Model Risk in Risk Analysis for No-Negative-Equity-Guarantees

Jr-Wei Huang, Sharon S. Yang, and Chuang-Chang Chang

Jr-Wei Huang
is an associate professor of risk management and insurance in the department of economics and deputy director of the Research Center of Hubei Financial Development and Financial Security at Hubei University of Economics in Wuhan, China.
jrwei.huang@icloud.com

Sharon S. Yang
is a professor in the Department of Money and Banking, a Research Fellow at the Risk and Insurance Research Center, and director of the NNIP & Nomura SITE Financial Center in the College of Commerce at National Chengchi University in Taipei City, Taiwan.
syang@nccu.edu.tw

Chuang-Chang Chang
is a professor in the Department of Finance at National Central University in Taoyuan City, Taiwan, and President of the Chung-Hua Institution for Economic Research, in Taipei City, Taiwan.
ccchang@cc.ncu.edu.tw

KEY FINDINGS

- We investigate model risk in risk analysis for No-Negative-Equity-Guarantees.
- Our numerical analyses reveal that the housing price risk, interest-rate risk, and longevity risk can affect the VaR and CTE of NNEGs, with the impact being as significant as that for housing risk.

ABSTRACT

Understanding the risk for No-Negative-Equity-Guarantees (NNEGs) requires the proper modeling of the housing return, interest rate, and mortality rate dynamics. This article investigates the model risk for the risk measures of NNEGs by calculating the Value-at-Risk (VaR) and Conditional-Tail-Expectation (CTE) from the provider perspective, with an emphasis on the housing price return model. Therefore, we propose a jump ARMA-GARCH model, according to nationwide house price return data in the UK. Interest rate and mortality rate dynamics are assumed to follow the CIR model (Cox et al. 1985) and the CBD model (Cairns et al. 2006) respectively. Our numerical analyses reveal that the housing price risk, interest-rate risk, and longevity risk can affect the VaR and CTE of NNEGs, with the impact being as significant as that for housing risk. The VaR and CTE of NNEGs will be greater for female borrowers than for male borrowers, essentially because females have a longer life expectancy. The proposed framework can help financial institutions manage the major three risk factors for NNEGs and assist in meeting the regulator's concerns.

TOPICS

*Quantitative methods, real estate, VAR and use of alternative risk measures of trading risk**

The continuing increase in life expectancy around the world demands urgent consideration of the ways in which the retirement incomes of the elderly can be increased in order to ensure the maintenance of an acceptable standard of living. Although pension systems have long been the primary financial resource for elderly people, aging populations and increases in longevity on a global scale have put pension and annuity providers in untenable positions, such that the response by many providers has been unavoidable reductions in pension benefits (Antolin 2007). About 75 percent of the elderly populations internationally are now considered to have inadequate income upon their retirement; thus, governments are faced with the growing challenge of financing such aging populations (Chen et al. 2010). Clearly, the development within the private markets of innovative financial products capable of increasing retirement income would be of significant benefit.

*All articles are now categorized by topics and subtopics. **View at PM-Research.com**.

Many elderly people are considered to be "cash poor and equity rich" (McCarthy et al. 2002; Rowlingson 2006; Shan 2011; Haurin et al. 2016; Pu et al. 2014; Alai et al. 2014; Davidoff 2015; Nakajima and Telyukova 2017). For example, nearly 10 million households with heads being age 65 or older had annual household incomes below $25,000 in 2009 (US Census Bureau 2012). However, the release equity owned by people over the age of 65 years was found to be £1,100 billion in the UK and the median value of mortgage-free homes in the early 2000s was found to be US $127,959, with more than 12.5 million elderly people having absolutely no mortgage debt (American Housing Survey 2005). On the other hand, Keenan (2010) and Bayer and Harper (2000) have pointed out the 78 percent to 92 percent of homeowners age 65 and older say that they would like to stay in their current residence "as long as possible."

Reverse mortgage loans (RMLs) offer a potential alternative financial resource capable of meeting current shortfalls in retirement income due to longevity risk (Tunaru 2007). Indeed, RMLs are designed exactly for this purpose, with homeowners receiving a lump-sum and/or annuity in exchange for the transfer of some, or all of the value of their house to a financial institution upon their death. The loan value is ultimately determined by the age of the borrower, the interest rate, and the value of the property. Such RMLs are available in several developed countries, including the US, the UK, France, Australia, Canada, Japan, and Korea, with the major advantage for homeowners being that they can receive cash without having to leave the property.

Due to the trend in population aging, a number of studies have focused on estimating the potential demand for RMLs. For example, Alai et al. (2014) show that RMLs dominate most equity release markets and Hanewald et al. (2016) also point out that RMLs have higher utility gains for homeowners and features that allow for higher lump-sum payouts and also provide downside protection for house prices. Nakajima and Telyukova (2017) find that households with lower incomes, better health, higher medical expenses, and those that do not have bequest motives are theoretically more likely to select RMLs.

There has been a growing literature addressing risk factors and capital adequacy of RMLs in recent years, including but not limited to Boehm and Ehrhardt (1994); Chinloy and Megbolugbe (1994); Szymanoski (1994); Rodda et al. (2004); Ma et al. (2007); Wang et al. (2008); Chen et al. (2010); Li et al. (2010); Sherris and Sun (2010); and Alai et al. (2014). However, little research has been done on risk analysis for RMLs from the provider's perspective, addressing the model risk. Hosty et al. (2008) and Ji et al. (2012) show that the most obvious risk of RMLs is the negative equity that such institutions may have to assume if the proceeds from the sale of the house prove to be less than the loan value paid out (No-Negative-Equity-Guarantees; NNEGs). RMLs differ from traditional mortgages, since the loans and accrued interest must be repaid when the borrower dies or leaves the house. The main risk factors involved in such products are the underlying value of the property, the interest rate, and the longevity of the homeowners (Kogure et al. 2014; Hanewald et al. 2016 and Tunaru 2017). The Prudential Regulation Authority (PRA) expressed concerns about understating the risk of NNEGs by issuing a supervisory statement 3/17 (SS3/17) in 2017. The PRA has four overriding principles in assessing the risk of NNEGs and the overall valuation of the RMLs: (1) securitizations where firms hold all tranches do not result in a reduction of risk to the firm; (2) the economic value of RML cash flows cannot be greater than either the value of an equivalent loan without a NNEG, or the present value of deferred possession of the property providing collateral; (3) the present value of deferred possession of property should be less than the value of immediate possession; and (4) the compensation for the risks retained by a firm as a result of the NNEGs must comprise more than the best estimate cost of the NNEGs. Therefore, the risk management has become a crucial element for RML providers in

the continuing development of the equity releasing market. Thus, the purpose of this research is to study the Value-at-Risk (VaR) and Conditional-Tail-Expectation (CTE) of NNEGs from the provider's perspective, considering the major risk factors of RMLs.

In the continuing development of the modeling for RMLs, the primary concern thus far has been house price risk (Kau et al. 1995). The uncertainty in the house price is the primary risk we need to consider. If the house price remains stagnant or grows at a lower rate than anticipated, the outstanding loan balance at maturity may exceed the sale proceed of the property. Lenders or their insurers may suffer from losses in this scenario (Alai et al. 2014). Traditionally, NNEGs models using the Black-Scholes (1973) approach have been introduced in several studies, based on the assumption that the house price process follows Geometric Brownian Motion (GBM) (Cunningham and Hendershott (1986); Kau et al. (1992); Ambrose and Buttimer (2000); Bardhan et al. (2006); Liao et al. (2008); and Huang et al. (2011)). However, the GBM assumption cannot accommodate many stylized facts; for example, the log-return of house prices is found to be autocorrelated, and there also may be volatility clustering (Li et al. 2010; Chen et al. 2010; and Kim and Li 2017), and jump diffusion (Kau and Keenan 1996; Kou 2002; Chen et al. 2010; Chang et al. 2011). We therefore examine the extant literature by taking these factors into consideration. In specific terms, we study the jump dynamics in house price returns based on an ARMA-GARCH specification that allows for both constant and dynamic jumps proposed by Chan and Maheu (2002) and Maheu and McMcurdy (2004) to model the housing model.

Interest-rate risk is another important risk factor in the analysis of NNEGs, since interest rates are a fundamental economic variable within any economy and cannot be treated as constant, particularly when relating to economic policies with long horizons, Thus, the incorporation of the feature of stochastic interest rates in the valuation of contingent claims has been proposed in numerous studies within the extant financial literature. Most RMLs feature adjustable interest rates, and the variation of interest rates imposes additional uncertainty on NNEG providers. We therefore employ the well-known CIR term structure model (Cox et al. 1985) to capture the interest rate dynamics in the VaR and CTE of NNEGs.

The longevity risk factor also needs to be considered in VaR and CTE of NNEGs, in case the borrower lives longer than expected. Given the uncertainty over improvement trends in long-term mortality, longevity risk has become a serious threat to RML providers, since it increases the payout period and the risks involved in issuing RMLs. In order to reflect this longevity risk, we consider a stochastic mortality model, employing the well-known CBD model (Carins et al. 2006) for VaR and CTE of NNEGs.

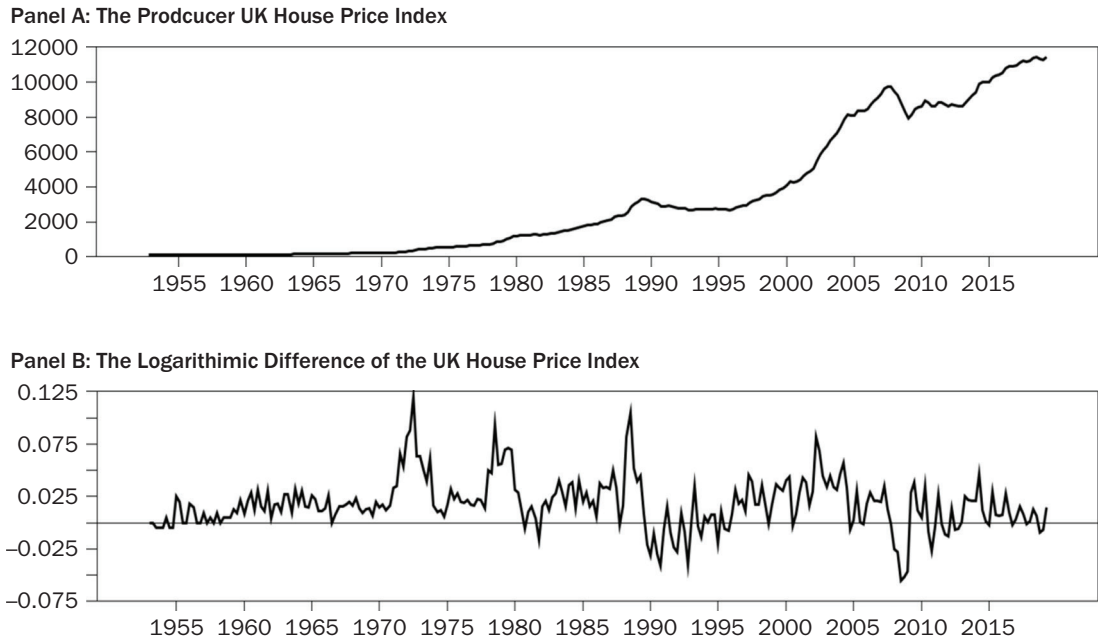
Our empirical analyses, we find that the jump ARMA-GARCH model with dynamic jump specifications provides the best fit, according to log-likelihood, Akaike information criteria (AIC), and Bayesian information criteria (BIC). The dynamic jump ARMA-GARCH model shows significant persistence in the conditional jump, which is indicated when computing the VaR and CTE of NNEGs. Thus, if we ignore the housing risk, interest rate risk, and longevity risk, we would underestimate the VaR and CTE in measuring the risk of NNEGs, which is as significant as that for housing risk.

In turn, we provide a general model in this study that allows for three stochastic components for VaR and CTE of NNEGs based on housing price return, interest rate, and mortality rate, contributing to the extant literature on RMLs in the following significant ways. First, our general model framework considers not only house price return dynamics, but also interest rate and mortality rate dynamics. Second, we also quantify risk measures such as VaR and CTE to illustrate the risk for NNEGs. Finally, the numerical results show that the housing risk for VaR and CTE of NNEGs may have a significant effect, a finding that has not been established previously.

The remainder of this article is organized as follows: We construct housing, interest rate, and morality models in the next section. We then outline the payoff of the

EXHIBIT 1

The Producer UK Nationwide House Price Index



NNEGs and define the risk measures of VaR and CTE for analyzing the risk of the NNEGs. A numerical investigation of the effects of housing, interest rate, and longevity risks is subsequently carried out on the VaR and CTE of NNEGs and the conclusions drawn from this study are presented in the final section.

MODELING HOUSING, INTEREST RATE, AND LONGEVITY RISK

To investigate housing, interest rate, and longevity risk and to outline the payoff structure of NNEGs, we need to specify the housing price, interest rate, and mortality dynamics.

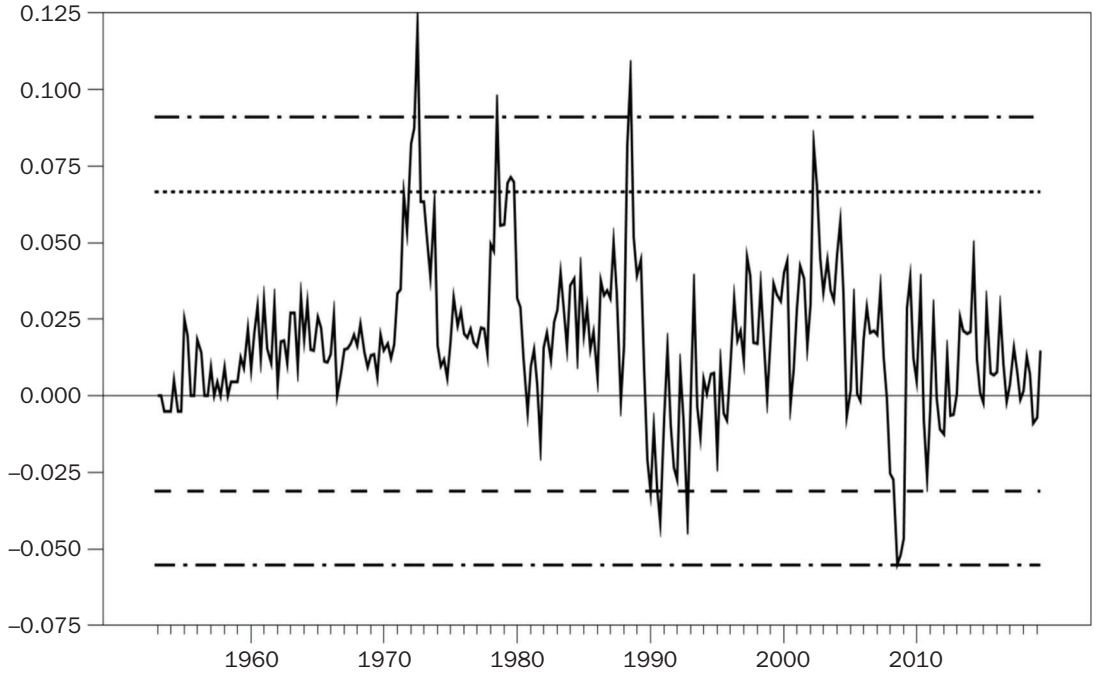
House Price Dynamics: Jump ARMA(s,m)-GARCH(p,q) Model

Li et al. (2010) and Chen et al. (2010) turned to the use of ARMA-EGARCH and ARMA-GARCH models in their approach to capturing autocorrelated and volatility clustering of house price dynamics in the UK equity release market and the US HECM program, respectively. The Producer UK House Price Index from Q4 1952 to Q2 2019 is illustrated in Exhibit 1, with these details being obtained from the Nationwide House Price Index (HPI).

Exhibit 2 shows the historical quarterly returns and reveals significant jump risk in the HPI. For example, the data shows that those quarterly housing price returns, change more than two standard deviations 17 times over the period. Further, we discover that those quarterly housing price returns change more than three standard deviations three times in the same time span. In particular, the greatest change of quarterly housing price returns is 12.21% in Q3 1972, while the lowest change is –5.5% in Q3 2008. The most significant downward jump occurred in 2008, following the outbreak of the subprime mortgage crisis. Given that the effects of such a downward jump are both systematic and non-diversifiable, Alai et al. (2014) have

EXHIBIT 2

Historical Quarterly Returns of the UK Nationwide House Price Index



pointed out the home price depreciation risk is only partially diversifiable. Pooling RMLs nationally only reduces the risk of a downturn in the regional housing market, but it cannot diversify the risk of a national economic recession. Nakajima and Telyukova (2017) also show that the great recession is likely to affect the RMLs market in both the short run and the long run. Therefore, in order to capture the autocorrelation, volatility clustering, and jump diffusion effects in the house price dynamics process, we consider the jump effect with house price return dynamics based upon an empirical investigation; our analysis involves the construction of a house price return model capable of capturing the properties of volatility clustering and both jump and autocorrelation effects under the Maheu and McCurdy (2004) framework.

We begin by investigating the house price returns data based on time-series analysis, and then develop the jump ARMA-GARCH model. Let $(\Omega;\Phi;P;(\Phi_t)_{t=0}^{T})$ be a complete probability space, where P is the data-generating probability measure, with specifications for the conditional mean and conditional variance. Let H_t denote the UK house price index and Y_t represent the house price return at time t. Y_t is defined as $\ln\left(\frac{H_t}{H_{t-1}}\right)$ and the proposed jump ARMA-GARCH model governing the return process is then expressed as:

$$Y_t = \ln\left(\frac{H_t}{H_{t-1}}\right) = \mu_t + \varepsilon_t, \tag{1}$$

The mean return follows an ARMA process as

$$u_t = c + \sum_{i=1}^{s} \vartheta_i Y_{t-i} + \sum_{j=1}^{m} \zeta_j \varepsilon_{t-j}, \tag{2}$$

where s is the order of the autocorrelation terms, m is the order of the moving average terms, ϑ_i is the ith-order autocorrelation coefficient, ζ_j is the jth-order moving average coefficient, and ε_t is the total returns innovation observable at time t, which is

$$\varepsilon_t = \varepsilon_{1,t} + \varepsilon_{2,t} \tag{3}$$

Extending from Maheu and McCurdy (2004),[1] we set two stochastic innovations in which the first component ($\varepsilon_{1,t}$) captures smoothly evolving changes in the conditional variance of returns and the second component ($\varepsilon_{2,t}$) causes infrequent large moves in returns, which are denoted as jumps. $\varepsilon_{1,t}$ is set as a mean-zero innovation ($E[\varepsilon_{1,t} \mid \Phi_{t-1}] = 0$), with a normal stochastic forcing process as

$$\varepsilon_{1,t} = \sqrt{h_t} z_t, \quad z_t \sim NID(0,1), \tag{4}$$

and h_t denotes the conditional variance of the innovations, given an information set of Φ_{t-1},

$$h_t = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{5}$$

where p is the order of the GARCH terms, q is the order of the ARCH term, α_i is the ith-order ARCH coefficient, and β_j is the jth-order GARCH coefficient. $\varepsilon_{1,t}$ is contemporaneously independent of $\varepsilon_{2,t}$. $\varepsilon_{2,t}$ is a jump innovation that is also conditionally mean zero ($E[\varepsilon_{2,t} \mid \Phi_{t-1}] = 0$) and we describe $\varepsilon_{2,t}$ in next subsection.

The Setting of Jump Dynamics

To capture the jump risk, the second component of innovation is employed to reflect the large change in price and is modeled as

$$\varepsilon_{2,t} = \sum_{k=1}^{N_t} V_{t,k} - \phi\lambda_t \quad V_{t,k} \sim NID(\phi, \theta^2) \qquad \text{for } k = 1,2,\ldots \tag{6}$$

where $V_{t,k}$ denotes the jump size for the kth jump with the jump size following the normal distribution with parameters (ϕ,θ^2) and N_t is the jump frequency from time $t-1$ to t, distributed as a Poisson process with a time-varying conditional intensity parameter (λ_t); that is:

$$P(N_t = j \mid \Phi_{t-1}) = \frac{\exp(-\lambda_t)\lambda_t^j}{j!}, j = 0,1,2\ldots, \tag{7}$$

where the parameter λ_t represents the mean and variance for the Poisson random variable, it is also referred to as the conditional jump intensity.

To facilitate our investigation of the jump effect on house price returns, we extend the work of Chan and Maheu (2002), Maheu and McCurdy (2004), and Daal et al. (2007) to specify λ_t as an ARMA form, which is

$$\lambda_t = \lambda_0 + \rho\lambda_{t-1} + \varsigma\psi_{t-1}, \tag{8}$$

[1] Maheu and McCurdy (2004) consider the jump setting under a constant conditional mean of the GARCH model. We deal with a jump ARMA-GARCH model and the likelihood function for parameter estimation is reconstructed.

where ρ measures jumps persistence. Since the ς variable measures the sensitivity of the jump frequency (λ_t) to past shocks (ψ_{t-1}), with ψ_{t-1} representing the unpredictable component affecting our inference on the conditional mean of the counting process, N_{t-1}, then this suggests corresponding changes. We also investigate the constant jump effect, which represents a special case of Equation (8) with the restriction of constant jump intensity ($\lambda_t = \lambda_0$); this is imposed by setting ρ = 0 and ς = 0. More details regarding the ARMA jump intensity can be found in Maheu and McCurdy (2004) and for the parameter estimation, see Appendix A.

Interest Rate Dynamic: CIR Model

To model interest rate risk, we employ the well-known CIR interest rate model (Cox et al. 1985), which results in the introduction of a 'square-root' term in the diffusion coefficient of the interest rate dynamics proposed by Vasicek (1977). The CIR model has been a benchmark in modeling interest rates for many years, essentially because of its analytical tractability, as well as the fact that, contrary to the Vasicek (1977) model, the interest rate is always positive. Under the CIR model, we assume that the time-t short rate, r_t, for a $(\Omega;\Phi;P;(\Phi_t)_{t=0}^{T})$ is a complete probability space, governed by the following equation:

$$dr_t = \alpha_r(\mu_r - r_t)dt + \sqrt{r_t}\sigma_r dW_{r,t} \tag{9}$$

where $\{W_{r,t}, t \geq 0\}$ is a standard Brownian Motion with parameters $\theta_r \equiv (\alpha_r, \mu_r, \sigma_r)$. The drift function $\alpha_r(\mu_r - r_t)$ is linear with a mean reversion property; that is, the interest rate, r_t, moves in the direction of its mean, μ_r, at speed α_r. The diffusion function, r_t σ_r^2, is proportional to the interest rate, r_t, which ensures that the process remains within a positive domain. Furthermore, if α_r, μ_r, and σ_r are all positive, and if $2\alpha_r\mu_r \geq \sigma_r^2$ holds, then we can also assume that r_t will remain positive.

Mortality Dynamics: CBD Model

To model mortality dynamics, as opposed to using the static mortality rate, we consider the longevity risk in NNEGs and employ the CBD model (Cairns et al. 2006) to project future mortality rates. The CBD model is attractive because it uses only a few parameters to obtain a good fit for the mortality probabilities of the elderly; thus, this model has been widely adopted as a way to deal with longevity risk for the elderly (Wang et al. 2010, Yang 2011). Since the reverse mortgage products are issued for the elderly, we also adopt the CBD model. Under the CBD model, the mortality rate for a person aged x dying before $x + 1$ valued in year t, denoted as $q(t,x)$, is projected by:

$$\text{logit } q(t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}), \tag{10}$$

where the parameter $\kappa_t^{(1)}$ represents the marginal effect of time on mortality rates, parameter $\kappa_t^{(2)}$ refers to the old age effect on mortality rates, and $\bar{x}$ is the mean age.[2] With the estimated values of $(\kappa_t^{(1)}, \kappa_t^{(2)})$, we can forecast the future mortality rates. In this study, we adopt Cairns et al.'s (2006) approach for estimating the parameters by using the least square method to fit the actual mortality curve and then project the $(\kappa_t^{(1)}, \kappa_t^{(2)})$ based upon a two-dimensional random walk with drift:

[2] We use the UK mortality data from 1950–2006 according to the human morality database (HMD) and the data ages cover ages 60 to 100. Therefore, the mean age is 80 in our model calibration.

$$\kappa_{t+1} = \kappa_t + \mu + CZ_{t+1} \tag{11}$$

where $\kappa_t = \left[\kappa_t^{(1)}, \kappa_t^{(2)}\right]'$ and μ is a constant 2×1 vector; C is a constant 2×2 upper triangular matrix and Z_t is a two-dimensional standard Gaussian process.

Equation (11) describes the dynamics of the random walk process κ_t under the real world probability measure, P, for projecting the mortality rate shown in Equation (10). Let $p(t,x)$ denote the projected one-year survival rate in year t based upon the CBD model, then the projected probability in year t that a borrower aged x will survive to age $x + s$ is calculated by

$$_s p(t,x) = p(t,x)p(t+1,x+1)\cdots p(t+s,x+s-1)\,. \tag{12}$$

OUTLINE AND MEASURING THE RISK OF NNEGS

The Structure of NNEGs

It has been accepted market practice within the UK for all RML products to include the provision of a NNEG. NNEGs protect borrowers by capping the redemption amount of the mortgage at the lesser of the face amount of the loan or the sale proceeds of the property.

Let's consider a 'roll-up' mortgage as an example.[3] We denote K_t as the outstanding balance of the loan and H_t represent the value of the mortgaged property. The amount repayable (outstanding balance) at time T is the sum of the principal, K, plus the interest accrued at a roll-up rate,[4] η_t; that is,

$$K_T = Ke^{\sum_{t=0}^{t=T-1} \nu_t}\,, \tag{13}$$

At the time that the loan becomes repayable, time T, if $H_t < K_t$, then the borrower pays H_t, and if $H_t > K_t$, then the borrower pays K_t. Once the loan is repaid, the provider receives an amount, K_t, plus the payoff of NNEGs (see Kogure et al. 2014), which is:

$$V_t = -Max[K_t - H_t, 0], \tag{14}$$

and

$$H_t = H_0 e^{\sum_{s=1}^{4t} Y_s} \tag{15}$$

where H_0 is the current housing price and Y_S is the quarterly log return of housing prices at time s. Equation (14) can be viewed as the future claim loss at time t for a NNEG. The total claim loss for a NNEG at issuance can be measured by calculating the present value of the future expected claim loss (*PVECL*), or

$$PVECL = \sum_{t=0}^{\omega - s_0 - 1} \exp\left(-\int_0^s r_t \; dt\right) {}_t p_{s_0} \times q_{s_0+t} \times V_t\,, \tag{16}$$

[3] Hosty et al. (2008) and Cho et al. (2015) have addressed the most common types of payment options for equity-release products are lump sum (roll-up), terms, lines of credit, modified terms (combining lines of credit and term payments), tenure, and modified tenure (combining lines of credit and tenure). Thus, they show that the roll-up mortgage has become the most popular payment option. Therefore, our ongoing analysis focuses on roll-up RMLs.

[4] The roll-up rate can be either fixed or floating. In addition, the initial principal is normally determined according to the housing value.

where ω is the final age, ${}_t p_{s_0}$ represents the t-year survival probability that an s_0-aged person will survive to age $s_0 + t$, q_{s_0+t} is the probability that a borrower aged s_0 at inception will die during the time interval t to $t + 1$, and r_t is stochastic interest rate. Thus, we can quantify the PVECL to measure the risk for NNEGs, which will be shown in next subsection.

VaR and CTE of the NNEGs

Pricing and risk analysis for a NNEG have clearly become two extremely important issues. The earlier studies have dealt with the risk-neutral valuation (market consistent approach) for pricing a NNEG using the conditional Esscher transform (see Li et al. 2010 and Chen et al. 2010, for example). In addition, Kogure et al. (2014) employ a Bayesian multivariate pricing framework. However, risk management is increasingly important for financial institutions, and quantifying risk under the real-world probability measure is a central task for managing it. Taking a different approach from the existing literature,[5] we propose that a risk analysis framework for a NNEG should consider the major three stochastic components. In addition, the VaR and CTE risk measures, also known as the tail VaR or expected shortfall, can both quantify risk in finance and insurance applications. The VaR generally focuses on the downside of the return distribution. According to Jorion (2000), the VaR is typically defined as the maximum expected loss of a portfolio over a specific holding period at a given significance level. It is a relatively recapitulative measure, in that it summarizes the potential change in the market value of a portfolio that stems from several sources of risk in a single number. Thus, the VaR methodology offers a comprehensive, compact advantage in terms of measuring market risk exposure. As an extension of the VaR concept, CTE includes a measure of the expected risk in the tail of the loss distribution; it is a coherent risk measure, in the sense of Artzner et al. (1999). To investigate the risk of NNEGs, we calculate the present value of total claim losses, as shown in Equation (16), and measure the VaR and CTE, using the following definitions of VaR and CTE in the PVECL.

First, let VaR_α denote the 100a% quantile of the present value of the future expected claim loss distribution, or 100a% VaR, which is

$$VaR_\alpha = \inf\left\{ l \in \mathbb{R} : F_{PVECL}(l) \geq \alpha \right\} \qquad \alpha \in (0,1), \tag{17}$$

where F_{PVECL} is the distribution function of the present value of claim loss. Second, to calculate the CTE, we use

$$CTE_\alpha = E^P(PVECL \mid PVECL > V_\alpha), \tag{18}$$

where E^P denotes the expectation under the physical measure P.

EMPIRICAL AND NUMERICAL ANALYSIS

Model Fitting

We examine the performance of the jump ARMA-GARCH model using time-series data from the Nationwide HPI, placing particular focus on an investigation into whether the conditional jump intensity is time-varying or constant. Our quarterly data period runs from Q4 1952 to Q2 2019, with a total of 267 quarterly observations. As a

[5] See Alai et al. 2014; Cho et al. 2015; Shao et al. 2015.

EXHIBIT 3

Summary Statistics, Q4 1952–Q2 2019

Variables	Mean	S.D.	Skewness[a,c]	Excess Kurtosis[a,c]	LB Q(32)[b] Stats
Y_t	0.0178	0.0243	0.5860***	2.2574***	374.70***
Y_t^2	0.0009	0.0016	4.2176***	23.4078***	262.42***

NOTES: [a]The skewness and excess kurtosis statistics include a test of the null hypotheses that each is zero (the population values if the series is i.i.d. Normal.). [b]The LB Q (32) statistics refer to the null hypothesis of no serial correlation with 32 lags. [c]***indicates significance at the 1%.

robustness check, we also examine the results for different data periods (from Q4 1962 to Q2 2019 and Q4 1972 to Q2 2019).

The summary statistics on the levels and squares of the log-return series are reported in Exhibit 3. The modified LB statistics (West and Cho 1995) show strong serial correlation in both the levels and the squares of the return series, a result that is consistent with that reported by Li et al. (2010).

We investigate the jump dynamics for both dynamic and constant jump models, using ARMA(1,1)-GARCH(1,1) models, with the parameters of these two jump ARMA-GARCH models being estimated by maximizing the conditional log-likelihood functions in Exhibit 4. The selection of the ARMA(1,1)-GARCH(1,1) models in the present study is based upon the Box-Jenkins approach.[6]

We evaluate the performance of the jump dynamics using log-likelihood, AIC, and BIC.[7] The log-likelihood, AIC, and BIC results indicate that the dynamic jump ARMA-GARCH model provides a better fit, with the persistence parameter (ρ) in this model being found to be 0.7495, with statistical significance. This finding suggests that a high probability of many (few) jumps will also tend to be followed by a similarly high probability of many (few) jumps.

In order to facilitate a thorough investigation in the present study of the importance of the jump effect in the modeling of house price returns, the existing models proposed in Chen et al. (2010) and Li et al. (2010)—which include the GBM, ARMA-GARCH, and ARMA-EGARCH models—are also fitted to exactly the same series of Nationwide HPI returns. We further compare the performance of the jump ARMA-GARCH model with other jump diffusion models, such as the Merton (1976) and Kou (2002) models, both of which allow for jump effects, but do not consider the effects of autocorrelation and volatility persistence.

The fitting results are presented for each of the different models in Exhibit 5.[8] Our empirical results indicate the superiority of the jump ARMA-GARCH model over the existing house price return models, with the dynamic jump ARMA-GARCH model demonstrating further improvements on each of the other models based upon the log-likelihood, AIC, and BIC values.

Although the jump effect is taken into consideration in the jump diffusion models, such as those proposed by Merton (1976) and Kou (2002), the performance of their models is nevertheless found to be inferior to that of the time-series models within which the effects of autocorrelation and volatility clustering are also taken into consideration; it therefore seems clear that a house price return model capable of simultaneously taking into consideration all three properties would represent an important contribution to this particular field of research.

As a check for the robustness of our results, we also investigate the model fit by considering different periods of the Nationwide HPI data in Exhibit 6. For both sub-periods, the dynamic jump ARMA-GARCH model is still found to outperform each of the other models.

The results reported in Exhibit 5 and Exhibit 6 confirm that the addition of jump dynamics improves the specification of the conditional distribution, as compared with

[6] Although not reported here, the parameter estimates of the models are available upon request.

[7] AIC = –2/obs. ln(likelihood) + 2/obs. × (No. of parameters) (Akaike 1973); BIC = –2/ obs. ln(likelihood) + ([No. of parameters] × ln[obs.])/obs.; obs. is the sample size.

[8] The stochastic processes of these models are available upon request.

EXHIBIT 4

Parameter Estimates and Model Fit of Constant and Dynamic Jump Models, Q4 1952–Q2 2019

	ARMA(1,1)-GARCH(1,1) Models*			
	Constant Jump		Dynamic Jump	
Parameters	Coeff.	S.E.	Coeff.	S.E.
Constant	4.73e-03***	1.65e-03	5.59e-03***	1.43e-03
ϑ_1	0.6445***	0.0707	0.6088***	0.0705
ζ_1	–0.0979	0.0901	–0.1002	0.0874
ω	8.76e-06	8.91e-06	7.83e-06	6.22e-06
α	0.2161***	0.0657	0.1805***	0.0522
β	0.7153***	0.0625	0.7556***	0.0485
λ_0	0.5042*	0.1589	0.1254*	0.0227
ρ	–	–	0.7495***	0.1782
ς	–	–	0.4440	0.3368
ϕ	0.0140	0.0112	0.0141*	7.27e-03
θ	0.0205*	0.0244	0.0221*	0.0126
AIC	–5.3902		–5.4071	
BIC	–5.1688		–5.1952	
Log-likelihood	689.9478		692.1150	

NOTE: *indicates significance at the 10% level and ***indicates significance at the 1% level.

EXHIBIT 5

Model Selection, Q4 1952–Q2 2019

Model	Log-Likelihood	AIC	BIC
Geometric Brownian Motion	610.8391	–4.5777	–4.5507
ARMA-GARCH	683.5855	–5.3405	–5.1105
ARMA-EGARCH	665.6008	–5.2000	–5.0483
Merton Jump	629.6252	–4.6964	–4.6290
Double Exponential Jump Diffusion	631.3421	–4.7123	–4.6974
Constant Jump ARMA-GARCH	689.9478	–5.3902	–5.1688
Dynamic Jump ARMA-GARCH	692.1150	–5.4071	–5.1952

the GBM, Merton jump, double exponential jump diffusion, ARMA-GARCH, ARMA-EGARCH, and constant jump ARMA-GARCH models. In addition, the persistence parameter (ρ) governing the jump dynamic is statistically significant. It clearly indicates that jump risk in housing returns is significant and critical for risk measures of NNEGs.[9]

Risk Analysis of NNEGs

In this section, we study the impacts of different risk factors on the VaR and CTE of NNEGs. The risk analysis of NNEGs depends upon the dynamics of house price returns, the interest rate, and mortality rates and we conduct 100,000 Monte Carlo simulations.

Regarding the NNEGs, we consider a floating roll-up mortgage, which is the most popular RMLs in the UK and the floating interest rate (ν_t) is set as being equal to the

[9]The persistence parameter (ρ) governing the jump model is estimated to be around 0.7495, with statistical significance. We didn't report the entire parameter estimates here, but they are available upon request.

EXHIBIT 6

Robustness Check of Model Selection

Model	Log-Likelihood	AIC	BIC
Panel A: Q41962–Q22019			
Geometric Brownian Motion	506.5634	–4.4651	–4.4348
ARMA-GARCH	559.2074	–5.1778	–4.9705
ARMA-EGARCH	546.3101	–5.0584	–4.8842
Merton Jump	519.3667	–4.5519	–4.4762
Double Exponential Jump Diffusion	527.1209	–4.8753	–4.5612
Constant Jump ARMA-GARCH	565.3306	–5.2345	–5.0105
Dynamic Jump ARMA-GARCH	566.3860	–5.2443	–5.0285
Panel B: Q41972–Q22019			
Geometric Brownian Motion	417.1806	–4.4443	–4.4296
ARMA-GARCH	449.2933	–5.1056	–4.8208
ARMA-EGARCH	436.6772	–4.9622	–4.7566
Merton Jump	418.7175	–4.4485	–4.3618
Double Exponential Jump Diffusion	420.5129	–4.4569	–4.3891
Constant Jump ARMA-GARCH	452.5170	–5.1422	–4.8778
Dynamic Jump ARMA-GARCH	452.6695	–5.1439	–4.9293

EXHIBIT 7

Base Assumption of Parameter Values for the Value-at-Risk and Conditional Tail Expectations

Parameters	Notation	Value
Initial Risk-free interest rate (%)	r	1.878
Roll-up rate (%)	ν	2.000
Amount of loan advanced at inception	K	30,000
Final age	ω	100
Initial property value for different ages, x, of borrowers (H_0)		
x = 60 Years		176,500
x = 70 Years		111,000
x = 80 Years		81,000
x = 90 Years		60,000

risk-free interest rate (r_t) plus a constant spread (ν_r), that is, $\nu_t = r_t + \nu_r$.[10] For comparison purposes, we follow Li et al. (2010) to set up the relevant assumptions for the NNEGs and list the information in Exhibit 7. In addition, the parameter estimates for the housing price return for the jump ARMA-GARCH model can refer to Exhibit 4 and for the interest rate and mortality rate models are in Exhibit 8 and Exhibit 9.[11]

To analyze the risk of NNEGs, we examine the tail risk by calculating the VaR and CTE of the NNEGs. The VaR and CTE at 95% and 99% significance levels are presented in Exhibit 10. As the borrower's age increases, the VaR and CTE decrease for different housing models and the VaR and CTE of NNEGs will be greater for female borrowers than male borrowers due to the fact that females have a longer life expectancy. Thus, in Exhibit 11 and Exhibit 12, we further examine the NNEGs when considering the tail risk. Comparing Exhibit 10 with Exhibit 12, we note some important findings. First, the NNEGs in terms of VaR and CTE differ significantly under the GBM, ARMA-GARCH,

[10] A fixed roll-up mortgage was considered in Lee et al. (2012).

[11] To be consistent, we employ the three-month T-bill interest rates from the period of Q4 1952 to Q2 2019 to estimate the parameters in CIR model.

EXHIBIT 8

CIR Model Estimation Results, Q4 1952–Q2 2019

α_r	μ_r	σ_r	Log-Likelihood
0.0067	0.0019	0.0367	3.9635

ARMA-EGARCH, Merton Jump, Double Exponential Jump Diffusion, and Constant Jump ARMA-GARCH housing model. Therefore, we cannot ignore the housing risk when measuring NNEGs risk. Second, the GBM assumption yields the lowest for VaR and CTE, which indicates that we would underestimate the risk of the NNEGs if we ignored the importance of the autocorrelation, volatility clustering, and jump diffusion in the housing return. In agreement with Bianchi (2015), the VaR and CTE strictly depend on the distribution assumptions of the model, and disregarding these stylized facts can result in underestimating the tail risk. Third, using the stochastic mortality rate and interest rate captures more extreme risk than a constant mortality rate and interest rate as shown in Exhibit 11 and Exhibit 12, respectively. Therefore, the constant mortality rate and constant interest rate assumption will lead to lower for VaR and CTE of NNEGs. Finally, as GBM is the benchmark model, the housing risk reveals the most significant effect on the VaR and CTE, compared with interest rate risk and longevity risk according to Exhibit 10 and Exhibit 12.

SENSITIVITY ANALYSIS ON THE RISK MEASURES FOR NNEGS

Sensitivity of the Interest Rate Model

To model the interest rate risk, we employ the well-known CIR interest rate model in this study. However, in the aftermath of the subprime crisis in several European countries, negative rates were observed e.g., in Sweden, Switzerland, and Denmark. Under such circumstances, we can use the Vasicek (1977) model instead of the

EXHIBIT 9A

Estimated Kappa Values for Male Samples

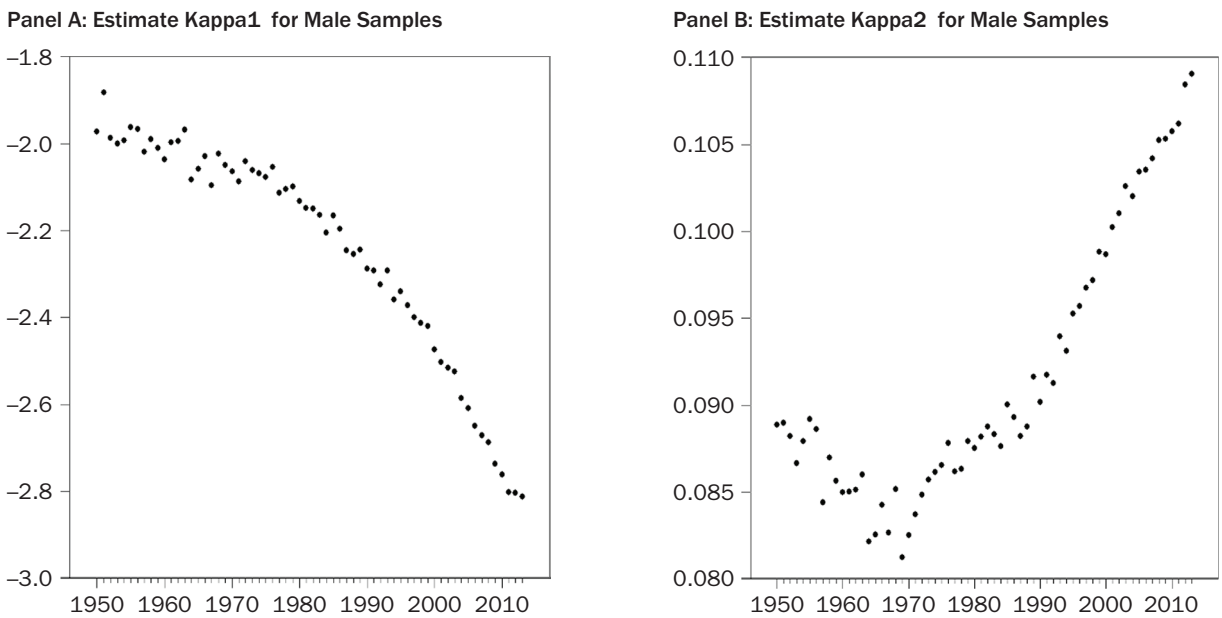
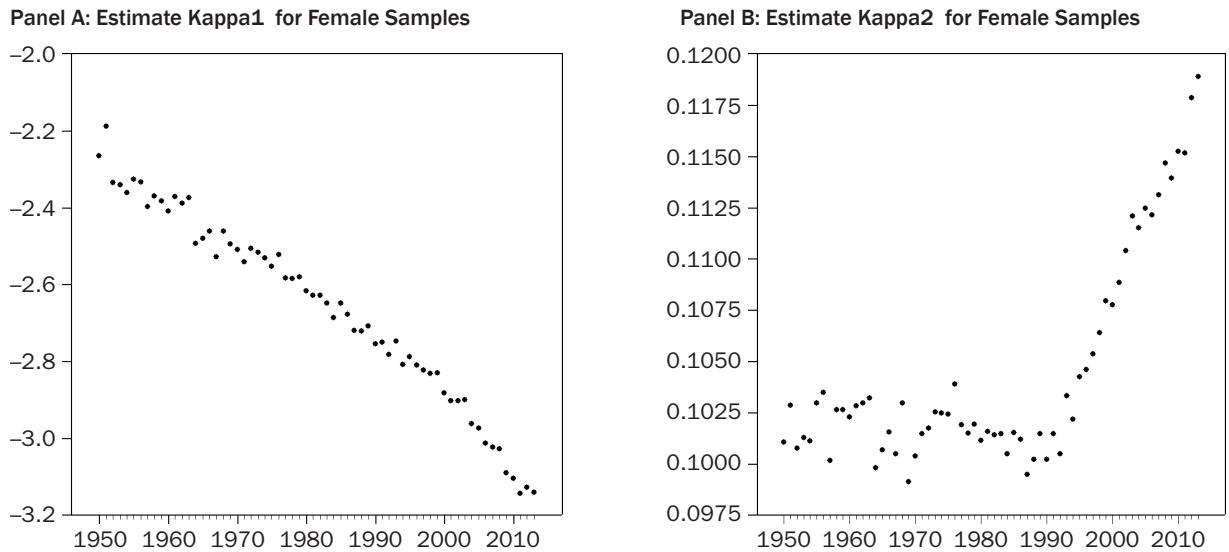


EXHIBIT 9B

Estimated Kappa Values for Female Samples



CIR model in capturing the interest rate dynamics. To demonstrate the feasibility of using a different model in the risk analysis framework, we further provide the numerical results based on the Vasicek model in Exhibit 13. Comparing Exhibit 12 with Exhibit 13, we find that there is not much difference for the VaR and CTE of NNEGs compared with using the CIR or Vasicek models. However, it shows that it is important to use the stochastic interest rate model to measure the risk of NNEGs, instead of using a constant interest rate assumption.

Sensitivity of the Type of Borrowers

To investigate the effect based on the type of borrowers, we further analyze the results for the joint borrowers, such as couples. We can adjust the PVECL from a single borrower to a joint borrower as follows.

$$PVECL = \sum_{t=0}^{\omega - s_0 - 1} \exp\left(-\int_0^s r_t \; dt\right) {}_{t|1}q_{\overline{s_0 s_1}} \times V_t \; , \tag{19}$$

where ${}_{t|1}q_{\overline{s_0 s_1}}$ is the probability that the last survivor dies within the $(t+1)$th year given survival up to the tth year for a male at age s_0 and female at age s_1. Traditionally, the mortality between husbands and wives is usually assumed to be independent. For example, Chia and Tsui (2004) adopt the Lee-Carter model to forecast cohort survival probability at each postretirement age for the household using the abridged life tables for Singapore, but they assume that the spouses' mortality is independent. Ji et al. (2012) compare the value of the NNEGs for joint borrowers under the independence assumption and the semi-Markov assumption. They show that the difference is insignificant. We also extend the independent assumption to calculate the joint survival function in this study, which is described in Appendix B.

We observe the VaR and CTE of NNEGs for the joint borrower in Exhibit 14. Comparing Exhibit 14 with Exhibit 10, it is very clear that the VaR and CTE of NNEGs are higher than those for the single borrower (male and female). Therefore, the type of borrower can affect the risk of NNEGs and our risk analysis incorporates the type of borrower for analyzing the RML risk.

EXHIBIT 10

Value-at-Risk and Conditional Tail Expectations for Different Housing Models

Model		60	70	80	90
Panel A: Male					
Model 1: Geometric Brownian Motion					
	VaR_{95}	2544.43	1471.60	696.33	337.28
	VaR_{99}	2745.25	1636.23	714.50	405.45
	CTE_{95}	2638.32	1570.27	733.13	420.99
	CTE_{99}	2810.49	2062.51	894.82	426.62
Model 2: ARMA-GARCH					
	VaR_{95}	5254.76	3793.24	2535.74	1249.47
	VaR_{99}	5589.76	4144.49	2863.38	1274.21
	CTE_{95}	5551.03	4346.79	2731.27	1281.21
	CTE_{99}	7009.09	4694.17	3159.25	1299.05
Model 3: ARMA-EGARCH					
	VaR_{95}	4956.53	3322.54	2291.16	1123.05
	VaR_{99}	5215.22	3938.45	2567.02	1147.12
	CTE_{95}	5323.49	3970.23	2501.02	1142.00
	CTE_{99}	6490.44	4295.30	2899.58	1164.88
Model 4: Merton Jump					
	VaR_{95}	3798.93	2422.71	1385.81	726.56
	VaR_{99}	3985.89	2670.97	1757.94	739.09
	CTE_{95}	3921.45	2860.84	1694.16	739.21
	CTE_{99}	4147.85	3201.76	1903.51	747.38
Model 5: Double Exponential Jump Diffusion					
	VaR_{95}	3427.98	2006.40	1028.63	586.83
	VaR_{99}	3537.46	2476.94	1223.17	601.30
	CTE_{95}	3670.08	2591.66	1223.85	600.06
	CTE_{99}	4058.16	2701.52	1378.29	615.22
Model 6: Constant Jump ARMA-GARCH					
	VaR_{95}	6087.92	4803.68	2839.88	1393.54
	VaR_{99}	6649.41	5245.30	3204.35	1423.05
	CTE_{95}	8508.87	5120.00	3430.09	1422.33
	CTE_{99}	9516.92	5326.69	3700.21	1476.43
Model 7: Dynamic Jump ARMA-GARCH					
	VaR_{95}	7057.15	5132.66	3162.82	1435.95
	VaR_{99}	7671.89	5486.08	3511.77	1489.65
	CTE_{95}	9820.95	5436.88	3842.65	1488.04
	CTE_{99}	10982.75	5767.82	4093.37	1536.01
Panel B: Female					
Model 1: Geometric Brownian Motion					
	VaR_{95}	3912.58	2426.84	1124.84	689.23
	VaR_{99}	4221.39	2698.35	1154.20	828.52
	CTE_{95}	4056.95	2589.57	1184.29	860.29
	CTE_{99}	4321.71	3401.33	1445.47	871.78
Model 2: ARMA-GARCH					
	VaR_{95}	6681.84	5349.94	3044.70	2003.46
	VaR_{99}	7107.82	5845.35	3438.11	2043.13
	CTE_{95}	7058.57	6130.67	3279.48	2054.35
	CTE_{99}	8912.61	6620.61	3793.37	2082.96

(continued)

EXHIBIT 10 *(continued)*

Value-at-Risk and Conditional Tail Expectations for Different Housing Models

Model		60	70	80	90
Model 3: ARMA-EGARCH					
	VaR_{95}	6358.92	4781.22	2844.20	1892.94
	VaR_{99}	6690.80	5667.53	3186.64	1933.50
	CTE_{95}	6829.70	5713.26	3104.71	1924.89
	CTE_{99}	8326.82	6181.05	3599.48	1963.45
Model 4: Merton Jump					
	VaR_{95}	5639.34	3897.40	1779.74	1419.16
	VaR_{99}	5916.88	4296.78	2257.66	1443.65
	CTE_{95}	5821.22	4602.23	2175.75	1443.87
	CTE_{99}	6157.30	5150.65	2444.62	1459.84
Model 5: Double Exponential Jump Diffusion					
	VaR_{95}	5397.25	3621.50	1619.71	1235.80
	VaR_{99}	5569.61	4470.82	1926.04	1266.27
	CTE_{95}	5778.42	4677.90	1927.11	1263.66
	CTE_{99}	6389.45	4876.20	2170.29	1295.58
Model 6: Constant Jump ARMA-GARCH					
	VaR_{95}	7618.53	6656.53	3480.53	2198.37
	VaR_{99}	8321.18	7268.48	3927.22	2244.92
	CTE_{95}	10648.15	7094.86	4203.88	2243.79
	CTE_{99}	11909.64	7381.26	4534.94	2329.12
Model 7: Dynamic Jump ARMA-GARCH					
	VaR_{95}	8818.15	7093.05	3787.20	2264.09
	VaR_{99}	9586.29	7581.46	4205.03	2348.76
	CTE_{95}	12271.62	7513.46	4601.23	2346.22
	CTE_{99}	13723.32	7970.81	4901.45	2421.85

CONCLUSIONS

In conjunction with the rapid growth in the RMLs market, there is growing demand for the development of effective risk management tools for these products. In the UK, RMLs are commonly sold with NNEGs protection, which caps the redemption amount at the lesser of the face amount of the loan, or the sale proceeds. Thus, it is crucial for the providers to have a strong understanding of the risk factors involved; house price, interest rate, and longevity risks can affect NNEGs in differing degrees. We extend the current literature by considering three risk factors in the risk management of NNEGs and by analyzing the corresponding effects.

Bianchi (2015) has pointed out the VaR and CTE strictly depend on the distribution assumption of the model. Thus, historical house price returns within the UK real estate market have autocorrelation, volatility clustering, and jump effects. It is extremely important for such providers to take these effects in house price returns into account when conducting a risk analysis of NNEGs. Despite this obvious requirement, this issue has not been dealt with in the prior literature, but it is examined in the present study using a jump ARMA-GARCH model. Furthermore, both interest rate and longevity risks can increase the probability of the home sale proceeds being less than the loan value paid out; hence, we also consider the CIR interest rate model and CBD mortality model to capture the respective interest rate and longevity risks in the VaR and CTE of NNEGs. Based upon our numerical analyses, we find that these three factors can affect the VaR and CTE of NNEGs, with the housing risk having the greatest impact, as

EXHIBIT 11

Effect of Mortality on Value-at-Risk and Conditional Tail Expectations

Model		60	70	80	90
Panel A: Male					
Deterministic Mortality Assumption					
Constant Jump ARMA-GARCH					
	VaR_{95}	5892.28	4655.00	2725.14	1311.57
	VaR_{99}	6435.72	5082.94	3074.88	1339.34
	CTE_{95}	8235.42	4961.52	3291.50	1338.67
	CTE_{99}	9211.08	5161.81	3550.71	1389.58
Dynamic Jump ARMA-GARCH					
	VaR_{95}	6820.59	4930.68	3019.52	1337.18
	VaR_{99}	7414.73	5270.20	3352.66	1387.19
	CTE_{95}	9491.76	5222.93	3668.55	1385.69
	CTE_{99}	10614.61	5540.85	3907.91	1430.36
CBD Model					
Constant Jump ARMA-GARCH					
	VaR_{95}	6087.92	4803.68	2839.88	1393.54
	VaR_{99}	6649.41	5245.30	3204.35	1423.05
	CTE_{95}	8508.87	5120.00	3430.09	1422.33
	CTE_{99}	9516.92	5326.69	3700.21	1476.43
Dynamic Jump ARMA-GARCH					
	VaR_{95}	7057.15	5132.66	3162.82	1435.95
	VaR_{99}	7671.89	5486.08	3511.77	1489.65
	CTE_{95}	9820.95	5436.88	3842.65	1488.04
	CTE_{99}	10982.75	5767.82	4093.37	1536.01
Panel B: Female					
Deterministic Mortality Assumption					
Constant Jump ARMA-GARCH					
	VaR_{95}	7365.35	6462.10	3327.54	2094.04
	VaR_{99}	8044.65	7056.17	3754.59	2138.38
	CTE_{95}	10294.28	6887.62	4019.09	2137.30
	CTE_{99}	11513.85	7165.66	4335.60	2218.59
Dynamic Jump ARMA-GARCH					
	VaR_{95}	8502.74	6855.43	3613.19	2150.12
	VaR_{99}	9243.41	7327.48	4011.83	2230.53
	CTE_{95}	11832.70	7261.76	4389.82	2228.12
	CTE_{99}	13232.47	7703.78	4676.24	2299.95
CBD Model					
Constant Jump ARMA-GARCH					
	VaR_{95}	7618.53	6656.53	3480.53	2198.37
	VaR_{99}	8321.18	7268.48	3927.22	2244.92
	CTE_{95}	10648.15	7094.86	4203.88	2243.79
	CTE_{99}	11909.64	7381.26	4534.94	2329.12
Dynamic Jump ARMA-GARCH					
	VaR_{95}	8818.15	7093.05	3787.20	2264.09
	VaR_{99}	9586.29	7581.46	4205.03	2348.76
	CTE_{95}	12271.62	7513.46	4601.23	2346.22
	CTE_{99}	13723.32	7970.81	4901.45	2421.85

EXHIBIT 12

Effect of Interest Rate Risk on Value-at-Risk And Conditional Tail Expectations

Model		60	70	80	90
Panel A: Male					
Constant Interest Rate Assumption					
Constant Jump ARMA-GARCH					
	VaR_{95}	5178.76	4117.44	2514.78	1274.31
	VaR_{99}	5656.39	4495.97	2837.52	1301.29
	CTE_{95}	7238.17	4388.57	3037.42	1300.64
	CTE_{99}	8095.68	4565.73	3276.62	1350.10
Dynamic Jump ARMA-GARCH					
	VaR_{95}	6045.23	4431.67	2763.63	1306.79
	VaR_{99}	6571.82	4736.83	3068.54	1355.66
	CTE_{95}	8412.74	4694.34	3357.65	1354.19
	CTE_{99}	9407.94	4980.08	3576.73	1397.85
CIR Model					
Constant Jump ARMA-GARCH					
	VaR_{95}	6087.92	4803.68	2839.88	1393.54
	VaR_{99}	6649.41	5245.30	3204.35	1423.05
	CTE_{95}	8508.87	5120.00	3430.09	1422.33
	CTE_{99}	9516.92	5326.69	3700.21	1476.43
Dynamic Jump ARMA-GARCH					
	VaR_{95}	7057.15	5132.66	3162.82	1435.95
	VaR_{99}	7671.89	5486.08	3511.77	1489.65
	CTE_{95}	9820.95	5436.88	3842.65	1488.04
	CTE_{99}	10982.75	5767.82	4093.37	1536.01
Panel B: Female					
Constant Interest Rate Assumption					
Constant Jump ARMA-GARCH					
	VaR_{95}	6674.85	5924.54	3098.05	2049.33
	VaR_{99}	7290.46	6469.20	3495.65	2092.72
	CTE_{95}	9329.19	6314.67	3741.91	2091.67
	CTE_{99}	10434.43	6569.58	4036.60	2171.22
Dynamic Jump ARMA-GARCH					
	VaR_{95}	7766.80	6356.42	3367.53	2081.75
	VaR_{99}	8443.36	6794.11	3739.07	2159.60
	CTE_{95}	10808.54	6733.17	4091.36	2157.26
	CTE_{99}	12087.16	7143.02	4358.31	2226.80
CIR Model					
Constant Jump ARMA-GARCH					
	VaR_{95}	7618.53	6656.53	3480.53	2198.37
	VaR_{99}	8321.18	7268.48	3927.22	2244.92
	CTE_{95}	10648.15	7094.86	4203.88	2243.79
	CTE_{99}	11909.64	7381.26	4534.94	2329.12
Dynamic Jump ARMA-GARCH					
	VaR_{95}	8818.15	7093.05	3787.20	2264.09
	VaR_{99}	9586.29	7581.46	4205.03	2348.76
	CTE_{95}	12271.62	7513.46	4601.23	2346.22
	CTE_{99}	13723.32	7970.81	4901.45	2421.85

EXHIBIT 13

Value-at-Risk and Conditional Tail Expectations under the Vasicek Model

Model		60	70	80	90
Panel A: Male					
Constant Jump ARMA-GARCH					
	VaR_{95}	6064.78	4786.22	2831.61	1390.51
	VaR_{99}	6624.13	5226.23	3195.01	1419.95
	CTE_{95}	8476.52	5101.38	3420.09	1419.24
	CTE_{99}	9480.75	5307.32	3689.43	1473.21
Dynamic Jump ARMA-GARCH					
	VaR_{95}	7031.39	5114.82	3152.66	1432.66
	VaR_{99}	7643.89	5467.01	3500.49	1486.24
	CTE_{95}	9785.11	5417.98	3830.30	1484.63
	CTE_{99}	10942.66	5747.77	4080.22	1532.49
Panel B: Female					
Constant Jump ARMA-GARCH					
	VaR_{95}	7594.51	6637.90	3470.80	2194.58
	VaR_{99}	8294.95	7248.14	3916.23	2241.04
	CTE_{95}	10614.58	7075.00	4192.12	2239.92
	CTE_{99}	11872.09	7360.61	4522.26	2325.11
Dynamic Jump ARMA-GARCH					
	VaR_{95}	8791.39	7074.31	3776.51	2259.45
	VaR_{99}	9557.20	7561.42	4193.17	2343.94
	CTE_{95}	12234.38	7493.60	4588.25	2341.41
	CTE_{99}	13681.68	7949.74	4887.62	2416.89

EXHIBIT 14

Value-at-Risk and Conditional Tail Expectations under Joint Life

Model		60	70	80	90
Constant Jump ARMA-GARCH					
	VaR_{95}	10797.94	4979.22	9028.36	2829.71
	VaR_{99}	11793.83	5618.25	9858.36	2889.62
	CTE_{95}	15091.90	6014.04	9622.86	2888.17
	CTE_{99}	16879.84	6487.66	10011.32	2998.01
Dynamic Jump ARMA-GARCH					
	VaR_{95}	12506.55	5475.22	9631.42	2914.89
	VaR_{99}	13595.99	6079.29	10294.61	3023.90
	CTE_{95}	17404.53	6652.08	10202.28	3020.63
	CTE_{99}	19463.44	7086.11	10823.29	3118.00

compared to interest rate risk and longevity risks. When analyzing the risk of NNEGs, it is crucial for issuers to identify the house price, interest rate, and longevity risks.

Because RMLs continue to increase in popularity and importance for aging societies globally, financial institutions and governments issuing such products must understand the impact of risk factors on NNEGs. The introduction of the house price, interest rate, and longevity risks and the corresponding risk analysis framework can help these providers assess the risks of NNEGs. Finally, in light of our analysis, from the perspective of risk management, the risk of NNEGs can be diversified under a portfolio of RMLs due to the dependence of the RMLs. However, although pooling RMLs in a nation can minimize the risk caused by a city economic recession, it cannot

diversify the risk under a national economic recession (Miao and Wang 2007; Zimmer 2015; Cummins and Trainar 2009; Pu et al. 2014). Therefore, understanding the dependence structure of house price returns is important in dealing with the portfolio of RMLs. It would be worthwhile to extend the VaR and CTE to measure the NNEG risk of the RML portfolio.

APPENDIX A

The parameters of the jump ARMA-GARCH model can be estimated using the maximum likelihood estimation (MLE) method. The construction of the likelihood function is described as follows. Let $F_n(\Theta)$ denote the log-likelihood function and Θ is the parameter set governing the jump ARMA-GARCH model, which implies $\Theta = (c,\vartheta_s,\zeta_m,w,\alpha_q,\beta_p,\lambda_0,\rho,\varsigma,\phi,\theta)$ We aim to find the optimal parameters (Θ^*) to maximize the log-likelihood function. The log-likelihood function can be expressed as

$$F_n(\Theta) := \sum_{t=1}^{N} \log f(Y_t|\Phi_{t-1},\Theta) \tag{A1}$$

The conditional on j jumps occurring the conditional density of returns is Gaussian,

$$f(Y_t|N_t = j,\Phi_{t-1},\Theta) = \frac{1}{\sqrt{2\pi(h_t + j\theta^2)}} \times \exp\left(-\frac{(Y_t - u_t + \phi\lambda_t - j\phi)^2}{2(h_t + j\theta^2)}\right). \tag{A2}$$

In Equation (A1), the conditional density of return at time t ($f(Y_t|\Phi_{t-1},\Theta)$) for calculating log-likelihood function can be obtained by integrating out the number of jumps as

$$\begin{aligned} f(Y_t|\Phi_{t-1},\Theta) &= \sum_{j=0}^{\infty} f(Y_t|N_t = j,\Phi_{t-1},\Theta)P(N_t = j|\Phi_{t-1},\Theta) \\ &= \sum_{j=0}^{\infty} \frac{1}{\sqrt{2\pi(h_t + j\theta^2)}} \times \exp\left(-\frac{(Y_t - u_t + \phi\lambda_t - j\phi)^2}{2(h_t + j\theta^2)}\right) \cdot \frac{\exp(-\lambda_t)\lambda_t^j}{j!} \end{aligned} \tag{A3}$$

where the conditional density of N_t ($P(N_t = j|\Phi_{t-1},\Theta)$) is shown in Equation (7). Since we assume the time-varying conditional intensity parameter λ_t follow an ARMA form as shown in Equation (8), we need to work out the past shock (ψ_{t-1}) that affects the inference on the conditional mean of the counting process first. ψ_{t-1} is defined as

$$\begin{aligned} \psi_{t-1} &= E[N_{t-1}|\Phi_{t-1},\Theta] - \lambda_{t-1} \\ &= \sum_{j=0}^{\infty} jP(N_{t-1} = j|\Phi_{t-1},\Theta) - \lambda_{t-1} \end{aligned} \tag{A4}$$

This expression could be estimated if $P(N_{t-1} = j \mid \Phi_{t-1},\Theta)$ are known. Following Maheu and McCurdy (2004), the ex post probability of the occurrence of j jumps at time $t-1$ can be inferred using Bayes' formula as follows.

$$\begin{aligned} E\left[N_{t-1}|\Phi_{t-1},\Theta\right] &= \sum_{j=0}^{\infty} jP(N_{t-1} = j|\Phi_{t-1},\Theta) \\ &= \sum_{j=0}^{\infty} j\frac{f(Y_{t-1}|N_{t-1} = j,\Phi_{t-2},\Theta)P(N_{t-1} = j|\Phi_{t-2},\Theta)}{f(Y_{t-1}|\Phi_{t-2},\Theta)} \\ &= \frac{\sum_{j=1}^{\infty} \frac{\exp(-\lambda_{t-1})\lambda_{t-1}^j}{j!} \frac{1}{\sqrt{2\pi(h_{t-1} + j\theta^2)}} \times \exp\left(-\frac{\left(Y_{t-1} - u_{t-1} + \phi\lambda_{t-1} - j\phi\right)^2}{2(h_{t-1} + j\theta^2)}\right)}{f(Y_{t-1}|\Phi_{t-2},\Theta)} \end{aligned} \tag{A5}$$

The details of Bayes' inference on calculating $E[N_{t-1} \mid \Phi_{t-1}, \Theta]$ is presented in Maheu and McCurdy (2004) Thus, by iterating on (8), (A3), and (A5), we can construct the log-likelihood function and obtain the maximum likelihood estimators. In addition, Equations (A3), (A4), and (A5) involve an infinite summation depending on the jumps.[12] We find that truncation of the infinite sum in the likelihood at 10 captures all the tail probabilities and gleans sufficient precision in the estimation procedure.

APPENDIX B

Suppose that the heads (s_0) and (s_1), belonging respectively to the gender *m* (males) and *f* (females), have remaining lifetime $T^m(s_0)$ and $T^f(s_1)$. Then the marginal survival functions are denoted by $S^m_{s_0}$ and $S^f_{s_1}$, respectively, so that, for all $t \geq 0$, ${}_tp^m_{s_0} = S^m_{s_0}(t) = P[T^m(s_0) > t]$ and ${}_tp^f_{s_1} = S^f_{s_1}(t) = P[T^f(s_1) > t]$. Therefore, its joint survival function, which both males at age s_0 and females at age s_1 are survival exceeding *t* years can be written as

$$ {}_tp_{s_0 s_1} = S_{s_0 s_1}(t,t) = \Pr[T^m(s_0) > t \text{ and } T^f(s_1) > t] = S^m_{s_0}(t) S^f_{s_1}(t) \tag{B1} $$

The probability of survival of the last spouse, which males at age s_0 and females at age s_1 is survival exceeding *t* years can be written as

$$ {}_tp_{\overline{s_0 s_1}} = 1 - {}_tq^m_{s_0}\, {}_tq^f_{s_1} = {}_tp^m_{s_0} + {}_tp^f_{s_1} - {}_tp^m_{s_0} \cdot {}_tp^f_{s_1} \tag{B2} $$

The probability that the last survivor dies within the $(t+1)$th year given survival up to the *t*th year for males at age s_0 and females at age s_1 can be computed as follows

$$ \begin{aligned} {}_{t|1}q_{\overline{s_0 s_1}} &= {}_tp_{\overline{s_0 s_1}} - {}_{t+1}p_{\overline{s_0 s_1}} \\ &= ({}_tp^m_{s_0} + {}_tp^f_{s_1} - {}_tp^m_{s_0} \cdot {}_tp^f_{s_1}) - ({}_{t+1}p^m_{s_0} + {}_{t+1}p^f_{s_1} - {}_{t+1}p^m_{s_0} \cdot {}_{t+1}p^f_{s_1}) \end{aligned} \tag{B3} $$

Finally, we can recall the PVECL expressed as

$$ PVECL = \sum_{t=0}^{\omega - s_0 - 1} \exp\left(-\int_0^s r_t \; dt\right) {}_{t|1}q_{\overline{s_0 s_1}} \times V_t, \tag{B4} $$

REFERENCES

Akaike, H. 1973. "Information Theory and an Extension of the Maximum Likelihood Principle." In B. N. Petrov and F. Csaki (eds.). *Proceedings of the 2*nd *International Symposium on Information Theory*, Akademia Kiado: Budapest, p. 267–281.

Alai, D. H., H. Chen, D. Cho, K. Hanewald, and M. Sherris. 2014. "Developing Equity Release Markets: Risk Analysis for Reverse Mortgages and Home Reversions." *North American Actuarial Journal* 18: 217–241.

Ambrose, B. W., and R. J. Buttimer. 2000. "Embedded Options in the Mortgage Contract." *Journal of Real Estate Finance and Economics* 21: 96–111.

American Housing Survey for the United States. 2005. Current Housing Reports, US Department of Housing and Urban Development and US Census Bureau. (August), p. 156.

Antolin, P. 2007. "Longevity Risk and Private Pensions." OECD Working Papers on Insurance and Private Pensions, No. 3.

[12] Equation (A3), (A4), and (A5) involve an infinite sum over the possible number of jumps, Nt. In practice, for our model estimated we found that the conditional Poisson distribution had zero probability in the tail for values of Nt ³10 and the likelihood and the parameter estimates converge.

Artzner, P., F. Delbaen, J. M. Eber, and D. Heath. 1999. "Coherent Measures of Risk." *Mathematical Finance* 9: 203–228.

Bardhan, A., R. Karapandža, and B. Urošević. 2006. "Valuing Mortgage Insurance Contracts in Emerging Market Economies." *Journal of Real Estate Finance and Economics* 32: 9–20.

Bayer, A., and L. Harper. 2000. "Fixing to Stay: A National Survey on Housing and Home Modification Issues." AARP Research Report, Washington, DC.

Bianchi, M. L. 2015. "Are the Log-Returns of Italian Open-End Mutual Funds Normally Distributed? A Risk Assessment Perspective." *Journal of Asset Management* 16: 437–449.

Black, F., and M. Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81: 637–57.

Boehm, T. P., and M. C. Ehrhardt. 1994. "Reverse Mortgage and Interest Rate Risk." *Journal of the American Real Estate and Urban Economics Association* 22: 387–408.

Cairns, A. J. G., D. Blake, and K. Dowd. 2006. "A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration." *Journal of Risk and Insurance* 73: 687–718.

Chan, W. H., and J. M. Maheu. 2002. "Conditional Jump Dynamics in Stock Market Returns." *Journal of Business and Economic Statistics* 20: 377–389.

Chang, C. C., W. Y. Huang, and S. D. Shyu. 2011. "Pricing Mortgage Insurance with Asymmetric Jump Risk and Default Risk: Evidence in the US Housing Market." *Journal of Real Estate Finance and Economics* 45: 846–868.

Chen, H., S. H. Cox, and S. S. Wang. 2010. "Is the Home Equity Conversion Mortgage in the United States Sustainable? Evidence from Pricing Mortgage Insurance Premiums and Non-Recourse Provisions Using the Conditional Esscher Transform." *Insurance: Mathematics and Economics* 46: 371–384.

Chen, M. C., C. C. Chang, S. K. Lin, and S. D. Shyu. 2010. "Estimation of Housing Price Jump Risks and Their Impact on the Valuation of Mortgage Insurance Contracts." *Journal of Risk and Insurance* 77: 399–422.

Chia, N., and A. Tsui. 2004. "Reverse Mortgages as Retirement Financing Instrument: An Option for "Asset-Rich and Cash-Poor." Working paper.

Chinloy, P., and I. F. Megbolugbe. 1994. "Reverse Mortgages: Contracting and Crossover Risk." *Real Estate Economics* 22: 367–386.

Cho, D., K. Hanewald, and M. Sherris. 2015. "Risk Analysis for Reverse Mortgages with Different Payout Designs." *Asia-Pacific Journal of Risk and Insurance* 9: 77–105.

Cox, J. C., J. E. Ingersoll, and S. A. Ross. 1985. "A Theory of the Term Structure of Interest Rates." *Econometrica* 53: 385–407.

Cummins, J. D., and P. Trainar. 2009. "Securitization, Insurance, and Reinsurance." *Journal of Risk and Insurance* 76: 463–492.

Cunningham, D. F., and P. F. Hendershott. 1986. "Pricing FHA Mortgage Default Insurance." NBER Working Paper No. 1382.

Daal, E., A. Naka, and J. S. Yu. 2007. "Volatility Clustering, Leverage Effects, and Jump Dynamics in the US and Emerging Asian Equity Markets." *Journal of Banking and Finance* 31: 2751–2769.

Davidoff, T. 2015. "Can "High Costs" Justify Weak Demand for the Home Equity Conversion Mortgage?" *The Review of Financial Studies* 28: 2365–2398.

Hanewald, K., T. Post, and M. Sherris. 2016. "Portfolio Choice in Retirement - What Is the Optimal Home Equity Release Product?" *The Journal of Risk and Insurance* 83: 421–446.

Haurin, D., C. Ma, S. Moulton, M. Schmeiser, J. Seligman, and W. Shi. 2016. "Spatial Variation in Reverse Mortgages Usage: House Price Dynamics and Consumer Selection." *Journal of Real Estate Finance and Economics* 53: 392–417.

Hosty, G. M., S. J. Groves, C. A. Murray, and M. Shah. 2008. "Pricing and Risk Capital in the Equity Release Market." *British Actuarial Journal* 14: 41–109.

Huang, H. C., C. W. Wang and Y. C. Miao. 2011. "Securitization of Crossover Risk in Reverse Mortgages." *Geneva Papers on Risk and Insurance–Issues and Practice* 36: 622–647.

Ji, M., M. Hardy, and J. Li. 2012. "A Semi-Markov Multiple State Model for ERM Terminations." *Annals of Actuarial Science* 6: 235–257.

Jorion, P. 2000. *Value at Risk: The New Benchmark of Managing Financial Risk*. McGraw-Hill, New York.

Kau, J., D. C. Keenan, W. J. Muller III, and J. F. Epperson. 1992. "A Generalized Valuation Model for Fixed-Rate Residential Mortgages." *Journal of Money, Credit and Banking* 24: 279–299.

Kau, J. B., and D. C. Keenan. 1996. "An Option-Theoretic Model of Catastrophes Applied to Mortgage Insurance." *Journal of Risk and Insurance* 63: 639–56.

Kau, J. B., D. C. Keenan, W. J. Muller, and J. F. Epperson. 1995. "The Valuation at Origination of Fixed Rate Mortgages with Default and Prepayment." *Journal of Real Estate Finance and Economics* 11: 5–36.

Keenan, T. A. 2010. *Home and Community Preferences of the 45+ Population*. AARP: Washington, DC.

Kim, J. H., and J. Li. 2017. "Risk-Neutral Valuation of the Non-Recourse Protection in ERMs: A Case Study for Korea." *Emerging Markets Review* 30: 133–154.

Kogure, A., J. Li, and S. Kamiya. 2014. "A Bayesian Multivariate Risk-Neutral Method for Pricing Reverse Mortgages." *North American Actuarial Journal* 18: 242–257.

Kou, S. G. 2002. "A Jump-Diffusion Model for Option Pricing." *Management Science* 48: 1086–1101.

Lee, Y. T., C. W. Wang, and H. C. Huang. 2012. "On the Valuation of Reverse Mortgages with Regular Tenure Payments." *Insurance: Mathematics and Economics* 51: 430–41.

Li, J. S. H., M. R. Hardy, and K. S. Tan. 2010. "On Pricing and Hedging the No- Negative-Equity Guarantee in Equity Release Mechanisms." *Journal of Risk and Insurance* 77: 449–522.

Liao, S. L., M. S. Tsai, and S. L. Chiang. 2008. Closed-Form Mortgage Valuation Using a Reduced-Form Model." *Real Estate Economics* 2: 313–47.

Ma, S., G. Kim, and K. Lew. 2007. "Estimating Reverse Mortgage Insurer's Risk Using Stochastic Models." Paper presented at the Asia-Pacific Risk and Insurance Association 2007 Annual Meeting.

Maheu, J. M., and T. H. McCurdy. 2004. "New Arrival, Jump Dynamics, and Volatility Components for Individual Stock Returns." *The Journal of Finance* 2: 755–793.

McCarthy, D., O. S. Mitchell, and J. Piggott. 2002. "Asset Rich and Cash Poor: Retirement Provision and Housing Policy in Singapore." *Journal of Pension Economics and Finance* 1: 197–222.

Merton, R. C. 1976. "Option Pricing When Underlying Stock Returns Are Discontinuous." *Journal of Financial Economics* 3: 125–44.

Miao, J., and N. Wang. 2007. "Investment, Consumption, and Hedging under Incomplete Markets." *Journal of Financial Economics* 86: 608–642.

Nakajima, M., and I. A. Telyukova. 2017. "Reverse Mortgage Loans: A Quantitative Analysis." *The Journal of Finance* 72: 911–950.

Rodda, D. T., K. Lam, and A. Youn. 2004. "Stochastic Modeling of Federal Housing Administration Home Equity Conversion Mortgages with Low-Cost Refinancing." *Real Estate Economics* 32: 589–617.

Rowlingson, K. 2006. "Living Poor to Die Rich or Spending the Kids' Inheritance? Attitudes to Assets and Inheritance in Later Life." *Journal of Social Policy* 35: 1795–192.

Pu, M., G. Z. Fan, and Y. Deng. 2014. "Breakeven Determination of Loan Limits for Reverse Mortgages under Information Asymmetry." *The Journal of Real Estate Finance and Economics* 48: 492–521.

Shan, H. 2011. "Reversing the Trend: The Recent Expansion of the Reverse Mortgage Market." *Real Estate Economics* 4: 743–68.

Shao, A. W., K. Hanewald, and M. Sherris. 2015. "Reverse Mortgage Pricing and Risk Analysis Allowing for Idiosyncratic House Price Risk and Longevity Risk." *Insurance: Mathematics and Economics* 63: 76–90.

Sherris, M., and D. Sun. 2010. "Risk-Based Capital and Pricing for Reverse Mortgages Revisited." University of New South Wales, Working paper.

Szymanoski, E. J. 199. "Risk and the Home Equity Conversion Mortgage." *Journal of the American Real Estate and Urban Economics Association* 22: 347–66.

Tunaru, R. 2017. *Real Estate Derivatives*. Oxford University Press, Oxford, Chapter 8.

US Census Bureau. 2012. US Statistical Abstract. http://www.census.gov/compendia/statab/cats/income_expenditures_poverty_wealth/household_income.html.

Vasicek, O. 1977. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Econometrics* 5: 177–188.

Wang, L., E. A. Valdez, and J. Piggott. 2008. "Securitization of Longevity Risk in Reverse Mortgages." *North American Actuarial Journal* 12: 345–371.

Wang, J. L., H. C. Huang, S. S. Yang, and J. T. Tsai. 2010. "An Optimal Product Mix for Hedging Longevity Risk in Life Insurance Companies: The Immunization Theory Approach." *Journal of Risk and Insurance* 77: 473–97.

West, K. D., and D. Cho. 1995. "The Predictive Ability of Several Models of Exchange Rate Volatility." *Journal of Econometrics* 69: 367–91.

Yang, S. S. 2011. "Securitization and Tranching: Longevity and House Price Risk for Reverse Mortgage Products." *The Geneva Papers on Risk and Insurance - Issues and Practice* 36: 648–674.

Zimmer, D. M. 2015. "Time-Varying Correlation in Housing Prices." *The Journal of Real Estate Finance and Economics* 51: 86–100.