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Option pricing under stock market cycles with jump risks: evidence from the S&P 500 index

Shin-Yun Wang, Ming-Che Chuang, Shih-Kuei Lin & So-De Shyu

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Abstract

This study incorporates the Markov switching model with return jumps to depict the behavior of stock returns. Based on the daily Standard & Poor's 500 index (hereafter SPX) and the daily closing price of the call option, we use the particle filtering algorithm to fit the parameter of the model. The joint log-likelihood evaluates the model performance: the weighted average log-likelihood with the rate of return of the SPX and the relative implied volatility root-mean-squared error for the SPX call options. The empirical results identify that the pricing model with jump risks improves the pricing performance to the median-term call options. According to the sensitivity analysis, option prices increase with the probability of remaining in the recession state but decrease with the probability of remaining in the expansion state. Moreover, the call option prices are positively associated with the volatility in each market state and the factors of jump risk.

Introduction

The normality of economic behavior is sometimes disrupted by dramatic events. To capture time-series behavior with business cycles, Hamilton (1990, 1989) pioneers the use of a Markov chain process to depict market state changes (also termed the *Markov switching model*). Since the introduction of Markov switching models to mainstream econometrics, they have received considerable attention from financial time-series analysis. There is a class of studies devoted to the forecasting of stock return, volatility, and the equity premium using Markov switching models.

Among these studies, that by Turner et al. (1989) represent the earliest example of applying the Markov switching technique to describe stock return behavior. They develop a two-regime Markov switching model whose transition probabilities remain constant. The main advantage of their model is an improvement in the accuracy of the stock return forecast under heteroskedasticity. Hamilton and Susmel (1994) distinguish a high-, median-, and low-volatility regime in stock return data, with the high-volatility regime being associated with economic recessions. Maheu and McCurdy (2000) also reach a similar conclusion, that volatilities are much higher in a bear market. Kim et al. (2004) develop a stock return model with a Markov switching volatility feedback effect to empirical test the positive relationship between the equity premium and stock market volatility. Extending the setting of regime-switching volatility, Kim et al. (2005) further examine the structural break in the equity premium based on Bayesian margin likelihood analysis. More recently, Chen (2007) investigates the asymmetric effects of monetary policy on stock returns using Markov switching models. Note that all of the above models based on the assumption that the dynamics of variables are continuous under a given regime. In brief, these studies ignore discrete effects in describing the behavior of economic variables.

To display the significance of such an effect, we take the Standard and Poor's 500 indexes (hereafter SPX) as the sample. From the empirical results, we observe that the index continued to grow during the period 1999–2000 due to the U.S. "new economy" effect, whereas in the second half of 2000, the economy faced the bursting of the dotcom bubble. The expansion from 2003 to mid-2007 is attributed to the effect of oil-shock-based inflation. However, the global subprime mortgage crisis occurred in 2008, leading to a recession. Such an undulating pattern for the index path is the so-called "stock market cycle," which is captured well by existing models. It is notable that fluctuations in the daily returns are visibly stronger, especially when abnormal events occur (e.g., the subprime mortgage crisis). The empirical results also identify the index returns to behave in a highly volatile manner within a short time period (e.g., one day). Such a dynamic is obviously not in line with the assumption that time-series variables act continuously. Therefore, the family of existing Markov switching models cannot explicitly capture the impacts of sudden shocks.

In 1976, Merton proposed the original type of jump-diffusion models. In his paper, he assumes that total changes in the asset prices may be divided into a normal variation part and an abnormal variation part. The former is modeled as a standard geometric Brownian motion with a fixed variance capturing continuous fluctuations in the prices because of strategic trading by informed or liquidity traders and market microstructure effects, whereas the latter is modeled as a counting process that reflects discrete effects due to unanticipated information released to the public. To capture the sudden shock under switching regimes, this paper combines Markov regime-switching processes with jump risks, and jump risks in the model are assumed to obey a Poisson process with a constant jump rate for the jump frequency and to follow a normal distribution for the jump sizes.

In the real world, the mean and volatility of a time series variable usually vary with market regimes. For instance, the mean level of the stock return is positive in a bull market but negative in a bear market, whereas its volatility is significantly higher during poor economic conditions. Such structural changes in the economic series cannot be captured by traditional models, which assume that all the observations are drawn from a Gaussian distribution with fixed mean and variance throughout the sample period. Moreover, based on our empirical results, it is found that the arrival of unanticipated abnormal events delivers sudden shocks to market states, leading the daily observations to behave in a highly volatile manner within a short time period. Such dynamics cannot be captured using existing models, which usually assume that variables act continuously under a given regime. To address this issue, we argue a Markov switching model with return jumps (MS-RJ). The MS-RJ model is especially useful for addressing financial phenomena, such as a leptokurtic feature of the asset return distribution, a volatility smile, and the volatility-clustering phenomenon.

In this paper, we use the joint log-likelihood to evaluate the model performance: the weighted average log-likelihood with the rate of return of the SPX and the relative implied volatility root-mean-squared error (RIVRMSE) for the SPX call options. For the log-likelihood for the rate of return of the SPX, we obtain the daily rate of returns for the SPX between January 5, 1999 and December 30, 2009 as a sample. Owing to the market state, jump risk, and stochastic volatility be unobservable, we employ an estimating and testing methodology with the particle filters algorithm (or called by sequential Monte Carlo algorithm) rather than the traditional maximum likelihood estimator for hidden states (market state, jump risks, and stochastic volatility). The employment of the particle filters algorithm over maximum likelihood estimation overcomes the problems of missing data and slow convergence. Because the Markov switching model comprises a Markov switching dynamics for the volatility and it can be categorized in the stochastic volatility models. It would be compared the goodness-of-fitting by others stochastic volatility models like the discrete-time GARCH family (Ornthanalai 2014; Christoffersen et al. 2012; Stentoft 2008; Heston and Nandi 2000) and the continuous-time stochastic volatility family (Bates 2012, 2000, 1996; Eraker 2004; Eraker et al. 2003; Bakshi et al. 1997; Heston 1993). Then, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) are used in this study to compare the fitting performance cross the Markov switching model with/without return jumps and others models. The empirical results show that the Markov switching model with return jumps exists the better goodness-of-fitting than other models. That is, the SPX market does have the features of market state switching and jump risks.

For the log-likelihood for the RIVRMSE of the SPX call options, based on the estimated parameters, we compute the call option price under each model with the characteristic function pricing framework which is a common method for the complex dynamics of the underlying price. Utilizing the daily closing price of SPX call options between January 04, 1999 and December 31, 2009 as a sample, we evaluate the pricing performance of each model with the RIVRMSE. The empirical results identify that the pricing model with jump risks can improve the pricing performance and the market state switching in the pricing formula applies to the median-term (the time-to-maturity is between 60 and 180 days) options.

There are contributions in this study: First, the time-series of SPX from January 5, 1999 through December 30, 2009 are characterized by properties such as the market cycles and jump risks. And, we use the particle filters algorithm to estimate the parameters of the Markov switching model with return jumps, because the occurrence of a market state and jump risks are unobservable, which leads to a latent data problem. Second, the closed solution of option pricing formula under the Markov switching model with return jumps is derived by the characteristic function pricing framework. Finally, because the Markov switching model is also a type of stochastic volatility model, we have compared the goodness-of-fitting for SPX and the pricing performance for SPX options cross others models.

The rest of this study is organized as follows. Section 1 presents stock return models, a Markov switching model with return jumps and the benchmark models. Section 2 estimates the model parameters using the particle filters algorithm and provides the empirical results for the SPX. Section 3 shows the European call option pricing formula under the Markov switching model with return jumps based on the no-arbitrage theorem and characteristic function pricing method. Numerical and empirical analyses are provided in Sect. 4. Section 5 draws the conclusion of this paper.

The model

Markov switching model with return-jumps (MS-RJ)

Consider a filtered probability space $\left\{\Omega, \mathbb{F}, \{\mathbb{F}(t)\}_{t=0}^{T}, \mathbb{P}\right\}$ where Ω is the set of all the possible outcomes, $\mathbb{F}$ is the sigma-field of subsets of Ω, $\{\mathbb{F}(t)\}_{t=0}^{T}$ is the sequence of the filtration at time t, and $\mathbb{P}$ is the real-world probability measure. The filtered probability space is generated by three components: First, the Gaussian white noise process, $Z^{\mathbb{P}}(t) \sim \mathrm{Normal}(0,1)$, representing the continuous noise of the stock market. Second, the compound Poisson process, $\sum_{n=1}^{\Delta N(t)} Y_n$, indicating the jump noise of the stock market. Note that the countable infinite sequence (or the jump size) $\{Y_n\}_{n=1}^{\infty}$ consists of independent and identically distributed (IID) random variables representing the n-times jump size ($Y_0 = 0$ presents no jumps in the market). For generalizability, the jump size follows a probability distribution $f_Y^{\mathbb{P}}(y)$ and its characteristic function (CF) is denoted by $g_Y^{\mathbb{P}}(\phi) = \mathrm{E}^{\mathbb{P}}\left(e^{i\phi Y_n}\right)$ where $i = \sqrt{-1}$ is the imaginary unit. And, the total number of jumps over the time period $(t-1, t]$ is counted by a Poisson process with the time-homogeneous intensity, i.e. $\Delta N(t) = N(t) - N(t-1) \sim \mathrm{Poisson}(\lambda)$. Third, the first-order Markov chain process exists the finite market states, i.e. $q(t) \in \{1, \cdots, M\}$ and the time-homogeneous transition probability matrix is

$$\mathcal{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1M} \\ p_{21} & p_{22} & \cdots & p_{2M} \\ \vdots & \vdots & & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MM} \end{bmatrix}, \quad (1)$$

where

$$p_{jk} = \mathbb{P}\left[q(t) = k | q(t-1) = j\right],$$

is a time-homogeneous transition probability for $\forall j, k \in \{1, \cdots, M\}$ and $t = 1, 2, \ldots, T$. Note that the sum of each entry in the same row must equal one; namely, $\sum_{k=1}^{M} p_{jk} = 1$ for $j = 1, \ldots, M$. To define the initial state probability, let $\xi_j = \mathbb{P}[q(1) = j]$, which satisfies $\sum_{j=1}^{M} \xi_j = 1$. In this study, we suppose that the probabilities of initial state are the same, i.e. $\xi_1 = \cdots = \xi_M = 1/M$. As usual, the four sources of randomness in the model, $\{Z^{\mathbb{P}}(t)\}_{t=1}^{T}$, $\{Y_n\}_{n=1}^{\infty}$, $\{\Delta N(t)\}_{t=1}^{T}$, and $\{q(t)\}_{t=1}^{T}$ are assumed to be mutually independent.

Definition 1

Consider a discontinuous trading economy. Under the $\mathbb{P}$ measure, given the filtration $\mathbb{F}(t-1)$, the dynamic process of stock returns, $R(t) = \ln[S(t)/S(t-1)]$, which governed by an MS-RJ model has the following explicit form:

$$R(t) = \mu_{q(t)} - \frac{1}{2}\sigma_{q(t)}^2 - \lambda\left[g_Y^{\mathbb{P}}(-i) - 1\right] + \sigma_{q(t)} Z^{\mathbb{P}}(t) + \sum_{n=1}^{\Delta N(t)} Y_n,$$

(2)

where $\mu_{q(t)}$ is the instantaneous mean level and $\sigma_{q(t)}$ instantaneous volatility at time t while the market state belongs to $q(t)$. Other notations are defined in the previous paragraph.

From Eq. (2), the volatilities of stock returns can be decomposed into the continuous variation and the jump variation. The continuous variation is modeled by a product of the Gaussian white noise and the regime-switching volatility, whereas a jump process reflecting the non-marginal effect of the information describes the jump variation. Our idea serves as an extension of Merton's jump-diffusion model. Note that $\sigma_{q(t)}^2/2$ and $\lambda\left[g_Y^{\mathbb{P}}(-i) - 1\right]$ are the convexity adjustment terms, which make the stock return equal to $\mu_{q(t)}$ given the market state $q(t)$ under the $\mathbb{P}$ measure, i.e. $\mathrm{E}_{\mathbb{F}(t-1)\vee q(t)}^{\mathbb{P}}[S(t)] = S(t-1)e^{\mu_{q(t)}}$, where $\mathrm{E}_{\mathbb{A}}^{\mathbb{P}}(\cdot)$ is the conditional expectation under the $\mathbb{P}$ measure given the condition $\mathbb{A}$.

The model of Eq. (2) can reduce in specific cases: First, if the market does not contain jump risks, $\lambda = 0$, the stock return follows a Markov switching model (MS) as shown by Duan et al. (2002), Hardy (2001). Second, if we exclude the market cycles, $\mu_1 = \cdots = \mu_M = \bar{\mu}$ and $\sigma_1 = \cdots = \sigma_M = \bar{\sigma}$, the dynamic of stock return reduces to the pure jump-diffusion model (denoted by GBM-RJ) as Merton (1976). Third, if the parameters are assumed by $\mu_1 = \cdots = \mu_M = \bar{\mu}$, $\sigma_1 = \cdots = \sigma_M = \bar{\sigma}$, and $\lambda = 0$, the dynamic process of the stock price is reduced to geometric Brownian motion (GBM).

Risk-neutral dynamics

Proposition 1

Related to Eq. (2), because of the noises from the continuous- and jump-components, $Z^{\mathbb{P}}(t)$ and $\sum_{n=1}^{\Delta N(t)} Y_n$ being independent, a conditional Randon-Nikodým derivative is assumed by the following form:

$$\frac{\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathbb{F}(t)}}{\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathbb{F}(t-1)}} = e^{-\frac{1}{2}h_{1,q(t)}^2\sigma_{q(t)}^2 + h_{1,q(t)}\sigma_{q(t)}Z^{\mathbb{P}}(t) - \lambda\left[g_Y^{\mathbb{P}}(-ih_{2,q(t)})-1\right] + h_{2,q(t)}\sum_{n=1}^{\Delta N(t)} Y_n},$$

(3)

where $h_{1,q(t)}$ and $h_{2,q(t)}$ are the parameters of change-of-measure for $Z^{\mathbb{P}}(t)$ and $\sum_{n=1}^{\Delta N(t)} Y_n$ with the market state $q(t)$. Thus, under the $\mathbb{Q}$ measure, given the market state $q(t)$, the distributions of the random variables are

$$Z^{\mathbb{Q}}(t) = Z^{\mathbb{P}}(t) - h_{1,q(t)}\sigma_{q(t)} \sim \text{Normal}(0,1),$$

(4)

$$\Delta N(t) \sim \text{Poisson}\left[\lambda g_Y^{\mathbb{P}}\left(-ih_{2,q(t)}\right)\right],$$

(5)

$$\{Y_n\}_{n=1}^{\infty} \overset{\text{IID}}{\sim} f_Y^{\mathbb{Q}}\left(y; h_{2,q(t)}\right), \quad f_Y^{\mathbb{Q}}\left(y; h_{2,q(t)}\right) = f_Y^{\mathbb{P}}(y)\frac{e^{h_{2,q(t)}y}}{g_Y^{\mathbb{P}}(-ih_{2,q(t)})}.$$

(6)

The parameters of change-of-measure, $h_{1,q(t)}$ and $h_{2,q(t)}$, can be seen as the market prices of risks for the noises from the continuous- and jump-components, $Z^{\mathbb{P}}(t)$ and $\sum_{n=1}^{\Delta N(t)} Y_n$. Also, for tractability, we assume that the jump size obeys a normal distribution with the constant mean θ and the variance ν^2. According to equations from (4) through (6), under the $\mathbb{Q}$ measure, the number of jumps and the jump size are distributed by $\Delta N(t) \sim \text{Poisson}\left[\lambda e^{h_{2,q(t)}\theta + \frac{h_{2,q(t)}^2\nu^2}{2}}\right]$ and $\{Y_n\}_{n=1}^{\infty} \overset{\text{IID}}{\sim} \text{Normal}\left(\theta + h_{2,q(t)}\nu^2, \nu^2\right)$, respectively.

Assume that the risk-free interest rate (r) is the constant for simplifying. According to the fundamental theorem of asset pricing which is developed by Harrison and Pliska (1983, 1981), based on the Eq. (3), the discounted stock price $e^{-rt}S(t)$ is a martingale under the $\mathbb{Q}$ measure. That is,

$$\mathrm{E}_{\mathbb{F}(t-1)}^{\mathbb{Q}}\left[e^{R(t)}\right] = \mathrm{E}_{\mathbb{F}(t-1)}^{\mathbb{Q}}\left[\frac{\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathbb{F}(t)}}{\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathbb{F}(t-1)}} e^{R(t)}\right] = e^r.$$

(7)

Proposition 2

Related to Eqs. (3) through (7), the martingale condition is equivalent to

$$r = \mu_{q(t)} + h_{1,q(t)}\sigma_{q(t)}^2 + \lambda\left[g_Y^{\mathbb{P}}\left[-i\left(1+h_{2,q(t)}\right)\right] - g_Y^{\mathbb{P}}\left(-ih_{2,q(t)}\right) - g_Y^{\mathbb{P}}\left(-i\right) + 1\right].$$

(8)

If we find the market prices of risks for the noises from the continuous- and jump-components, $h_{1,q(t)}$ and $h_{2,q(t)}$, to satisfy the martingale condition, the $\mathbb{Q}$ measure is called by the risk-neutral probability measure.

Note that the Eq. (8) identifies that the market is incomplete, i.e. there are infinite risk-neutral probability measures. The total risk premium can be decomposed into the risk premium from continuous component $-h_{1,q(t)}\sigma_{q(t)}^2$ and the risk premium from jump component $-\lambda\left[g_Y^{\mathbb{P}}\left[-i\left(1+h_{2,q(t)}\right)\right] - g_Y^{\mathbb{P}}\left(-ih_{2,q(t)}\right) - g_Y^{\mathbb{P}}\left(-i\right) + 1\right]$.

In special cases, if jump risk is diversifiable (i.e. $h_{2,q(t)} = 0$), the jump risk premium is zero, and the distributions of jump term (the compound Poisson) are the same between the $\mathbb{P}$ measure and the $\mathbb{Q}$ measure. It is often called by a Merton measure. If jump risk is non-diversifiable (i.e. $h_{2,q(t)} \neq 0$), and the distributions of jump term are altered. Gerber and Shiu (1994) select a specific $\mathbb{Q}$ measure with $h_{1,q(t)} = h_{2,q(t)}$, denoted the Gerber-Shiu (or Esscher) measure.

Definition 2

Consider a discontinuous trading economy. Assume that the jump risk premiums are the same in the market state, i.e. $h_{2,1} = \cdots = h_{2,M} = h_2$. Under the $\mathbb{Q}$ measure, given the filtration $\mathbb{F}\left(t-1\right)$, the dynamic process of stock returns, $R\left(t\right) = \ln[S\left(t\right)/S\left(t-1\right)]$, which governed by an MS-RJ model has the following explicit form:

$$R\left(t\right) = r - \frac{1}{2}\sigma_{q(t)}^2 - \lambda^{\mathbb{Q}}\left[g_Y^{\mathbb{Q}}\left(-i\right) - 1\right] + \sigma_{q(t)}Z^{\mathbb{Q}}\left(t\right) + \sum_{n=1}^{\Delta N(t)} Y_n,$$

(9)

where

$$\lambda^{\mathbb{Q}} = \lambda g_Y^{\mathbb{P}}\left(-ih_2\right), \quad g_Y^{\mathbb{Q}}\left(\phi\right) = \frac{g_Y^{\mathbb{P}}(\phi - ih_2)}{g_Y^{\mathbb{P}}(-ih_2)},$$

and other notations are defined in Eq. (2).

Note that $\sigma_{q(t)}^2/2$ and $\lambda^{\mathbb{Q}}\left[g_Y^{\mathbb{Q}}\left(1\right) - 1\right]$ are the convexity adjustment terms, which make the stock return equal to r given the market state $q\left(t\right)$ under the $\mathbb{Q}$ measure, i.e. $\mathrm{E}_{\mathbb{F}(t-1)\vee q(t)}^{\mathbb{Q}}\left[S\left(t\right)\right] = S\left(t-1\right)e^r$.To simplify the analysis, we assume that only two market state in the stock market like the model shown in Lin, Lin, and Wu (2015).

European call options pricing formula under the MS-RJ model

Theorem 1

Assume that the dynamics of stock return obeys the MS-RJ model under the $\mathbb{Q}$ *measure like the Eq.* (9). *Consider a* T*-maturity European call option with the strike price* (K). *Given the filtration* $\mathbb{F}(t)$, *the option pricing formula is*

$$\text{Call}\,(t;K,T)=\sum_{m=0}^{T-t}\frac{p_{21}\psi_{T,m|q(t)=1}+p_{12}\psi_{T,m|q(t)=2}}{p_{12}+p_{21}}\left[S\left(t\right)\Pi_{1,m}-Ke^{-r(T-t)}\Pi_{2,m}\right],$$

(10)

where $\psi_{T,m|q(t)=j}$ *presents the probability that the stock market belongs to the state 1 for* m *days while the time-to-maturity is* $T-t$ *days given the initial market state* $q\left(t\right)=j$. *It is proved in* Duan et al. (2002) *with a hidden Markov Chain. Others notations*

$$\Pi_{1,m}=\frac{1}{2}+\frac{e^{-r(T-t)}}{\pi S\left(t\right)}\cdot\text{Re}\left(\int_{0}^{\infty}\frac{e^{-i\phi\ln K}\,g_{X|\mathbb{G}(t)}^{\mathbb{Q}}\left(\phi-i\right)}{i\phi}d\phi\right),$$

$$\Pi_{2,m}=\frac{1}{2}+\frac{1}{\pi}\cdot\text{Re}\left(\int_{0}^{\infty}\frac{e^{-i\phi\ln K}\,g_{X|\mathbb{G}(t)}^{\mathbb{Q}}\left(\phi\right)}{i\phi}d\phi\right).$$

are lists in "Appendix A". *In this paper, the numerical integrations of* $\Pi_{1,m}$ *and* $\Pi_{2,m}$ *are approximated by the Gauss-Laguerre quadrature.*

Equation (10) can reduce in specific cases: First, if the market does not contain jump risks, $\lambda=0$, the pricing formula under the MS model is reduced to Elliott et al. (2005), Duan et al. (2002), Hardy (2001). Second, if we exclude the market cycles, $\mu_1=\cdots=\mu_M=\bar{\mu}$ and $\sigma_1=\cdots=\sigma_M=\bar{\sigma}$, the pricing formula reduces to Merton (1976)'s model. Third, if the parameters are assumed by $\mu_1=\cdots=\mu_M=\bar{\mu}$, $\sigma_1=\cdots=\sigma_M=\bar{\sigma}$, and $\lambda=0$, the pricing formula becomes to Black and Scholes (1973)'s model. The other pricing model with the stochastic volatility under the discrete- and continuous-time framework is shown in "Appendix B".

Parameters fitting

Joint estimation

Numerous studies on stock index options perform joint estimations to fit the model parameters with the prices of the underlying stock and stock options like Chang et al. (2018) and Ornthanalai (2014). The key is that the price difference between the stock index and stock options mutually affect each other. Therefore, we also employ the joint estimation to fit the model parameters and the market price of risk for each model.

Let's talk about the log-likelihood function of the stock index dynamics first. We take the MS-RJ model as an example. Owing to the MS-RJ model include the latent/unobservable components (the market state and the return jump), it is very difficult to estimate the model parameters by maximizing the incomplete log-likelihood function from the observable data (the stock return). The particle filter (PF) algorithm is the common method for estimating the model parameters like the Markov switching family, the Lévy-jump family, and the stochastic volatility family. Through sampling and re-sampling the particles (the latent components), we can compute the complete log-likelihood function for the stock index dynamics.

Given the observable stock returns $\mathbf{R} = \{R(1), \cdots, R(T)\}$, the latent components like the stock market states and the total jump sizes are denoted by $\mathbf{q} = \{q(1), \cdots, q(T)\}$ and $\mathbf{Jump} = \left\{\sum_{n=1}^{\Delta N(1)} Y_n, \cdots, \sum_{n=1}^{\Delta N(T)} Y_n\right\}$. And, the set of parameters under the MS-RJ model is indicated by Φ, i.e.

$$\Phi = \{p_{12}, p_{21}, \mu_1, \mu_2, \sigma_1, \sigma_2, \theta, \nu, \lambda, h\},$$
(11)

with the boundaries of parameters $0 \le p_{12}, p_{21} \le 1$, $-\infty < \mu_1, \mu_2, \theta < \infty, 0 \le \sigma_1, \sigma_2, \nu, \lambda < \infty$. There are two steps for handling the PF algorithm: The Monte-Carlo filtering and the resampling.

First, for processing the Monte-Carlo filtering, we obtain $M \in \mathbb{Z}^+$ independent samples for each random variables, $q(t) = \{1, 2\}$ with $u(t)|_{q(t-1)=j} \sim \text{Bernoulli}(p_{j1})$, $\Delta N(t) \sim \text{Poisson}(\lambda)$, $\{Y_n\}_{n=1}^{\infty} \overset{\text{IID}}{\sim} \text{Normal}(\theta, \nu^2)$ from the corresponding distributions. Then, given the m-th particle of market state and return jump and the parameters of the k-th iteration, the probability weight of stock return is computed by

$$w^{(m)}\left(t|\Phi^{(k)}\right)=\mathbb{P}\left[R(t)\,|q^{(m)}(t)=j,Jump^{(m)}=a,\Phi^{(k)}\right]$$
$$=\frac{1}{\sqrt{2\pi}\sigma_j}\exp\left\{-\frac{\left[R(t)-\mu_j-\frac{\sigma_j^2}{2}-\lambda\left(e^{\theta+\frac{\nu^2}{2}}-1\right)+a\right]^2}{2\sigma_j^2}\right\}.$$

(12)

Second, the resampling focuses on filtering the independent particles obtained from the Monte-Carlo filtering. That is, according to the corresponding weight, we exclude the particles with the lower weight (i.e. the lower importance) and add in the more important particles from the set of the original particles. The new particles can depict latent components (the market states and the return jumps) more exactly. Given the parameters of the k-th iteration and the re-sampling particles, the complete log-likelihood, $\mathrm{LLF}_{\mathrm{return}}(\Phi|\mathbf{R})$, is shown as follows:

$$\mathrm{LLF}_{\mathrm{return}}(\Phi|\mathbf{R})=\sum_{t=1}^{T}\sum_{m=1}^{M}\frac{1}{M}\ln w^{(m)}\left(t|\Phi^{(k)}\right),$$

(13)

On the other hand, the information from the stock options market is also important. Let $\zeta_n(t)$ be the relative error of the implied volatility at time t, i.e.

$$\zeta_n(t)=\frac{\mathrm{IV}_n^M(t)-\mathrm{IV}(t;T_n,S_n,K_n,O_n,\Phi_t)}{\mathrm{IV}_n^M(t)}\sim\mathrm{Normal}\left(0,\sigma_\zeta^2\right).$$

(14)

where $\mathrm{IV}_n^M(t)$ represents the market implied volatility call option, and $\mathrm{IV}(t;T_n,S_n,K_n,O_n,\Phi)$ is the theoretic implied volatility of call option given the settle date t, maturity T_n, the current price of the underlying stock index S_n, the strike price K_n, the theoretic premium of the call option O_n, and the estimated parameters. In addition, the implied volatility is determined by

$$\mathrm{IV}(t;T_n,S_n,K_n,O_n,\Phi_t)=\left\{x\in\mathbb{R}^+|\mathrm{Call}(t;K,T,\Phi_t)=\mathrm{BS}(t;K,T,x)\right\},$$

(15)

where $\mathrm{BS}(t;K,T,x)$ is the theoretic premium of the call option under the Black–Scholes' model, i.e.

$$\mathrm{BS}(t;K,T,x)=S(t)\Psi[d_1(x)]-Ke^{-r(T-t)}\Psi[d_2(x)],$$
$$d_1(x)=\frac{\ln\frac{S(t)}{K}+\left(r+\frac{1}{2}x^2\right)(T-t)}{x\sqrt{T-t}},$$
$$d_2(x)=\frac{\ln\frac{S(t)}{K}+\left(r-\frac{1}{2}x^2\right)(T-t)}{x\sqrt{T-t}},$$

and $\Psi(\cdot)$ is the cumulative density function of standard normal distribution. Equation (15) evaluates the models based on the pricing errors between the market volatility and the model volatility. Given the daily implied volatilities, we can calculate the log-likelihood for the options pricing errors as follows:

$$\mathrm{LLF}_{\mathrm{option}}\left(\Phi|\mathbf{IV}\right)=\sum_{t=1}^{T}\sum_{n=1}^{N}\left[-\frac{1}{2}\ln\left(2\pi\sigma_{\zeta}^{2}\right)-\frac{\zeta_{n}^{2}\left(t\right)}{2\sigma_{\zeta}^{2}}\right].$$

(16)

In practice, because of the sample sizes of the stock return and the options being different significantly, we assign the following weighted average log-likelihood when calculating the joint log-likelihood,

$$\mathrm{LLF}_{\mathrm{Joint}}\left(\Phi|\mathbf{R},\mathbf{IV}\right)=\frac{T+N}{2}\cdot\frac{\mathrm{LLF}_{\mathrm{return}}\left(\Phi|\mathbf{R}\right)}{T}+\frac{T+N}{2}\cdot\frac{\mathrm{LLF}_{\mathrm{option}}\left(\Phi|\mathbf{IV}\right)}{T}.$$

(17)

Finally, we obtain the parameter estimates in the $(k+1)$-th iteration by maximizing $\mathrm{LLF}_{\mathrm{Joint}}\left(\Phi|\mathbf{R},\mathbf{IV}\right)$, employing the re-sampling particles from the m-th iteration, i.e.

$$\hat{\Phi}^{(k+1)}=\arg\max_{\Phi}\mathrm{LLF}_{\mathrm{Joint}}\left(\Phi|\mathbf{R},\mathbf{IV}\right).$$

(18)

Substituting $\hat{\Phi}^{(m+1)}$ into the Monte-Carlo filtering and the resampling, then we can perform the $(k+2)$-th iteration. Similarly, we iterate these steps until the maximum likelihood estimators are convergent.

Description of data

The data include the daily Standard & Poor's 500 indexes (SPX) from January 5, 1999 through December 30, 2009, and the daily closing prices of the call option with the sample period covers between January 02, 2004 and December 31, 2009. These data are obtained from the Datastream database. The data on sudden shocks in SPX daily returns are provided in Table 1. Weak, median, and strong shocks are defined as occurring when the daily observation is over or below the single, double, and triple standard deviation, respectively, of those computed during the full period. From top graphics of Fig. 1, we observe that the index continued to grow during the period 1999–2000 due to the U.S. "new economy" effect, whereas in the second half of 2000, the economy faced the bursting of the dotcom bubble. The expansion from 2003 to mid-2007 is attributed to the effect of oil-shock-based inflation. However, the global subprime mortgage crisis occurred in 2008, leading to a recession. Such an undulating pattern for the index path is the so-called "stock market cycle," which is captured well by existing models.

Table 1 Sudden shocks in the daily returns

Fig. 1

Time-series of the stock index and the rate of return. Note that the data include the daily Standard & Poor's 500 indexes (SPX) from January 5, 1999 through December 30, 2009. This figure plots the time-series of the daily closing index and log-return for SPX

Compared to the bottom graphics of Fig. 1, it is notable that fluctuations in the daily returns are visibly stronger, especially when abnormal events occur (e.g., the subprime mortgage crisis). The numbers shown in Table 1 provide a clearer insight into the sudden shocks. For example, the ratio of strong-shock observations to the total is 1.5913%, which approximates the likelihood of a strong sudden shock. Further, the means and variances of returns under shocks are also lower and higher, respectively, relative to those computed by the full sample. These facts suggest that as abnormal events strike the market, the index returns behave in a highly volatile manner within a short time period (e.g., one day). Such a dynamic is obviously not in line with the assumption that time-series variables act continuously. Therefore, the family of existing Markov switching models cannot explicitly capture the impacts of sudden shocks.

The data on the daily closing price of the SPX options with 6 to 360 days-to-maturity and 0.90 to 1.10 spot-strike-ratio (K/S) are reserved. After filtering the sample, there are a total of 204,351 available observations for call options. The sample of SPX options is divided into 15 categories, i.e. three types of days-to-maturity (short-term for < 60; median-term for 60–180; long-term for 180–360) and five types of moneyness (K/S) (deep-in-the-money for 0.90–0.94; in-the-money for 0.94–0.98; at-the-money for 0.98–1.02; out-the-money for 1.02–1.06; deep-out-the-money for 1.06–1.10). Obviously, there exists the volatility smile for the short-term options and the volatility smirk for the median- and long-term options. Moreover, the average call price from $3.66 for short-term and out-the-money options to $154.67 for long-term and in-the-money. Table 2 lists the sample properties of SPX call options like the average implied volatility and the average closed price for each category.

Table 2 Sample properties of SPX call options

Fitting results

The particle filter algorithm as shown in Sect. 2.1 estimates the parameters under the MS, MS-RJ, GBM-RJ, SV, SV-RJ, and SV-RJ-VJ models while the parameters under the GBM and GARCH model are estimated by the maximum likelihood estimation. Table 3 presents the estimated parameters, the joint log-likelihood, the Akaike information criterion (AIC), the Bayesian information criterion (BIC) for each model. Under the MS-RJ model, the transition probabilities of SPX return (p_{12}, p_{21}) are 0.5439 and 0.4491, respectively, and it implies that the market state belongs to the expansion (market state 2) with the more probability than recession (market state 1) during the sample period. In a recession, the mean stock return μ_1 is – 0.0022 with a standard deviation σ_1 of 0.0128. In contrast, in expansion, the mean stock return μ_2 is 0.0019 with a standard deviation σ_2 of 0.0067. The volatility of stock returns in the expansion is more stable than that in recession. For the jump risks, the jump frequency λ is 0.1256, and it means that an average number of 8 days per jump. And, the mean of jump size θ is 0.0021 implying that the release of unanticipated information on average causes returns to increase but not statistical significance. The standard deviation of jump size ν is 0.0254. The market price of risk h is – 0.0330. From the bottom of Table 3, the MS-RJ model exists the maximum joint log-likelihood function and minimum AIC/BIC, and it tells that the MS-RJ model is the better goodness-of-fitting and the pricing performance than other models.

Table 3 The estimating and testing under each model

Figure 2 shows that the conditional probability of recession (market state 1) and the conditional probability of jump calculated by the posterior distributions, and the daily jump size estimated by the particle filter algorithm given the estimated parameters of MS-RJ model. The top graphics of Fig. 2 indicates that the probability of recession from 1999 to 2002 is high because of the dot-com bubble and that the probability of recession from 2003 to 2007 is low. Therefore, there is a transition of states from 2002 to 2003. In 2008, as the financial crisis progressed, the probability of a recession in 2008 also increased. There is also a switch of states from 2007 to 2008. The middle and bottom graphics of Fig. 2 shows that the high probability of jumps and the large jump size from 2000 to 2003 and in 2007 and 2008, consistent with the events of the dot-com bubble in 2000, the September 11 attacks in 2001, the end of the Iraq war in 2003, the Yen carry trade in 2007, and the financial crisis in 2008.

Fig. 2

Time-Series of the conditional probability of recession, the conditional probability of jump, and the jump size. Note that the data include the daily Standard & Poor’s 500 indexes (SPX) from January 5, 1999 to December 30, 2009. Based on the estimated parameters of MS-RJ model, the conditional probability of recession (market state 1) and the conditional probability of jump are calculated by the posterior distributions. Moreover, the daily jump size is estimated by the particle filter algorithm

The validity of the stock index option

Option pricing performance

According to the empirical results of the previous section, given the full sample, we know that the MS-RJ model can improve the goodness-of-fitting and the pricing performance. However, we are also interested in the performance of each model in different periods. In implementing the in-sample option pricing procedure, we re-fit the parameters daily with the past 1250 daily rates of return and the current closing prices of the SPX call options. The joint estimation process is shown in Sect. 3.1. For assessing the option pricing performances of models, like Ornthanalai (2014), we report the relative implied volatility root-mean-squared error (RIVRMSE) at a specific date t. That is,

$$\text{RIVRMSE}(t) = \sqrt{\frac{1}{N(t)}\sum_{n=1}^{N(t)}\left(\frac{\text{IV}_n^M(t) - \text{IV}\left[t; T_n, S_n, K_n, O_n, \Phi(t)\right]}{\text{IV}_n^M(t)}\right)^2},$$

(19)

where $N(t)$ denotes the number of call options at time t, $\text{IV}_n^M(t)$ and $\text{IV}\left[t; T_n, S_n, K_n, O_n, \Phi(t)\right]$ are defined by Eqs. (14) and (15), and the daily re-fitted parameters.

Table 4 reports the daily average of the RIVRMSE computed by Eq. (19). Let's observe the short-term categories. The MS-RJ model exists the smallest RIVRMSE for the deep-in-the-money options (2.14%) while the GARCH-RJ model demonstrates the superior pricing performance for the in-the-money (3.22%), the at-the-money (4.40%), the out-the-money (4.99%), and the deep-out-the-money options (3.78%). For the median-term call options, the RIVRMSE is the smallest under the MS-RJ model in the categories of deep-in-the-money (2.90%), the in-the-money (3.78%), the at-the-money (5.35%), and the out-the-money (6.48%). And the GARCH-RJ model also performs the smaller pricing errors than others model in the category of deep-out-the-money (6.73%). Finally, the MS-RJ model and SV-RJ-VJ model exist the best pricing performance in each category. There are three types of the stochastic volatility model in Table 4: the MS family, the GARCH family, and the SV family. Based on empirical results, the GARCH-RJ model, the MS-RJ model, and the SV-RJ-VJ model enhance the pricing performance for the short-, the medium-, and the long-term options, respectively.

Table 4 In-sample pricing errors

Next, we discuss the evaluation of option pricing model for each model during the sub-period of the sample. Given the re-fitted parameters day-by-day as the same as Table 4, the time-series of monthly averaged RIVRMSE of short-, median-, and long-term call options are graphed in Figs. 3, 4, and 5, respectively. In the category of the short-term and deep-in-the-money call options in Fig. 3, the RIVRMSEs of the MS-RJ model is smaller than other models between initial-2004 and mid-2007. During the same sub-period, the GARCH-RJ model improves the pricing error for in-the-money, at-the-money, out-the-money, and deep-out-the-money call options. Between mid-2007 and mid-2009 (period of the financial crisis), the SV-RJ-RJ model exhibits better pricing performance than other models for deep-in-the-money, in-the-money, at-the-money call options. In addition, the pricing error of SV-RJ-VJ model increases after late-2009. Overall, the MS-RJ model exists a smaller deviation of RIVRMSE than other models. They can provide stable pricing performance during the sample period of this paper.

Fig. 3

Time-Series of monthly averaged RIVRMSE for short-term call options. Note that we re-fit the parameters daily with the past 1250 daily rates of return and the current closing prices of the SPX call options. The joint estimation process is shown in Sect. 3.1. This figure plots the time-series of monthly averaged RIVRMSE, computed by Eq. (19), for short-term call options

Fig. 4

Time-Series of monthly averaged RIVRMSE for median-term call options Note that we re-fit the parameters daily with the past 1250 daily rates of return and the current closing prices of the SPX call options. The joint estimation process is shown in Sect. 3.1. This figure plots the time-series of monthly averaged RIVRMSE, computed by Eq. (19), for median-term call options

Fig. 5

Time-Series of monthly averaged RIVRMSE for median-term call options. Note that we re-fit the parameters daily with the past 1250 daily rates of return and the current closing prices of the SPX call options. The joint estimation process is shown in Sect. 3.1. This figure plots the time-series of monthly averaged RIVRMSE, computed by Eq. (19), for long-term call options

In the category of the median-term deep-in-the-money, in-the-money, at-the-money, out-the-money call options in Fig. 4, the RIVRMSEs of the MS-RJ model is also lower than other models between initial-2004 and mid-2007. During the same sub-period, the GARCH-RJ model improves the pricing error for the deep-out-the-money call options. However, after the financial crisis in 2008, the SV-RJ-VJ model exhibits better pricing performance than other models. Similar empirical results are shown in Fig. 5. Overall, before the financial crisis in 2008, the GARCH-RJ model, the MS-RJ model, and the SV-RJ-VJ model strengthen the pricing performance for the short-, the medium-, and the long-term options, respectively. And, the SV-RJ-VJ model performs less pricing errors than other models after the financial crisis.

On the other hand, in implementing the out-of-sample option pricing procedure, we compute the current model implied volatility of call option relying on the day-by-day re-fitted parameters of the previous one day. Table 5 reports the daily time-series of RIVRMSE computed by Eq. (24). For the category of the short term and deep-in-the-money, the out-of-sample pricing error measure under the MS-RJ model exists the minimum error. The ranks cross nine models in each category are as similar to the results of Table 4.

Table 5 Out-of-sample pricing errors

Volatility smile

Based on the option price, asset price, strike price, risk-free interest rate, and maturity, the implied volatility is computed under the Black–Scholes model. Harvey and Whaley (1992) indicate that implied volatility changes with the expectation and the changes in the market. Therefore, implied volatility not only represents current market prices but also reflects market expectations. The volatility curve is plotted under different ratios of the strike to spot price. Because the implied volatility for at-the-money options is smaller than that for out-the-money or in-the-money options, the volatility curve is convex and thus named a volatility smile. However, Fleming, Ostdiek, and Whaley (1995), Harvey and Whaley (1992), Schwert (1989) note that the increasing speed of implied volatility for in-the-money options is faster than that for out-the-money options, making the volatility smile more of a smirk.

Figure 6 plots the implied volatility the 1, 2, 6, and 12 months-to-maturity call options with different cases of moneyness given the current stock indexes, the annual riskless rates at some trading dates, and the daily estimated parameters of MS-RJ model. In 2004 and 2005, the implied volatility for call options at the lower strikes is lower than the implied volatility at higher strikes. It suggests that in-the-money calls are more costly compared to out-of-the-money calls, and that is called by volatility smirk. The long-term call options exhibit similar patterns in other years.

Fig. 6

Implied volatility curve of a call option under the MS-RJ model. Note that this figure plots the implied volatility the 1, 2, 6, and 12 months-to-maturity call options with different cases of moneyness given the current stock indexes, the annual riskless rates at some trading dates, and the daily estimated parameters of MS-RJ model

Next, Fig. 7 indicates the implied volatility the 1 months-to-maturity call options with different cases of moneyness given the current stock indexes, the annual riskless rates at some trading dates, and the daily estimated parameters of each model. These graphs show that the implied volatilities under MS and MS-RJ models are lower than the implied volatilities under GBM and GBM-RJ models. It suggests that if the call option premium computed by the model without the regime-switching pattern would be over-estimated. Nevertheless, in 2009, the implied volatilities of deep-in-the-money and deep-out-the-money call options under the MS-RJ model are higher than the implied volatilities under the GBM model. Overall, the MS-RJ model can improve the fitting of volatility smile and pricing performance.

Fig. 7

Implied volatility curve of call option cross models. Note that this figure plots the implied volatility the 1 months-to-maturity call options with different cases of moneyness given the current stock indexes, the annual riskless rates at some trading dates, and the daily estimated parameters of each model

Sensitivity analysis

Because the S&P500 market exhibit stock market state switching, this section performs sensitivity analysis for the estimated parameters. Based on the current price ($100), the strike price ($100), the annual riskless rate (2%), the days-to-maturity (30 days), and the estimated parameters of MS-RJ model shown in Table 2, Figs. 8 and 9 plot the sensitivity analysis of European call prices for the market state switching and jump risk under the MS-RJ model.

Fig. 8

The impact of pricing cycles for call option premium under the MS-RJ model. Based on the current price ($100), the strike price ($100), the days-to-maturity (30 days), and the estimated parameters of MS-RJ model shown in Table 2, this figure plots the sensitivity analysis of European call prices for the market state switching under the MS-RJ model

Fig. 9

The impact of jump risk for call option premium under the MS-RJ model. Based on the current price ($100), the strike price ($100), the days-to-maturity (30 days), and the estimated parameters of MS-RJ model shown in Table 2, this figure plots the sensitivity analysis of European call prices for the jump risk under the MS-RJ model

According to the sensitivity analysis in Fig. 8, there is a positive relationship between volatility and call option value in state 1 and state 2, holding other parameters fixed, implying that larger volatility indicates a higher probability of increasing stock prices and thus a higher call price. In addition, there is a negative relationship between p_{12} and the call value, other parameters held constant, because the volatility of state 1 (higher volatility state) is sustained when p_{12} is close to 0. A higher value of p_{12} implies a lower probability that the economy will switch from state 1 to state 2. That is, in the long term, a longer duration of state 1, which features higher volatility, implies a higher call value. On the contrary, there is a positive relationship between p_{21} and the value of the call option. A lower value of p_{21} implies a lower probability that the economy will switch from state 2 to state 1. In the long term, a longer duration of state 2, which has lower volatility, implies a lower value of the call option.

This paper also discusses the influence of jump volatility on call prices. Figure 9 illustrates the sensitivity analysis of the impact of jump size and jump frequency on call prices. Other factors held constant, there is a U-shape relationship between average jump size and call prices. Because of the total volatility of the rate of return, $\sqrt{\sigma_{q(t)}^2 + \lambda^{\mathbb{Q}}\left[\theta^2 + (1+h_2)\,\nu^2\right]}$, is positively associated with the absolute mean of jump size, the standard deviation of jump size, and the jump frequency, the larger jump risk implies greater volatility of stock at expiration and thus a higher call price.

Conclusion

This study proposes a Markov switching model with return jumps (MS-RJ) to price European options. To capture the dynamics of stock returns over expansion-recession cycles and the occurrences of abnormal events in financial markets, we assume that the index return follows the MS-RJ model. In this study, we show that compared with the discrete- and continuous-time stochastic volatility models, jump-diffusion model, and geometric Brownian motion, the MS-RJ model is better able to explain the dynamics of S&P 500 stock indices than others models. In addition, both the MS-RJ model and MS model can address the leptokurtic feature of the asset return distribution, the volatility smile, and the volatility clustering phenomenon.

For the goodness-of-fitting for each model, we obtain the daily rate of returns for the SPX between January 5, 1999 and December 30, 2009 as a sample. Owing to the market state, jump risk, and stochastic volatility be unobservable, we employ the particle filters algorithm to estimate the model parameters. Comparing to the discrete-time GARCH family and the continuous-time stochastic volatility family, the MS-RJ model exists the better goodness-of-fitting than other models. That is, the SPX market does have the features of market state switching and jump risks. On the other hand, for the pricing error for the SPX call options, we utilize the daily closed price of SPX call options between January 04, 1999 and December 31, 2009 as a sample, then the empirical results identify that the pricing model with jump risks can improve the pricing performance, and the market state switching in the pricing formula applies to the short-term at-the-money, median- and long-term in-the-money options.

Furthermore, according to the sensitivity analysis, option prices increase with the probability of remaining in the recession state but decrease with the probability of remaining in the expansion state. Moreover, increases in the standard deviation (in either state), the absolute mean jump size, the standard deviation of jump sizes, and the mean jump frequency all increase option prices.

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Author information

Affiliations

1. Department of Finance, National Dong Hwa University, Hualien, Taiwan, ROC

   Shin-Yun Wang

2. Department of Finance, Feng Chia University, Taichung, Taiwan, ROC

   Ming-Che Chuang

3. Department of Money and Banking & Risk and Insurance Research Center, College of Commerce, National Chengchi University, NO. 64, Sec.2, ZhiNan Rd., Wenshan District, Taipei City, 11605, Taiwan, ROC

   Shih-Kuei Lin

4. Department of Accounting Information, Takming University of Science and Technology, Taipei, Taiwan, ROC

So-De Shyu

Corresponding author

Correspondence to Shih-Kuei Lin.

Additional information

Publisher's Note

Appendices

Appendix A: European call option pricing formula

Assume that the jump risk premiums are the same in the market state 1 and 2, i.e. $h_{2,1} = h_{2,2} = h_2$. Given the filtration $\mathbb{F}(t)$, the days belonging to the market state 1 (denoted by m), the stock return between time t and T, $\ln S(T) - \ln S(t) = \sum_{u=t+1}^{T} R(u)$, under the $\mathbb{Q}$ measure is

$$\ln \frac{S(T)}{S(t)} = r(T-t) - \frac{1}{2}V_m - \lambda^{\mathbb{Q}}\left[g_Y^{\mathbb{Q}}(-i) - 1\right](T-t) + \sqrt{V_m}\varepsilon^{\mathbb{Q}}(t) + \sum_{n=1}^{N(T-t)} Y_n,$$

(20)

where $V_m = \sigma_1^2 k + \sigma_2^2 (T-t-m)$ is the variance with m days visiting the market state 1 between time t and T. And, the random variables under the $\mathbb{Q}$ measure: $\varepsilon^{\mathbb{Q}}(t) \sim \text{Normal}(0,1)$ is the Gaussian noise process, $N(T-t) \sim \text{Poisson}\left[\lambda^{\mathbb{Q}}(T-t)\right]$ is the number of jumps during $(t, T]$, and $Y_n \overset{\text{IID}}{\sim} f_Y^{\mathbb{Q}}(y; h_2)$ is the jump size. Let the random variable X be the log-price of stock at time T, i.e. $X = \ln S(T) = \ln S(t) + \sum_{u=t+1}^{T} R(u)$, and $\mathbb{G}(t) = \mathbb{F}(t) \vee m$ be enlarged filtration. Then, we can derive its conditional characteristic function as follow:

$$g_{X|\mathbb{G}(t)}^{\mathbb{Q}}(\phi) = \mathrm{E}_{\mathbb{G}(t)}^{\mathbb{Q}}\left(e^{i\phi X}\right) = e^{i\phi\left(\ln S(t) + r(T-t) - \frac{1}{2}V_m - \lambda^{\mathbb{Q}}\left[g_Y^{\mathbb{Q}}(-i)-1\right](T-t)\right) - \frac{\phi^2}{2}V_m + \lambda^{\mathbb{Q}}\left[g_Y^{\mathbb{Q}}(\phi)-1\right](T-t)}.$$

(21)

Consider a T-maturity European call option with the strike price K. Given the filtration $\mathbb{F}(t)$, the option pricing formula is

$$\begin{aligned}\text{Call}(t;K,T) &= \mathrm{E}^{\mathbb{Q}}_{\mathbb{F}(t)}\left[e^{-r(T-t)}\left(e^{X}-e^{\ln K}\right)1_{\{X>\ln K\}}\right]\\ &= e^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}_{\mathbb{F}(t)}\left[\mathrm{E}^{\mathbb{Q}}_{\mathbb{G}(t)}\left(e^{X}1_{\{X>\ln K\}}\right)\right]-Ke^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}_{\mathbb{F}(t)}\left[\mathrm{E}^{\mathbb{Q}}_{\mathbb{G}(t)}\left(1_{\{X>\ln K\}}\right)\right],\end{aligned} \tag{22}$$

where the second equation is derived by the law of iterated expectation. By the inverse Fourier transform of the characteristic function, we can rewrite Eq. (22) as follows:

$$\begin{aligned}\mathrm{E}^{\mathbb{Q}}_{\mathbb{G}(t)}\left(e^{X}1_{\{X>\ln K\}}\right) &= \int_{\ln K}^{\infty} e^{x} f^{\mathbb{Q}}_{X|\mathbb{G}(t)}(x)\,dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\ln K}^{\infty} e^{-i(\phi+i)x} g^{\mathbb{Q}}_{X|\mathbb{G}(t)}(\phi)\,dx d\phi\\ &= S(t)\,e^{r(T-t)}\left[\frac{1}{2}+\frac{e^{-r(T-t)}}{\pi S(t)}\cdot \mathrm{Re}\left(\int_{0}^{\infty}\frac{e^{-i\phi\ln K} g^{\mathbb{Q}}_{X|\mathbb{G}(t)}(\phi-i)}{i\phi}d\phi\right)\right]\\ &:= S(t)\,e^{r(T-t)}\Pi_{1,m},\end{aligned} \tag{23}$$

$$\begin{aligned}\mathrm{E}^{\mathbb{Q}}_{\mathbb{G}(t)}\left(1_{\{X>\ln K\}}\right) &= \int_{\ln K}^{\infty} f^{\mathbb{Q}}_{X|\mathbb{G}(t)}(x)\,dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\ln K}^{\infty} e^{-i\phi x} g^{\mathbb{Q}}_{X|\mathbb{G}(t)}(\phi)\,dx d\phi\\ &= \frac{1}{2}+\frac{1}{\pi}\cdot \mathrm{Re}\left(\int_{0}^{\infty}\frac{e^{-i\phi\ln K} g^{\mathbb{Q}}_{X|\mathbb{G}(t)}(\phi)}{i\phi}d\phi\right) := \mathit{\Pi}_{2,m},\end{aligned} \tag{24}$$

The second equations in Eqs. (23) and (24) are computed by the Fubini theorem, and the third equations in Eqs. (23) and (24) are derived by the property of complex conjugate. A similar proof can be shown in Bakshi and Madan (2000). Therefore, the pricing formula in Eq. (22) is rewritten by

$$\begin{aligned}\text{Call}(t;K,T) &= S(t)\,\mathrm{E}^{\mathbb{Q}}_{\mathbb{F}(t)}(\Pi_{1,m}) - Ke^{-r(T-t)}\mathrm{E}^{\mathbb{Q}}_{\mathbb{F}(t)}(\Pi_{2,m})\\ &= \sum_{m=0}^{T-t}\frac{p_{21}\psi_{T,m|q(t)=1}+p_{12}\psi_{T,k|q(t)=2}}{p_{12}+p_{21}}\left[S(t)\,\Pi_{1,m}-Ke^{-r(T-t)}\Pi_{2,m}\right],\end{aligned} \tag{25}$$

where $\psi_{T,m|q(t)=j}$ presents the probability that the stock market belongs to the state 1 for k days while the time-to-maturity is $T-t$ days given the initial market state $q(t)=j$. It is proved in Duan, Popova, and Ritchken (2002) with a hidden Markov Chain.

Appendix B: Benchmark models: stochastic volatility processes

Our model comprises a Markov switching dynamics for the volatility and it can be categorized in the stochastic volatility models. Under the discrete-time framework, Li (2019), Ornthanalai (2014), Christoffersen et al. (2012), Wu (2006) Heston and Nandi (2000) employ the affine GARCH models to depict the stock return and its variance. That is,

$$R(t) = \mu - \frac{1}{2}V(t) - \lambda\left[g_Y^{\mathbb{P}}(-i) - 1\right] + \sqrt{V(t)}Z^{\mathbb{P}}(t) + \sum_{n=1}^{\Delta N(t)} Y_n,$$

(26)

$$V(t) = \sigma^2 + \gamma\left[Z^{\mathbb{P}}(t-1) - \eta\sqrt{V(t-1)}\right]^2 + \xi V(t-1),$$

(27)

where μ is the mean of stock return, $V(t)$ is the conditional variance of stock return at time t, σ^2 is the intercept term of the variance equation, γ is the coefficient of leverage effect, η measures the level of asymmetric leverage effect between the good and bad events, ξ is the parameter of volatility clustering effect. The variance is s stationary process if and only if $\gamma\eta^2 + \xi < 1$ is satisfied. Also, note that $V(t)/2$ and $\lambda\left[g_Y^{\mathbb{Q}}(-i) - 1\right]$ are the convexity adjustment terms, which make the stock return equal to μ under the $\mathbb{P}$ measure, i.e. $\mathrm{E}^{\mathbb{P}}_{\mathbb{F}(t-1)\vee q(t)}[S(t)] = S(t-1)e^{\mu_{q(t)}}$. Equations (26) and (27) are called by the GARCH model with return jumps (GARCH-RJ). If there are no jumps, $\lambda = 0$, Eqs. (26) and (27) reduce to Heston and Nandi (2000)'s GARCH model.

On the other hand, the stock return is also measured under the continuous-time framework like Bates (2012, 2000, 1996), Eraker (2004), Eraker et al. (2003), Bakshi et al. (1997), Eisenberg and Jarrow (1994), Heston (1993). The dynamic of stock log-price under the stochastic volatility model is

$$d\ln S(t) = \left(\mu - \frac{1}{2}V(t) - \lambda\left[g_Y^{\mathbb{P}}(-i) - 1\right]\right)dt + \sqrt{V(t)}dW^{\mathbb{P}}(t) + d\sum_{n=1}^{N(t)} Y_n,$$

(28)

$$dV(t) = (1-\xi)\left[\sigma^2 - V(t)\right]dt + \gamma\sqrt{V(t)}d\hat{W}^{\mathbb{P}}(t) + d\sum_{n=1}^{\hat{N}(t)} \hat{Y}_n,$$

(29)

where μ is the instantaneous mean, $V(t)$ is the instantaneous variance at time t, $1-\xi$ is the mean-reverting speed, σ^2 is the long-run mean level, and γ is the volatility of variance. Moreover, $W^{\mathbb{P}}(t) \sim \text{Normal}(0,t)$ and $\hat{W}^{\mathbb{P}}(t) \sim \text{Normal}(0,t)$ are the correlated Brownian motions with the correlation coefficient ρ. $N(t) \sim \text{Poisson}(\lambda t)$ and $\hat{N}(t) \sim \text{Poisson}\left(\hat{\lambda} t\right)$ are the independent number of jumps for the return and the variance with the time-homogeneous arrival rate. $Y_n \overset{\text{IID}}{\sim} \text{Normal}(\theta, \nu^2)$ and $\hat{Y}_n \overset{\text{IID}}{\sim} \text{Exponential}\left(\hat{\theta}\right)$ are the independent jump sizes for the return- and the variance-jumps, respectively. Equations (28) and (29) are called by the stochastic volatility model with return-jumps and volatility-jumps (SV-RJ-VJ). If there are no jumps for volatility, $\hat{\lambda} = 0$, the model reduces to the stochastic volatility model with return jumps (SV-RJ) like Bates (2012, 2000) and Bakshi et al. (1997). Next, if the stock market does not exist the jump risks for the return and volatility, $\lambda = \hat{\lambda} = 0$, the dynamic of stock log-price degenerates to Heston (1993)'s SV model. For comparing under the same assumption, in this paper, we suppose that the jump frequencies of return and volatility are time-homogeneous under the model with jumps.

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