edu.tw.

$\mathcal{A}$. A subset $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is called a *shift space* if X, denoted by $X = \mathsf{X}_{\mathcal{F}}$, consists of configurations which avoid patterns from some set $\mathcal{F}$ of patterns and the patterns avoided in $\mathcal{F}$ are modulo translation. A shift space is called a *shift of finite type* (SFT) if $\mathcal{F}$ is a finite set; $\mathbb{Z}^d$ acts on X by translation of configurations, making X a symbolic dynamical system. One of the many motivations in studying symbolic dynamical systems is that it helps for the research of hyperbolic topological dynamical systems. The interested reader can consult standard literature such as [8, 22].

While almost all properties of $\mathbb{Z}$-SFTs are decidable (cf. [17]), research of $\mathbb{Z}^d$-SFTs for $d \geq 2$ shows that many undecidability issues have been seen in it. It is even undecidable if a $\mathbb{Z}^d$-SFT is nonempty [7]. Different kinds of mixing properties have been introduced for examining the existence and density of periodic configurations [9]. A straightforward generalization of $\mathbb{Z}^d$-SFTs is considering SFTs on G which is associated with some algebraic structure. Since the d-dimensional lattice $\mathbb{Z}^d$ is a finitely generated abelian group, it is natural we start with the cases where G is a finitely generated nonabelian group or a finitely generated free monoid. Whenever G is a free monoid, it has been demonstrated that many such issues do not arise. For instance, the conjugacy between two irreducible G-SFTs (i.e., SFTs over G) is decidable [1]; furthermore, nonemptiness, extensibility, and the existence of periodic configurations are decidable for G-SFTs [2, 4]. Aside from the dynamical point of view, the phenomena from the computational perspective are also fruitful [3, 20, 21].

For the case where $G = \mathbb{Z}$, the topological entropy of a G-SFT relates to the spectral radius of an integral matrix, and the set of entropies of G-SFTs is the set of logarithms of Perron numbers [16, 17]. When $G = \mathbb{Z}^d$ for $d \geq 2$, the entropy of a G-SFT is a right recursively enumerable number which may not be algebraic and is not computable in general [15, 18, 19]. However, the story is quite different when G is a free monoid.

Suppose that G is a finitely generated free monoid. Let Σ be a finite set which generates G. An element $g \in G$ is called an *i-word* provided the minimal expression of $g = g_1 g_2 \cdots g_i$ for some $g_1, \ldots, g_i \in \Sigma$. For $n \in \mathbb{N}$, let Γ_n denote the set of n-blocks in a G-SFT, where an n-block is a pattern whose support consists of all i-words in G for $i \leq n$. Petersen and Salama developed an algorithm to estimate the tree shift

topological entropy of *hom-shifts* over G [20]. A hom-shift, roughly speaking, is a G-SFT which is isotropic and symmetric; alternatively, a hom-shift is determined by the same rule in each direction (herein, a direction means a generator of G). For instance, a d-dimensional golden mean shift is a hom-shift. The interested reader is referred to [11]. In [3], the authors showed that the topological entropy of a G-SFT is achieved by an infinite series provided $|\mathcal{A}| = d = 2$.

This paper considers the *entropy dimension*, which is known as *entropy* in [5], of G-SFTs (defined in Equation (2), see [10] for more details) for the case where G is a finitely generated nonabelian monoid. It is known that the entropy dimension of a G-shift space (i.e., a shift space over G) with positive topological entropy is $\ln d$ if G is a free monoid with d generators (cf. [6, 5, 4]). In other words, the research of entropy dimension is related to revealing zero entropy systems. Meanwhile, it remains open for the existence of a zero topological entropy G-shift space with full entropy dimension. In this paper, a necessary and sufficient condition for a G-SFT having full entropy dimension is addressed, which provides a criterion for determining whether a G-SFT has zero entropy. Zero entropy systems have drawn a lot of attention lately; many $\mathbb{Z}^d$-actions with zero entropy exhibit diverse complexities. See [10, 12, 13, 14] and the references therein for more details. This elucidation extends the computation of entropy dimension of G-SFTs to the case where G is a monoid with *finite representation* (see Definition 2.1 and Theorem 4.2). Roughly speaking, a monoid G with finite representation means that the Cayley graph of G is spatially periodic; that is, $\{gG : g \in G\}$ is finite. What is more, the algorithm extends to the case where G has finitely many *free-followers* (defined in Equation (5), see Section 6).

When G is a free monoid with d generators, the set of entropy dimensions of G-SFTs is a finite subset of the set of logarithms of Perron numbers less than or equal to d. More explicitly, the set of entropy dimensions of G-SFTs is

$$\{\ln \rho_M : \rho_M \text{ is the spectral radius of some matrix } M \in D\},$$

where D consists of $k \times k$ nonnegative integral matrices M satisfying $\sum_{j=1}^{k} M(i,j) \leq d$ for all i and $k = |\mathcal{A}|$. The interested reader is referred to [6] for more details. This paper extends these results to SFTs over monoids having finitely many free-followers. Theorems 5.1 and 5.3 demonstrate that the entropy dimension of a

G-SFT is related to the maximal spectral radius of a collection of integral matrices which are constrained by the structure of the Cayley graph of G. To be precise, let $\mathcal{M}$ be the set consisting of

$$M = \begin{pmatrix} C_1 & C_2 & C_3 & \cdots & C_d \\ I & 0 & \cdots & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & I & 0 \end{pmatrix}$$

for some $l \times l$ matrices C_i, $l \leq k$, satisfying $\sum_{q=1}^{l} C_i(p,q) \leq \xi_i$ for all $1 \leq i \leq d$, $1 \leq p \leq l$, where ξ_i represents the numbers of leaves of the Cayley graph of G. The entropy dimension spectrum of G-SFTs is

$$H = \{\ln \rho_M : \rho_M \text{ is the spectral radius of } M \in \mathcal{M}\}.$$

We end this section with a summary of the remainder of the paper. Whenever G is a monoid such that a matrix A determines the relationships among the generators of G and G has *finite representation* (see Section 2), the coefficients of the characteristic polynomial of A relate to the number of leaves of the Cayley graph of G (Theorem 3.1). After revealing an algorithm for the computation of the entropy dimension (Theorem 4.3), the set of entropy dimensions consists of the spectral radius of integral matrices being such that every coefficient of the characteristic polynomial of A is an upper bound of the sum of a corresponding row (Theorems 5.1 and 5.3). Furthermore, Section 6 extends the algorithm to the case where G has finitely many free-followers.

2. Definition and Notation

Let d be a positive integer. A *semigroup* is a set $G = \langle \Sigma | R \rangle$ together with a binary operation which is closed and associative, where $\Sigma = \{s_1, \ldots, s_d\}$ is the set of generators and R is a set of equivalences which describe the relations among the generators. A *monoid* is a semigroup with an identity element e.

Given a finite set of generators $\Sigma = \{s_1, s_2, \ldots, s_d\}$ and a $d \times d$ binary matrix A, a monoid G of the form $G = \langle \Sigma | R_A \rangle$ means that $s_i s_j = s_i$ is an equivalence relation in R_A if and only if $A(i,j) = 0$. Alternatively, s_i is a *right* (resp. *left*) *free generator* if and only if $A(i,j) = 1$ (resp. $A(j,i) = 1$) for $1 \leq j \leq d$. For example, the generators

of the monoid $G = \langle \Sigma | R_A \rangle$ with

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

satisfy the equivalences

$$s_1^2 = s_1, s_2 s_1 = s_2, s_2^2 = s_2, \text{ and } s_3 \text{ is a free generator.}$$

Let Σ_r (resp. Σ_ℓ) denote the set of right (resp. left) free generators of G. For each $g \in G$, the *length* $|g|$ indicates the number of generators used in its minimal presentation; that is,

$$|g| = \min\{j : g = g_1 g_2 \cdots g_j, g_i \in \Sigma \text{ for } 1 \le i \le j\}.$$

Hence, the length of $g \in G$ is the (minimum) number of generators used including repetitions, and e is the only element having zero length. We call g an n-word if $|g| = n$.

Suppose that $C = (V, E)$ is the (right) Cayley graph of G. Define a subgraph $D = (V_D, E_D) \subseteq C$, which is induced by V_D, as follows.

(i) $g \in V \setminus V_D$ if and only if $g = g' a g''$ for some $a \in \Sigma_r$ and $g'' \in V$ with $|g''| \ge 1$;

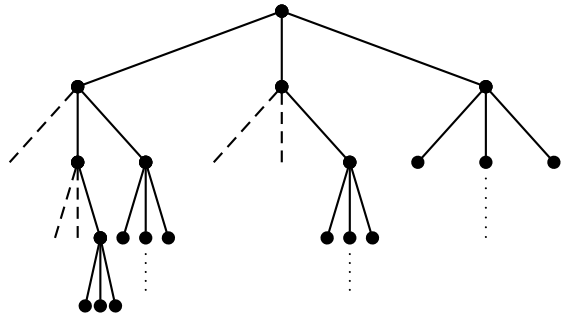
(ii) $(g, g') \in E_D$ if and only if $(g, g') \in E$ and $g, g' \in V_D$.

Definition 2.1. Suppose $G = \langle \Sigma | R \rangle$ is a finitely generated monoid with generating set Σ. Let D be the Cayley subgraph of G defined as above. Then G is said to have a *finite representation* if D is a finite graph.

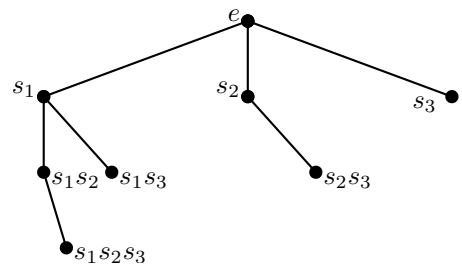
For $H_1, H_2 \subseteq G$, denote $H_1 \simeq H_2$ if the Cayley graph of H_1 is isomorphic to the Cayley graph of H_2. It is known that $\simeq$ is an equivalence relation. Observe that G having a finite representation implies that $\{[gG]_\simeq : g \in G\}$ is a finite set. Roughly speaking, G acts on G periodically. Furthermore, $gG \simeq G$ if $g \in V_D$ and $gs \notin V_D$ for all $s \in \Sigma$. For the rest of this paper, $G = \langle \Sigma | R_A \rangle$ denotes a monoid with a finite representation unless otherwise stated. See Example 2.2.

Example 2.2. Let $d = 3$ and $\Sigma = \{s_1, s_2, s_3\}$. The relations among generators of the monoid $G = \langle \Sigma | R_A \rangle$ with

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



(A) Part of Cayley graph of monoid G in Example 2.2.



(B) Finite representation of the Cayley graph C of Example 2.2.

FIGURE 1. The Cayley graph of monoid G in Example 2.2 has a finite representation. The generators s_1, s_2, s_3 satisfy the equivalences $s_1^2 = s_1$ and $s_2s_1 = s_2^2 = s_2$.

are

$$s_1^2 = s_1, s_2s_1 = s_2, \text{ and } s_2^2 = s_2.$$

It follows that s_3 is the only free generator and $\{[gG]_{\simeq}\}_{g\in G}$ is finite. Therefore, G has a finite representation with

$$V_D = \{e, s_1, s_2, s_3, s_1s_2, s_1s_3, s_2s_3, s_1s_2s_3\}.$$

See Figure 1 for the Cayley graph of G and its finite subgraph.

Let $\mathcal{A}$ be a finite alphabet. A *configuration* (or *labeled tree*) is a function $t : G \to \mathcal{A}$. For each $g \in G$, $t_g = t(g)$ denotes the label attached to the vertex g of the Cayley graph of G. We call a configuration a labeled tree since all the monoids considered in this paper can be represented as trees. The *full shift* $\mathcal{A}^G$ is the set of all labeled trees, and the *(left) shift action* $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is defined as $(\sigma_g t)_{g'} := \sigma(g,t)_{g'} = t_{gg'}$ for $g, g' \in G$. For each $n \geq 0$, let $\Delta_n = \{g \in G : |g| \leq n\}$. An *$n$-block* is a function $\tau : \Delta_n \to \mathcal{A}$ and we say that τ is a block if τ is an n-block for

some $n \in \mathbb{N}$. A labeled tree t *accepts* an n-block τ if there exists $g \in G$ such that $t_{gg'} = \tau_{g'}$ for all $g' \in \Delta_n$; otherwise, t *avoids* τ. A *G-shift space* is a set $X \subseteq \mathcal{A}^G$ of all labeled trees which avoid all of a certain set of blocks. More explicitly,

$$X = \mathsf{X}_{\mathcal{F}} = \{t \in \mathcal{A}^G : t \text{ avoids } \tau \text{ for all } \tau \in \mathcal{F}\}$$

for some set of blocks $\mathcal{F}$. A G-shift space X is called a *G-shift of finite type* (G-SFT) if $X = \mathsf{X}_{\mathcal{F}}$ for some finite $\mathcal{F}$.

3. Characterization of Finite Representation

In this section, we establish the correspondence between the coefficients of the characteristic polynomial of A and ∂V_D, where $g \in \partial V_D$ if and only if $g \in V_D$ and $gs \notin V_D$ for all $s \in \Sigma$. For each $n \in \mathbb{N}$, let

$$\xi_n = \#\{g \in G : |g = g_1 \cdots g_n| = n, g_i \in \Sigma, \text{ and } g_n \text{ is the only right free generator}\}.$$

Theorem 3.1 reveals $\{\xi_n\}$ plays an important role in the characteristic polynomial of A.

Theorem 3.1. *Suppose $G = \langle \Sigma | R_A \rangle$ is a monoid determined by a binary matrix A. Then the characteristic polynomial of A is*

$$f(\lambda) = \lambda^d - \sum_{i=1}^{d} \xi_i \lambda^{d-i}. \tag{1}$$

Before proving Theorem 3.1, it is essential to characterize the structure of the Cayley graph of G. Let

$$P_n = \{g \in G : |g| = n + 1 \text{ and } g_1 = g_{n+1}\}$$

and

$$\Xi_n = \{g \in P_n : g_n \text{ is the only right free generator}\}$$

be the sets of periodic $(n+1)$-words and periodic $(n+1)$-words whose second to last symbol is the unique right free generator, respectively. It follows immediately that $|P_n| = \mathrm{tr}(A^n)$ and $|\Xi_n| = \xi_n$.

Lemma 3.2. *For each $(n+1)$-word $g = g_1 g_2 \cdots g_n g_1 \in P_n$, there exists $1 \le i \le n$ such that g_i is a right free generator.*

Proof. Suppose not, it comes immediately that $(g_1 \cdots g_n)^m \in G$ is a vertex of D for all $m \in \mathbb{N}$, which contradicts that $|V_D| < \infty$. The proof is complete. □

Lemma 3.3. *For each positive integer $n > d$, $\xi_n = 0$. That is, every $(n+1)$-word contains at least one right free generator.*

Proof. Suppose there exists $n > d$ and $g = g_1 \cdots g_n \in G$ such that g_n is the only free generator. The pigeonhole principle asserts that $g_i = g_j$ for some $1 \le i < j \le d+1$. Lemma 3.2 demonstrates that there exists $i \le \ell \le j$ such that g_ℓ is a right free generator, which is a contradiction. This derives the desired result. □

Let

$$T(\Xi_n) = \{g_i \cdots g_n g_1 \cdots g_i : i = 1, \ldots, n, g = g_1 g_2 \cdots g_n g_1 \in \Xi_n\}$$

collect the circular shifts of all elements of Ξ_n. Observe that $|T(\Xi_n)| = n\xi_n$. Let

$$L(P_m, \Xi_n) = \{g_1 \cdots g_\alpha h_1 \cdots h_n g_{\alpha+1} \cdots g_{m+1} : \alpha = \min_i \{i \le m : g_i \in \Sigma_r\}, g \in P_m, h \in \Xi_n\}.$$

That is, $L(P_m, \Xi_n)$ consists of words obtained by inserting the first n digits of every $h = h_1 \cdots h_{n+1} \in \Xi_n$ in a periodic $(m+1)$-word g right after the first right free generator of g. Obviously, $L(P_m, \Xi_n) \subseteq P_{m+n}$.

Lemma 3.4. *Suppose $x = x_1 \cdots x_{n+1} \in P_n$ contains at least two right free generators. Let x_{r_1} and x_{r_2} be the first and second free generators, respectively. Then*

$$r_2 - r_1 = l \quad \textit{if and only if} \quad x \in L(P_{n-l}, \Xi_l).$$

Proof. If $x \in L(P_{n-l}, \Xi_l)$, then $x = g_1 \cdots g_{r_1} h_1 \cdots h_l g_{r_1+1} \cdots g_{n-l+1}$ for some $g \in P_{n-l}, h \in \Xi_l$, where $r_1 = \min\{i \le n-l : g_i \in \Sigma_r\}$. Since $h \in \Xi_l$, $x_{r_1+l} = h_l$ is the second right free generator in x. This concludes that $r_2 - r_1 = (r_1 + l) - r_1 = l$.

For each $x = x_1 \cdots x_{n+1} \in P_n$ which contains at least two right free generators, let $g = x_1 \cdots x_{r_1} x_{r_1+l+1} \cdots x_{n+1}$ and let $h = x_{r_1+1} \cdots x_{r_1+l} x_{r_1+1}$. Since $g_{r_1} = x_{r_1} \in \Sigma_r$, $x_{r_1} x_{r_1+l-1}$ is a 2-word. Furthermore, $x \in P_n$ and $x_1 = x_{n+1}$ indicate that $g_1 = g_{n-l+1}$. Alternatively, $g = g_1 \cdots g_{n-l+1} \in P_{n-l}$. Similarly, $h_l = x_{r_1+l} \in \Sigma_r$ shows that $x_{r_1+l} x_{r_1+1}$ is also a 2-word. The fact of $x_{r_2} = x_{r_1+l}$ being the second right free generator implies that $h_1, \ldots, h_{l-1}, h_{l+1} \notin \Sigma_r$, $h_l \in \Sigma_r$, and $h_{l+1} = h_1$. Hence, $h \in \Xi_l$. We conclude that $x \in L(P_{n-l}, \Xi_l)$. This completes the proof. □

Lemma 3.4 shows that $L(P_{n-l}, \Xi_l) \bigcap L(P_{n-m}, \Xi_m) = \varnothing$ if and only if $l \ne m$. Proposition 3.5, additionally, reveals a partition of P_n.

Proposition 3.5. *For each $n \in \mathbb{N}$, $\{L(P_{n-i}, \Xi_i)\}_{i=1}^{n-1} \bigcup \{T(\Xi_n)\}$ forms a partition of P_n.*

Proof. Obviously, $L(P_{n-i}, \Xi_i) \bigcap T(\Xi_n) = \varnothing$ for $1 \le i \le n-1$, since every element of $T(\Xi_n)$ has exactly one free generator, while $L(P_{n-i}, \Xi_i)$ consists of words which contain at least two free generators. The desired result comes immediately from observing that

$$P_n = \bigcup_{i=1}^{n-1} L(P_{n-i}, \Xi_i) \bigcup T(\Xi_n).$$

Indeed, the definitions of $L(P_{n-i}, \Xi_i)$ and $T(\Xi_n)$ indicate that

$$\bigcup_{i=1}^{n-1} L(P_{n-i}, \Xi_i) \bigcup T(\Xi_n) \subseteq P_n.$$

For each $x \in P_n$, $x \in T(\Xi_n)$ if x has exactly one free generator. Otherwise, x has x_{r_1} and x_{r_2} as its first two free generators for some $r_1 < r_2$. Let $l = r_2 - r_1$. Lemma 3.4 shows that $x \in L(P_{n-l}, \Xi_l)$. The proof is complete. □

Example 3.6. Let us continue with Example 2.2. Recall that

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and s_3 is the only right free generator of G. Then $\Xi_1 = \{s_3 s_3\} = T(\Xi_1) = P_1$. Since Ξ_2 consists of words of the form $u_1 s_3 u_1$ for $u_1 \neq s_3$,

$$\Xi_2 = \{s_1 s_3 s_1, s_2 s_3 s_2\} \quad \text{and} \quad T(\Xi_2) = \{s_1 s_3 s_1, s_3 s_1 s_3, s_2 s_3 s_2, s_3 s_2 s_3\}.$$

As defined above, $L(P_1, \Xi_1) = \{s_3 s_3 s_3\}$ collects the words obtained by inserting the first word of Ξ_1 in each word of P_1 right after the first right free generator. It follows that

$$P_2 = \{s_1 s_3 s_1, s_2 s_3 s_2, s_3 s_1 s_3, s_3 s_2 s_3, s_3 s_3 s_3\} = T(\Xi_2) \bigcup L(P_1, \Xi_1).$$

Similarly, $\Xi_3 = \{s_1 s_2 s_3 s_1\}$ and $T(\Xi_3) = \{s_1 s_2 s_3 s_1, s_2 s_3 s_1 s_2, s_3 s_1 s_2 s_3\}$.

$$L(P_2, \Xi_1) = \{s_1 s_3 s_3 s_1, s_2 s_3 s_3 s_2, s_3 s_3 s_1 s_3, s_3 s_3 s_2 s_3, s_3 s_3 s_3 s_3\},$$
$$L(P_1, \Xi_2) = \{s_3 s_1 s_3 s_3, s_2 s_3 s_3 s_3\}.$$

Then

$$\begin{aligned} P_3 = \{&s_1 s_2 s_3 s_1, s_1 s_3 s_3 s_1, s_2 s_3 s_1 s_2, s_2 s_3 s_3 s_2, s_3 s_1 s_2 s_3, \\ &s_3 s_1 s_3 s_3, s_3 s_2 s_3 s_3, s_3 s_3 s_1 s_3, s_3 s_3 s_2 s_3, s_3 s_3 s_3 s_3\} \\ = T(\Xi_3) &\bigcup L(P_2, \Xi_1) \bigcup L(P_1, \Xi_2). \end{aligned}$$

Furthermore, $\Xi_n = \varnothing$ for $n \ge 4$.

For each real $d\times d$ matrix A, there is a recursive formula for the coefficients of the characteristic polynomial of A; more explicitly, $f(\lambda) = \det(A - \lambda I) = \sum_{i=0}^{d} b_i \lambda^{d-i}$, where

$$\begin{aligned}
&b_0 = (-1)^d, \quad b_1 = -(-1)^d A_1, \quad b_2 = -\frac{1}{2}(b_1 A_1 + (-1)^d A_2),\\
&b_3 = -\frac{1}{3}(b_2 A_1 + b_1 A_2 + (-1)^d A_3), \ldots,\\
&b_i = -\frac{1}{i}(b_{i-1} A_1 + b_{i-2} A_2 + \cdots + b_1 A_{i-1} + (-1)^d A_i), \ldots,\\
&b_d = -\frac{1}{d}(b_{d-1} A_1 + b_{d-2} A_2 + \cdots + b_1 A_{d-1} + (-1)^d A_d),
\end{aligned}$$

and A_i is the trace of A^i for $1 \le i \le d$ (cf. [23, p.303-305]).

Proof of Theorem 3.1. Proposition 3.5 shows that, for $n \in \mathbb{N}$,

$$|P_n| = |T(\Xi_n)| + \sum_{i=1}^{n-1} |L(P_i, \Xi_{n-i})|;$$

that is, $A_1 = \xi_1$ and $A_n = n\xi_n + \sum_{i=1}^{n-1} A_i \xi_{n-i}$ for $n \ge 2$. Since $\xi_n = 0$ for $n > d$, Equation (1) follows from

$$\xi_n = \frac{1}{n}(A_n - \sum_{i=1}^{n-1} A_i \xi_{n-i}), \quad 1 \le n \le d,$$

and the recursive formula of the coefficients of the characteristic polynomial of A. □

4. Entropy Dimension of Shift Spaces on Monoids

Suppose that X is a G-shift space with alphabet $\mathcal{A}$. Let $\Gamma_n^{[g]}(X)$ denote the set of n-blocks of X rooted at g; that is, the support of each block of $\Gamma_n^{[g]}(X)$ is $g\Delta_n$. Let $\gamma_n^{[g]}$ denote the cardinality of $\Gamma_n^{[g]}(X)$. The *entropy dimension* of X is defined as

$$\overline{\mathcal{D}^e}(X) = \limsup_{n\to\infty} \frac{\ln\ln \gamma_n(X)}{n}, \tag{2}$$

where $\gamma_n(X) = \gamma_n^{[e]}(X)$. The rest of this paper omits X from the notation when it causes no confusion.

For each $a \in \mathcal{A}$, let $\Gamma_{a,n}^{[g]} \subseteq \Gamma_n^{[g]}$ consist of all the n-blocks rooted at g and labeled a at the root. A symbol a is *essential*[1] if $\gamma_{a,n} = |\Gamma_{a,n}| \ge 2$ for some $n \in \mathbb{N}$; otherwise,

[1]In one-dimensional symbolic dynamical systems, a graph presentation of an SFT is called *essential* if there is no stranded vertex [17]. In other words, every vertex has its contribution in the corresponding SFT. This paper extends the terminology to alphabet of a G-SFT X since a symbol is essential if it contributes to the complexity of X.

a is an *inessential* symbol. Proposition 4.1 indicates that only the essential symbols contribute to the entropy dimension.

Proposition 4.1 (See [6])**.** *Suppose that X is a G-SFT. Then*

$$\overline{\mathcal{D}^e}(X) = \lim_{n\to\infty} \frac{\ln \sum_{i=1}^{k} \ln \gamma_{i,n}}{n} = \lim_{n\to\infty} \frac{\ln \sum_{i\in\mathcal{E}} \ln \gamma_{i,n}}{n}, \tag{3}$$

where $\mathcal{E} \subseteq \mathcal{A}$ denotes the set of essential symbols and $k = |\mathcal{A}|$.

Theorem 4.2 shows that, whenever every symbol is essential, the entropy dimension of a G-SFT is the logarithm of the spectral radius of A (recall that $G = \langle \Sigma | R_A \rangle$ is determined by a $d \times d$ matrix A, see Section 2).

Theorem 4.2. *Suppose that X is a G-SFT and every symbol is essential. Then $\overline{\mathcal{D}^e}(X) = \ln \rho_A$, where ρ_A is the spectral radius of A.*

Ban and Chang [6] developed an algorithm for computing the entropy dimension of G-SFTs, where G is a finitely generated free monoid. The algorithm extends to $G = \langle \Sigma | R_A \rangle$ with finite representation via analogous argument. For the sake of brevity, this section rephrases the main ideas and propositions of the algorithm in [6] via Example 4.4.

In [4], the authors showed that every SFT over a free monoid $F_d = \langle s_1, \ldots, s_d | \rangle$ is topologically conjugate to an F_d-SFT defined as

$$\mathsf{X}_{\mathbf{A}} = \{x \in \mathcal{A}^{F_d} : A_i(x_g, x_{gs_i}) = 1 \text{ for } 1 \le i \le d, g \in F_d\}, \tag{4}$$

where $\mathbf{A} = \{A_1, A_2, \ldots, A_d\}$ is a collection of binary matrices indexed by $\mathcal{A}$. In this paper, we focus on G-SFTs defined by some $\mathbf{A} = \{A_1, A_2, \ldots, A_d\}$. Since G has a finite representation, the cardinality of n-blocks of each G-SFT is related to a *recurrence representation* (or *system of nonlinear recurrence equations*, SNRE) of the form

$$\gamma_{i,n} = \sum c_{\mathbf{j}} \gamma_{1,n-1}^{j_{1,1}} \cdots \gamma_{k,n-1}^{j_{k,1}} \gamma_{1,n-2}^{j_{1,2}} \cdots \gamma_{k,n-2}^{j_{k,2}} \cdots \gamma_{1,n-l}^{j_{1,l}} \cdots \gamma_{k,n-l}^{j_{k,l}}$$

for some $l \in \mathbb{N}$, where $c_{\mathbf{j}} \in \mathbb{N}$, $\mathbf{j} = (j_{1,1}, \ldots, j_{k,1}, \ldots, j_{1,l}, \ldots, j_{k,l})$, and $1 \le i \le k = |\mathcal{A}|$. A *subsystem* of X is of the form

$$\gamma_{i,n} = \sum c'_{\mathbf{j}} \gamma_{1,n-1}^{j_{1,1}} \cdots \gamma_{k,n-1}^{j_{k,1}} \gamma_{1,n-2}^{j_{1,2}} \cdots \gamma_{k,n-2}^{j_{k,2}} \cdots \gamma_{1,n-l}^{j_{1,l}} \cdots \gamma_{k,n-l}^{j_{k,l}}$$

such that $0 \le c'_{\mathbf{j}} \le c_{\mathbf{j}}$ for each $\mathbf{j}$ and $1 \le i \le k$, and a *simple subsystem* of X is of the form

$$\gamma_{i,n} = \gamma_{1,n-1}^{j_{1,1}} \cdots \gamma_{k,n-1}^{j_{k,1}} \gamma_{1,n-2}^{j_{1,2}} \cdots \gamma_{k,n-2}^{j_{k,2}} \cdots \gamma_{1,n-l}^{j_{1,l}} \cdots \gamma_{k,n-l}^{j_{k,l}}$$

for some $j_{1,1}, \ldots, j_{k,1}, \ldots, j_{1,l}, \ldots, j_{k,l}$ and $1 \le i \le k$. Take logarithm on the above equation and let

$$\theta_n = (\ln \gamma_{1,n}, \ldots, \ln \gamma_{k,n}, \ln \gamma_{1;n-1}, \ldots, \ln \gamma_{k,n-1}, \ldots, \ln \gamma_{1,n-l+1}, \ldots, \ln \gamma_{k,n-l+1})',$$

where v' is the transpose of v. Then there exists a $kl \times kl$ matrix M called *adjacency matrix* (of the simple subsystem) such that $\theta_n = M\theta_{n-1}$ for $n \ge l+1$. Theorem 4.3 reveals that the entropy dimension of X is related to the maximum spectral radius among the adjacency matrices of simple subsystems of X.

Theorem 4.3 (See [6])**.** *Suppose that X is a G-SFT. Then*

$$\overline{\mathcal{D}^e}(X) = \max\{\ln \rho_M : M \text{ is the adjacency matrix of a simple subsystem of } X\},$$

where ρ_M denotes the spectral radius of M.

Example 4.4. Let G be the monoid defined in Example 2.2. Suppose that X is a hom-shift on G determined by a $k \times k$ binary matrix T; that is, for each labeled tree $t \in X$ and $g \in G$, a pattern (t_g, t_{gs_i}) is allowable if and only if $T(t_g, t_{gs_i}) = 1$. For instance, consider the case where $k = 2$ and $T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. A hom-shift defined by T is a full G-shift space and $\overline{\mathcal{D}^e}(X) = \ln \rho_A$. This example shows that the above algorithm derives the desired result.

It follows from s_3 being a free generator that, for $i = 1, 2$, $\gamma_{i,n}^{[g]} = \gamma_{i,n}$ if $g = g's_3$ for some $g, g' \in G$. Hence, for $i = 1, 2$,

$$\begin{aligned} \gamma_{i,n} &= (\gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]})(\gamma_{1,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_2]})(\gamma_{1,n-1}^{[s_3]} + \gamma_{2,n-1}^{[s_3]}) \\ &= (\gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]})(\gamma_{1,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_2]})(\gamma_{1,n-1} + \gamma_{2,n-1}). \end{aligned}$$

Combining

$$\begin{aligned} \gamma_{i,n-1}^{[s_1]} &= (\gamma_{1,n-2}^{[s_1 s_2]} + \gamma_{2,n-2}^{[s_1 s_2]})(\gamma_{1,n-2}^{[s_1 s_3]} + \gamma_{2,n-2}^{[s_1 s_3]}) = (\gamma_{1,n-2}^{[s_1 s_2]} + \gamma_{2,n-2}^{[s_1 s_2]})(\gamma_{1,n-2} + \gamma_{2,n-2}), \\ \gamma_{i,n-1}^{[s_2]} &= \gamma_{1,n-2}^{[s_2 s_3]} + \gamma_{2,n-2}^{[s_2 s_3]} = \gamma_{1,n-2} + \gamma_{2,n-2}, \end{aligned}$$

with

$$\gamma_{i,n-1}^{[s_1 s_2]} = \gamma_{1,n-3}^{[s_1 s_2 s_3]} + \gamma_{2,n-3}^{[s_1 s_2 s_3]} = \gamma_{1,n-3} + \gamma_{2,n-3}$$

derives that

$$\gamma_{i,n} = (4\gamma_{1,n-3}\gamma_{1,n-2} + 4\gamma_{2,n-3}\gamma_{1,n-2} + 4\gamma_{1,n-3}\gamma_{2,n-2} + 4\gamma_{2,n-3}\gamma_{2,n-2})\cdot (2\gamma_{1,n-2} + 2\gamma_{2,n-2})(\gamma_{1,n-1} + \gamma_{2,n-1}).$$

Let

$$\theta_n = (\ln\gamma_{1,n}, \ln\gamma_{2,n}, \ln\gamma_{1,n-1}, \ln\gamma_{2,n-1}, \ln\gamma_{1,n-2}, \ln\gamma_{2,n-2})'.$$

For every simple subsystem of X, the corresponding adjacency matrix is of the form

$$M = \begin{pmatrix} B_1 & B_2 & B_3 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix},$$

where B_l is a 2×2 matrix that satisfies $\sum_{q=1}^{2} B_l(p,q) = \xi_l$ for all $l = 1,2,3$, $p = 1,2$. That is, $\theta_n = M\theta_{n-1}$ for $n \geq 3$. Let

$$v = (\rho_A^2, \rho_A^2, \rho_A, \rho_A, 1, 1)'.$$

Observe that $Mv = \rho_A v$. Perron-Frobenius Theorem demonstrates that ρ_A is also the spectral radius of M. In other words, $\overline{\mathcal{D}^e}(X) = \ln\rho_A$.

Proof of Theorem 4.2. The proof focuses on the case where X is a G-SFT determined by $k\times k$ binary matrices $A_1,\ldots,A_d$ for the sake of clarity, the proof of the general case is analogous. In this case, for each labeled tree $t \in X$ and $g \in G$, (t_g, t_{gs_l}) is allowable if and only if $A_l(t_g, t_{gs_l}) = 1$ for $1 \leq l \leq d$.

Write $A_l = (a_{l;i_1,i_2})$ for $1 \leq l \leq d, 1 \leq i_1, i_2 \leq k$. Since $\gamma_{i,n}^{[gs_l]} = \gamma_{i,n}$ for all $1 \leq i \leq k, n \in \mathbb{N}$, and $g \in G$ provided s_l is a free generator, we can derive that

$$\begin{aligned}\gamma_{i,n} &= \prod_{s_l\in\Sigma}\Big(\sum_{j_1=1}^{k} a_{s_l;i,j_1}\gamma_{j_1,n-1}^{[s_l]}\Big)\\ &= \prod_{s_l\notin\Sigma_r}\Big(\sum_{j_1=1}^{k} a_{s_l;i,j_1}\gamma_{j_1,n-1}^{[s_l]}\Big)\prod_{s_l\in\Sigma_r}\Big(\sum_{j_1=1}^{k} a_{s_l;i,j_1}\gamma_{j_1,n-1}^{[s_l]}\Big)\\ &= \prod_{s_l\notin\Sigma_r}\Big(\sum_{j_1=1}^{k} a_{s_l;i,j_1}\gamma_{j_1,n-1}^{[s_l]}\Big)\prod_{s_l\in\Sigma_r}\Big(\sum_{j_1=1}^{k} a_{s_l;i,j_1}\gamma_{j_1,n-1}\Big).\end{aligned}$$

Observe that $f_1 = \prod_{s_l\in\Sigma_r}(\sum_{j_1=1}^{k} a_{s_l;i,j_1}\gamma_{j_1,n-1})$ is a polynomial of degree ξ_1 over $\gamma_{1,n-1},\ldots,\gamma_{k,n-1}$.

Similarly, for each s_l which is not a free generator,

$$\gamma_{j_1,n-1}^{[s_l]} = \prod_{s_l s_m\in G}\Big(\sum_{j_2=1}^{k} a_{s_m;j_1,j_2}\gamma_{j_2,n-2}^{[s_l s_m]}\Big)$$

implies that

$$\gamma_{i,n} = f_1 \cdot \prod_{s_l s_m \in G, s_l \notin \Sigma_r} \Big(\sum_{j_1, j_2 = 1}^{k} a_{s_l;i,j_1} a_{s_m;j_1,j_2} \gamma_{j_2,n-2}^{[s_l s_m]} \Big).$$

Herein, we refer to $gs \in G$ as $gs \neq g$, where $g \in G$ and $s \in \Sigma$; equivalently, g and gs are two different vertices in the Cayley graph of G. Let

$$\begin{aligned} f_2 &= \prod_{s_l s_m \in G, s_l \notin \Sigma_r, s_m \in \Sigma_r} \Big(\sum_{j_1, j_2 = 1}^{k} a_{s_l;i,j_1} a_{s_m;j_1,j_2} \gamma_{j_2,n-2}^{[s_l s_m]} \Big) \\ &= \prod_{s_l s_m \in G, s_l \notin \Sigma_r, s_m \in \Sigma_r} \Big(\sum_{j_1, j_2 = 1}^{k} a_{s_l;i,j_1} a_{s_m;j_1,j_2} \gamma_{j_2,n-2} \Big). \end{aligned}$$

Then f_2 is a polynomial of degree ξ_2. Repeating the same process decompose $\gamma_{i,n} = f_1 f_2 \cdots f_\ell$, where $\ell = \max\{j : \xi_j \neq 0\} \leq d$, and f_j is a polynomial of degree ξ_j over $\gamma_{1,n-j}, \ldots, \gamma_{k,n-j}$ for $1 \leq j \leq \ell$.

Let

$$\theta_n = (\ln \gamma_{1,n}, \cdots, \ln \gamma_{k,n}, \ln \gamma_{1,n-1}, \cdots, \ln \gamma_{k,n-1}, \cdots, \ln \gamma_{1,n-d+1}, \cdots, \ln \gamma_{k,n-d+1})'.$$

For each simple subsystem of X, there exists

$$M = \begin{pmatrix} B_1 & B_2 & B_3 & \cdots & B_\ell \\ I & 0 & \cdots & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & 0 \end{pmatrix},$$

where B_j is a $k \times k$ nonnegative integral matrix satisfies $\sum_{q=1}^{k} B_j(p,q) = \xi_i$ for all $1 \leq j \leq \ell, 1 \leq p \leq k$, such that M is the corresponding adjacency matrix (note that $\xi_j = 0$ for $j > \ell$). That is, $\theta_n = M\theta_{n-1}$ is the designated simple subsystem. Let $v = (\rho_A^{\ell-1} \cdots \rho_A 1)' \otimes \mathbf{1}_k$, where $\otimes$ is the Kronecker product and $\mathbf{1}_k \in \mathbb{R}^k$ is the vector consisting of 1's. It follows immediately that $Mv = \rho_A v$. Perron-Frobenius Theorem implies that ρ_A is also the spectral radius of M. Hence, $\overline{\mathcal{D}^e}(X) = \ln \rho_A$.

This completes the proof. □

Remark 4.5. For the general cases, Proposition 4.1 demonstrates that Theorem 4.3 holds if the rows and columns of matrix M indexed by inessential symbols are eliminated.

5. Entropy Dimension Spectrum of G-SFTs

Theorem 4.2 reveals that the entropy dimension of G-SFTs is $\ln \rho_A$ whenever every symbol is essential. This section extends to the general case and gives the complete characterization of entropy dimension spectrum of G-SFTs.

Let $\mathbb{Z}_+$ be the set of nonnegative integers. For $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^d$, define $\mathbf{m} \preceq \mathbf{n}$ if $m_i \leq n_i$ for $1 \leq i \leq d$, and $\mathbf{m} \prec \mathbf{n}$ if $\mathbf{m} \preceq \mathbf{n}$ and $\mathbf{m} \neq \mathbf{n}$. Theorem 5.1 characterizes the entropy dimension spectrum (i.e., the set of entropy spectrum) of G-SFTs for the case where $k = 2$.

Theorem 5.1. *Suppose that $k = 2$. Let $\xi = (\xi_1, \ldots, \xi_d)$. The entropy dimension spectrum of G-SFTs is*

$$H = \{\ln \lambda : \lambda = \max\{x : x^d - \sum_{i=1}^{d} \alpha_i x^{d-i} = 0\} \text{ for some } \alpha \in \mathbb{Z}_+^d, \alpha \preceq \xi\}.$$

Proof. Obviously, if two symbols are inessential, then the entropy dimension is 0; Theorem 4.2 indicates the entropy dimension is $\ln \rho_A$ and $\rho_A = \max\{x : x^d - \sum_{i=1}^{d} \xi_i x^{d-i} = 0\}$ if every symbol is essential. It suffices to consider the case where $1 \in \mathcal{A}$ is essential and $2 \in \mathcal{A}$ is inessential.

Similar to the discussion in Example 4.4, write $\gamma_{1,n} = f_1 f_2 \cdots f_d$, where

$$f_1 = \prod_{u_1 \in \Sigma_r} \Big(\sum_{j_1=1}^{2} a_{u_1;1,j_1} \gamma_{j_1,n-1} \Big)$$

and

$$f_i = \prod_{u_1 \cdots u_i \in G, u_1, \ldots, u_{i-1} \notin \Sigma_r, u_i \in \Sigma_r} \Big(\sum_{j_1, \ldots, j_i = 1}^{2} a_{u_1;1,j_1} a_{u_2;j_1,j_2} \cdots a_{u_i;j_{i-1},j_i} \gamma_{j_i,n-i} \Big)$$

for $2 \leq i \leq d$, and f_i is a polynomial of degree ξ_i. Hence, every simple subsystem of X is of the form

$$\gamma_{1,n} = \gamma_{1,n-1}^{\eta_1} \gamma_{2,n-1}^{\tau_1} \gamma_{1,n-2}^{\eta_2} \gamma_{2,n-2}^{\tau_2} \cdots \gamma_{1,n-d}^{\eta_d} \gamma_{2,n-d}^{\tau_d},$$
$$\gamma_{2,n} = \gamma_{2,n-1}^{\xi_1} \gamma_{2,n-2}^{\xi_2} \cdots \gamma_{2,n-d}^{\xi_d},$$

where $\eta_i + \tau_i = \xi_i$ for $1 \leq i \leq d$.

Let $\theta_n = (\ln\gamma_{1,n}, \ln\gamma_{2,n}, \ln\gamma_{1,n-1}, \ln\gamma_{2,n-1}, \cdots, \ln\gamma_{1,n-d+1}, \ln\gamma_{2,n-d+1})'$, and let

$$M = \begin{pmatrix} \eta_1 & \tau_1 & \eta_2 & \tau_2 & \cdots & \eta_d & \tau_d \\ 0 & \xi_1 & 0 & \xi_2 & \cdots & 0 & \xi_d \\ 1 & & & & & 0 & 0 \\ & 1 & & & & \vdots & \vdots \\ & & \ddots & & & \vdots & \vdots \\ & & & \ddots & & \vdots & \vdots \\ & & & & 1 & 0 & 0 \end{pmatrix}.$$

Then the simple subsystem is $\theta_n = M\theta_{n-1}$. Since 2 is inessential, the entropy dimension of such a simple subsystem is $\ln\lambda$, where λ is the spectral radius of

$$\overline{M} = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_d \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix}.$$

A straightforward examination elaborates that $\lambda = \max\{x : x^d - \sum_{i=1}^d \eta_i x^{d-i} = 0\}$. This derives

$$H \subseteq \{\ln\lambda : \lambda = \max\{x : x^d - \sum_{i=1}^d \alpha_i x^{d-i} = 0\} \text{ for some } \alpha \in \mathbb{Z}_+^d, \alpha \le \xi\}.$$

To show that, for each $\alpha \in \mathbb{Z}_+^d$ satisfying $\alpha \le \xi$, there exists a G-SFT such that $\overline{\mathcal{D}^e}X = \ln\lambda$ with $\lambda = \max\{x : x^d - \sum_{i=1}^d \alpha_i x^{d-i} = 0\}$, construct a one-step G-SFT as follows. Without loss of generality, assume that $\xi_i > 0$ for $i \le d$. The symbol 2 is inessential in the following construction, thus it suffices to mention where to label 1.

For $n \in \mathbb{N}$, let $S_1 = \{e\}$ and, for $n \ge 2$, let

$$S_n = \{g = g_1 \cdots g_{n-1} : gs \in G \text{ for some } s \in \Sigma_r, g_i \notin \Sigma_r \text{ for } 1 \le i \le n-1\}.$$

Observe that $S_n = \varnothing$ if and only if $n > d$ (under the assumption that $\xi_n = 0$ if and only if $n > d$). Let

$$\overline{S}_n = \{gs : g \in S_n, s \in \Sigma, gs \in G\}.$$

Then $\bigcup_{i=1}^d \overline{S}_i$ is the set of supports of two-blocks of X up to shift. For $n = 1$, let $B_1 \subseteq \mathcal{A}^{\overline{S}_1}$ consists of 1-blocks ϕ which satisfy $\phi_g = 1$ if and only if

$$g \in S_1 \bigcup \Sigma \setminus \Sigma_r \quad \text{and} \quad |\{g \in \Sigma_r : \phi_g = 1\}| = \alpha_1.$$

In other words, each pattern of B_1 labels 1 at, except from the root and non-free generators, arbitrary α_1 free generators. This makes $\max\{p : \gamma_{1,n-1}^p | \gamma_{1,n}\} = \alpha_1$.

Analogously, let $B_2 \subseteq \mathcal{A}^{\overline{S}_2}$ consists of 1-blocks ϕ which satisfy $\phi_g = 1$ if and only if

$$g \in S_2 \bigcup \{g's \in \overline{S}_2 : s \notin \Sigma_r\} \quad \text{and} \quad |\{g's \in \overline{S}_2 : s \in \Sigma_r, \phi_{g's} = 1\}| = \alpha_2.$$

Then $\max\{p : \gamma^p_{1,n-2} | \gamma_{1,n}\} = \alpha_2$. Repeating the same process to construct B_i for $i \leq d$ makes

$$\max\{p : \gamma^p_{1,n-i} | \gamma_{1,n}\} = \alpha_i \quad \text{for} \quad 1 \leq i \leq d.$$

For each subset $G' \subseteq G$ such that G' forms the support of a 1-block, observe that there exists $g \in G$ ended in free generator and $1 \leq i \leq d$ such that $G' = gG''$ for some $G'' \subseteq \overline{S}_i$. Then each labeled pattern of support G' follows the same rule as determined in $\overline{S}_i$. Notably, such a pattern is still in B_i.

Therefore, every simple subsystem of X generated by $B = \bigcup_{i=1}^d B_i$ is of the form

$$\gamma_{1,n} = c \cdot \gamma^{\alpha_1}_{1,n-1} \gamma^{\beta_1}_{2,n-1} \gamma^{\alpha_2}_{1,n-2} \gamma^{\beta_2}_{2,n-2} \cdots \gamma^{\alpha_d}_{1,n-d} \gamma^{\beta_d}_{2,n-d}, \quad \gamma_{2,n} = \gamma^d_{2,n-1},$$

where c is a constant, and $\alpha_i + \beta_i = \xi_i$ for all i. A straightforward examination indicates that $\overline{\mathcal{D}^e} X = \ln \lambda$ with $\lambda = \max\{x : x^d - \sum_{i=1}^d \alpha_i x^{d-i} = 0\}$.

The proof is complete. □

Remark 5.2. Notably, $\xi_n = 0$ for $n \geq 2$ if and only if G is a free monoid. If this is the case, then $H = \{0, \ln 2, \ldots, \ln d\}$.

Theorem 5.3 extends Theorem 5.1 to the general case where $k \geq 2$. The proof is similar, thus it is omitted.

Theorem 5.3. *Let $\mathcal{M}$ be the set consisting of*

$$M = \begin{pmatrix} C_1 & C_2 & C_3 & \cdots & C_d \\ I & 0 & \cdots & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & I & 0 \end{pmatrix}$$

for some $l \times l$ matrices C_i, $l \leq k$, satisfying $\sum_{q=1}^l C_i(p,q) \leq \xi_i$ for all $1 \leq i \leq d$, $1 \leq p \leq l$. The entropy dimension spectrum of G-SFTs is

$$H = \{\ln \rho_M : \rho_M \text{ is the spectral radius of } M \in \mathcal{M}\}.$$

Corollary 5.4 follows from the proof of Theorem 5.1, and elaborates a necessary and sufficient condition of a G-SFT achieved full entropy dimension.

Corollary 5.4. *Suppose that X is a G-SFT. Then $\overline{\mathcal{D}^e}(X) = \ln \rho_A$ if and only if the essential symbols form a subshift on right free generators; that is, for each $s \in \Sigma_r$ and ϕ is a one-block with support* $\operatorname{supp}(\phi) = s\Sigma$, *$\phi_g$ is essential for $g \in \operatorname{supp}(\phi)$.*

Proof. It suffices to consider the case where $k = 2$ since the demonstration of the general case is analogous but more complicated. Recall that, in the proof of Theorem 5.1, every simple subsystem of X is of the form

$$\gamma_{1,n} = \gamma_{1,n-1}^{\eta_1}\gamma_{2,n-1}^{\tau_1}\gamma_{1,n-2}^{\eta_2}\gamma_{2,n-2}^{\tau_2}\cdots\gamma_{1,n-d}^{\eta_d}\gamma_{2,n-d}^{\tau_d},$$
$$\gamma_{2,n} = \gamma_{1,n-1}^{\delta_1}\gamma_{2,n-1}^{\iota_1}\gamma_{1,n-2}^{\delta_2}\gamma_{2,n-2}^{\iota_2}\cdots\gamma_{1,n-d}^{\delta_d}\gamma_{2,n-d}^{\iota_d},$$

where $\eta_i + \tau_i = \delta_i + \iota_i = \xi_i$ for $1 \le i \le d$. In other words, $\overline{\mathcal{D}^e}(X) = \ln \lambda$, where λ is the spectral radius of one of the following matrix, which depends on the essential symbols.

$$M_1 = \begin{pmatrix} \eta_1 & \tau_1 & \eta_2 & \tau_2 & \cdots & \eta_d & \tau_d \\ \delta_1 & \iota_1 & \delta_2 & \iota_2 & \cdots & \delta_d & \iota_d \\ 1 & & & & & 0 & 0 \\ & 1 & & & & \vdots & \vdots \\ & & \ddots & & & \vdots & \vdots \\ & & & \ddots & & \vdots & \vdots \\ & & & & 1 & 0 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_d \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} \iota_1 & \iota_2 & \cdots & \iota_d \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix}.$$

It follows that $\lambda = \rho_A$ if and only if exactly one of the following three conditions holds.

a. (Case M_1) Two symbols are essential.

b. (Case M_2) Symbol 1 is essential and $\eta_i = \xi_i$ for $1 \le i \le d$.

c. (Case M_3) Symbol 2 is essential and $\iota_i = \xi_i$ for $1 \le i \le d$.

This completes the proof. □

6. Groups with Finitely Many Free-Followers

Suppose that G is a monoid. For each $g \in G$, define the *free-follower set* (free-follower for short) of g as

$$F_g = \{g' \in G : |gg'| = |g| + |g'|\}. \tag{5}$$

Set $F = \{F_g : g \in G\}$. Then G *has finitely many free-followers* if F is finite. It is easily seen that if G is a finitely generated free monoid or has finite representation, then G has finitely many free-followers. The investigation in Sections 4 and 5 extends to the case where G has finitely many free-followers via analogous elaboration. This section, rather than rephrasing every result in the previous two sections, presents an example to address how to compute the entropy dimension of a G-SFT (G has finitely many free-followers herein) for the sake of brevity.

Suppose that $d = k = 2$. In this case, $\Sigma = \{s_1, s_2\}$ and $\mathcal{A} = \{1, 2\}$. Let $G = \langle \Sigma | R \rangle$ be the monoid with $R = \{s_2 s_1^{2i+1} s_2 = s_2\}_{i \geq 0}$. It follows that G has finitely many free-followers. Indeed, let

$$F_{s_1} = \{e, s_1, s_2, s_1^2, s_1 s_2, s_2 s_1, s_2^2, \ldots\} = G,$$
$$F_{s_2} = \{e, s_1, s_2, s_1^2, s_2 s_1, s_2^2, s_1^3, s_1^2 s_2, \ldots\} = \{s_1^n\}_{n \geq 0} \bigcup \{s_1^{2i} s_2 g : g = s_1^n, s_1^{2j} s_2^n, i, j, n \geq 0\},$$
$$F_{s_2 s_1} = \{e, s_1, s_1^2, s_1 s_2, s_1^3, \ldots\} = \{s_1^n\}_{n \geq 0} \bigcup \{s_1^{2i+1} s_2 g : g = s_1^n, s_1^{2j} s_2^n, i, j, n \geq 0\}.$$

An examination indicates that, for each $g \in G$,

$$F_g = \begin{cases} F_{s_1}, & g = s_1^n; \\ F_{s_2}, & g \text{ ends in } s_2 s_1^{2i}, i \geq 0; \\ F_{s_2 s_1}, & g \text{ ends in } s_2 s_1^{2i+1}, i \geq 0. \end{cases}$$

A straightforward examination elaborates that there is a one-to-one correspondence between the monoid G and the set of finite words of one-dimensional even-shift.

Let X be a hom-shift on G determined by $T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Alternatively, X is a full G-shift; it follows immediately that $\overline{\mathcal{D}^e}(X) = \ln \lambda$, where $\lambda = \frac{1+\sqrt{5}}{2}$ satisfies $\lambda^2 - \lambda - 1 = 0$. The following shows that the algorithm in Section 4 derives the same result.

Observe that $\gamma_{i,n}^{[g]} = \gamma_{i,n}$ for $i = 1, 2$ since $F_{s_1^j} = G$ for $j \in \mathbb{N}$. For $i = 1, 2$,

$$\begin{aligned} \gamma_{i,n} &= (\gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]})(\gamma_{1,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_2]}) \\ &= (\gamma_{1,n-1} + \gamma_{2,n-1})(\gamma_{1,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_2]}). \end{aligned}$$

Also, $F_{s_2^2} = F_{s_2}$ and $F_{s_2 s_1^2} = F_{s_2}$ imply that

$$\begin{aligned} \gamma_{i,n-1}^{[s_2]} &= (\gamma_{1,n-2}^{[s_2 s_1]} + \gamma_{2,n-2}^{[s_2 s_1]})(\gamma_{1,n-2}^{[s_2 s_2]} + \gamma_{2,n-2}^{[s_2 s_2]}) \\ &= (\gamma_{1,n-2}^{[s_2 s_1]} + \gamma_{2,n-2}^{[s_2 s_1]})(\gamma_{1,n-2}^{[s_2]} + \gamma_{2,n-2}^{[s_2]}), \\ \gamma_{i,n-2}^{[s_2 s_1]} &= \gamma_{1,n-3}^{[s_2 s_1^2]} + \gamma_{2,n-3}^{[s_2 s_1^2]} = \gamma_{1,n-3}^{[s_2]} + \gamma_{2,n-3}^{[s_2]}. \end{aligned}$$

Hence, the SNRE of X is

$$\gamma_{i,n} = 2(\gamma_{1,n-1} + \gamma_{2,n-1})(\gamma_{1,n-2}^{[s_2]} + \gamma_{2,n-2}^{[s_2]})(\gamma_{1,n-3}^{[s_2]} + \gamma_{2,n-3}^{[s_2]})$$

for $i = 1, 2$. Let $\theta_n = (\ln \gamma_{1,n}^{[s_2]}, \ln \gamma_{2,n}^{[s_2]}, \ln \gamma_{1,n-1}^{[s_2]}, \ln \gamma_{2,n-1}^{[s_2]})'$ and let

$$M = \begin{pmatrix} \eta_1 & \tau_1 & \eta_2 & \tau_2 \\ \delta_1 & \iota_1 & \delta_2 & \iota_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then, every simple subsystem of the invariant system $\ln \gamma_{1,n}^{[s_2]}, \ln \gamma_{2,n}^{[s_2]}$ is of the form $\theta_n = M\theta_{n-1}$ with $\eta_j + \tau_j = \delta_j + \iota_j = 1$ for $1 \le j \le 2$. It follows that $\ln \gamma_{i,n}^{[s_2]} \approx e^{\lambda n}$ for $i = 1, 2$ and n large enough.

Furthermore, every simple subsystem of X is of the form

$$\ln \gamma_{1,n} \approx \eta \ln \gamma_{1,n-1} + \tau \ln \gamma_{2,n-1} + e^{(n-2)\lambda} + e^{(n-3)\lambda},$$
$$\ln \gamma_{2,n} \approx \delta \ln \gamma_{1,n-1} + \iota \ln \gamma_{2,n-1} + e^{(n-2)\lambda} + e^{(n-3)\lambda},$$

where $\eta + \tau = \delta + \iota = 1$. A straightforward examination shows that

$$\overline{\mathcal{D}^e}(X) = \lim_{n\to\infty} \frac{\ln(\ln \gamma_{1,n} + \ln \gamma_{2,n})}{n} = \ln \lambda.$$

This concludes the desired result.

Acknowledgments

This work is partially supported by the Ministry of Science and Technology, ROC (Contract No MOST 107-2115-M-259 -001 -MY2 and 107-2115-M-390 -002 -MY2).

References

[1] N. Aubrun and M.-P. Béal, *Tree-shifts of finite type*, Theor. Comput. Sci. **459** (2012), 16–25.

[2] J.-C. Ban and C.-H. Chang, *Mixing properties of tree-shifts*, J. Math. Phys. **58** (2017), 112702.

[3] ———, *On the topological entropy of subshifts of finite type on free semigroups*, preprint, 2017.

[4] ———, *Tree-shifts: Irreducibility, mixing, and chaos of tree-shifts*, Trans. Am. Math. Soc. **369** (2017), 8389–8407.

[5] ———, *Tree-shifts: The entropy of tree-shifts of finite type*, Nonlinearity **30** (2017), 2785–2804.

[6] ———, *Characterization for entropy of shifts of finite type on Cayley trees*, J. Stat. Mech. Theory Exp. (2020), to appear, arXiv:1705.03138.

[7] R. Berger, *The undecidability of the domino problem*, Mem. Amer. Math. Soc. **66** (1966).

[8] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer-Verlag, Berlin-New York, 1975.

[9] M. Boyle, R. Pavlov, and M. Schraudner, *Multidimensional sofic shifts without separation and their factors*, Trans. Am. Math. Soc. **362** (2010), 4617–4653.

[10] M. Carvalho, *Entropy dimension of dynamical systems*, Port. Math. **54** (1997), 19–40.

[11] N. Chandgotia and B. Marcus, *Mixing properties for hom-shifts and the distance between walks on associated graphs*, Pacific J. Math. **294** (2018), 41–69.

[12] W.-C. Cheng and B. Li, *Zero entropy systems*, J. Stat. Phys. **140** (2010), 1006–1021.

[13] D. Dou, W. Huang, and K. K. Park, *Entropy dimension of topological dynamical systems*, Trans. Am. Math. Soc. **363** (2011), 659–680.

[14] ———, *Entropy dimension of measure preserving systems*, 2018, arXiv:1312.7225.

[15] M. Hochman and T. Meyerovitch, *A characterization of the entropies of multidimensional shifts of finite type*, Ann. of Math. **171** (2010), 2011–2038.

[16] D. Lind, *The entropies of topological Markov shifts and a related class of algebraic integers*, Ergodic Theory Dynam. Systems **4** (1984), no. 02, 283–300.

[17] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[18] B. Marcus and R. Pavlov, *Approximating entropy for a class of $\mathbb{Z}^2$ Markov random fields and pressure for a class of functions on $\mathbb{Z}^2$ shifts of finite type*, Ergodic Theory Dynam. Systems **33** (2013), 186–220.

[19] R. Pavlov and M. Schraudner, *Classification of sofic projective subdynamics of multidimensional shifts of finite type*, Trans. Am. Math. Soc. **367** (2015), 3371–3421.

[20] K. Petersen and I. Salama, *Tree shift topological entropy*, Theoret. Comput. Sci. **743** (2018), 64–71.

[21] S. T. Piantadosi, *Symbolic dynamics on free groups*, Discrete Contin. Dyn. Syst. **20** (2008), 725–738.

[22] D. Ruelle, *Thermodynamic formalism: The mathematical structures of classical equilibrium statistical mechanics*, Addison-Wesley Publishing Co., Reading, Mass., 1978.

[23] L. A. Zadeh and C. A. Desoer, *Linear system theory: The state approach*, McGraw-Hill, New York, 1963.