國立政治大學 110 學年度碩士班暨碩士在職專班招生考試試題

考試科目	微積分	系所別	應用數學系	考試時間	2 月 5 日(星期五)第三節

Show all your work and carefully justify all your answers. Answers without explanation will not receive any score.

1. (14 points) Consider the following function:

$$f(x) = \begin{cases} 3x-1, & x<1; \\ 2x, & x \geq 1. \end{cases}$$

Use the ϵ-δ definition of the limit to show that $\lim_{x\to 1} f(x) = 2$.

2. Consider

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) (3 points) Find the partial derivatives at $(0,0)$ and the gradient vector $\nabla f(0,0)$.

(b) (6 points) Use the definition of the directional derivative to find $D_{\mathbf{u}}(0,0)$ for all unit vectors $\mathbf{u} = \langle a, b\rangle$.

(c) (6 points) Is f continuous at $(0,0)$? How about differentiability?

3. (a) (6 points) Let $f(x)$ be a continuous function. Show that if $f(x)$ has two local maxima, it also must have a local minima.

(b) (8 points) Show that part (a) becomes wrong for a function of two variables by finding all local maxima and minima of the function:

$$f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2.$$

4. (a) (7 points) Evaluate

$$\int \frac{x}{(x+1)(x+2)} \mathrm{d}x.$$

(b) (7 points) Evaluate

$$\int_0^1 \frac{\mathrm{d}x}{(2-x)\sqrt{1-x}}.$$

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

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5. (a) (7 points) Use a suitable change of variable to compute

$$\iint_R (x+y)^2 e^{x^2-y^2}\,\mathrm{d}A,$$

where R is the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$.

(b) (7 points) Evaluate

$$\iiint_{x^2+y^2+z^2\le 1} e^{(x^2+y^2+z^2)^{3/2}}\,\mathrm{d}V.$$

6. Consider the series:

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)}.$$

(a) (3 points) Find radius and interval of convergence.

(b) (6 points) Show that for $|x|<1$:

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)} = \frac{1-x}{x^2}\ln(1-x)+\frac{1}{x}. \qquad (1)$$

(c) (6 points) The function on the right-hand side of (1) has a removable discontinuity at $x=1$. Remove it and then show that (1) also holds for $x=1$.

7. (a) (6 points) Consider

$$f(x)=\int_{\tan x}^{x/4} \sin(t^2)\,\mathrm{d}t.$$

Compute $f'(\pi)$.

(b) (8 points) Suppose that $f(0)=0$ and

$$f'(\ln x)=\begin{cases} 1, & \text{if } 0<x\le 1;\\ x, & \text{if } 1<x<\infty. \end{cases}$$

Find $f(x)$.

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國立政治大學 110 學年度碩士班暨碩士在職專班招生考試試題

考 試 科 目	線性代數	系 所 別	應用數學系	考 試 時 間	2 月 5 日(五) 第四節

註：1. 計算題必須要有計算過程，否則沒有分數。

2. 證明題若用到其他定理結果，請簡述該定理敘述。

3. 假設 F 是一個 field，若對任意 $n \in N$ 且 $a \in F\backslash\{0\}$ 皆有 $na \neq 0$，稱 F 的 characteristic 是 0。若存在 $p \in N$ 是質數使得對任意的 $a \in F$ 皆有 $pa = 0$，則稱 F 的 characteristic 是 p。

1. (15%) Prove that there exists a linear transformation $T: R^2 \to R^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?
2. (15%) Find linear transformations $U, T: F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$.
3. (15%) Let $A, B \in M_{n\times n}(F)$ be such that $AB = -BA$. Prove that if n is odd and F is not a field of characteristic two, then A or B is not invertible.
4. (20%) Prove that every invertible matrix is a product of elementary matrices.
5. (20%) For

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Find an expression for A^n, where n is an arbitrary positive integer.

6. (15%) Let T be a linear operator on an inner product space V, and suppose that

$$\|T(x)\| = \|x\|$$

for all x. Prove that T is one-to-one.

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