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Inventory renewal for a perishable product: Economies of scale and age-dependent demand

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Abstract

We consider a finite-horizon, periodic-review system for a perishable product at a retailer that faces stochastic, age-dependent demand and loses excess demand, if any. For this system, we build three models that capture two sources of unfilled demand, insufficient inventory and inappropriate age, and penalize them at different rates. The models are characterized by increasing replenishment flexibility, and the goal of each model is to identify when to place an order and the quantity whenever an order is placed. In the first model, reorder intervals are equal. In the second model, reorder intervals can vary across orders. In the third model, reorder intervals continue to remain flexible but the retailer can also partially salvage her inventory whenever she has excess inventory. Using the models, we explore the effect of lost-sales penalties on the structure of the optimal value function. We find that the inventory-related cost in a period may lack convexity if the ratio of penalties for stockout and high age is below a threshold, which percolates to the value function as well. We also identify properties of the optimal replenishment policy for the three models. Finally, we conduct numerical experiments to identify the marginal value added by the flexibility in reorder intervals and the option to partially salvage inventory as a function of model parameters.

KEYWORDS

age-dependent demand, dynamic programming, inventory control, periodic review model, perishable products

1 | INTRODUCTION

In the United States, around eight million tons of food is wasted every year in supermarkets, grocery stores and distribution centers, according to an estimate by ReFED, a nonprofit focused on reducing food waste (ReFED, 2016). The value of the wasted food is estimated to be $18.2 billion. Apart from causing a massive hit on the bottomline of the retailers, the waste also produces a significant social cost in the form of landfills. For brick-and-mortar retailers such as Wal-Mart, there is an additional reason to reduce cost of wastage: increasingly intense competition from Amazon.com.

Indeed, the importance of reducing food waste has not gone unnoticed in the retail sector, and the retailers have tried a number of solutions such as dynamic pricing, better forecasting and extension of shelf life (Jaszczyk, 2019; ReFED, 2018). Out of these, shelf life extension is a particularly appealing solution since it alleviates perishability, the root cause of the problem. One way in which shelf life can be extended is through better packaging technology. For example, if beef is vacuum packed in multilayer plastic, its shelf life can be increased to 45 days as compared to a life of 3 to 7 days when it is packed in polystyrene foam trays (Gray, 2018). Another packaging technique is modification of the atmosphere. Putting sweet peppers in a bag with modified atmosphere can increase its shelf life from 4 to 7 days (Flexible Packaging Association, 2014). Smarter packaging, however, is not the only way to increase the shelf life. Apeel Sciences, a California-based startup, has developed a coating

made from organic materials that can be applied on fruits and vegetables to extend their shelf life (McGrath, 2018; Strom, 2016). The articles report that one of the company's products can more than double the shelf life of cassava, which otherwise starts decaying within 1 to 2 days.

While greater shelf life definitely provides relief to retailers, the relief may be partial at best since an increase in shelf life may not be completely translatable to an increase in shelf duration, defined as the time between the arrival of a fresh unit and its removal from shelf if it remains unsold. There are two reasons for it. One, in spite of growing awareness and concern among consumers on food wastage, they still want the produce to be fresh. Naturally, this constrains retailers from increasing the shelf duration to the maximum. More so, since perishables and their appearance play an important role in driving store traffic (Tsiros & Heilman, 2005). Another reason is that the more time food spends in a store, the more it may be moved around, which may make it appear worn out.

Indeed, complete exploitation of increased lifetime of bakery products was cited as one of the reasons when Interstate Bakeries Corp., the largest baker in the United States, filed for bankruptcy in 2004 (Adamy, 2004). As the lifetime of several bakery products increased due to innovations, Interstate sought to take full advantage by maximizing the shelf duration of its products. In particular, it increased the number of days its famous brands of bread was sold from 3 to 7 days, which was the new shelf-life of bread. The company expected several benefits such as reduced spoilage and savings in operational costs due to greater economies of scale. Yet contrary to the company's expectation, the strategy did not help. The longer bread stayed on the shelf, the more frequently it was moved by store personnel and customers. This made the bread look shelf worn even though it was still consumable. The net result was a reduction in the demand of many popular products. In contrast, a competitor of Interstate, Flower Foods, utilizing the same technology, chose to increase the shelf duration of bread from 3 to 4 days instead of full 7 days, yet its sales did not decrease. In fact, Flower Foods' sales increased by 5% (Adamy, 2004). These examples clearly illustrate the importance of the decision regarding shelf duration and form the motivation for this study.

We consider a period-review model over finite horizon for a retailer that faces stochastic, age-dependent demand for a perishable product. The retailer faces two decisions: when to replace old inventory with fresh inventory and the order quantity for the fresh inventory. Whenever the retailer places an order, it salvages any unsold inventory and incurs a fixed cost. Since demand is stochastic, it is possible that demand exceeds inventory between two orders, in which case, excess demand is lost. Demand may also get lost when a customer chooses not to purchase the product when he finds the inventory to be too old. We model both sources of unfilled demand separately with possibly different penalties associated with them. Overall, the trade-offs faced by the retailer are economies of scale in order placement, which encourage large order sizes, versus deterioration of demand with age, which makes short shelf duration and small order sizes more attractive.

An example of a firm that discards old inventory while placing an order is Chesapeake Bagels (Ferguson & Koenigsberg, 2007). Per Li, Cheang, and Lim (2012), this practice is also commonly deployed in supermarkets in Hong Kong. In contrast, Bruegger's continues to sell older bagels alongside new bagels after receiving an order. As Li et al. (2012) note, both practices are common and deemed to be alternatives of each other. One reason retailers choose to discard old inventory while restocking shelves is their anxiety about store image if they sell both old and new inventory simultaneously. Managers also worry that it may cause customers to behave strategically. Yet another reason lies in the practice of ordering identical quantities in each replenishment cycle. Since replenishment interval is also constant, fresh inventory is sufficient to satisfy all demand, making old inventory expedient.

Given the above context, we develop three models to analyze the retailer's decisions. In the first model, which we refer to as the fixed cycle model, the reorder interval is assumed to be fixed. This model is simple to deploy and is suitable for products whose demand has relatively low volatility. In the second model, the reorder interval is flexible, and we refer to this model as the flexible replenishment model. This model is thus more suitable for products with volatile demand. In the third model, we examine a partial salvage strategy in the presence of flexible replenishment intervals. We refer to this model as the partial salvage model. The model allows for situations in which the retailer has an option to partially salvage inventory whenever she has an excess amount of it.

Using the three models, we probe several research questions, which are listed as follows:

1. Does the inventory-related cost function, which includes both types of lost-sales costs, have useful structural properties such as convexity?
2. When the replenishment interval is fixed, what should be its value and how much quantity should be ordered?
3. When the replenishment interval is flexible, when should an order be placed and what should be the associated quantity?
4. When it is possible to partially salvage inventory, how does the optimal replenishment policy change compared to when it is not possible to do so?
5. What are the incremental cost improvements due to flexible replenishment intervals and the flexibility to partially salvage inventory?

A summary of key insights obtained during the analysis of the above questions is as follows. We find that the inventory-related cost function may not always be convex in the amount of initial inventory and derive a sufficient condition under which it is convex. The condition requires that the ratio of penalties for stockout and high age be above a

threshold. Although the result is useful in establishing many properties of the value functions in the three models, it does not imply convexity of the value functions in them. In particular, for the fixed cycle model, we find that the convexity of the optimal value function requires that the ratio of two penalties not only be above a threshold but also be less than another threshold. When the ratio is within the thresholds as well as in another special case, we develop bounds on the optimal order quantity. We also develop a heuristic to determine the length of a cycle.

The structure of the optimal value function is more complex for the flexible replenishment model, and the value function appears to lack useful properties such as monotonicity and convexity. Still, we are able to show the existence of two thresholds such that an order is placed in a period if on-hand inventory at the beginning of the period is less than the smaller threshold or greater than the larger threshold. We also develop bounds on the thresholds as well as the optimal order quantity. In contrast, in the partial salvage model, there exists a single threshold such that an order is placed in a period if and only if the on-hand inventory at the beginning of the period is less than the threshold. Moreover, the value function for the partial salvage model has additional structure in the form of monotonicity. Finally, flexible replenishment intervals provide significantly more cost reduction (0%–6.4%) than the flexibility to partially salvage inventory (<0.1%).

A summary of the rest of the paper is as follows. In Sections 3 to 5, we develop and analyze the fixed cycle, flexible replenishment and partial salvage models, respectively. In Section 6, we present results of numerical experiments to determine the cost reduction due to the flexible replenishment and partial salvage strategies as a function of several model parameters such as variance of demand. In Section 7, we briefly discuss two extensions. In the first extension, we develop an approach to model the case in which quality of the product may deteriorate randomly over time. In the second extension, we analyze the scenario in which the leadtime is equal to one period. We conclude and summarize our findings in Section 8. We begin by positioning this work in the existing literature in the following section.

2 | LITERATURE REVIEW

We review two streams of literature here. Studies in the first stream primarily consider replenishment decision, that is, when to place an order and in what quantity, for perishable products in a retail setting. On the other hand, the second stream includes studies that consider other issues such as pricing to improve the profit of a retailer from the sale of perishables.

The literature on replenishment decisions in a retail setting can be further classified into two groups depending upon the modeling approach deployed. In the first approach, lifetime of the product is assumed to be fixed and known. Once the product reaches the end of its usable lifetime, it becomes unfit for consumption and must be discarded (perhaps for a cost) or salvaged. In the second approach, inventory is assumed to perish at a uniform rate. When the lifetime is fixed and known, the retailer may either salvage old inventory while replenishing stock (eg, Li et al. 2012) or may sell inventories of different ages simultaneously (eg, Li, Yu, and Wu, 2016). In the latter scenario, a model also needs to consider the order in which inventories of different ages are sold. Li et al. (2016) assume the order to be last-in, first-out (or, LIFO) since consumers select the freshest unit available when all units are sold at the same price in a retail setting. (An alternative is first-in, first-out or FIFO, which we discuss shortly.)

On the other hand, when inventory is assumed to perish at a uniform rate, Nahmias (1977), Friedman and Hoch (1978) and Chu, Hsu, and Shen (2005) examine a periodic review model for lot sizing. This approach has also been utilized in numerous continuous-time models; see Raafat (1991), Goyal and Giri (2001), and Bakker, Riezebos, and Teunter (2012) for reviews.

We note that there also exists a large class of papers that assume FIFO consumption of inventory for a perishable product with fixed and known lifetime. Such models are applicable in scenarios where the inventory manager controls the order in which inventory of different ages is sold or consumed. The reason is that the FIFO order minimizes spoilage due to expiry, which any inventory manager would prefer. An example of such a setting is blood bank. Assuming the FIFO issuance policy, Fries (1975) and Nahmias (1975) characterize the form of the optimal policy when excess demand is lost and backordered, respectively. A few other papers that have analyzed models when the issuing policy is FIFO are Brodheim, Derman, and Prastacos (1975), Li, Lim, and Rodrigues (2009), Chen, Pang, and Pan (2014), and Minner and Transchel (2017).

Three excellent reviews of literature on inventory control of products with fixed and known lifetime are Nahmias (1982), and more recently, Karaesmen, Scheller–Wolf, and Deniz (2011) and Bakker et al. (2012). Within this literature, the studies that are of particular relevance to us are the ones that consider age-dependent demand. Abouee-Mehrizi, Baron, Berman, and Chen (2019) consider multiple demand classes that have different freshness requirement. Moreover, the demand classes may have nonidentical penalties for not fulfilling demand. Chen, Li, Yang, and Zhou (2018) also study a similar problem. Within this stream, the paper that comes closest to this study is Li et al. (2012). Similar to us, they consider a retailer that discards old inventory when she places a new order. However, their model and results are significantly different from us, and a list of key differences is as follows. One, they consider an infinite-horizon model with average-cost criterion whereas we consider a finite-horizon dynamic program. Two, their objective is to develop a computational procedure to identify optimal replenishment, pricing and salvaging decisions. In contrast, we aim to develop insights on when to replace old inventory while

differentiating between lost sales due to old inventory and a stockout. Overall, we contribute to this literature by capturing both sources of unfilled demand, insufficient inventory and high age, separately.

We now discuss the literature on the management of perishables in a retail setting. Studies in this stream have considered a wide variety of research issues such as in-store logistics, shelf management, value of information, and markdown strategy. As an example of a study on in-store logistics, Reiner, Teller, and Kotzab (2013) examine the logistics of dairy products inside 202 stores of a supermarket chain to benchmark processes and to identify ways to improve them. Similarly, Li, Yu, and Wu (2017) develop insights on shelf management by analyzing the replenishment decision for a product that is sold with two packaging technologies that result in differential lifetimes; one packaging technology results in greater lifetime than the other technology, though it is more expensive as well. In another example with relevance for shelf management, Blackburn and Scudder (2009) develop a model to identify the harvesting batch size and pick rate for fresh produce such as melons to minimize the total cost of picking the produce, which includes the loss in freshness, and transporting it from farm to retail store.

Many studies have sought to determine the value of information that can potentially reduce deterioration and spoilage of perishable products. Two examples are Ketzenberg and Ferguson (2008) and Ketzenberg, Bloemhof, and Gaukler (2015). Ketzenberg and Ferguson (2008) develop a framework to identify the value of sharing information and centralized control (as opposed to decentralized decision-making) for a supply chain consisting of a supplier and a retailer. Similarly, Ketzenberg et al. (2015) develop a model to determine the value of information regarding time and temperature history for a perishable product with random lifetime and demonstrate the application of their model for fish.

Lastly, since the value of perishable products reduces over time, many retailers consider reducing prices for old items. Accordingly, multiple studies have developed models to derive insights on the optimal markdown strategy. Ferguson and Koenigsberg (2007) develop a two-period model to analyze pricing and replenishment strategy when new and leftover units from the first period compete in the second period. Hu, Shum, and Yu (2015) incorporate strategic behavior of customers, exhibited in the form of forward buying, to determine how many leftover units to sell at markdown price and the quantity to order using a stochastic dynamic program.

For an excellent review of literature on retail operations, see Mou, Robb, and DeHoratius (2018), which also discusses issues related to perishables in a grocery store.

By examining when to renew inventory in the face of declining demand with increasing age, the findings of this study will be useful for better management of perishables in a retail setting. Thus, the study contributes to the literature in the above stream as well. Overall, this is the first paper to derive insights on the replenishment interval and order quantity for a perishable product that is renewed periodically and whose demand depends on age.

Having positioned this article in the extant literature, we now proceed to defining the notation and describing the fixed cycle model in the following section.

3 | FIXED CYCLE MODEL

In the fixed cycle model, the retailer places an order once every R periods.[1,2] (The lifetime of the product S is greater than or equal to R.) While placing the order, she incurs a fixed cost K and variable cost at the rate of c per unit ordered. The lead-time is zero, so the order is delivered immediately. At the same time, the retailer salvages any unsold old inventory through a clearance sale at a price of w_R per unit, where the subscript (R) in w_R stands for the age at the time of clearance. Similar to Li et al. (2016), we assume that the demand at clearance price is so large that all leftover inventory gets sold rapidly and that the demand at clearance price does not impact the demand for fresh inventory.[3] The clearance price is nonnegative and decreases with the age of inventory at the time of clearance, that is, $w_{R_1} \geq w_{R_2}$ for $R_1 < R_2$.

3.1 | Demand

Whenever the product is in stock, the demand $D_{t,s}$ for inventory whose age is equal to s at the beginning of a generic period t is equal to:

$$D_{t,s} = D_t f(s), \tag{3.1}$$

where D_t is a random variable. The terms D_t and $f(s)$ may be interpreted as the total number of customers who arrive in period t and the fraction of customers that decide to purchase the product after observing the age, respectively. The factor $f(s)$ applies so long as the product remains in stock. If and when the product runs out of stock, the customers who arrive subsequently cannot observe the age of the product. Even though a fraction $f(s)$ of such customers would not have purchased the product had it been available, their loss of demand is *actually* caused by a lack of stock and not old age. Therefore, we penalize the loss of demand of all the customers who arrive subsequent to a stockout at the same rate, as we discuss in the following subsection.

We make the following assumptions on D_t and $f(s)$:

[1] See Table 1 for a summary of notation.
[2] R may also be interpreted as *shelf-duration*, defined in Section 1.
[3] It is possible that instead of a clearance sale, the retailer may salvage the leftover inventory, for example, to a thrift store or just discard it. Bread may be salvaged to thrift stores at heavy discounts and meat may be salvaged to chemical manufacturers, who may extract useful compounds from it. On the other hand, if the retailer just discards the leftover inventory, the clearance price will be equal to 0. In either case, the model remains unchanged.

TABLE 1 Summary of notation

$D_{t,s}$	**Demand in period t when age of inventory at the beginning of the period is s (subscript t is omitted in the paper unless necessary)**
$D^{(s)}$	Sum of demands of ages 0 through s, that is, $D_0 + D_1 + \cdots + D_s$
s	Age of inventory at the beginning of a period
$f(s)$	Fraction of customers who purchase the product when age is s
G_s	CDF of $D^{(s)}$
S	Lifetime of the product
T	Number of periods in the planning horizon
R	Replenishment interval when cycle length is fixed
x	Inventory at the beginning of a generic period
z	Amount of inventory salvaged in a period when partial salvage is permitted
q	Inventory postsalvage in a period when partial salvage is permitted
y	Order quantity
y_R^*	Optimal order quantity in fixed cycle model when cycle length is R
c	Unit purchasing cost
K	Fixed cost of order placement
h	Unit holding cost
p_1	Unit penalty cost for each customer walking away due to stale inventory
p_2	Unit penalty cost for each unsatisfied customer due to stockout
w_s	Salvage value when age of inventory is s
$L_s(x)$	Total inventory-related cost in a period when x units of age s are on hand
$v_t(s, x)$	Optimal cost from period t onward when x units of age s are on hand
$V_R(y)$	Cost of a cycle in fixed cycle model when order quantity is y
$v_t^1(s,x)$	Optimal cost from period t onward in flexible replenishment and partial salvage models when no order is placed in period t
$v_t^2(s,x)$	Optimal cost from period t onward in flexible replenishment and partial salvage models when an order is placed in period t

Assumption 1 1. For any t, D_t is a continuous random variable with density. Furthermore, $\{D_t\}$ are identically and independently distributed.

2. $f(s)$ is a deterministic and decreasing function of s that takes values in the interval [0, 1] such that $f(0) = 1$.[4]

We omit subscript t from $D_{t,s}$ and D_t, unless necessary, in the rest of the paper to keep notation simple. Furthermore, we note that it is possible to extend our results to more general demand functions, for example, $f(s)$ may be random and correlated with D_t, since our results only require that $D_{t,s}$ be decreasing in s.

The functional form of $f(s)$ should be determined by the characteristics of the product (Tsiros & Heilman, 2005). One such characteristic is product quality risk, which is defined as the health risk associated with consuming a product as its expiry date nears. If the product quality risk for a product is high (eg, beef, chicken), Tsiros and Heilman (2005) find that the willingness-to-pay, which can be interpreted as a proxy for willingness to purchase at full price,[5] decreases exponentially as expiry date approaches. In contrast, for items whose product quality risk is perceived to be lower (eg, carrots), the willingness-to-pay decreases linearly. Another factor is the controllability of aging. For milk, aging is difficult to control, and so the willingness-to-pay 1 day before expiry is lower than vegetables such as lettuce whose aging can be controlled (to some extent) through refrigeration.

3.2 | Cost model

We consider three types of costs related to the amount of inventory and its age in a period. The first type is the holding cost. As per the convention, this cost is charged proportional to the amount of inventory at the end of the period. Therefore, it is equal to $hE[x - D_s]^+$, where $x \geq 0$ is the available inventory to satisfy demand at the beginning of the period, h is the unit holding cost and $[\cdot]^+ = \max[\cdot, 0]$.

The second and third types of costs arise from unsatisfied customers. A customer may not purchase the product because either there is no stock on-hand or age of the inventory is too high for him. We use p_1 to denote the unit cost of a lost sale due to high age of inventory and p_2 to denote the unit cost of a lost sale due to stockout.

The expression for the lost-sales cost depends on whether or not inventory is stocked out in the period. Suppose first that a stockout occurs in the period, that is, $D_s > x$. This is only possible if $f(s) > 0$, so we assume that to be the case as well. Let $\mathcal{D} \in [0, D)$ be the number of customers who arrive before

[4]Throughout this paper, we use the terms "increasing" and "decreasing" to mean weakly increasing and weakly decreasing, respectively.

[5]As Tsiros and Heilman (2005) state. "… The way that WTP (willingness to pay) decreases throughout the course of the product's shelf life provides information about consumers' perception of how product quality deteriorates over time …."

the stock runs out. Using the definition of $\mathcal{D}$,

$$\mathcal{D}f(s) = x.$$

This means that the number of customers who do not purchase the product because of unacceptable age is equal to $\mathcal{D}(1 - f(s)) = \frac{x(1-f(s))}{f(s)}$. On the other hand, the number of customers who do not purchase the product because of unavailable stock is equal to $(D - \mathcal{D}) = D - \frac{x}{f(s)}$. Conversely, when there is no stockout, lost sales occur only due to unacceptable age. The number of such customers is equal to $D - D_s = D(1 - f(s))$.

Putting both cases together, the total cost of lost sales when $f(s) > 0$ is equal to

$$\mathbf{1}(D_s > x)\left(p_1\left(\frac{x(1-f(s))}{f(s)}\right) + p_2\left(D - \frac{x}{f(s)}\right)\right)$$
$$+ \mathbf{1}(D_s \leq x)p_1 D(1 - f(s)),$$

where $\mathbf{1}(\cdot)$ is an indicator function. On the other hand, if $f(s) = 0$, all demand is lost and every lost sale is due to unacceptable age. Hence, the total lost-sale cost in this case is equal to $p_1 D$.

Bringing all the terms together, the expected inventory-related cost in a period is equal to

$$\begin{aligned} L_s(x) &= hE[x - D_s]^+ \\ &\quad + E\left(\mathbf{1}(D_s > x)\left(p_1\left(\frac{x(1-f(s))}{f(s)}\right) + p_2\left(D - \frac{x}{f(s)}\right)\right)\right. \\ &\quad \left. + \mathbf{1}(D_s \leq x)p_1 D(1 - f(s))\right), \quad f(s) > 0 \\ &= hx + p_1 E(D), \quad f(s) = 0. \end{aligned} \tag{3.2}$$

We assume that both p_1 and p_2 include the list price, so we do not need to include a revenue term in the cost model. We illustrate the cost model using the following example.

> **Example 1** Let $D = 50$ (ie, realized value of D), $x = 20$, and $s = 2$ in a period. Furthermore, $f(2) = 0.8$. Given this information, the number of customers who would find the age of inventory acceptable is equal to $D_s = D \cdot f(s) = 40$. Since $D_s > x$, a stockout occurs. The number of customers who arrive before the stockout occurs is equal to $\mathcal{D} = \frac{x}{f(s)} = 25$. Of these, 20 purchase the product and five find the age of the inventory unacceptable and their demand is lost. The remaining 25 customers arrive after the stockout; their demand also remains unfulfilled.
>
> The inventory-related cost incurred in the period is thus equal to $5p_1 + 25p_2$.

Although the two types of lost-sale penalties do not seem to have been compared in the literature, it is possible to draw some conclusions based on the many studies that have analyzed the behavior of customers in the presence of a stockout. Based on this literature, we believe that in general the two penalties should not be equal to each other. To see this, note that each penalty is equal to the sum of short- and long-term costs associated with a dissatisfied customer (Anderson, Fitzsimons, & Simester, 2006). The short-term cost is the expected foregone revenue due to a customer not purchasing the product, and the long-term cost is the expected loss of future business.

When a customer chooses to not purchase an old unit or is unable to purchase the product due to stockout, he has many choices such as switching to another size, brand or flavor, postponement, and purchasing the product from another store. Clearly, the most damaging choice from the retailer's perspective is the switch to another store. Although many factors that influence a customer's decision have been listed in the existing literature, for example, nature of the product (hedonic vs utilitarian), urgency of need, and perceived product risk (Emmelhainz, Stock, & Emmelhainz, 1991; Sloot, Verhoef, & Franses, 2005), no study has listed circumstance (old age or stockout) as a factor. Intuitively as well, what a customer chooses to do when he is unable to purchase the product should not have a strong relationship with the reason for no purchase. Thus, contribution of the short-term cost should be identical for both p_1 and p_2.

In contrast, the long-term cost, which is driven by a loss of customer confidence or damage to brand image, is likely to be different in the two scenarios since the sentiments felt by the customers are different. When a customer faces a stockout, he forms an impression of incompetence on the part of the retailer (Peterson, 2018). On the other hand, selling old or nearly expired food creates an impression of carelessness or lack of concern for customers' health (Palmer, 2019). This difference in sentiments may lead to different actions in the long run in the two scenarios, which implies differential costs (and hence penalties) in the two scenarios.

Even though the penalties can take different values, the methodology to estimate them should be the same. Two research studies that discuss such methods are Emmelhainz et al. (1991) and Anderson et al. (2006). The first study uses a survey to identify the choices made by customers in case of a stockout. This can be used to estimate the short-term costs associated with stockout and old age. The second study tracked both short- and long-term shopping behavior of a control group of customers. This approach can be used to identify the aggregate cost, which includes both short- and long-term costs, associated with an upset customer.

Since p_1 and p_2 may take different values, we have considered general values for them in this study. The analysis and the structure of optimal policy, however, depends on their relative values. This is reflected in our first result, in which we show that $L_s(x)$ is a convex function of x for a given s provided p_2 is sufficiently large relative to p_1. We also show that the expression for $L_s(x)$ can be simplified when $f(s) > 0$. The result is as follows.

Proposition 1 *Consider the expression in (3.2).*

1. *An alternative expression for $L_s(x)$ is as follows*

$$L_s(x) = hE[x - D_s]^+ + \pi_s E[D_s - x]^+ + \frac{1-f(s)}{f(s)} p_1 E[D_s],$$

where $\pi_s = p_1 + \frac{p_2 - p_1}{f(s)}$. *Furthermore, if* $p_2 \geq p_1(1-f(s)) - hf(s)$, *then* $L_s(x)$ *is convex in* x *for any given* s.

2. *If* $p_1 = p_2$, $L_{s+1}(x) \geq L_s(x)$, $s+1 < S$.

Proofs of all the results are available in Appendix S1, Supporting Information.

In the proof of part 1, we show that the cost of lost sales due to age can be simplified to $\frac{1-f(s)}{f(s)} p_1 E \min(x, D_s)$, which is increasing in x. In other words, lower inventory reduces this cost. This is intuitive since the smaller the inventory, the faster it gets consumed, which leads to smaller number of disappointed customers and hence lower penalty. In contrast, the cost of lost sales due to insufficient inventory is equal to $p_2 E(D_s - x)^+$, which decreases with x. The component of the sum of the two lost-sales costs that depends on x is thus equal to $\pi_s E(D_s - x)^+$, where $\pi_s = p_1 + \frac{p_2 - p_1}{f(s)}$. Observe that for large enough p_1, π_s becomes negative. In that case, the total lost-sales cost (and hence the inventory cost function $L_s(x)$) becomes increasing in x. This is in contrast to the traditional inventory models in which lost-sales cost always decreases with additional inventory.

Not only does $L_s(x)$ become monotone, but it may also cease to be convex when π_s is negative.[6] A necessary and sufficient condition to ensure that π_s remains strictly positive for all $s \in \{0,1,2, \ldots, S-1\}$ is that $p_2 > p_1(1-f(S-1))$. In general, three possibilities arise depending upon the relative values of p_1 and p_2.

1. For $p_2 \in (0, p_1(1-f(S-1))]$, π_s is strictly positive for at least one value of s, but it is also negative for at least one value of s. Furthermore, π_s decreases with s. Let $\tilde{s}$ be the smallest value of s for which $\pi_s \leq 0$.

2. For $p_2 \in (p_1(1-f(S-1)), p_1]$, $\pi_s > 0$ for all $s \leq S-1$. In particular, when $p_2 = p_1$, π_s is independent of s. As before, π_s decreases with s.

3. Finally, for $p_2 \in (p_1, \infty)$, $\pi_s > 0$ for all $s \leq S-1$. But now, π_s increases with s.

Part 2 of the proposition provides a sufficient condition for inventory-related costs to increase with age for a given amount of inventory. The result hints at the possibility that it is not necessary for such costs to always increase with age. Whereas the holding cost always increases with age (since demand decreases with age), the lost-sales cost may, in fact, decrease. For example, if x is small, then $E(D_s - x)^+ \approx E(D_s) - x$ and $E(D_{s+1} - x)^+ \approx E(D_{s+1}) - x$. Also, we can ignore the holding cost. Therefore,

$$L_s(x) \approx p_2 E(D) - \pi_s x > p_2 E(D) - \pi_{s+1} x \equiv L_{s+1}(x)$$

when $\pi_{s+1} > \pi_s$ (equivalently, $p_2 > p_1$ and $f(s+1) < f(s)$).

An explanation for the above observation is as follows. Since $f(s+1) < f(s)$, more customers find the age of the inventory unsatisfactory and walk away when the age is $s+1$ compared to when the age is s (assuming identical levels of demand D in both scenarios). This means that fewer customers observe a stockout in the period in which inventory of age $s+1$ is sold. Since the penalty for lost sales due to stockout is higher than the penalty for losing customers due to higher age, the total lost-sales cost is smaller when the age is higher.

We are now ready to state the cost model for the planning horizon. We will formulate the problem in two stages. In the first stage, we will analyze the order quantity decision for a given reorder interval R. In the second stage, we will explore the optimal reorder interval. Accordingly, let $v_t(s, x)$ be the optimal cost from period t through the end of horizon when x units of age s are on hand for a given R. (We have omitted an explicit dependence of R on $v_t(s, x)$ for brevity.) The expression for $v_t(s, x)$ depends on whether an order is placed in period t or not. When no order is placed in period t,

$$v_t(s,x) = L_s(x) + E[v_{t+1}(s+1, (x - D_s)^+)],$$

where T is the length of the planning horizon. On the other hand, if an order is placed in period t,

$$v_t(R,x) = -w_R x + \min_{y \geq 0} \{K\mathbf{1}(y > 0) + L_0(y) + cy + Ev_{t+1}(1, (y - D_0)^+)\},$$

where $v_{T+1}(s, x) = -w_s x$. We assume that it is suboptimal to not place an order even though there is nothing on hand. The assumption is stated formally as follows.

Assumption 2 There exists a $y > 0$ such that $K + cy + L_0(y) - w_1 E(y - D_0)^+ < p_2 E(D)$.

One implication of the above assumption is that the set of feasible values for R is not empty. ($R = 1$ is feasible due to the assumption.) An empty set for R would imply the order quantity to be zero, which, apart from being a trivial solution, suggests the possibility of incorrect estimation of model parameters.

3.3 | Structure of the optimal solution

The structure of the optimal solution depends on the relationship between the imputed cost of lost sales (π_s) and age of inventory (s). When π_s decreases with s, provided it remains positive, the optimal value function $v_t(s, x)$ is convex in on-hand inventory. The convexity property continues to hold even when the imputed lost-sales cost increases with age provided the cost does not increase too rapidly. More precisely, we find that if $0 \leq \pi_{s+1} \leq \pi_s + h$ for any $s \in \{0, 1, \ldots, R-2\}$

[6] $L_s(x)$ becomes a concave function of x when $\pi_s < -h$.

and $\pi_{R-1} \geq w_R - h$, then the optimal value function is convex. The convexity of v_t implies that the function of order quantity, $L_0(y) + cy + Ev_{t+1}(1, (y - D_0)^+)$, is also convex in it.

We state the result in the following proposition.

Proposition 2 *Let $0 \leq \pi_{s+1} \leq \pi_s + h$ for any $s \in \{0, 1, \ldots, R-2\}$ and $\pi_{R-1} \geq w_R - h$. The optimal cost function $v_t(s, x)$ is convex in x for any s.*

Note that when both assumptions together get violated, the value function becomes concave. This implies that the optimal value of y will be at one of the boundary points (0 or $+\infty$) depending upon the value of c, which makes the problem uninteresting. Thus, the assumptions are not only necessary for a nice structure, but are also crucial to make the problem meaningful.

The condition $\pi_{s+1} \in [0, \pi_s + h]$ can be equivalently stated as $p_2 \in [p_1(1 - f(R-1)), p_1 + hF(R-1)]$, where

$$F(R-1) = \min_{s \in \{0,1,\ldots,R-2 : f(s) > f(s+1)\}} \frac{f(s)f(s+1)}{f(s) - f(s+1)},$$

As an example, when $f(s)$ is linear, the minimand decreases in s. Therefore, $F(R-1) = \frac{f(R-1)f(R-2)}{f(R-2) - f(R-1)}$. Note that the expression for $F(R-1)$ assumes that there exists at least one value of $s \in \{0, 1, \ldots, R-2\}$ for which $f(s) < f(s+1)$. If $f(s) = f(s+1)$ for all values of $s \in \{0, 1, \ldots, R-2\}$, then we set $F(R-1) = \infty$.

Consider now the scenario in which either $\pi_{s+1} \notin [0, \pi_s + h]$ for some s or $\pi_{R-1} < w_R - h$. In this case, the function $L_0(y) + cy + Ev_{t+1}(1, (y - D_0)^+)$ may be neither convex nor concave and it may have multiple local minima. (We say that a function $\zeta : \mathfrak{R} \to \mathfrak{R}$ has multiple local minima if there exist at least two local minima $w_1^* < w_2^*$ such that either $\zeta(w_1^*) \neq \zeta(w_2^*)$ or there exists $w_3 \in (w_1^*, w_2^*)$ such that $\zeta(w_3) \neq \zeta(w_1^*)$.) To see this result, without loss of generality, we consider the cost incurred over one cycle. The cost incurred over one cycle is a suitable objective since the choice of y affects the cost of only the following $R-1$ periods. We denote this cost by $V_R(y)$, and it is equal to

$$V_R(y) = K + \sum_{i=0}^{R-1} E\left(L_i\left(\left(y - \sum_{j=0}^{i-1} D_j\right)^+\right)\right) + cy - w_R E\left(y - \sum_{j=0}^{R-1} D_j\right)^+.$$

Let the optimal order quantity be denoted by y_R^*. In the following proposition, we characterize the equation that y_R^* satisfies and prove that the equation has at least one solution. We also obtain simple bounds on the optimal order quantity in two cases: $\pi_{s+1} > \pi_s + h$ for all s and $\pi_{s+1} \leq \pi_s + h$ for all s.

Proposition 3 *1. The optimal order quantity y_R^* satisfies the following equation:*

$$c - \pi_0 + \sum_{s=0}^{R-2} (h + \pi_s - \pi_{s+1}) G_s(y) + (h - w_R + \pi_{R-1}) G_{R-1}(y) = 0, \tag{3.3}$$

where $G_s(y) = P(D_0 + \cdots + D_{s-1} + D_s \leq y)$. Moreover, the above equation has at least one solution.

2. Let $h + \pi_{R-1} \geq w_R$. When $\pi_{s+1} \leq \pi_s + h$ for all s

$$G_0^{-1}\left(\frac{\pi_0 - c}{\pi_0 + Rh - w_R}\right) \leq y_R^* \leq G_{R-1}^{-1}\left(\frac{\pi_0 - c}{\pi_0 + Rh - w_R}\right).$$

On the other hand, when $\pi_{s+1} > \pi_s + h$ for all s,

$$G_{R-1}^{-1}\left(\frac{\pi_0 - c}{\pi_0 + Rh - w_R}\right) \leq y_R^* \leq G_{R-1}^{-1}\left(\frac{\pi_{R-1} - (R-1)h - c}{\pi_{R-1} + h - w_R}\right).$$

To derive the equation in part 1, we expand $L_s(\cdot)$ using the alternative expression stated in Proposition 1. Subsequently, we simplify the terms corresponding to the lost-sales cost. On the other hand, the bounds in part 2 are obtained using the observation that $G_{s+1}(y) \leq G_s(y)$ for any $s \in \{0, 1, \ldots, R-2\}$. Using the same approach, it is possible to obtain simple bounds on the optimal order quantity in two other (more general) cases: One, when $\pi_{s+1} > \pi_s + h$ for $s \geq s_0$ where s_0 is an arbitrary integer lying in $\{1, 2, \ldots, R-2\}$ and $\pi_{s+1} \leq \pi_s + h$ otherwise, and two, $\pi_{s+1} < \pi_s + h$ for $s \geq s_0$ (where s_0 is as before) and $\pi_{s+1} \geq \pi_s + h$ otherwise.

The assumption $h + \pi_{R-1} \geq w_R$ is useful in deriving the lower bound when $\pi_{s+1} \leq \pi_s + h$ for all s. However, it is possible to derive another lower bound when the assumption is violated. The details are available in the proof.

We will now use (3.3) to explain why $V_R(y)$ may not be convex and why it may have multiple local minima when $\pi_{s+1} \notin [0, \pi_s + h]$ or $\pi_{R-1} < w_R - h$. To keep analysis simple, consider this equation for $R = 2$:

$$c - \pi_0 + (h + \pi_0 - \pi_1) G_0(y) + (h - w_2 + \pi_1) G_1(y) = 0. \tag{3.4}$$

Thus,

$$\frac{d^2 V_2(y)}{dy^2} = (h + \pi_0 - \pi_1)\psi_0(y) + (h - w_2 + \pi_1)\psi_1(y),$$

where ψ_0 and ψ_1 are the densities of D_0 and $D_0 + D_1$, respectively. Suppose first that $\pi_1 > \pi_0 + h$. For large enough π_1, the RHS may become negative, which would imply a lack of convexity. Similarly, when $\pi_1 < w_2 - h$, for small enough π_1, the RHS may, once again, become negative. However, when $w_2 - h \leq \pi_1 \leq \pi_0 + h$, the above expression is always positive. To see the possibility of multiple local minima, we restate (3.4) as

$$\pi_0 - c + (\pi_1 - h - \pi_0) G_0(y) = (h - w_2 + \pi_1) G_1(y).$$

For $\pi_1 > \pi_0 + h$ and $h + \pi_1 > w_2$, both the LHS and RHS increase in y, and they may be equal at several places. Similarly, both the LHS and RHS decrease in y if $\pi_1 < w_2 - h < 0$. Once again, the LHS and RHS may be equal at several

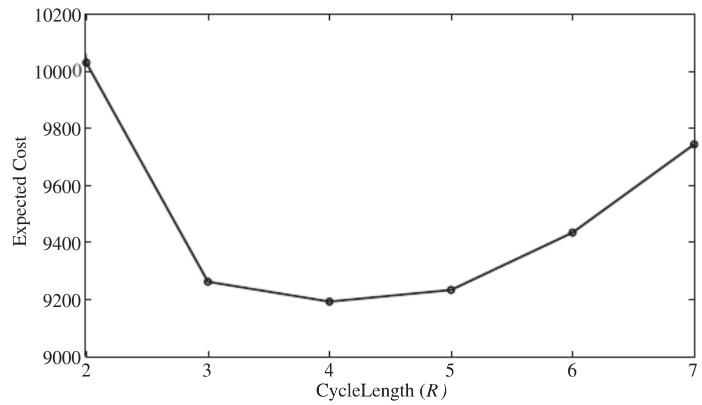


FIGURE 1 Expected cost over planning horizon as a function of cycle length (R). We take $p_1 = p_2 = 12$ and $K = 250$. All other parameters are same as in Table 2

places. The above equation also provides another glimpse of why there is a single minimum in order quantity when $\pi_1 \in (0, \pi_0 + h)$ and $h + \pi_1 > w_2$. Under these conditions, the LHS is decreasing and the RHS is increasing in y, which results in a single local minimum.

We next discuss how the cycle length (R) affects $V_R(y_R^*)$ and y_R^*. We state these relationships in the following three remarks.

1. It is not necessary that y_R^* increases in R. For example, if w_{R-1} is significantly larger than w_R, then it is possible that $y_{R-1}^* > y_R^*$. However, $w_{R-1} \gg w_R$ is not a necessary condition for $y_{R-1}^* > y_R^*$. Even if the salvage value is uniformly zero (ie, $w_i = 0$ for all i), y_R^* may still be smaller than y_{R-1}^*. One scenario in which this may occur is when demand in the last period in the cycle is likely to be low (due to low $f(R-1)$).
2. The optimal cycle cost, $V_R(y_R^*)$, increases in R. There are two main drivers of this result. One, the inventory-related costs increase with R. Even though the optimal order quantity may not be monotone with respect to R (as noted in the first remark), the total cost of carrying inventory and lost sales increases with R. Two, the salvage value of leftover inventory decreases with R. Together these factors cause $V_R(y_R^*)$ to increase with R.
3. $V_R(y_R^*)$ is not necessarily convex in R.

We present examples below to illustrate the nonmonotone behavior of y_R^* when $w_i \neq 0$ and possible nonconvexity of $V_R(y_R^*)$. Following the examples, we analytically establish two other observations that we discussed above: the increasing nature of $V_R(y_R^*)$, and the nonmonotone behavior of y_R^* when $w_i \equiv 0$.

Example 2 1. If $D \sim N(10, 2)$, $f(1) = 0.5$, $c = 5$, $h = 1$, $p_1 = 6$, $p_2 = 7$, $w_1 = 3.8$, $w_2 = 1.5$, then $y_2^* = 9.61 < y_1^* = 9.88$.

2. Suppose D is deterministic such that $D = 100$ and $f(s) = \exp\left(-\frac{s}{2}\right)$. $V_3(y_3^*) - V_2(y_2^*) = 73.6h > V_4(y_4^*) - V_3(y_3^*) = 66.9h$.

Proposition 4 *1. $V_R(y_R^*)$ increases with R.*

2. Suppose $w_i \equiv 0$ and $p_1 = p_2$. There exists an $\varepsilon > 0$ such that for all $f(R) \leq \varepsilon$, $y_R^* \leq y_{R+1}^*$.

3.4 | Optimal cycle length

The importance of the correct cycle length decision is apparent from the difference in fortunes of Interstate Bakeries and Flower Foods, as we discussed in Section 1. As the lifetime of bread (S) increased from 3 to 7 days, Interstate chose to increase the cycle length R automatically from 3 to 7 days while Flower Foods selected a smaller value of R of 4 days. For Interstate, the reduction in demand due to old inventory caused far more damage to its sales than the savings in fixed costs due to less frequent replenishment. See Figure 1 for a sample plot on how a sub-optimal selection of R could lead to reduction in expected profit. (In the figure, we have set $p_1 = p_2 =$ unit price, so the expected cost can be interpreted as negative of expected profit.)

With this motivation, we next discuss the computation of the optimal value of R. Given R, the total cost incurred over the planning horizon may be approximated by $\frac{T \cdot V_R(y_R^*)}{R}$. Let R^* be used to denote the optimal value of R. We first show that R^* cannot be greater than $\tilde{s}$ whenever $\tilde{s} \leq S - 1$. (Recall that $\tilde{s}$ is the smallest value of s for which $\pi_s \leq 0$.)

Proposition 5 $R^* \leq \tilde{s}$.

One implication of the above proposition is that when $p_2 \in (0, p_1(1 - f(1))]$, $\tilde{s} = 1$ and so it is optimal to place an order every period.

In general, the computation of R^* will be relatively straightforward if $\frac{V_R(y_R^*)}{R}$ is convex in R. However, this ratio may not be convex. For one example, consider the setting given in Example 2, part 2. For $h = 1$ and $K = 20$, $\frac{V_2(y_2^*)}{2} - \frac{V_1(y_1^*)}{1} = 20.33 > 11.08 = \frac{V_3(y_3^*)}{3} - \frac{V_2(y_2^*)}{2}$.

A weaker property than convexity but still conducive to the optimization is quasi-convexity. With quasi-convexity of $\frac{V_R(y_R^*)}{R}$, multiple local minima are ruled out, and the determination of R^* is easier. Indeed, $\frac{V_R(y_R^*)}{R}$ will be quasi-convex provided $V_R(y_R^*)$ is convex (Boyd & Vandenberghe, 2004, p. 103). However, as noted above, $V_R(y_R^*)$ may not always be convex.

In the absence of convexity or quasi-convexity of $\frac{V_R(y_R^*)}{R}$, one heuristic that could be used is to take the smallest local optimal solution; that is, identify the smallest R such that $\frac{V_{R+1}(y_{R+1}^*)}{R+1} - \frac{V_R(y_R^*)}{R} > 0$. If no such R exists, which would imply that $\frac{V_R(y_R^*)}{R}$ strictly decreases in R for all feasible values of R, then take the cycle length equal to S. The heuristic appears to perform well when we tested it computationally. For 25 different parameter combinations, the heuristic identified the optimal value of cycle length with 100% accuracy.

4 | FLEXIBLE REPLENISHMENT MODEL

In this section, we model and analyze a flexible replenishment strategy. In this strategy, the retailer can place an order in *any* period, so the time between two successive orders is not necessarily identical. In each period, the retailer pursues one of the following two options: Either retain old inventory or place an order and sell old inventory at clearance price. Once the age of inventory becomes equal to the lifetime of the product, an order is necessarily placed. We refer to the option in which no order is placed as Strategy 1. The other option in which an order is placed is referred to as Strategy 2.

With this setup, the optimal cost from period t through the end of horizon is

$$v_t(s,x) = \begin{cases} \min\{v_t^1(s,x), v_t^2(s,x)\}, & \text{for } s < S, \\ v_t^2(s,x), & \text{for } s = S, \\ -w_s x, & \text{for } t = T+1, \end{cases}$$

where

$$v_t^1(s,x) = L_s(x) + Ev_{t+1}(s+1,(x-D_s)^+), \tag{4.5}$$

is the optimal cost from period t through the end of horizon when Strategy 1 is used in period t and

$$v_t^2(s,x) = -w_s x + \min_{y\geq 0}\{K\mathbf{1}(y>0) + cy + L_0(y) + Ev_{t+1}(1,(y-D_0)^+)\} \tag{4.6}$$

is the optimal cost from period t through the end of horizon when Strategy 2 is followed in period t. The term $L_s(x)$ in (4.5) and (4.6) is as defined in (3.2).

To make analysis interesting, we assume the following.

Assumption 3 $v_t^2(s,0) < v_t^1(s,0)$. Moreover, if for some x, $v_t^1(s,x) = v_t^2(s,x)$, then the tie is broken in favor of Strategy 2.

The assumption ensures that an order is necessarily placed when there is nothing on-hand. If the assumption regarding relative values of $v_t^2(s,0)$ and $v_t^1(2,0)$ is violated for some t, then it may be optimal not to place an order in the period and lose all the demand. Thus, the assumption helps in avoiding trivial outcomes. Moreover, any inventory system with such an outcome is likely incorrectly estimating the cost parameters. Remaining assumptions remain the same as in Section 3.

Observe that similar to the model in Section 3, the above model becomes a multi-period newsvendor model when $S = 1$. Furthermore, when demand is deterministic (ie, $D \equiv d$), the fixed cycle and flexible replenishment models produce identical results. Since the flexibility to place orders is more useful when demand is more volatile, flexible replenishment is likely to be more useful in volatile demand environments.

4.1 | Analysis

For the flexible replenishment model, the optimal cost function $v_t(s,x)$ is not necessarily convex in x. To see this, observe that the optimal cost for either strategy is convex for $t = T$ (assuming $s < S$). However, the minimum of two convex functions is not necessarily convex, so the optimal cost function for $t = T$ or any other period is not necessarily convex. This is also illustrated in Figure 2A, which shows a sample plot of the optimal cost function. The figure also shows that the optimal cost function appears to lack properties, such as quasi-convexity or monotonicity, that are useful in establishing a structure for the optimal policy. This makes analysis inherently difficult, though we are still able to derive several results on the structure of the optimal policy.

4.1.1 | Inventory renewal

We begin by showing that if the age of inventory in a period is equal to $\tilde{s}$ (and so $\pi_{\tilde{s}} \leq 0$), then it is optimal to follow Strategy 2 in the period. In other words, if the imputed shortage cost is negative, then it is optimal to renew the inventory. The result is stated in the following proposition.

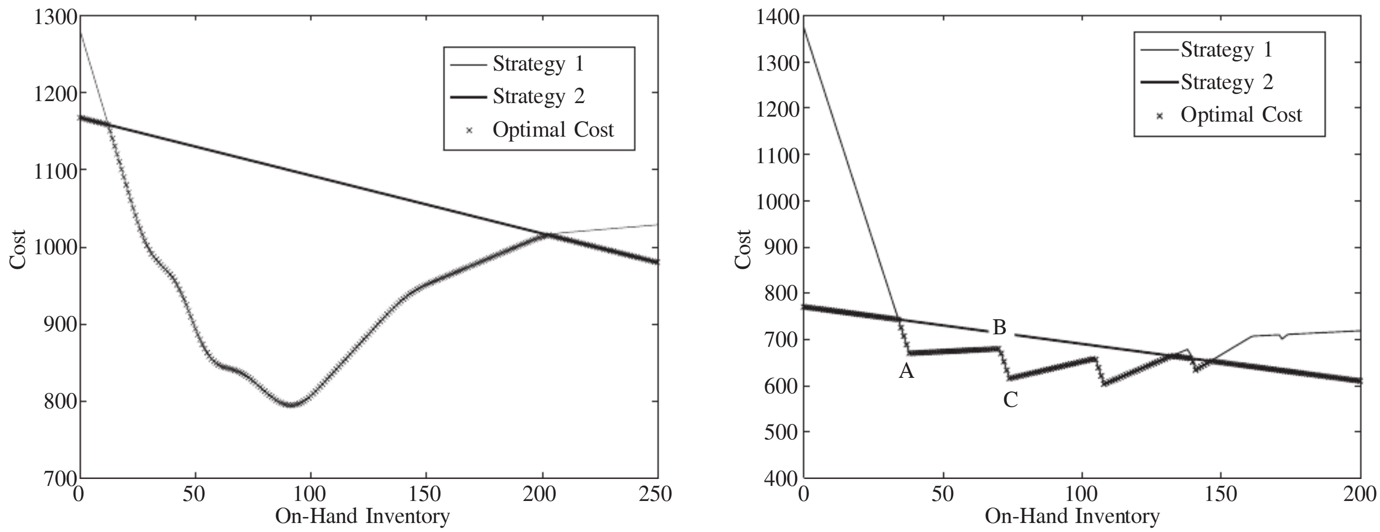
Proposition 6 *If $s = \tilde{s}$ in period t, then Strategy 2 is optimal in the period.*

Recall that if $p_2 \leq p_1(1-f(1))$, then $\tilde{s} = 1$. Thus, similar to the Fixed Cycle model, an order must be placed every period.

For the rest of this section, we assume that $s < \tilde{s}$. In the following proposition, we state two properties of the optimal policy when the lifetime is equal to two periods.

Proposition 7 *Let $S = 2$, and let $\pi_1 > 0$.*

1. *If $h + \pi_1 \geq w_2$, then $v_t^1(s,\cdot)$ is convex. Conversely, if $h + \pi_1 \leq w_2$, then $v_t^1(s,\cdot)$ is concave and decreasing.*



(a) (b)

FIGURE 2 For (A), $K = 100$, $h = 1$, $c = 5$, $p_1 = 10$, $p_2 = 10$, $D = (40 + \xi)$, $f(s) = 2 - \exp(0.1s)$, $w_s \equiv 0.75$, $t = 5$, $s = 2$, $T = 8$, $S = 6$, $\xi \sim N(0, 10)$ truncated at ± 10. For (B), $K = 30$, $h = 1$, $c = 2$, $p_1 = 6$, $p_2 = 18$, $D \equiv 40$, $f(s) = 2 - \exp(0.04s)$, $S = 6$, $t = 2$, $s = 1$, $T = 8$.

2. *Either* $v_t^2(s,x) < v_t^1(s,x)$ $\forall x$, *in which case Strategy 2 is followed for every x, or there exist two thresholds* x_t^l *and* x_t^u *such that Strategy 1 is followed for all values of* $x \in (x_t^l, x_t^u)$ *and Strategy 2 otherwise.*

When v_t^1 is concave, it always remains above v_t^2. On the other hand, when it is convex, it may either remain above v_t^2 for all x or meet v_t^2 twice. When the two functions meet twice, Strategy 2 is optimal when the initial inventory is either very low or very high. When the inventory is very low, it may be insufficient to satisfy demand in that period, so the retailer prefers to place an order. On the other hand, when inventory is very high, the cost of holding inventory may outweigh the possible future gain (through sale of that inventory) that could be obtained by retaining it. Therefore, it may be optimal for the retailer to write-off the old inventory through a clearance sale and make a fresh start through a new order.

We next discuss the general case in which the lifetime could be more than two periods. In the following theorem, we characterize the form of the optimal policy for general lifetime values. Similar to the case when $S = 2$, there exist two thresholds such that an order is placed when the amount of inventory at the beginning of a period is less than the lower threshold or greater than the upper threshold. However, it is not necessary that no order is placed between the two thresholds.

Theorem 1 *Suppose that the distribution of D has a bounded support, that is, there exists* $M < \infty$ *such that* $0 \le D \le M$. *If* $v_t^1(s,x) < v_t^2(s,x)$ *for some x, then there exists a lower threshold* $x_t^l(s)$ *and an upper threshold* $x_t^u(s)$ *such that Strategy 2 is followed when* $x \le x_t^l(s)$ *and* $x \ge x_t^u(s)$.

Although Strategy 1 may not be followed between $x_t^l(s)$ and $x_t^u(s)$, our extensive numerical experiments show this to be the case usually. They indicate that only when demand is deterministic or has very low volatility that an order might be placed when on-hand inventory lies between the two thresholds. In other words, when demand is sufficiently volatile, only Strategy 1 is likely to be optimal between $x_t^l(s)$ and $x_t^u(s)$.

To explain this observation, consider Figure 2B. In this figure, we have plotted optimal costs corresponding to Strategy 1, Strategy 2, and their minimum as a function of inventory. To construct this example, we have assumed the demand to be deterministic.

Observe that the plot corresponding to Strategy 1 has several peaks and troughs. Consider the trough point denoted by A. This point corresponds to the inventory required to satisfy exactly one period's demand. If the inventory at the beginning of the period is less than A, the retailer will lose sales. When the inventory increases beyond A, it is more than sufficient to satisfy one period's demand but not enough to satisfy demands of two periods. Thus, any remaining inventory will have to be salvaged in the next period when an order will be placed. This is true for all the inventory values between points A and B. For such values, the cost increases at a rate equal to the difference of holding cost rate and salvage value $(h - w_s)$. When the inventory increases beyond point B, it is now more beneficial to use it to satisfy a second period's demand than salvaging it entirely. For inventory values between points B and C, where C corresponds to the amount of inventory required to satisfy demands of two periods, the retailer will have to incur some lost-sales. As the inventory increases beyond C, the retailer once again has to salvage some of it while placing an order two periods later until it reaches the next peak.

As can be seen from Figure 2B, multiple points of intersection between $v_t^1(s,\cdot)$ and $v_t^2(s,\cdot)$ may arise due to the zigzag

nature of the plot for $v_t^1(s,\cdot)$. Sharp peaks and troughs arise since demand is fully known. This ceases to be true when demand is not deterministic. As a result, the curve becomes more "smooth" and peaks and troughs gradually vanish. Our experiments indicate that relatively mild values of demand uncertainty result in a smooth curve with mild or no peaks and troughs. In such a case, it is very likely that only Strategy 1 is followed between $x^l(s)$ and $x^u(s)$.

However, since it is not certain that Strategy 1 will necessarily be followed between points $x_t^l(s)$ and $x_t^u(s)$, we identify a set of inventory values where Strategy 1 is sure to be optimal. We accomplish this objective by developing two upper bounds on $v_t^1(s,x)$. These bounds are convex functions of x and thus intersect $v_t^2(s,x)$ at none or two points since $v_t^2(s,x)$ is a linear function of x for any given s. When a bound intersects with $v_t^2(s,x)$ at two points, the values of x that lie between the two points follow Strategy 1. We also note that the left intersection point of $v_t^2(s,x)$ with either bound provides an upper bound on $x_t^l(s)$. Similarly, the right intersection point of $v_t^2(s,x)$ with either bound provides a lower bound on $x_t^u(s,x)$.

We obtain the first upper bound on $v_t^1(s,x)$ by imposing the restriction that Strategy 1 be followed in period $t+1$ and Strategy 2 be followed in period $t+2$. (Clearly, the bound is feasible only when $s+1<\tilde{s}$.) Similarly, the second upper bound is obtained by imposing the requirement that Strategy 2 be followed in period $t+1$.

We state the result formally in the following proposition.

Proposition 8 *Let $t<T$. Let $B_t^1(s,x)$ and $B_t^2(s,x)$ be defined as follows:*

$$B_t^1(s,x)=L_s(x)+EL_{s+1}((x-D_s)^+)+Ev_{t+2}^2(s+2,((x-D_s)^+-D_{s+1})^+),\quad s+1<\min(\tilde{s},S)$$

where $\pi_{s+1}\in[0,\pi_s+h]$ and $h+\pi_{s+1}\geq w_{s+2}$, and

$$B_t^2(s,x)=L_s(x)+Ev_{t+1}^2(s+1,(x-D_s)^+),\quad s<\min(\tilde{s},S),$$

where $h+\pi_s\geq w_{s+1}$. $B_t^1(s,\cdot)$ and $B_t^2(s,\cdot)$ are convex. Thus, these functions intersect with $v_t^2(s,x)$ at none or two points. When $B_t^i(s,x)$ intersects with $v_t^2(s,x)$ at two points, let the left and right intersection points be denoted by ℓ^i and u^i, respectively. If $B_t^i(s,x)$ does not intersect with $v_t^2(s,x)$, set $\ell^i=+\infty$ and $u^i=0$. Then, $x_t^l(s)$ is bounded from above by $\min(\ell^1,\ell^2)$ *and $x_t^u(s)$ is bounded from below by* $\max(u^1,u^2)$.

Observe that it is possible to develop more bounds similar to the B_t^1 by requiring that Strategy 1 is followed in next k, $k\geq 2$, periods and Strategy 2 is followed in $(k+1)$-th period.

Similar to Proposition 2, the assumptions on cost parameters are crucial to ensure the convexity of the two functions. If, for example, $h+\pi_{s+1}\leq w_{s+2}$ and $h+\pi_s<\pi_{s+1}$, then $B_t^1(s,x)$ becomes concave and decreasing. Furthermore, the function will always remain above v_t^2.

To examine how close the bounds are to the optimal thresholds, we computed the thresholds, $\min(\ell^1,\ell^2)$ and $\max(u^1,u^2)$, for 25 different parameter combinations. For a sample of results, see Table S1 in Appendix S1. It is clear from the sample data that the bounds are either equal to the thresholds or very close to them. This means that the bounds, which are easier to compute, can be heuristically used as substitutes for the thresholds.

4.1.2 | Order quantity

We note that the computation of the optimal order quantity is time intensive due to lack of a simple structure of the cost function. It is also possible that there exist multiple order quantities that minimize the cost locally. However, the computational effort can be reduced if the search is restricted to an interval. With this objective, we state an upper bound and a lower bound on the optimal order quantity in the following proposition.

Proposition 9 *Suppose that $\pi_{s+1}\leq\pi_s+h$ for all $s\in\{0,1,\ldots,S-2\}$ and $\pi_s\geq(s-1)h+w_1$ for all $s\in\{0,\ldots,S-1\}$. If an order is placed in period t, $t\leq T-(S-1)$, then the optimal order quantity is bounded from above by $G_{S-1}^{-1}\left(\frac{\pi_0-c}{\pi_0+h-w_1}\right)$ and from below by $G_0^{-1}\left(\frac{\pi_0-c}{\pi_0+Sh-w_S}\right)$.*

Both bounds can be interpreted in terms of the critical ratio used in the newsvendor model. The lower bound is equivalent to the critical ratio for a newsvendor model with demand D_0, one-period underage cost of π_0-c, but with the maximum possible overage cost of $c+Sh-w_S$. On the other hand, the upper bound is equivalent to the critical ratio with maximum possible demand $D^{(S-1)}$, maximum possible underage cost of π_0-c, but lowest possible overage cost of $c+h-w_1$. In Table S3 in Appendix S1, we report the bounds as a function of multiple model parameters along with the optimal order quantity at $t=10$. The table shows that the upper bound is closer to the optimal order quantity than the lower bound.

Since the computation of the optimal order quantity is not easy, an alternative is to use a heuristic. One such heuristic is the use of the optimal order quantity for the fixed cycle model. Not only is this heuristic relatively easy to implement, it also appears to perform well. Upon testing for accuracy, we found that the heuristic results in a cost that is within 0%-2.3% of the optimal cost (see Table S2 in Appendix S1).

Before we close this section, we note that the flexible replenishment strategy can be alternatively modeled using a renewal process, which can be solved using the machinery of fractional programming. This approach is also applicable to the partial salvage strategy. For two examples of this approach

for inventory control models, see Li et al. (2012) and Feng and Xiao (2000).

5 | PARTIAL SALVAGE MODEL

In the previous section, we learned that when the retailer has excessive amount of inventory, she may be better off selling all of it at clearance price to save on inventory holding costs. Since the flexible replenishment model only permits either no salvage or full salvage, it may be even more cost effective to salvage only a few units instead of all of them and continue to sell the remaining inventory at list price without placing a new order. The strategy of partially salvaging extra inventory not only saves carrying costs, but may also result in fetching higher salvage revenue if some units are unlikely to be sold before the next order is placed. In this section, we analyze such a model in which some inventory may be salvaged at the beginning of each period in which Strategy 1 is followed.

Inventory write-offs or disposals for both perishable or non-perishable products are quite common in the real world. This is also reflected in the inventory control literature, as many papers have modeled disposal of inventory. Rosenfield (1989) develops an approach using a continuous review model to identify when and in what quantity to salvage some inventory when demand is stochastic. He also looks at the case when the product is perishable. Another paper that models inventory disposal is Çetinkaya and Parlar (2010). They consider a one-time disposal during the transition to a different replenishment policy. Finally, Li et al. (2016), which is also an excellent source for a summary of the papers that model inventory disposal, consider a periodic-review model for a perishable product with the possibility of clearance sale in every period. The key difference between that paper and our study is that they assume a LIFO issuance policy, while the consideration of an issuance policy is not required in our model.

We now discuss the modeling of partial salvage. For this purpose, we include a new decision variable z, which represents the partial salvage quantity, into the formulation. The modified expression for one-period expected cost is as follows:

$$L_s(x,z) = -w_s z + hE[x-z-D_s]^+ + \pi_s E[D_s - x + z]^+ + \frac{1-f(s)}{f(s)} p_1 E[D_s],$$

where we have included z as an argument in the definition of L_s.

Given the definition of $L_s(x, z)$, we now state the formulations of Strategies 1 and 2 as follows:

$$v_t^1(s,x) = \min_{0\le z\le x} \{L_s(x,z) + E[v_{t+1}(s+1,(x-z-D_s)^+)]\}, \tag{5.7}$$

$$v_t^2(s,x) = -w_s x + \min_{y\ge 0} \{K\mathbf{1}(y>0) + cy + L_0(y,0) + E[v_{t+1}(1,(y-D_0)^+)]\}. \tag{5.8}$$

Observe that the formulation for Strategy 2 remains unchanged. The analysis is simplified if we substitute $x - z = q$ into the formulation. The variable q may be interpreted as the quantity remaining after the partial salvage. With this substitution, the new formulation is

$$v_t^1(s,x) = \min_{0\le q\le x} \{Q_s(x,q) + E[v_{t+1}(s+1,(q-D_s)^+)]\},$$
$$v_t^2(s,x) = -w_s x + \min_{y\ge 0} \{K\mathbf{1}(y>0) + cy + Q_0(y,y) + E[v_{t+1}(1,(y-D_0)^+)]\},$$

where

$$Q_s(x,q) = -w_s(x-q) + hE[q-D_s]^+ + \pi_s E[D_s - q]^+ + \frac{1-f(s)}{f(s)} p_1 E[D_s].$$

Observe that $Q_s(x, q) = L_s(x, z)$, where $q = x - z$. The optimal cost function $v_t(s, x)$ remains the same as in Section 4, that is,

$$v_t(s,x) = \begin{cases} \min\{v_t^1(s,x), v_t^2(s,x)\}, & \text{for } s<S, \\ v_t^2(s,x), & \text{for } s=S, \\ -w_s x, & \text{for } t=T+1. \end{cases}$$

Similar to the flexible replenishment model, we assume that $v_t^2(s,0) < v_t^1(s,0)$ to ensure that an order is necessarily placed when there is nothing on hand. The remaining assumptions remain the same as for the fixed cycle model.

Compared to the flexible replenishment model, the optimal cost function for the partial salvage model has simpler structure. It still lacks convexity for the same reasons as in Section 4, but it is now decreasing in x. The difference arises because $v_t^1(s,x)$ is now decreasing in x. (See subsection A.11 for a proof.) Since $v_t^2(s,x)$, which remains unchanged, is also decreasing, the optimal cost function is decreasing in x. The monotonicity of v_t makes analysis considerably easier.

5.1 | Analysis

5.1.1 | Inventory renewal

We now turn our attention to establishing the form of the optimal replenishment policy. To accomplish this objective, we transform the cost functions as follows: $g_t^i(s,x) = v_t^i(s,x) + w_s x$ *for* $i = 1,2$, and let $g_t(s,x) = \min\{g_t^1(s,x), g_t^2(s,x)\}$. It is easy to see that $g_t(s, x) = v_t(s, x) + w_s x$. By definition,

$$\begin{aligned} g_t^1(s,x) &= \min_{0\le q\le x} \{w_s q + hE[q-D_s]^+ + \pi_s E[D_s-q]^+ + p_1\frac{(1-f(s))}{f(s)}E(D_s) \\ &\quad + E[g_{t+1}(s+1,(q-D_s)^+) - w_{s+1}(q-D_s)^+]\} \\ &= \min_{0\le q\le x} \{w_s q + (h-w_{s+1})E[q-D_s]^+ \\ &\quad + p_1\frac{(1-f(s))}{f(s)}E(D_s) + \pi_s E[D_s-q]^+ \\ &\quad + E[g_{t+1}(s+1,(q-D_s)^+)]\} \\ &=: \min_{0\le q\le x} J(q), \end{aligned}$$

and

$$g_t^2(s,x) = v_t^2(s,x) + w_s x = \min_{y\geq 0}\{K\mathbf{1}(y>0) + cy + L_0(y,0) + E[v_{t+1}(1,(y-D_0)^+)]\}.$$

Note that $g_t^2(s,x)$ is independent of both x and s, so we define it as a constant H. If $H \leq g_t^1(s,x)$ for all x, then it is optimal to always follow Strategy 2. One scenario in which this condition is satisfied is when $s = \widetilde{s}$. (The argument is similar to proof of Proposition 6.) The more interesting case occurs when $H > g_t^1(s,x)$ for some x. We state the optimal policy for that case in the following theorem.

Theorem 2 *Suppose q_x^* is the constrained minimizer of $J(q)$ over $q\in[0, x]$ and let q^* be the global minimizer of $J(q)$. Furthermore, suppose that $H > g_t^1(s,q^*)$.*

1. There exists a threshold $x_t^l(s)$ such that an order is placed if and only if $x \leq x_t^l(s)$.

2. Suppose Strategy 1 is followed. Then, the quantity $x - q_x^*$ is salvaged. Furthermore, for all $x \geq q^*$, the quantity $x - q^*$ is salvaged.

The above theorem states that there exists a threshold ($x_t^l(s)$) such that an order is placed in a period if and only if the inventory at the beginning of the period is less than or equal to the threshold. Otherwise, no order is placed and some inventory may be salvaged if necessary. When the inventory exceeds a certain threshold (q^*), it is optimal to salvage every unit in excess of the threshold. Even when the inventory lies between $x_t^l(s)$ and q^*, it may still be optimal to salvage some inventory. If it is optimal to salvage some inventory, its purpose would be to bring the inventory level to that local minimum of $J(q)$ lying within the interval $(x_t^l(s), x)$ that has the least cost.

Similar to Proposition 8, it is possible to obtain an upper bound on $x_t^l(s)$ by requiring that Strategy 2 be followed in period $t+1$. The corresponding cost function $\widetilde{B}_t(s,x)$ is as follows:

$$\widetilde{B}_t(s,x) = \min_{0\leq q\leq x} w_s q + (h - w_{s+1})E(q-D_s)^+ + p_1(1-f(s))E(D) + \pi_s E(D_s - q)^+ + Eg_{t+1}^2(s+1,(q-D_s)^+).$$

Since g_{t+1}^2 is a constant and the minimand of q is convex, $\widetilde{B}_t(s,x)$ is convex and decreasing in x (assuming $h+\pi_s \geq w_{s+1}$). Since $g_{t+1}^2(s,x) \geq g_{t+1}(s,x)$, $\widetilde{B}_t(s,x) \geq g_t^1(s,x)$. As a consequence, it meets $g_t^2(s,x)$ at most once, and its meeting point provides an upper bound on $x_t^l(s)$.

5.1.2 | Order quantity

Similar to the flexible replenishment model, the computation of the order quantity is challenging. With some modifications, the proof of Proposition 9 can be adapted to the partial salvage model. Thus, the bounds derived in Proposition 9 remain valid. Moreover, the optimal order quantity for the fixed cycle model can be used as a heuristic. Upon testing, we find that the heuristic results in a cost that is within 0%-1.81% of the optimal cost (see Table S2 in Appendix S1).

6 | NUMERICAL EXPERIMENTS

In this section, we numerically compare the optimal costs of the flexible replenishment and partial salvage models to that of the fixed cycle model. The purpose of these experiments is to estimate the value of flexible replenishment and partial salvage strategies. A second objective is to conduct a sensitivity analysis to understand how the cost improvements are affected by model parameters. Finally, we compare the magnitude of expected lost-sales due to old age and stockout for the three models.

We first describe the experimental setup. We use the following function to model the effect of product age on demand:

$$f(s) = 2 - \exp(\alpha s), \quad \alpha > 0.$$

Observe that this function is concave, which implies that the reduction in the fraction of customers who do not want to purchase the product due to its age is slow when age is small, but accelerates as age increases. The parameter α determines the rate at which the expected demand declines with respect to age. For small values of α, the demand reduction as age increases is relatively small, vice versa, this reduction occurs more and more rapidly as α increases. This is why we refer to α as age sensitivity of demand in the rest of this section.

Given the expression for $f(s)$, the demand in a period is equal to

$$D_s = D(2 - \exp(\alpha s)).$$

We take D to have a truncated normal distribution with mean a and variance σ^2; the truncation occurs at 0 and $2a$. The values of a and σ and other parameters used in our experiments are listed in Table 2.

In Table 3, we report the percent reduction in optimal costs of the flexible replenishment and partial salvage models with respect to the fixed cycle model as a function of lost-sales parameters p_1 and p_2, volatility of demand σ, age-sensitivity of demand α and fixed cost of order placement K. We use the following formulae to compute the percent reductions:

$$\frac{\text{Optimal cost of fixed cycle model - Optimal cost of flexible replenishment model}}{\text{Optimal cost of fixed cycle model}} \times 100\%,$$

and

$$\frac{\text{Optimal cost of fixed cycle model - Optimal cost of partial salvage model}}{\text{Optimal cost of fixed cycle model}} \times 100\%.$$

In Table S4 in Appendix S1, we present the same metrics for $p_1 = 16, p_2 = 10$.

Key observations from both tables are as follows.

TABLE 2 Default parameter values used for Table 3

T	a	σ	K	c	h	w_s	p_1	p_2	α	S
30	30	10	200	5	1	0.8-0.1 s	10	16	0.1	6

TABLE 3 Percent cost improvement for the flexible replenishment and partial salvage models with respect to the fixed cycle model

p_1	8	10	12	14	16	18
Flexible replenishment	5.69	5.13	4.14	3.41	2.99	2.82
Partial salvage	5.77	5.17	4.16	3.43	3.01	2.83
p_2	8	10	12	14	16	18
Flexible replenishment	0.00	2.42	3.63	4.48	5.13	5.66
Partial salvage	0.00	2.42	3.65	4.50	5.17	5.70
α	0.09	0.095	0.1	0.11	0.12	0.13
Flexible replenishment	5.06	5.12	5.13	5.16	4.95	4.79
Partial salvage	5.09	5.15	5.17	5.20	5.02	4.87
σ	8	9	10	11	12	13
Flexible replenishment	3.87	4.56	5.13	5.63	6.06	6.42
Partial salvage	3.90	4.59	5.17	5.67	6.11	6.47
K	125	150	175	200	225	250
Flexible replenishment	4.59	4.69	4.95	5.13	5.07	4.96
Partial salvage	4.60	4.71	4.97	5.17	5.11	5.02

1. The tables demonstrate that the optimal cost of the flexible replenishment model is lower than that of the fixed cycle model by 0%-6.4%. The tables also show that the optimal cost of the partial salvage model is only marginally better (less than 0.1%) than that of the flexible replenishment model. This small difference implies that the marginal value added by the use of the partial salvage strategy is significantly less than the marginal value added by the flexibility in placing an order based on the inventory level.

2. The tables also show that the reduction in cost in flexible replenishment and partial salvage models increases with p_2 and σ, but, interestingly, the trend reverses with respect to p_1. To explain this observation, first consider p_2. As p_2 increases, the value of $\pi_s = p_1 + \frac{p_2 - p_1}{f(s)}$ increases. Since π_s is the imputed cost of lost-sales, a higher value for it results in higher order quantities in each of the three models. An increase in σ also leads to higher order quantities due to greater safety stock requirement. However, higher order quantities increase the risk of excessive inventory. Since both flexible replenishment and partial salvage strategies manage inventory risk better, their cost advantage improves as p_2 and σ increase. On the other hand, as p_1 increases, π_s decreases. The reduction in cost improvements in flexible replenishment and partial salvage models with respect to p_1 is thus consistent.

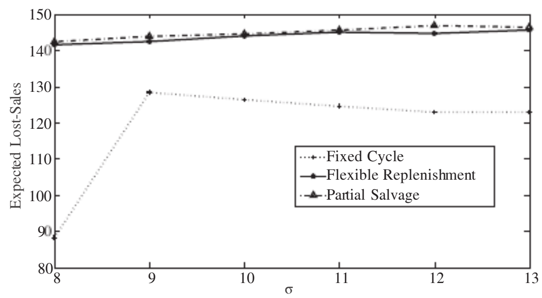
3. The cost improvements due to flexible replenishment and partial salvage strategies are nonmonotonic with respect to α and K. In fact, for both parameters, the values of p_1 and p_2 determine the shape of the percent cost reduction curve. For both parameters, this curve has a quasi-concave shape when $p_1 = 10$ and $p_2 = 16$. However, it becomes decreasing when $p_1 = 16$, $p_2 = 10$. (We have also observed a quasi-convex shape for some values of p_1 and p_2.) Even though the optimal cost function for each model shows a clear trend (increasing) with respect to both parameters, there does not appear to be a consistent trend in the percent cost reduction.

One key factor that plays a role here is relatively rapid changes in the optimal cycle length (R^*) for the fixed cycle model as either α or K increases, which leads to a sharp change in the cost components (eg, holding cost) for the model. Such changes may influence the shape of the curve.

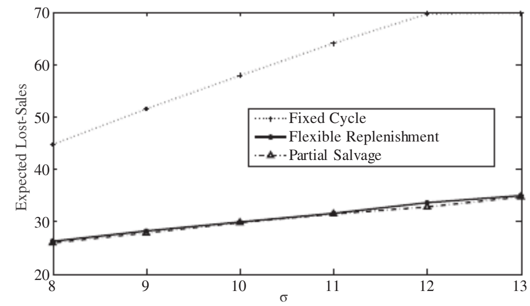
4. In general, greater number of lost-sales due to age occur in the flexible replenishment and partial salvage models compared to the fixed cycle model. In contrast, lost-sales due to stockout are more numerous in the fixed cycle model. (See Figure 3 for an example.) Out of the 52 parameter combinations considered in Table 3 and Table S4, in only three cases, the expected lost-sales due to age is greater for the fixed cycle model. Similarly, the expected lost-sales due to stockout is smaller for the fixed cycle model in none of the cases. At the same time, the expected quantity salvaged in the fixed cycle model is significantly larger than the other two models. Together these observations confirm that supply–demand mismatch is larger in the fixed cycle model. However, this mismatch is beneficial in some way for customers since whenever they see the inventory, it is fresher than the other two models.

7 | EXTENSIONS

In this section, we discuss two extensions to our models, random lifetime and nonzero lead-time.



(a) Aggregate expected lost-sales due to age over planning horizon



(b) Aggregate expected lost-sales due to stockout over planning horizon

FIGURE 3 All parameters are as in Table 2

7.1 | Random lifetime

So far, our analysis is based on an implicit assumption that the product quality does not deteriorate randomly over time. In other words, age of the product is a good *predictor* of its quality. If, however, product quality declines randomly over time, demand may decrease depending upon product quality rather than age, and the lifetime becomes random. In this subsection, we describe an approach to model such a scenario.

To model this phenomenon, we now let s to denote the quality level at the beginning of a period. The random variable s can take a value in the set $\{0,1,2, \ldots, S\}$ such that a high value of the state indicates greater deterioration. The product is fresh when s is equal to 0 and perishes when s becomes equal to S.

We assume that the process $\{s_t\}$ evolves according to a discrete-time Markov chain with the following transition probability matrix:

$$P(s_{t+1} = j | s_t = i) = \begin{cases} r_{ij} & \text{for } j \geq i, \\ 0 & \text{otherwise.} \end{cases}$$

State S is an absorbing state, so $r_{SS} = 1$. We further assume that the evolution of chain $\{s_t\}$ is independent of D in any period. Observe that the case in which age determines the demand is a special case of the above model such that r_{ij} is equal to 1 for $j = i + 1$ and 0 for other values of j.

The revision in the demand model also necessitates a revision in the cost formulations. For the flexible replenishment and partial salvage models, the revisions are relatively straightforward. For example, the formulation of the flexible replenishment model can now be stated as

$$v_t(s,x) = \begin{cases} \min\{v_t^1(s,x), v_t^2(s,x)\}, & \text{for } s < S \\ v_t^2(s,x), & \text{for } s = S \\ -w_s x, & \text{for } t = T+1 \end{cases}$$

where

$$v_t^1(s,x) = L_s(x) + \sum_{j=s}^{S} r_{sj} E v_{t+1}(j,(x-D_s)^+) \tag{7.9}$$

and

$$v_t^2(s,x) = -w_s x + \min_{y \geq 0} \left\{ K\mathbf{1}(y>0) + cy + L_0(y) + \sum_{j=0}^{S} r_{0j} E v_{t+1}(j,(y-D_0)^+) \right\}. \tag{7.10}$$

However, the fixed cycle model requires an additional clarification: Can a new order be placed if inventory reaches state S even before the completion of a cycle? In the spirit of the fixed cycle length assumption, we assume that a new order will be placed only at the end of every R periods even if inventory has reached the end of its life before. The revised model formulation for a given R is as follows:

$$v_t(s,x) = \begin{cases} L_s(x) + \sum_{j \geq s} r_{sj} E[v_{t+1}(j,(x-D_s)^+)], & s < S, t \neq nR+1, \\ -w_S x + p_2 E(D) + E v_{t+1}(S,0), & s = S, t \neq nR+1, \\ -w_s x + \min_{y \geq 0}\{K\mathbf{1}(y>0) + L_0(y) + cy \\ \quad + \sum_{j \geq 0} r_{0j} E v_{t+1}(j,(y-D_0)^+)\}, & t = nR+1, \end{cases}$$

where n is a positive integer and $v_{T+1}(s,x) = -w_s x$..

While some of the results, for example, Proposition 1, remain entirely unaffected, all the other results continue to hold, though the proofs may require some modification. One result whose proof requires a major modification is part 1 of Proposition 4. We restate this result and its proof in Appendix S1 (subsection A.13) to illustrate the changes required in re-proving the results.

7.2 | Nonzero lead time

In this subsection, we briefly discuss the case in which the lead-time for order placement is equal to one period.

For the fixed cycle model in Section 3, the optimal decisions remain unchanged. The only change required will be that orders are placed one period in advance.

For the flexible replenishment model, the formulation now becomes:

$$v_t(s,x) = \begin{cases} \min\{v_t^1(s,x), v_t^2(s,x)\}, & \text{for } s < S, \\ v_t^2(s,x), & \text{for } s = S-1, \\ v_t^1(s,x), & \text{for } t = T, \\ -w_s x, & \text{for } t = T+1, \end{cases}$$

where

$$v_t^1(s,x) = L_s(x) + E v_{t+1}(s+1,(x-D_s)^+), \tag{7.11}$$

and

$$v_t^2(s,x) = L_s(x) - w_{s+1} E(x - D_s)^+ + \min_{y \ge 0} \{K\mathbf{1}(y > 0) + cy + Ev_{t+1}(0,y)\}. \quad (7.12)$$

Observe that the formulation changes only for Strategy 2. Owing to the revision, the cost function for Strategy 2 is no longer linear in x; it now is convex in x, provided $\pi_s \ge w_{s+1} - h$. As for the optimal cost function, it may lack convexity just as in the case when lead time is zero. However, we have verified that a result similar to Theorem 1 continues to hold.

For the partial salvage model, we assume that the partial salvage decision (when Strategy 1 is followed) continues to be taken at the beginning of a period. With this assumption, the formulation for only Strategy 2 changes, and the new formulation is identical to (7.12). (The formulation for Strategy 1 remains same as in (5.7).) Therefore, $v_t^2(s,x)$ is convex for the partial salvage model as well. The nonmonotonicity of $v_t^2(s,x)$ implies that the optimal cost function may also not be monotone. Additionally, we have proved that there now exist two thresholds such that it is optimal to use Strategy 1 on the left of the smaller threshold and on the right of the larger threshold. Of course, it may be optimal to use Strategy 1 for some inventory values lying between the two thresholds as well. (We omit a formal statement of the results for brevity.)

8 | CONCLUSION AND FUTURE DIRECTIONS

We develop three models to obtain insights on when to replace old inventory and the quantity of fresh inventory to order for a perishable product while taking into account the reduction of demand with age. In the first model, the replenishment interval is static. In the second model, the replenishment interval is flexible and the retailer can choose to place an order in any period to replace aged inventory with fresh inventory. In the third model, the replenishment interval continues to remain flexible, but the retailer may also salvage a few units whenever it has excess inventory. A key contribution of the paper is to consider differential costs of lost-sales caused by stockout and customers walking away due to staleness of inventory.

A list of major insights from the study is as follows.

1. The total lost-sales penalty cost, which is the sum of lost-sales costs due to stockouts and customers not buying the product due to staleness, can be written as the product of an age-dependent imputed cost and the excess of demand over inventory. Thus, it has the same functional form as shortage cost in traditional inventory models. The imputed lost-sales cost may be both positive and negative depending upon the rate at which demand declines with age and the relative values of two lost-sales penalties. In particular, when the imputed cost is negative, the inventory-related cost function, which includes the holding cost and the total lost-sales penalty cost, may not be convex.

2. When the replenishment interval is fixed, the value function need not be convex in order quantity, unlike in traditional inventory control models. The convexity requires that the imputed lost-sales cost to be not only positive, but also to not increase too rapidly with age. We develop bounds on the optimal order quantity when the value function is convex as well as when it is not convex.

The cycle cost as a function of cycle length also does not appear to have a useful structure for an easy computation of the optimal cycle length. We show that a cycle does not include a period with negative imputed lost-sales penalty. We also develop heuristics to identify cycle length. The accuracy of the heuristic is 100% in the computational experiments.

3. When the replenishment interval is permitted to be flexible, the optimal cost function appears to lack important properties such as convexity and monotonicity that are useful in identifying the characteristics of the optimal decisions. We show existence of two thresholds such that it is optimal to place an order when the inventory at the beginning of a period is either less than the lower threshold or greater than the upper threshold. Furthermore, we develop bounds on the thresholds that are easy to compute. In the numerical experiments, we find that the bounds are tight.

Numerical experiments show that flexible replenishment may reduce cost by 0%-6.4% over the planning horizon.

4. In addition to flexible cycle length, when the retailer has the flexibility to reduce inventory level through partial salvage, the corresponding optimal value function has more structure. In particular, the value function now decreases in inventory and there exists a threshold such that an order is placed if and only if the inventory is less than the threshold. The partial salvage strategy, however, contributes little in reducing the cost further on top of the flexible replenishment, and the marginal cost improvement due to it is less than 0.1%.

One direction for future research is estimation of the penalties associated with customers walking away due to stockout and unsatisfactory age. In particular, as our analysis illustrates, it will be useful for retailers to know how the two penalties compare with each other for different products and whether one penalty always dominates the other. This knowledge on relative values of the penalties is necessary not only for effective inventory management of perishables, as we note in the paper, but also for effective resource allocation to increase customer satisfaction.

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SUPPORTING INFORMATION

Additional supporting information may be found online in the Supporting Information section at the end of the article.

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