國立政治大學 109 學年度 碩士暨碩士在職專班 招生考試試題

考試科目	微積分	系所別	應用數學系	考試時間	2 月 7 日 (五) 第三節

1. (20 points) Evaluate the limits.
   (a) (6 points) $\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)}$
   (b) (6 points) $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2}$
   (c) (8 points) $\lim_{x\to\infty} \frac{(x+2)^{1/x}-x^{1/x}}{(x+3)^{1/x}-x^{1/x}}$

2. (32 points) Evaluate the integrals.
   (a) (8 points) $\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x \cos x}dx$
   (b) (8 points) $\int_{-1}^{0} \frac{e^{1/x}}{x^3}dx$
   (c) (8 points) $\int_{1}^{\infty} \frac{\tan^{-1} x}{x^2}dx$
   (d) (8 points) $\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

3. (8 points) Find the volume of solid obtained by rotating the region bounded by the following curves about the $x-$axis:
$$x=-3y^2+12y-9,\quad x=0.$$

4. (8 points) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$; $\mathbf{F}(x,y,z)=(x+y^2)\mathbf{i}+xz\,\mathbf{j}+(y+z)\mathbf{k}$ and $\mathbf{r}(t)=t^2\mathbf{i}+t^3\mathbf{j}-2t\mathbf{k},\ 0\le t\le 2$.

5. (10 points) If $z=\frac{1}{x}[f(x-y)+g(c+y)]$, show that
$$\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right)=x^2\frac{\partial^2 z}{\partial y^2}.$$

6. (12 points) $F(x)$ is the absolute value function if $F(x)=|x|$.
   (a) (6 points) Prove that if f is a continuous function on an interval, then so is the absolute function $|f|$.
   (b) (6 points) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.

7. (10 points) Given any series $\sum a_n$, we define a series $\sum a_n^+$ whose terms are all the positive terms of $\sum a_n$ and a series $\sum a_n^-$ whose terms are all the negative terms of $\sum a_n$. To be specific, we let
$$a_n^+=\frac{a_n+|a_n|}{2},\qquad a_n^-=\frac{a_n-|a_n|}{2}.$$
If $\sum a_n$ is absolutely convergent, show that both of the series $\sum a_n^+$ and $\sum a_n^-$ are convergent.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

國立政治大學 109 學年度 碩士暨碩士在職專班 招生考試試題

考 試 科 目	線性代數	系 所 別	應用數學系	考 試 時 間	2 月 7 日(五) 第四節

Notations

R: the set of real numbers.

$P_n(F)$: the set of polynomials of degree at most n.

$M_{m\times n}(F)$: the set of $m\times n$ matrices with entries in F.

R(T): the range of T.

N(T): the kernel of T.

A_{ij}: the ij-th element of a matrix A.

Show all your works

1. (15%) Let V be a finite-dimensional vector space over a field F, and let $T: V \to V$ be linear. If
$$\operatorname{rank}(T) = \operatorname{rank}(T^2)$$
Prove that $R(T) \cap N(T) = \{0\}$.

2. (20%) Two linear operators T and U on a finite-dimensional vector space V are called *simultaneously diagonalizable* if there exists an ordered basis β for V such that $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Prove that if T and U are diagonalizable linear operators on a finite-dimensional vector space V such that $TU = UT$, then T and U are simultaneously diagonalizable.

3. (15%) In R^2, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where T is the reflection of R^2 about L.

4. (15%) If $A \in M_{n\times n}(F)$ and B is a matrix obtained from A by interchanging any two rows of A, then
$$\det(B) = -\det(A)$$

5. (20%) Let
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$
Find the general formula of $\sum_{i,j=1}^{2} A_{ij}^{n}$ for any positive integer n.

6. (15%) Let $T: P_2(R) \to P_2(R)$ be defined by
$$T(f(x)) = f(x) + f'(x) + f''(x)$$
Where $f'(x)$ and $f''(x)$ denote the first and second derivatives of $f(x)$. Is T invertible? (Examine your answer), if it is, please find T^{-1}.

備 註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。