Received March 13, 2020, accepted March 28, 2020, date of publication April 21, 2020, date of current version May 28, 2020.
Digital Object Identifier 10.1109/ACCESS.2020.2989400

Loss-Based Control Charts for Monitoring Non-Normal Process Data

SU-FEN YANG [1] AND LIJUAN SHEN [2]
[1]Department of Statistics, National Chengchi University, Taipei 116, Taiwan
[2]Singapore-ETH Centre, Singapore 138602
Corresponding author: Su-Fen Yang (yang@mail2.nccu.tw)

This work was supported in part by the Ministry of Science and Technology, Taiwan, under Grant 106-2118-M-004-001-MY2, and in part by the National Natural Science Foundation of China under Grant 71801168.

ABSTRACT Quality and loss of products are crucial factors in competitive companies, and firms widely adopt a loss function to measure the loss caused by a deviation in the quality variable from the target value. From Taguchi's view point, it is important to monitor any deviation from the process target value. While most existing studies assume the quality variable follows a normal distribution, the distribution can in fact be skewed or deviate from normal in practice. This paper thus proposes loss-based control charts for monitoring the quality loss location or equivalently the deviation of the quality variable from the target value under a skew-normal distribution. We consider the exponentially weighted moving average (EWMA) average loss control chart, which illustrates the best performance in detecting an out-of-control loss location for a process with a left-skewed distribution. Numerical analysis demonstrates that the proposed EWMA average loss chart always performs better than the existing median loss chart for both left-skewed and right-skewed distributions. A numerical example illustrates the application of the proposed EWMA average loss control chart.

INDEX TERMS Control chart, loss function, process control, run length, skew-normal distribution.

I. INTRODUCTION

Control charts are commonly-used tools in process signal detection to improve the quality of manufacturing and service processes, yet in the past few years, increasing attention has been paid to the application of control charts to service industries. See, for example, Tsung *et al.* [1], Ning *et al.* [2], and Yang and Wu [3]. While a normal distribution has been widely employed in practice to fit data, some data of real examples in psychology, reliability, telecommunications, environment, climatology, sciences services, education, finance, and health insurance often exhibit moderate to strong asymmetry as well as light or heavy tails (for example, see Bono Cabré [4]). In most situations, quality data from the service sector do not follow a normal distribution. Clearly, fitting a normal distribution to such data is not appropriate, and the commonly-used Shewhart variables control charts that depend on a normality assumption are not suitable. A list of statistical process control research for dealing with non-normal data includes Amin *et al.* [5], Chakraborti *et al.* [6], Altukife [7], Bakir [8], Li *et al.* [9], Zou and Tsung [10], Graham *et al.* [11], Yang and Wu [3], Abbas *et al.* [12].

The associate editor coordinating the review of this manuscript and approving it for publication was Roberto Pietrantuono.

The quality loss function is a popular method for measuring the loss of quality caused by variations in a product or service. Sullivan [13] emphasized the importance of monitoring deviations from the target value. Taguchi [14] proposed the quadratic loss function of quality variable from the target value. Changes in the process mean and/or dispersion lead to a variation of the loss. A few loss control charts have been proposed to monitor process loss. For example, Wu and Tian [15] and Wu *et al.* [16] suggested the weighted loss function chart and adaptive loss-function- based control charts, but they assumed that the in-control process mean equals the target, and that the quality variable follows a normal distribution. Yang [17] and Lu [18] examined loss-based control charts, assuming that the in-control process mean may not equal the target under the normality and non-normalty distributions when simultaneously monitoring the process mean and dispersion.

A major drawback of loss-based control charts is that most of them assume the quality variable follows a normal distribution. Hence, this paper focuses on discussing loss-based control charts under a skew-normal distribution. Yang *et al.* [19] proposed using the median loss instead of the average loss to simultaneously monitor changes in the process location and/or dispersion when the distribution of a process is not symmetric, but rather left-skewed or right-skewed. Their median loss (ML) chart illustrated the best out-of-control detection performance for the left-skewed distributed process. Even under a normal distribution, they showed that the resulting out-of-control detection performance of the ML chart performs better than the average loss (AL) chart in Yang [17] except for very small shifts in process mean. Yang and Lu [20] proposed the median loss control chart with an unbiased average run length in order to monitor the process loss center. The exponentially weighted moving average (EWMA) is an effective alternative to the Shewhart-type control chart, and may be used when small shifts occur in the process parameter (for example, see Montgomery [21]). However, the properties of the average loss control chart have not been discussed for the skew-normal distributed process. We are interested in knowing whether the EWMA-ALSN control charts have better out-of-control detection performance than that of the ML chart for a skew-normal distributed process. Our paper thus considers fixing the sample size and sampling time interval.

The rest of the paper runs as follows. Section II derives the sampling distribution of the average loss for a process with a skew-normal distribution. Section III designs the EWMA-ALSN charts and lists their control limits for various sample sizes and shape parameter of a skew-normal distribution under a predetermined in-control average run length (ARL_0). Hence, their out-of-control detection performance for small to moderate shifts in the difference of process location and target and/or dispersion can be evaluated using out-of–control average run length (ARL_1) under the specified process shifts. Section III also compares out-of-control detection performance among the EWMA-ALSN and existing ML charts in Yang *et al.* [19]. Section IV illustrates the application of the proposed charts using the Roberts IQ score. Section V summarizes the findings and provides a recommendation.

II. THE DERIVATION OF AVERAGE LOSS DISTRIBUTION

The skew normal (SN) distribution is an extension of the normal distribution, allowing for the presence of skewness. Helguero [22] was the pioneer of the skew normal (SN) distribution and formulated the genesis of non-normal distributions via a selection mechanism, leading to a departure from normality. Many researchers, for example, Azzalini [23], Azzalini and Valle [24], Azzalini [25], Azzalini [26], and Azzalini and Capitanio [27] contributed to the development of the theory of SN distributions.

Assume the random variable X follows an in-control skew-normal distribution with location parameter $\xi_0 \in (-\infty, \infty)$, scale parameter $a_0 \in (0, \infty)$, and shape parameter $b \in (-\infty, \infty)$. In other words, $X \sim SN(\xi_0, a_0, b)$. From Azzalini [26], the probability density function (pdf) of X is

$$f_X(x) = \frac{2}{a_0}\phi\left(\frac{x-\xi_0}{a_0}\right)\Phi\left(b\frac{x-\xi_0}{a_0}\right), \quad x \in (-\infty, \infty), \tag{1}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are respectively the pdf and cumulated distribution function (cdf) of the standard normal distribution. If $b = 0$, then the skew-normal distribution reduces to the traditional normal distribution with mean ξ_0 and standard deviation a_0. The distribution is right-skewed for $b > 0$ and left-skewed for $b < 0$.

The cdf of the skew-normal random variable X is

$$F_X(x) = \Phi\left(\frac{x-\xi_0}{a_0}\right) - \frac{1}{\pi}\int_0^b \frac{\exp\left\{-\frac{1}{2}\left(\frac{x-\xi_0}{a_0}\right)^2(1+y^2)\right\}}{1+y^2}dy, \quad x \in (-\infty, \infty). \tag{2}$$

The expectation and variance of X are

$$\mu_o = \xi_0 + a_0\frac{b}{\sqrt{1+b^2}}\sqrt{\frac{2}{\pi}} \quad \text{and}$$

$$\sigma_0^2 = a_0^2\left[1 - \frac{2b^2}{\pi(1+b^2)}\right],$$

respectively.

Hence, if we know μ_0, σ_0^2, and shape b, then we obtain:

$$\xi_0 = \mu_0 - \frac{\sqrt{2}b\sigma_0}{\sqrt{(1-b^2)\pi - 2b^2}},$$

$$a_0 = \frac{\sigma_0}{\sqrt{1-2b^2/((1+b^2)\pi)}}. \tag{3}$$

We also consider that the in-control process mean may not be the target, and let δ_3 denote the dispersion parameter that satisfies $\mu_0 - T = \delta_3\sigma_0$, where $\delta_3 \in R$.

Suppose that X^* is the quality characteristic from the out-of-control process, and $X^* \sim SN(\xi^*, a^*, b)$ with mean $\mu_1 = \mu_0 + \delta_1\sigma_0$. Here, δ_1 is the mean shift scale, $\delta_1 \neq 0$, and standard deviation $\sigma_1 = \delta_2\sigma_0\delta_2$, where δ_2 is the standard deviation shift scale, for $\delta_2 > 1$.

We thus have:

$$\xi^* = \mu_0 + \delta_1\sigma_0 - \frac{\sqrt{2}b\delta_2\sigma_0}{\sqrt{(1+b^2)\pi - 2b^2}},$$

$$a^* = \delta_2\sigma_0/\sqrt{1-\frac{2b^2}{\pi(1+b^2)}}. \tag{4}$$

We define the Taguchi loss function as $L = k(X-T)^2$. Without loss of generality, we set $k = 1$. Let $X_i, i = 1, 2, \ldots, n$, be a random sample from the incontrol distribution of $SN(\xi, a, b)$. We further define the sample erage loss (AL) as

$$AL = \frac{1}{n}\sum^{n}(X_i - T)^2 = \frac{(n-1)}{n}S_x^2 + (\bar{X} - T)^2. \tag{5}$$

Edgeworth [28] derived the Edgeworth expansion, which relates the cdf of a random variable having expectation zero and variance 1 to the cumulative density function (cdf) of the

TABLE 1. The hermite polynomial.

Order	Hermite polynomial
$\mathrm{He}_0(z)$	1
$\mathrm{He}_1(z)$	z
$\mathrm{He}_2(z)$	z^2-1
$\mathrm{He}_3(z)$	$z\left(z^2-3\right)$
$\mathrm{He}_4(z)$	z^4-6z^2+3
$\mathrm{He}_5(z)$	$z\times\left(z^4-10z^2+15\right)$

standard normal distribution using the Chebyshev-Hermite polynomials.

We obtain the r^{th} moments of $L_i=(X_i-T)^2$ by

$$M_r=\frac{2}{a_0}\int_0^{\infty}(x-T)^{2r}\phi\left(\frac{x-\xi_0}{a_0}\right)\Phi\left(b\cdot\frac{x-\xi_0}{a_0}\right)dx,\quad r=1,2,\ldots,\tag{6}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the probability density function and cdf of the standard normal distribution, respectively. The expectation and the standard deviation of L (μ_L and σ_L) can be obtained by the moments of L, that is $\mu_L=M_1$ and $\sigma_L=\sqrt{M_2-M_1^2}$.

Define $Z_n=\sqrt{n}\left(AL-\mu_L\right)/\sigma_L$. Thus we approximate the cdf of Z_n by the Edgeworth expansion, which is expressed as:

$$F_{z_n}(z)\approx\Phi(z)-\frac{1}{\sqrt{n}}\left(\frac{1}{6}\lambda_3\Phi^{(3)}(z)\right)+\frac{1}{n}\left(\frac{1}{24}\lambda_4\Phi^{(4)}(z)+\frac{1}{72}\lambda_3^2\Phi^{(6)}(z)\right),\tag{7}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the probability density function and cdf of the standard normal distribution, respectively.

Note that $\Phi^{(r)}(z)=(-1)^{r-1}\mathrm{He}_{r-1}(z)\phi(z)$, where $\mathrm{He}_{r-1}(z)$ is the Chebyshev-Hermite polynomial. One can obtain $\mathrm{He}_{r-1}(\cdot)$by

$$\mathrm{He}_{r-1}(z)=-\frac{1}{\phi(z)}\left\{\frac{d}{dz}\left[\mathrm{He}_{r-2}(z)\phi(z)\right]\right\}.\tag{8}$$

Table 1 lists the Hermite polynomials obtained from (8).

To obtain λ_r, one can use the relation $\lambda_r=\kappa_r/\sigma_L^r$, where $\boldsymbol{\kappa}_r$ is the r^{th} cumulant of L. From Hall [29], the cumulants of L can be obtained from the moments of L as shown in Table 2.

The first step to construct the ALSN chart is to find the distribution of AL when X follows a skew-normal distribution. However, the exact distribution of AL is not available. Our study uses Edgeworth expansion (for example, see Hall [29]) to approximate the AL distribution.

The approximate pdf of Z_n can be obtained by differentiating (7) as:

$$f_{Z_n}(z)\approx\phi(z)-\frac{1}{\sqrt{n}}\left(\frac{1}{6}\lambda_3\Phi^{(4)}(z)\right)+\frac{1}{n}\left(\frac{1}{24}\lambda_4\Phi^{(5)}(z)+\frac{1}{72}\lambda_3^2\Phi^{(7)}(z)\right).\tag{9}$$

TABLE 2. The cumulants of $L=(X-T)^2$.

Order	Hermite polynomial
κ_1	M_1
κ_2	$M_2-\kappa_1^2$
κ_3	$M_3-3\kappa_2\kappa_1-\kappa_1^3$
κ_4	$M_4-4\kappa_3\kappa_1-3\kappa_2^2-6\kappa_2\kappa_1^2-\kappa_1^4$

The cdf and pdf of AL can therefore be obtained by the following.

$$\begin{aligned}F_{AL}(t)&=P(AL\le t)=P\left(Z_n\le\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\\&=F_{Z_n}\left(\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\\&\approx\Phi\left(\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\\&\quad-\frac{1}{\sqrt{n}}\left(\frac{1}{6}\lambda_3\Phi^{(3)}\left(\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\right)\\&\quad+\frac{1}{n}\left(\frac{1}{24}\lambda_4\Phi^{(4)}\left(\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\right.\\&\quad\left.+\frac{1}{72}\lambda_3^2\Phi^{(6)}\left(\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\right),\end{aligned}\tag{10}$$

as well as:

$$\begin{aligned}f_{AL}(t)&=\frac{d}{dt}F_{AL}(t)=f_{Z_n}\left(\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\frac{\sqrt{n}}{\sigma_L}\\&\approx\frac{\sqrt{n}}{\sigma_L}\phi\left(\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\\&\quad-\frac{1}{\sigma_L}\left(\frac{1}{6}\lambda_3\Phi^{(4)}\left(\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\right)\\&\quad+\frac{1}{\sqrt{n}\sigma_L}\left(\frac{1}{24}\lambda_4\Phi^{(5)}\left(\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\right.\\&\quad\left.+uy7\frac{1}{72}\lambda_3^2\Phi^{(7)}\left(\frac{\sqrt{n}\left(t-\mu_L\right)}{\sigma_L}\right)\right).\end{aligned}\tag{11}$$

Let μ_{AL} and σ_{AL} be the expectation and standard deviation of AL, respectively. Thus, we arrive at:

$$\mu_{AL}=E(AL)=\int_0^{\infty}tf_{AL}(t)dt,\tag{12}$$

and

$$\sigma_{AL}=\sqrt{E(AL-\mu_{AL})^2}.\tag{13}$$

The accuracy of this approximation is examined by Pearson's χ^2 goodness-of-fit test. We consider sample sizes $n=5,11$, $\delta_3=1$, $\mu_0=0$, $\sigma_0=1$, and $b=-500,0,500$ and simulate $m(=1000,2000)$ samples from $SN(\xi_0,a_0,b)$ with each n, so as to calculate the m random samples of AL and then fit them with the approximated cdf given in (10). Table 3 lists the p-values of the χ^2 test. We see the test reveals that the approximated cdf has no significant difference from the cdf using Monte Carlo simulation. Fig. 1 illustrates that their colf curves are very close for $n=11$, $m=2000$, and $b=-500$. Moreover, the accuracy improves for larger sample sizes.

TABLE 3. P-value of the Pearson χ^2 goodness-of-fit Test for Edgeworth expansion.

b	n	Number of samples (m) 2000	1000
500	5	0.1712	0.2493
-500	11	0.6475	0.3345
0	5	0.1816	0.3838
0	11	0.1866	0.1519
500	5	0.1554	0.2622
500	11	0.7499	0.9357

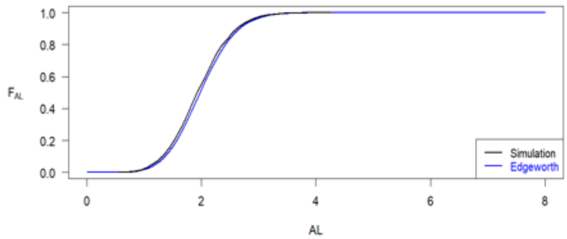

FIGURE 1. The cdf curves of *AL* by Edgeworth expansion and Monte Carlo simulation.

As noted before, X^* is the quality characteristic from the out-of-control process, while $X^* \sim SN(\xi^*, a^*, b)$ with mean $\mu_1 = \mu_0 + \delta_1\sigma_0$ and standard deviation $\sigma_1 = \delta_2\sigma_0$. That is,

$$\xi^* = \mu_1 - \frac{\sqrt{2}b\sigma_1}{\sqrt{(1+b^2)\,\pi - 2b^2}} \text{ and } a^* = \frac{\sigma_1}{\sqrt{1 - \frac{2b^2}{\pi(1+b^2)}}}.$$

We denote the out-of-control average loss as $AL^* = \sum_{i=1}^{n}\left(X_i^* - T\right)^2/n$. Following the proving procedure of the cdf of AL (eq. (10)), we can derive the cdf of AL^*.

III. DERIVATION OF THE EXPONENTIAL WEIGHTED MOVING AVERAGE ALSN CONTROL CHART

To better detect small and moderate shifts in the process average loss, we propose an EWMA average loss (EWMA-ALSN) control chart with monitoring statistic $\text{EWMA}_{AL,t}$ at time t, as follows.

Define $\text{EWMA}_{AL,t}$ as the monitoring statistic of the EWMA-ALSN chart at time t, which is in the form of

$$\text{EWMA}_{AL,t} = \lambda \cdot AL_t + (1-\lambda)\text{EWMA}_{AL,t-1}, \quad (14)$$

where $\lambda \in (0, 1)$ is the smoothing parameter. When $\lambda = 1$, the EWMA-ALSN chart reduces to the average loss (ALSN) chart.

The in-control mean of $\text{EWMA}_{AL,t}$ is μ_{AL}, and its standard deviation is σ_{AL} as time, t, approaches infinity.

A. THE CONTROL LIMITS OF THE EWMA-ALSN CUNA-ALLSN CHART

Based on the mean and variance of the monitoring statistic, $\text{EWMA}_{AL,t}$, we can construct the EWMA-ALSN chart. The design parameters of the proposed chart are sample size n sampling time interval h, and coefficients (k_1, k_2) of the upper and lower control limits. We consider the n to be fixed, and h is one time unit. Given the derived μ_{AL} in (12) and σ_{AL} in (13), and fixing the predetermined in-control average run length ($\text{ARL}_0 = \ell$), we may determine the upper and lower control limits (UCL, LCL) of the proposed EWMA-ALSN control chart as follows.

TABLE 4. The control limits of the EWMA-ALSN chart with $\text{ARL}_0 = 370.4$.

b	λ	k1	k2	UCL	LCL
-500	1	2.678	3.577	4.718	0.000
	0.2	2.701	3.086	2.782	1.316
	0.05	2.420	2.566	2.312	1.706
-2	1	2.799	3.849	5.647	0.167
	0.2	2.448	3.239	3.023	1.227
	0.05	2.322	2.663	2.404	1.648
0	1	3.776	4.045	6.431	0.114
	0.2	2.293	3.373	3.232	1.163
	0.05	2.328	2.638	2.463	1.592
2	1	4.087	4.219	7.377	0.040
	0.2	2.068	3.535	3.502	1.122
	0.05	2.245	2.721	2.555	1.542
500	1	4.189	4.293	8.324	0.000
	0.2	1.942	3.613	3.774	1.046
	0.05	2.234	2.715	2.640	1.473

$$\text{UCL} = \mu_{AL} + k_2\sigma_{AL}\sqrt{\frac{\lambda}{2-\lambda}},$$
$$\text{LCL} = \mu_{AL} - k_1\sigma_{AL}\sqrt{\frac{\lambda}{2-\lambda}}, \quad (15)$$

where k_1 and k_2 are determined by the Monte Carlo simulation such that $\text{ARL}_0 = \ell$. If the monitoring statistic, $\text{EWMA}_{AL,t}$, falls outside of UCL or LCL then the process is deemed to be out-of-control.

The Monte Carlo simulation method is applied to determine the chart parameters k_1 and k_2 so as to meet the predetermined in-control average run length, namely $\text{ARL}_0 = \ell$ with λ.

Table 4 lists the coefficients (k_1, k_2) of LCL and UCL as well as the corresponding LCL and UCL of the EWMA-ALSN chart by setting $b = -500, -2, 0, 2, 500$, $\mu_0 = 0$ and $\sigma_0 = 1$, $\delta_3 = 1$, $\lambda = 0.05, 0.2, 0.4, 1.0$, $n = 5$ and $\text{ARL}_0 = 370.4$. We find that the control limits are more symmetric when λ is small, for example, $\lambda = 0.05$. Furthermore, the width of the chart increases when λ or b increases.

B. PERFORMANCE COMPARISON AMONG THE EWMA-AL, ALSN AND ML CHARTS

Using the resulting control limits in Table 4, we adopt ARL_1 to measure the out-of-control detection performance of the proposed EWMA-ALSN chart. In order to measure the spread of the run length distribution, we consider the standard deviation of run length (SDRL). Using Monte Carlo simulation, we calculate ARL_{1s} and SDRL_s. Here, we assume the process shifts (δ_1 and δ_2) are known or specified. When δ_1 and δ_2 are unknown or unable to specified, the expected ARL (EARL) can be employed as a performance metric (for example, see Teoh *et al.* [30]).

Montgomery [21] recommended to adopt $\lambda = 0.05$, $\lambda = 0.1$ or $\lambda = 0.2$ to detect smaller shifts. Here, we adopt $\lambda = 0.05$ and 0.2 for detecting small and moderate shifts and comparing detection performance with existing loss control charts for a process with a skew-normal distribution.

TABLE 5. The ARL_1 of the EWMA-ALSN, ALSN and ML charts with $ARL_0 = 370.4$.

δ_1	δ_2	Type	$b=-500$	$b=-2$	$b=0$	$b=2$	$b=500$
		(1)	2.77	3.58	3.83	4.46	5.18
			0.85	1.08	1.32	4.24	1.95
		(2)	1.99	2.58	2.87	3.87	4.35
1	1		0.82	1.08	1.38	1.84	1.46
		(3)	1.75	3.00	4.72	7.76	7.87
			1.15	2.47	3.16	7.27	7.72
		(4)	2.06	4.76	8.15	14.13	22.21
		(1)	1.23	1.46	1.66	1.87	2.08
			0.42	0.52	0.32	0.53	0.60
		(2)	1.04	1.13	1.17	1.23	1.48
2	1		0.20	0.34	0.45	0.54	0.58
		(3)	1.01	1.04	1.10	1.59	1.50
			0.11	0.20	0.32	0.54	0.87
		(4)	1.07	1.16	1.31	1.59	2.31
		(1)	1.00	1.00	1.03	1.10	1.32
			0.06	0.13	0.21	0.53	0.47
		(2)	1.00	1.00	0.05	1.06	1.00
3	1		0.02	0.03	1.00	0.24	0.18
		(3)	1.00	1.00	1.00	1.00	1.00
			0.01	0.01	0.01	0.02	0.01
		(4)	1.00	1.00	1.00	1.01	1.01
		(1)	6.64	8.37	9.93	12.02	15.15
			3.12	5.08	5.69	7.76	9.95
		(2)	5.25	7.31	8.83	12.42	15.44
0	1.5		3.24	5.08	7.04	9.98	12.71
		(3)	7.51	12.23	14.46	16.50	17.84
			7.01	10.23	13.62	16.03	17.30
		(4)	7.49	24.35	37.32	55.99	69.24
		(1)	2.98	3.58	4.13	5.23	5.58
			1.28	1.77	2.14	4.34	3.64
		(2)	2.22	2.84	3.22	4.40	5.33
0	2		1.15	1.65	2.12	2.92	3.87
		(3)	2.02	2.87	3.72	4.72	7.74
			1.44	2.32	3.19	4.34	4.26
		(4)	3.01	6.73	10.22	16.91	26.02
		(1)	1.50	1.72	1.93	2.24	2.76
			0.64	0.83	0.95	1.70	1.48
		(2)	1.24	1.45	1.60	1.93	2.33
0	3		0.48	0.68	0.82	1.06	1.40
		(3)	1.14	1.32	1.49	1.82	2.25
			0.40	0.66	0.82	1.22	1.67
		(4)	1.60	2.39	3.01	4.17	9.48
		(1)	1.65	1.98	2.26	2.59	2.97
			0.67	0.88	1.05	1.31	1.64
		(2)	1.28	1.58	1.74	2.16	1.52
1	2		0.52	0.74	0.94	1.22	1.50
		(3)	1.15	1.42	1.79	2.16	2.61
			0.44	0.77	1.12	1.59	2.07
		(4)	1.58	2.39	3.36	4.88	6.87
		(1)	1.15	1.29	1.44	1.61	1.81
			0.36	0.51	1.19	0.70	0.83
		(2)	1.05	1.13	1.15	1.26	1.58
2	2		0.22	0.35	0.46	0.59	0.73
		(3)	1.02	1.07	1.13	1.26	1.43
			0.15	0.26	0.39	0.57	0.79
		(4)	1.17	1.34	1.55	1.87	2.52
		(1)	1.02	1.05	1.10	1.21	1.28
			0.14	0.20	0.73	0.42	0.50
		(2)	1.01	1.01	1.03	1.08	1.15
3	2		0.07	0.12	0.19	0.31	0.36
		(3)	1.00	1.01	1.02	1.03	1.06
			0.05	0.08	0.12	1.71	0.42
		(4)	1.04	1.07	1.11	1.16	1.33
		(1)	1.21	1.40	1.57	1.87	2.26
			0.43	0.61	0.51	0.82	1.18
		(2)	1.09	1.22	1.34	1.45	1.88
1	3		0.29	0.46	0.60	0.53	0.36
		(3)	1.01	1.14	1.28	1.45	1.72
			0.23	0.40	0.57	0.48	1.05
		(4)	1.31	1.71	2.16	2.98	4.92

TABLE 5. *(Continued.)* The ARL_1 of the EWMA-ALSN, ALSN and ML charts with $ARL_0 = 370.4$.

δ_1	δ_2	Type	$b=-500$	$b=-2$	$b=0$	$b=2$	$b=500$
		(1)	1.07	1.17	1.27	1.46	1.69
			0.26	0.41	0.51	0.43	0.81
		(2)	1.03	1.07	1.15	1.21	1.43
2	3		0.17	0.28	0.40	0.31	0.69
		(3)	1.01	1.04	1.11	1.19	1.34
			0.12	0.21	0.32	0.24	0.64
		(4)	1.15	1.13	1.21	1.33	1.61
		(1)	1.02	1.06	1.10	1.19	1.32
			0.14	0.24	0.32	0.43	0.52
		(2)	1.01	1.02	1.05	1.07	1.19
3	3		0.09	0.15	0.22	0.31	0.42
		(3)	1.01	1.01	1.03	1.06	1.12
			0.06	0.10	0.16	0.24	0.36
		(4)	1.07	1.13	1.21	1.33	1.61

In Table 5, the "Type" columns (1), (2) and (3) respectively illustrate ARL_1 (first row) and SDRL (second row) of the EWMA-ALSN chart with $\lambda = 0.05$, EWMA-ALSN chart with $\lambda = 0.2$, and ALSN chart (or EWMA-ALSN chart with $\lambda = 1.0$); (4) shows ARL_1 of the existing ML chart in Yang *et al.* [19] for $\delta_1 = 1.0, 2.0, 3.0$, $\delta_2 = 1, 1.5, 2.0, 3.0$, $b = -500, -2, 0, 2, 500$, $ARL_0 = 370.4$, $n = 5$, $\delta_3 = 1$, $\mu_0 = 0$, and $\sigma_0 = 1$. In "Type" columns (1), (2) and (3), we can see that, under $b = -500, -2, 0, 2, 500$, respectively, ARL_1 decreases when δ_1 and/or δ_2 increases. These results look reasonable. When b increases, ARL_1 increases. On the other hand, the EWMA-ALSN and ALSN charts perform better when $b < 0$ rather than when $b \geq 0$. This indicates that EWMA-ALSN with $\lambda = 0.05$ or 0.2 and ALSN charts perform the best for the left-skew distribution compared to right-skew or symmetric distribution. We further compare ARL_{1s} among the proposed EWMA-ALSN, ALSN and ML charts. We see that the EWMA-ALSN chart with $\lambda = 0.2$ performs the best when the process has small changes in location or any dispersion, like $\delta_1 < 2$ or $\delta_2 \leq 3.0$. On the other hand, the detection performance between the EWMA-ALSN and ALSN charts has no significant difference for moderate to large changes in location and/or dispersion, like $\delta_1 \geq 2$ and/or $\delta_2 \geq 2.0$. We further see that the EWMA-ALSN chart always performs much better than the ML chart for small changes in location and small to large changes in dispersion, like $\delta_1 < 2$ and $\delta_2 > 1$. On the other hand, the detection performance between the EWMA-ALSN and ML charts has no significant difference for moderate to large changes in location and dispersion, like $\delta_1 \geq 2$ and $\delta_2 \geq 1.0$.

Based on the results in Table 5, we conclude that the EWMA-ALSN chart performs the best among the three control charts when the process follows a skew-normal distribution. On the other hand, the EWMA-ALSN chart performs better than the ALSN chart for small changes in process location and dispersion, and the EWMA-ALSN chart performs better than the ML chart for small changes in location and small to large changes in dispersion. Hence, we recommend using the proposed EWMA-ALSN chart to monitor the loss location or the process location and/or dispersion by replacing the proposed ALSN chart or the existing ML chart when the process follows a skew-normal distribution.

TABLE 6. Seventeen subgroups of IQ scores for white males hired.

t	$X1$	$X2$	$X3$	$X4$	$X5$	$\text{EWMA}_{AL,t}$
1	124	106	108	112	113	1.717
2	122	100	108	108	94	1.582
3	102	120	101	118	113	1.412
4	117	100	106	111	107	1.201
5	112	120	102	135	125	1.456
6	98	121	117	124	114	1.423
7	103	122	122	113	113	1.321
8	104	103	113	120	106	1.148
9	132	106	112	118	113	1.206
10	112	112	121	113	107	1.048
11	122	103	97	116	114	1.025
12	131	94	112	108	118	1.175
13	112	116	113	111	122	1.052
14	112	136	116	108	112	1.197
15	85	117	109	104	129	1.427
16	140	106	115	109	122	1.665
17	108	119	121	108	116	1.465

IV. ILLUSTRATIVE EXAMPLE

This section demonstrates an application of our proposed control charts using a real data set collected from the Roberts IQ score [31]. The IQ score data give the Otis IQ scores for 87 white males and 52 non-white males hired by a large insurance company in 1971. Brown [32] showed that the IQ data of the 87 white males follow a skew-normal distribution with estimated location $\hat{\xi}_0 = 105.78$, scale $\hat{a}_0 = 11.94$ and shape $\hat{b} = 1.14$. In other words, the mean is 118.39 and standard deviation is 9.53. However, the IQ data of the 52 non-white males follow a skew normal distribution with estimated $\hat{\xi}_0 = 106.62$, $\hat{a}_0 = 8.266$, and $\hat{b} = 0$. Hence, it follows a normal distribution.

We take 85 IQ data from all IQ data of 87 white males hired and regard them as the in-control data with population mean 118.39 and standard deviation 9.53. The in-control 85 IQ data are grouped into 17 subgroups with sample of size 5 (Table 6). Furthermore, we take 50 IQ data grouped into 10 subgroups with sample of size 5 (Table 7). We set the target of IQ score (T) to be 109.39, and so the scale of the deviation from target value is $\delta_3 = 1$. We define the loss as the deviation of the Q score from the target value, $L = (X - T)^2$. Hence, using the proposed loss location monitoring approach in Section III, we determine $UCL = 3.367$ and $LCL = 1.143$ of the EWMA-ALSN chart with $\lambda = 0.2$ and $\text{ARL}_0 = 370.4$. The in-control subgroup statistics, $\text{EWMA}_{\text{AL},t}$, of the seventeen subgroups are listed in Table 6 and plotted in Fig. 2. Although the subgroup numbers 10,11, and 13 of the in-control samples fall below LCL (very close to LCL), they are in-control subgroups or false alarms. Furthermore, we calculate the out-of-control subgroup statistics, $\text{EWMA}_{AL^*,t}$, of the ten subgroups of IQ score data for the non-white males. The subgroup statistics, $\text{EWMA}_{AL^*,t}$, are listed in Table 7 and plotted in Fig. 3. We find nine out of ten fall outside the LCL. This indicates that deviation of the IQ score from the target or loss location of the IQ score data of the non-white males is significantly different from that of white males hired.

In this example, the IQ score data of white males hired follow a skew-normal distribution with $b > 0$, but not for the IQ score data of non-white males hired. The result of the example shows that the proposed EWMAALSN chart may effectively detect the out-of-control loss location or equivalently the deviation of the IQ score from the target for the non-white male IQ scores with a normal (or skew-normal distribution with $b = 0$) distribution. The EWMA-ALSN chart is thus recommended to monitor the out-of-control loss location or the deviation of quality variable from the target for a process with a skewed distribution.

TABLE 7. Ten subgroups of IQ scores for non-white males hired.

t	X1	X2	X3	X4	X5	$\text{EWMA}^*_{A,t}$
1	91	102	100	117	122	1.497
2	115	97	109	108	104	1.102
3	108	118	103	123	123	1.139
4	103	106	102	118	100	0.783
5	103	107	108	107	97	0.439
6	95	119	102	108	103	0.206
7	102	112	99	116	114	0.154
8	102	111	104	122	103	0.103
9	111	101	91	99	121	−0.096
10	97	109	106	102	104	−0.298

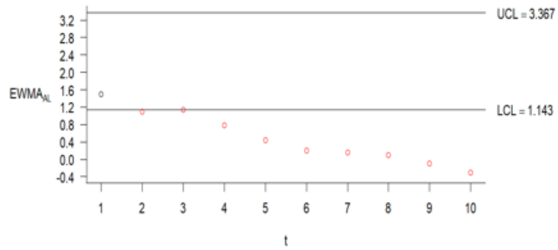

FIGURE 2. Plotting points for subgroups of white males.

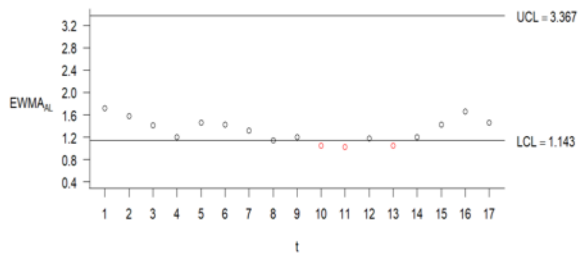

FIGURE 3. Plotting points for subgroups of non-white males.

V. CONCLUSIONS

This study has proposed ALSN and EWMA-ALSN charts, based on the derived approximate distributions of an average loss statistic, in order to monitor the process loss location, or equivalently, the deviation of the quality variable from the target when the process exhibits a skew-normal distribution. Compared to the ALSN chart, we find that the out-of-control detection performance of the EWMA-ALSN chart performs better for small changes in process location and/or dispersion. Furthermore, compared to the existing loss control chart - such as the median loss (ML) chart - when the process has a skew-normal distribution, the newly proposed ALSN and EWMA-ALSN charts always perform better for detecting the out-of-control process.

In a real example of detecting out-of-control IQ scores with a skew-normal distribution for non-white males hired in a large insurance company, we demonstrate that the proposed EWMA-ALSN control chart performs well. We thus recommend using this new EWMA-ALSN chart to efficiently monitor shifts in process location and/or dispersion when the process variable follows a skew-normal distribution.

REFERENCES

[1] F. Tsung, Y. Li, and M. Jin, "Statistical process control for multistage manufacturing and service operations: A review and some extensions," *Int. J. Services Oper. Informat.*, vol. 3, no. 2, pp. 191–204, 2008.

[2] X. Ning, Y. Shang, and F. Tsung, "Statistical process control techniques for service processes: A review," in *Proc. 6th Int. Conf. Service Syst. Service Manage.*, Xiamen, China, 2009, pp. 927–931.

[3] S.-F. Yang and S.-H. Wu, "A double sampling scheme for process variability monitoring," *Qual. Rel. Eng. Int.*, vol. 33, no. 8, pp. 2193–2204, Dec. 2017.

[4] R. B. Cabré, C. Blanca, M. José, J. Arnau, and J. Gómez-Benito, "Non-normal distributions commonly used in health, education, and social sciences: A systematic review," *Frontiers Psychol.*, vol. 8, pp. 1–6, Sep. 2017.

[5] R. W. Amin, M. R. Reynolds, and B. Saad, "Nonparametric quality control charts based on the sign statistic," *Commun. Statist. Theory Methods*, vol. 24, no. 6, pp. 1597–1623, Jan. 1995.

[6] S. Chakraborti, P. Van Der Laan, and S. T. Bakir, "Nonparametric control charts: An overview and some results," *J. Qual. Technol.*, vol. 33, no. 3, pp. 304–315, Jul. 2001.

[7] F. S. Altukife, "A new nonparametric control chart based on the observations exceeding the grand median," *Pakistan J. Statist.*, vol. 19, no. 3, pp. 343–352, 2003.

[8] S. T. Bakir, "Distribution-free quality control charts based on Signed-Rank-Like statistics," *Commun. Statist.-Theory Methods*, vol. 35, no. 4, pp. 743–757, May 2006.

[9] S.-Y. Li, L.-C. Tang, and S.-H. Ng, "Nonparametric CUSUM and EWMA control charts for detecting mean shifts," *J. Qual. Technol.*, vol. 42, no. 2, pp. 209–226, Apr. 2010.

[10] C. Zou and F. Tsung, "Likelihood ratio-based distribution-free EWMA control charts," *J. Qual. Technol.*, vol. 42, no. 2, pp. 174–196, Apr. 2010.

[11] M. A. Graham, S. W. Human, and S. Chakraborti, "A phase i nonparametric shewhart-type control chart based on the median," *J. Appl. Statist.*, vol. 37, no. 11, pp. 1795–1813, Nov. 2010.

[12] T. Abbas, B. Zaman, A. Atir, M. Riaz, and S. Akbar Abbasi, "On improved dispersion control charts under ranked set schemes for normal and non-normal processes," *Qual. Rel. Eng. Int.*, vol. 35, no. 5, pp. 1313–1341, Jul. 2019.

[13] L. P. Sullivan, "Reducing variability: A new approach to quality," *Quality Prog.*, vol. 17, no. 7, pp. 15–21, 1984.

[14] G. Taguchi, "Introduction to quality engineering: Designing quality into products and processes," Asian Productiv. Org., Tokyo, Japan, Tech. Rep., 1986.

[15] Z. Wu and Y. Tian, "Weighted-loss-function control charts," *Int. J. Adv. Manuf. Technol.*, vol. 31, nos. 1–2, pp. 107–115, Oct. 2006.

[16] Z. Wu, P. Wang, and Q. Wang, "A loss function-based adaptive control chart for monitoring the process mean and variance," *Int. J. Adv. Manuf. Technol.*, vol. 40, nos. 9–10, pp. 948–959, Feb. 2009.

[17] S.-F. Yang, "Using a new VSI EWMA average loss control chart to monitor changes in the difference between the process mean and target and/or the process variability," *Appl. Math. Model.*, vol. 37, nos. 16–17, pp. 7973–7982, Sep. 2013.

[18] S. W. Lu, "Loss control charts under skewnormal population," M.S. thesis, Dept. Stat., Nat. Chengchi Univ., Taipei, Taiwan, Jun. 2014.

[19] S.-F. Yang, R. Zhou, and S.-W. Lu, "A median loss control chart for monitoring quality loss under skewed distributions," *J. Stat. Comput. Simul.*, vol. 87, no. 17, pp. 3241–3260, Nov. 2017.

[20] S.-F. Yang and S. W. Lu, "A new loss-based control chart," *J. Qual.*, vol. 27, no. 2, pp. 69–82, 2020.

[21] D. C. Montgomery, *Introduction to Statistical Quality Control*. Hoboken, NJ, USA: Wiley, 2019.

[22] F. D. Helguero, "Interpretazione fillotassica dei poligoni fitostatistici," in *Contribuzioni Alla Biologia Vegetale*, vol. 4, A. Borzi, Ed. Palermo, Italy: Tipografia Priulla, 1908, pp. 169–191.

[23] A. Azzalini, "A class of distributions which includes the normal ones," *Scandin. J. Statist.*, vol. 12, no. 2, pp. 171–178, 1985.

[24] A. Azzalini and A. D. Valle, "The multivariate skew-normal distribution," *Biometrika*, vol. 83, no. 4, pp. 715–726, 1996.

[25] A. Azzalini and A. Capitanio, "Statistical applications of the multivariate skew normal distribution," *J. Roy. Stat. Soc. B, Statist. Methodol.*, vol. 61, no. 3, pp. 579–602, 1999.

[26] A. Azzalini, "The skew-normal distribution and related multivariate families," *Scandin. J. Statist.*, vol. 32, no. 2, pp. 159–188, 2005.

[27] A. Azzalini, *The Skew-Normal and Related Families*, vol. 3. Cambridge, U.K.: Cambridge Univ. Press, 2014.

[28] F. Y. Edgeworth, "The law of error," *Trans. Cambridge Philos. Soc.*, vol. 20, pp. 113–141, Sep. 1905.

[29] P. Hall, *The Bootstrap and Edgeworth Expansion*. New York, NY, USA: Springer, 2013.

[30] W. L. Teoh, J. K. Chong, M. B. Khoo, P. Castagliola, and W. C. Yeong, "Optimal designs of the variable sample size chart based on median run length and expected median run length," *Qual. Rel. Eng. Int.*, vol. 33, no. 1, pp. 121–134, 2017.

[31] H. V. Roberts, *Data Analysis for Managers With Minitab*. Redwood City, CA, USA: Scientific Press, 1991.

[32] N. D. Brown, "Reliability studies of the skew normal distribution," M.S. thesis, Dept. Math., Univ. Maine, Orono, ME, USA, May 2001.

SU-FEN YANG was born in Taichung, Taiwan. She received the B.S. degree from National Chung Hsing University, Taiwan, the M.S. degree from National Tsing Hua University, Taiwan, and the Ph.D. degree in statistics from the University of California at Riverside, Riverside, CA, USA. She was a Professor, from 1997 to 2008, and has been a Distinguished Professor, since 2008. She is the author of a book *Quality Management* and over 150 articles published in international journals and conferences. Her research interests are mainly in statistical process control, quality engineering, and applied statistics. She was a recipient of Outstanding Research from National Chengchi University and the Ministry of Science and Technology, Taiwan, and the Best Paper Award from Asian Network for Quality. She has been a member of the Quality Management Committee of the Ministry of Economic Affairs, Bureau of Standards, Taiwan, and the Committee of Chinese Society for Quality, and an Associate Editor of the *Journal of Quality* and the *Journal of the Chinese Institute of Industrial Engineers*. She has reviewed many articles in many international journals.

LIJUAN SHEN received the Ph.D. degree from Beihang University, China. She is currently a Postdoctoral Researcher with the Singapore-ETH Centre.