國立政治大學 106 學年度 轉學生 招生考試試題

考試科目	微積分（一）：以微分為主	系所別	應用數學系二年級	考試時間	7 月 6 日(四)第二節

1. Find the derivatives of the following functions

$$\text{(a) (10\%) } f(x) = \tan^{-1}(\sin^{-1}\sqrt{x^2+1}).$$

$$\text{(b) (10\%) } f(x) = \int_{\log_{10}(2x+1)}^{3^{\sqrt{tan3x}}} \cos(t^4+1)\,dt.$$

2. Evaluate the limits.

$$\text{(a) (10\%) } \lim_{y\to\infty}\left(1+\frac{4\ln x}{x^2}\right)^{\frac{x^2}{3\ln x}}.$$

$$\text{(b) (10\%) } \lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right).$$

$$\text{(c) (10\%) } \lim_{x\to 0}\frac{\left(1-\cos(3x)\right)^2\csc(5t)\tan^2 t}{t^2\sin^3(2t)}.$$

$$\text{(d) (10\%) } \lim_{(x,y)\to(0,0)}\frac{xy\sqrt{\sin y}}{x^2+2y^2}.$$

3. (10%) Let $z = f(u,v)$ where $u = xy$ and $v = y/x$, where f has continuous second partial derivatives, show that

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = -4uv\frac{\partial^2 z}{\partial u\partial v} + 2v\frac{\partial z}{\partial v}.$$

4. (10%) Let $a_1 = 1$, and $a_{n+1} = a_n + n^{-\frac{7}{2}}\sum_{m-1}^{n} m^2$ for $n \geq 1$. Determine whether the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent or divergent.

5. (10%) Find the Maclaurin series for $f(x) = \ln(4-x)$ and find its radius of convergence.

6. (10%) Find the extreme values of $f(x,y) = e^{-xy}$ on the region $x^2+4y^2 \leq 1$.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

考試科目	微積分（二）：以積分為主	系所別	應用數學系二年級	考試時間	7月6日(四) 第四節

Problem 1 Evaluate the integrals.

(1) (10%) $\displaystyle\int_1^\infty \frac{\sqrt{\ln x}}{x^2}\,dx.$

(2) (10%) $\displaystyle\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx.$

(3) (10%) $\displaystyle\int (\tan^{-1} x)^2\,dx.$

(4) (10%) $\displaystyle\int_C F\cdot d\mathbf{r},$ where $F(x,y,z) = yze^{xz}\mathbf{i} + e^{xz}\mathbf{j} + xye^{xz}\mathbf{k}$,
$C : \mathbf{r}(t) = (t^2+1)\mathbf{i} + (t^2-1)\mathbf{j} + (t^2-2t)\mathbf{k}$, $0 \le t \le 2$.

(5) (10%) $\displaystyle\iiint_E x\,dV,$ where E is the solid tetrahedron bounded by the four planes $x=0$, $x=2y$, $z=-2$ and $x+2y+z=2$.

Problem 2 Determine whether the series is convergent or divergent

(a) (10%) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln n(\ln\ln n)^2}.$

(b) (10%) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n)!}.$

Problem 3 (10%) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n(2x-3)^n}{\sqrt{n+3}}.$$

Problem 4 (10%) Find $\alpha \in \mathbb{R}$ such that $\int_0^\infty \frac{1-\cos x}{x^\alpha}\,dx$ is convergence.

Problem 5 (10%) Evaluate the line integral

$$\int_C F\cdot d\vec{r}, \quad \text{where } F(x,y) = \frac{-y\vec{i}+x\vec{j}}{(x^2+y^2)}$$

and C is any positive oriented simple closed curve and enclosed the origin.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

考 試 科 目	微積分	系 所 別	應用數學系三年級	考 試 時 間	7 月 6 日(四) 第二節

Problem 1 Evaluate the limits

$$(1)\,(10\%)\quad \lim_{x\to 0^+}\left(\frac{1}{x\ln 3}-\frac{1}{3^x-1}\right).$$

$$(2)\,(10\%)\quad \lim_{t\to 0}\frac{\left(1-\cos(3t)\right)^2\tan(2t)\csc t}{t^2\sin^2(5t)}.$$

Problem 2 Evaluate the integrals.

$$(1)\,(10\%)\quad \int_0^\infty \frac{3x^3-x^2+6x-4}{(x^2+4)(x^2+2)}\,dx.$$

$$(2)\,(10\%)\quad \int_0^\infty \frac{\sin 3x}{x}\,dx.$$

$$(3)\,(10\%)\quad \iint_D \frac{y}{x^2+1}\,dA,\quad \text{where } D=\{(x,y):0\le x\le 4,\ 0\le y\le \sqrt{x}\}.$$

$$(4)\,(10\%)\quad \iint_R \frac{x-2y}{3x-y}\,dA\quad \text{where } R \text{ is the parallelogram enclosed by}$$

the lines $x-2y=0$, $2-2y=4$, $3x-y=1$ and $3x-y=8$.

Problem 3 (10%) Evaluate the derivative:

$$\frac{d}{dx}\left(\int_{\sin^2 2x}^{\sec^2 2x} e^{-t^2}\,dt\right).$$

Problem 4 (10%) If $\lim_{n\to\infty}a_n=a$ for some $a\in\mathbb{R}$, show that $\frac{1}{n}\sum_{k=1}^n a_k\to a$ as $n\to\infty$.

Problem 5 (10%) Find $s\in\mathbb{R}$ such that $\sum_{n=0}^\infty\left(1-f(n)\right)$ is convergence where $f(n)=\prod_{j=n+1}^\infty(1-j^{-s})^j$.

Problem 6 (10%) Let $F(x,y)=\sin y\,\mathbf{i}+(x\cos y+\cos z)\,\mathbf{j}-y\sin z\,\mathbf{k}$. Evaluate $\int_C F\cdot d\mathbf{r}$, wher C is the curve given by $\mathbf{r}(t)=\sin t\,\mathbf{i}+t\,\mathbf{j}+2t\,\mathbf{k}$, $0\le t\le \pi/2$.

備 註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

考 試 科 目	線性代數	系 所 別	應用數學系三年級	考 試 時 間	7 月 6 日(四) 第 四 節

Please show all your work.

1. **(15%)** Consider the 3×3 Vandermonde matrix

$$V=\begin{bmatrix}1 & x_1 & x_1^2\\ 1 & x_2 & x_2^2\\ 1 & x_3 & x_3^2\end{bmatrix}.$$

What condition must the scalars x_1, x_2 and x_3 satisfy in order for V to be nonsingular? Justify your answer.

2. **(15%)** Let A be a $n\times n$ matrix with characteristic polynomial

$$p(\lambda)=(-1)^n\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0.$$

Show that if $a_0\neq 0$, then A is nonsingular and $A^{-1}=q(A)$ for some polynomial q of degree less than n.

3. **(15%)** Let $W=\{(2,-1,-2,4),(-2,1,-5,5),(-1,3,7,11)\}$ be a subset of the inner product space R^4 over the field R. Find an orthonormal basis β for the subspace spanned by the vectors in W.

4. **(15%)** Let T be a linear operator on an inner product space V, and let W be an T-invariant subspace of V. Show that $W^\perp$ is T^*-invariant where $W^\perp$ denotes the orthogonal complement of W and T^* denotes the adjoint of T.

5. **(20%)** Let V be an inner product space, and let $y,z\in V$. Define $T:V\to V$ by $T(x)=\langle x,y\rangle z$ for all $x\in V$. First prove that T is linear. Then show that the adjoint T^* exists, and find an explicit expression for it.

6. **(20%)** Let T be an upper triangular matrix with distinct diagonal entries (i.e., $T_{ii}\neq T_{jj}$ whenever $i\neq j$). Show that there is an upper triangular matrix R that diagonalizes T.

備 註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。