

A Strong Law of Large Numbers for Random Elements in Banach Spaces

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Abstract. Let \mathcal{B} be a real separable Banach space of Rademacher type p ($1 \leq p \leq 2$). In this article, we study the Chung's strong law of large numbers for a sequence of independent \mathcal{B} -valued random elements. We generalize the results of Hu and Taylor, Jardaš et al. and Woyczyński.

Keywords: Law of large numbers; Random element; Banach space.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables. The Chung's strong law of large numbers (SLLN) for random variables states that if $EX_n = 0$ for $n \geq 1$; and $0 < a_n \uparrow \infty$ and φ is a positive, even and continuous function such that

$$\frac{\varphi(t)}{|t|} \uparrow \quad \text{and} \quad \frac{\varphi(t)}{t^2} \downarrow \quad \text{as } |t| \uparrow, \quad (1)$$

then

$$\sum_{n=1}^{\infty} \frac{E\varphi(X_n)}{\varphi(a_n)} < \infty$$

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implies $\sum_{n=1}^{\infty} \frac{X_n}{a_n}$ converges almost surely. Hu and Taylor [4] proved Chung's SLLN for arrays of rowwise independent random variables. Jardas et al. [5] extended Chung's SLLN to a sequence of independent random variables weighted by a positive increasing sequence of real numbers, by using a sequence ϕ_n of positive Borel functions satisfying conditions which are weaker than Chung's condition (1). Woyczyński [7] generalized Chung's result for random elements in Banach spaces. The results in this article are generalizations of those in [4, 5, 7].

Let \mathcal{B} be a real separable Banach space with norm $\|\cdot\|$. We say that X is a \mathcal{B} -valued random element if X is a Borel measurable function defined on a probability space (Ω, \mathcal{F}, P) and taking values in \mathcal{B} .

Let $\{\gamma_i, 1 \leq i \leq n\}$ be a symmetric Bernoulli family, that is, $\{\gamma_i, 1 \leq i \leq n\}$ is a family of independent and identically distributed random variables with $P\{\gamma_1 = -1\} = P\{\gamma_1 = 1\} = \frac{1}{2}$. Then \mathcal{B} is said to be of Rademacher type p ($1 \leq p \leq 2$) if there exists a constant $0 < C < \infty$ such that

$$E \left\| \sum_{i=1}^n \gamma_i X_i \right\|^p \leq C \sum_{i=1}^n E \|X_i\|^p$$

for any finite collection $\{X_1, \dots, X_n\}$ of \mathcal{B} -valued random elements. Hoffmann-Jørgensen and Pisier [3] proved that a real separable Banach space \mathcal{B} is of Rademacher type p if and only if there exists a constant $0 < C < \infty$ such that

$$E \left\| \sum_{i=1}^n X_i \right\|^p \leq C \sum_{i=1}^n E \|X_i\|^p \quad (2)$$

where $\{X_1, \dots, X_n\}$ is any finite collection of independent \mathcal{B} -valued random elements with mean zero and finite p -th moment. Throughout this article we assume $1 \leq p \leq 2$.

We organize this article as the following:

In Section 2, we state the main results of this article. In Section 3, we state and prove some lemmas which will be needed in Section 4. In Section 4, we give the proofs of the main theorems.

2. Main Results

Let $\{\phi_n, n \geq 1\}$ be a sequence of positive Borel functions which satisfy the following property:

There are $C_n, D_n > 0$, $\alpha_n \geq 1$, $\beta_n \leq p$, $n \geq 1$, such that for $u \geq v$,

$$C_n \frac{u^{\alpha_n}}{v^{\alpha_n}} \leq \frac{\phi_n(u)}{\phi_n(v)} \leq D_n \frac{u^{\beta_n}}{v^{\beta_n}}.$$

The main results of this article are the following theorems.

Theorem 2.1. *Let \mathcal{B} be a real separable Banach space of Rademacher type p , and $\{X_n, n \geq 1\}$ independent \mathcal{B} -valued random elements with $EX_n = 0$. Then for every sequence $\{a_n, n \geq 1\}$ of positive numbers with*

$$\sum_{n=1}^{\infty} A_n \frac{E\phi_n(\|X_n\|)}{\phi_n(a_n)} < \infty, \quad \text{where } A_n = \max\left\{\frac{1}{C_n}, C_n D_n\right\},$$

$\sum_{n=1}^{\infty} \frac{X_n}{a_n}$ converges a.s.

Theorem 2.2. *Let \mathcal{B} be a real separable Banach space of Rademacher type p , and let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$, $k_n \rightarrow \infty$ as $n \rightarrow \infty$, be an array of rowwise independent \mathcal{B} -valued random elements with $EX_{ni} = 0$ for every ni . Then for every array $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ of positive numbers with*

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} A_n \frac{E\phi_n(\|X_{ni}\|)}{\phi_n(a_{ni})} < \infty, \quad \text{where } A_n = \max\left\{\frac{1}{C_n}, C_n D_n\right\},$$

$\sum_{i=1}^{k_n} \frac{X_{ni}}{a_{ni}}$ converges to 0 a.s.

3. Preliminary Lemmas

The following lemmas will be needed in the next section.

Lemma 3.1. [1, p. 103] *Let X, Y be independent \mathcal{B} -valued random elements with $E\|X\|^r < \infty$ and $E\|Y\|^r < \infty$ for some $r \geq 1$. If $EX = 0$, then*

$$E\|X + Y\|^r \geq E\|Y\|^r.$$

Lemma 3.2. *Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be two sequences of \mathcal{B} -valued random elements. If $\sum_{n=1}^{\infty} P\{X_n \neq Y_n\} < \infty$, then $\sum_{n=1}^{\infty} (X_n - Y_n)$ converges a.s.*

Proof. The proof is just as the same as for random variables (see [2, p. 108]). ■

Lemma 3.3. *Let X be a \mathcal{B} -valued random element with $EX = 0$ and $E\|X\|^r < \infty$, $r \geq 1$. Suppose that X^* is the symmetrized version of X , i.e., $X^* = X - Y$, where X and Y are independent and have the same distribution. Then*

$$E\|X\|^r \leq E\|X^*\|^r \leq 2^p E\|X\|^r.$$

Proof. Let $\psi(X) = \|X\|^r$ for $r \geq 1$. Then ψ is a convex function. Since X^* is the symmetrized version of X , by the conditional expectation ([8, p. 43]),

$$E\{X^*|X\} = E\{X - Y|X\} = X$$

and by the conditional Jensen's inequality ([1, p. 110]), we have that

$$\|X\|^r = \|E\{X^*|X\}\|^r \leq E\{\|X^*\|^r|X\}.$$

Therefore

$$E\|X\|^r \leq E\{E\{\|X^*\|^r|X\}\} = E\|X^*\|^r. \quad \blacksquare$$

Lemma 3.4. (Kahane inequality [6]) *Let $\{X_i, 1 \leq i \leq n\}$ be \mathcal{B} -valued random elements. Then for $1 \leq r, s < \infty$, there exists $K > 0$ such that*

$$\left(E \left\| \sum_{i=1}^n \gamma_i X_i \right\|^r\right)^{\frac{1}{r}} \leq K \cdot \left(E \left\| \sum_{i=1}^n \gamma_i X_i \right\|^s\right)^{\frac{1}{s}}$$

where $\{\gamma_i, 1 \leq i \leq n\}$ is a symmetric Bernoulli sequence.

The following two lemmas are the extensions of the Kolmogorov's inequality and the Kolmogorov's three series theorem, respectively, in Banach spaces.

Lemma 3.5. *Let $\{X_n, n \geq 1\}$ be independent \mathcal{B} -valued random elements with $EX_n = 0$ and $E\|X_n\|^r < \infty$, $r \geq 1$, for all n . Then for every $\varepsilon > 0$*

$$P\left\{\max_{1 \leq j \leq n} \|S_j\| > \varepsilon\right\} \leq \frac{E\|S_n\|^r}{\varepsilon^r}, \quad \text{where } S_n = \sum_{i=1}^n X_i.$$

Proof. Fix $\varepsilon > 0$. Let

$$\Delta = \{\omega : \max_{1 \leq j \leq n} \|S_j(\omega)\| > \varepsilon\}.$$

Put

$$\Delta_k = \{\omega : \max_{1 \leq j \leq k-1} \|S_j(\omega)\| \leq \varepsilon, \|S_k(\omega)\| > \varepsilon\}$$

(for $k = 1$, $\max_{1 \leq j \leq k-1} \|S_j(\omega)\|$ is taken to be 0). These Δ_k 's are disjoint and $\Delta = \bigcup_{k=1}^n \Delta_k$. Since $\{X_n, n \geq 1\}$ is independent, $EX_n = 0$, and $E\|X_n\|^r < \infty$ for all n , by Lemma 3.1, we have that

$$\begin{aligned} E\|S_n\|^r &\geq E\|S_k\|^r \\ &\geq \sum_{k=1}^n \int_{\Delta_k} \|S_k\|^r dP \\ &\geq \varepsilon^r P(\Delta). \end{aligned}$$

Hence

$$P\left\{\max_{1 \leq j \leq n} \|S_j\| > \varepsilon\right\} \leq \frac{E\|S_n\|^r}{\varepsilon^r}. \quad \blacksquare$$

Lemma 3.6. Let $\{X_n, n \geq 1\}$ be a sequence of independent \mathcal{B} -valued random elements with $E\|X_n\|^r < \infty$, $r \geq 1$. Define for a fixed constant $\alpha > 0$,

$$Y_n(\omega) = \begin{cases} X_n(\omega), & \text{if } \|X_n(\omega)\| \leq \alpha; \\ 0, & \text{if } \|X_n(\omega)\| > \alpha. \end{cases}$$

Then the series $\sum_{n=1}^{\infty} X_n$ converges if the following three series all converge:

- (i) $\sum_{n=1}^{\infty} P\{X_n \neq Y_n\}$,
- (ii) $\sum_{n=1}^{\infty} EY_n$,
- (iii) $\sum_{n=1}^{\infty} E\|Y_n - EY_n\|^r$.

Proof. Suppose that (i), (ii) and (iii) all converge. Let $Z_n = Y_n - EY_n$. Then $EZ_n = 0$ and $E\|Z_n\|^r < \infty$. By Lemma 3.5, for all $m \in \mathbf{N}$,

$$P\left\{\max_{n \leq j \leq n'} \left\|\sum_{k=n}^j Z_k\right\| > \frac{1}{m}\right\} \leq m^r E\left\|\sum_{k=n}^{n'} Z_k\right\|^r.$$

So

$$P\left\{\max_{n \leq j \leq n'} \left\|\sum_{k=n}^j Z_k\right\| \leq \frac{1}{m}\right\} \geq 1 - m^r E\left\|\sum_{k=n}^{n'} Z_k\right\|^r \geq 1 - Cm^r \sum_{k=n}^{n'} E\|Z_k\|^r.$$

By (iii),

$$\sum_{k=n}^{n'} E\|Y_k - EY_k\|^r \longrightarrow 0, \quad \text{as } n, n' \longrightarrow \infty.$$

Then

$$\lim_{n \rightarrow \infty} \lim_{n' \rightarrow \infty} P\left\{\max_{n \leq j \leq n'} \left\|\sum_{k=n}^j (Y_k - EY_k)\right\| \leq \frac{1}{m}\right\} = 1.$$

By Borel-Cantelli Lemma, $\sum_{n=1}^{\infty} (Y_n - EY_n)$ converges a.s. Also by (ii), we get that $\sum_{n=1}^{\infty} Y_n$ converges a.s. In addition, by (i) and Lemma 3.2, $\sum_{n=1}^{\infty} (Y_n - X_n)$ converges a.s. Then almost sure convergence of $\sum_{n=1}^{\infty} Y_n$ implies almost sure convergence of $\sum_{n=1}^{\infty} X_n$. \blacksquare

4. Proofs

Proof of Theorem 2.1. Let $Y_n = X_n I_{\{\|X_n\| \leq a_n\}}$ and $Z_n = X_n I_{\{\|X_n\| > a_n\}}$ for all

n , where I is the indicator function. First,

$$\begin{aligned} \sum_{n=1}^{\infty} P \left\{ \frac{X_n}{a_n} \neq \frac{Y_n}{a_n} \right\} &\leq \sum_{n=1}^{\infty} \left\| \frac{Z_n}{a_n} \right\| P \{ \|X_n\| > a_n \} \\ &\leq \sum_{n=1}^{\infty} E \left\| \frac{Z_n}{a_n} \right\|^{\alpha_n} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{C_n} E \left[\frac{\phi_n(\|Z_n\|)}{\phi_n(a_n)} \right] \\ &\leq \sum_{n=1}^{\infty} A_n \frac{E\phi_n(\|X_n\|)}{\phi_n(a_n)} < \infty. \end{aligned}$$

Next, since $X_n = Y_n + Z_n$ and $EX_n = 0$, $\sum_{n=1}^{\infty} E \left[\frac{Y_n}{a_n} \right] = -\sum_{n=1}^{\infty} E \left[\frac{Z_n}{a_n} \right]$ provided the limits exist. For each $N \in \mathbf{N}$,

$$E \left\| -\sum_{n=1}^N \frac{Z_n}{a_n} \right\| \leq \sum_{n=1}^N E \left\| \frac{Z_n}{a_n} \right\|^{\alpha_n} \leq \sum_{n=1}^N A_n \frac{E\phi_n(\|X_n\|)}{\phi_n(a_n)} < \infty,$$

so $-\sum_{n=1}^N E \left[\frac{Z_n}{a_n} \right]$ exists for all $N \in \mathbf{N}$. Now by the convergence of $\sum_{n=1}^{\infty} A_n \cdot \frac{E\phi_n(\|X_n\|)}{\phi_n(a_n)}$, we know that $\{-\sum_{n=1}^N E \left[\frac{Z_n}{a_n} \right], N \geq 1\}$ is a Cauchy sequence in the Banach space, hence $-\sum_{n=1}^{\infty} E \left[\frac{Z_n}{a_n} \right]$, the limit of the sequence, exists. Therefore $\sum_{n=1}^{\infty} E \left[\frac{Y_n}{a_n} \right]$ converges. Finally,

$$\begin{aligned} \sum_{n=1}^{\infty} E \left\| \frac{Y_n}{a_n} - E \left[\frac{Y_n}{a_n} \right] \right\|^p &\leq 2^p C \sum_{n=1}^{\infty} E \left\| \frac{Y_n}{a_n} \right\|^{\beta_n} \\ &\leq 2^p C \sum_{n=1}^{\infty} D_n \frac{E\phi_n(\|Y_n\|)}{\phi_n(a_n)} \\ &\leq 2^p C \sum_{n=1}^{\infty} A_n \frac{E\phi_n(\|X_n\|)}{\phi_n(a_n)} < \infty. \end{aligned}$$

By Lemma 3.6, convergence of the three series implies almost sure convergence of $\sum_{n=1}^{\infty} \frac{X_n}{a_n}$. The proof is complete. \blacksquare

Proof of Theorem 2.2. Let $Y_{ni} = X_{ni}I_{\{\|X_{ni}\| \leq a_{ni}\}}$ and $Z_{ni} = X_{ni}I_{\{\|X_{ni}\| > a_{ni}\}}$ for all n and all i . For each $\varepsilon > 0$,

$$\begin{aligned} &P \left\{ \bigcup_{n=m}^{\infty} \left\{ \left\| \sum_{i=1}^{k_n} \left(\frac{Z_{ni}}{a_{ni}} - E \left[\frac{Z_{ni}}{a_{ni}} \right] \right) \right\| > \varepsilon \right\} \right\} \\ &\leq \sum_{n=m}^{\infty} P \left\{ \left\| \sum_{i=1}^{k_n} \left(\frac{Z_{ni}}{a_{ni}} - E \left[\frac{Z_{ni}}{a_{ni}} \right] \right) \right\| > \varepsilon \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{\varepsilon} \sum_{n=m}^{\infty} \sum_{i=1}^{k_n} E \left\| \frac{Z_{ni}}{a_{ni}} \right\| \\
&\leq \frac{2}{\varepsilon} \sum_{n=m}^{\infty} \sum_{i=1}^{k_n} E \left\| \frac{Z_{ni}}{a_{ni}} \right\|^{\alpha_n} \\
&\leq \frac{2}{\varepsilon} \sum_{n=m}^{\infty} \sum_{i=1}^{k_n} A_n \frac{E\phi_n(\|X_{ni}\|)}{\phi_n(a_{ni})}.
\end{aligned}$$

From the hypothesis, we have

$$\lim_{m \rightarrow \infty} P \left\{ \bigcup_{n=m}^{\infty} \left\{ \left\| \sum_{i=1}^{k_n} \left(\frac{Z_{ni}}{a_{ni}} - E \left[\frac{Z_{ni}}{a_{ni}} \right] \right) \right\| > \varepsilon \right\} \right\} = 0.$$

So

$$\left\| \sum_{i=1}^{k_n} \frac{Z_{ni}}{a_{ni}} - E \left[\sum_{i=1}^{k_n} \frac{Z_{ni}}{a_{ni}} \right] \right\| \longrightarrow 0 \quad \text{a.s.}$$

Then

$$\sum_{i=1}^{k_n} \frac{Z_{ni}}{a_{ni}} - E \left[\sum_{i=1}^{k_n} \frac{Z_{ni}}{a_{ni}} \right] \quad \text{converges to 0 a.s.}$$

Similarly, for each $\varepsilon > 0$,

$$\begin{aligned}
&P \left\{ \bigcup_{n=m}^{\infty} \left\{ \left\| \sum_{i=1}^{k_n} \left(\frac{Y_{ni}}{a_{ni}} - E \left[\frac{Y_{ni}}{a_{ni}} \right] \right) \right\| > \varepsilon \right\} \right\} \\
&\leq \sum_{n=m}^{\infty} P \left\{ \left\| \sum_{i=1}^{k_n} \left(\frac{Y_{ni}}{a_{ni}} - E \left[\frac{Y_{ni}}{a_{ni}} \right] \right) \right\| > \varepsilon \right\} \\
&\leq \frac{2C}{\varepsilon^p} \sum_{n=m}^{\infty} \sum_{i=1}^{k_n} E \left\| \frac{Y_{ni}}{a_{ni}} \right\|^p \\
&\leq \frac{2C}{\varepsilon^p} \sum_{n=m}^{\infty} \sum_{i=1}^{k_n} E \left\| \frac{Y_{ni}}{a_{ni}} \right\|^{\beta_n} \\
&\leq \frac{2C}{\varepsilon^p} \sum_{n=m}^{\infty} \sum_{i=1}^{k_n} A_n \frac{E\phi_n(\|X_{ni}\|)}{\phi_n(a_{ni})}.
\end{aligned}$$

We have that

$$\lim_{m \rightarrow \infty} P \left\{ \bigcup_{n=m}^{\infty} \left\{ \left\| \sum_{i=1}^{k_n} \left(\frac{Y_{ni}}{a_{ni}} - E \left[\frac{Y_{ni}}{a_{ni}} \right] \right) \right\| > \varepsilon \right\} \right\} = 0.$$

So

$$\left\| \sum_{i=1}^{k_n} \frac{Y_{ni}}{a_{ni}} - E \left[\sum_{i=1}^{k_n} \frac{Y_{ni}}{a_{ni}} \right] \right\| \longrightarrow 0 \quad \text{a.s.}$$

Then

$$\sum_{i=1}^{k_n} \frac{Y_{ni}}{a_{ni}} - E \left[\sum_{i=1}^{k_n} \frac{Y_{ni}}{a_{ni}} \right] \quad \text{converges to 0 a.s.}$$

Since $EX_{ni} = 0$, therefore $\sum_{i=1}^{k_n} \frac{X_{ni}}{a_{ni}}$ converges to 0 a.s. ■

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