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Valuation of callable accreting interest rate swaps: Least squares Monte-Carlo method under Hull-White interest rate model

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ABSTRACT

Using the Hull-White interest rate model, this paper proposes a valuation method of callable accreting interest rate swap (CAIRS) and how it can be used for managing the risk of zero callable bonds (ZCBs). Firstly, CAIRS can be decomposed into accreting payer interest rate swaps and Bermudan options. Considering the financial valuation of both components, the former can be valued directly while the latter has no close-form due to its early exercise characteristics. Using the Least Squares Monte-Carlo method (LSM) proposed by Longstaff and Schwartz (2001), we find that the two options embedded in ZCB and CAIRS have the same exercise strategy since the terms of the swaps will include the bonds in practice. However, the cash flow of risk management in swaps and bonds can differ when considering the time value. Hence, CAIRS is not the best financial instrument for managing risks of ZCB under the current design.

1. Introduction

Interest Rate Swaps are commonly used to hedge interest rate risk in financial markets. When a company issues a floating interest rate bond, it can purchase the interest rate swap, receives the floating interest rate from the swap bank, and the company pays a fixed interest rate to the swap bank. The company's financing cost is locked at the time of issuing the bonds, without interference from interest rate risk. If the company expects a downward trend in future interest rates and has issued fixed-rate bonds, it can control the interest rate risk by purchasing the interest rate of the buyer.

In 2014, the Financial Supervisory Commission began to promote financial import substitution in Taiwan. The 2014 amended insurance law allows international bonds that are issued domestically to not be included in the foreign investment ceiling of 45% in the insurance industry. Since then, the development of international bonds in Taiwan have been growing. These bonds are denominated in United States (US) dollars, and the majority are issued as Zero Callable Bonds (ZCBs). In terms of ZCBs, Callable Accreting Interest Rate Swap (CAIRS) is commonly used as a risk management tool (for example, see Abken, 1991; Roos, 2016; Kienitz and Caspers, 2017). In 2017, the insurance industry suffered a loss from the exchange rate due to the excessive issuance of international bonds in Taiwan. The amendment of the Insurance Act Section 146-4 in 2018 stipulated that the sum of foreign investment and international bond issued shall not exceed 65% of the total amount. While the revised act will impact the development of the international bond market in Taiwan, the extent of the impact is difficult to determine. Therefore, there is a necessity to study the evaluation of this bond and the risk management tool.

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As empathized earlier, ZCB is the largest traded bond instrument in the international bonds issued in Taiwan and the CAIRS is the most often used financial risk management tool. Section 2 is the literature review on the evaluation method of the CAIRS. Section 3 is the introduction to the CAIRS, its decomposition and suitability analysis as a ZCB risk management instrument. Section 4 is the methodology, including the Hull and White model and LSM method. Section 5 is the empirical results, including the use of data in the Hull and White model parameter estimation and results, evaluation results and sensitivity analysis of the commodities. Section 6 is the conclusion of this study.

2. Literature review

Hull (2003) discussed the cancellation of the interest rate swap as the optional swap contract. When the interest rate swap has multiple cancellable rights, the commodity can be regarded as the addition of a common interest rate swap and Bermudan options. The example given by Hull supposed that Microsoft has entered into a five years swap with semiannual payments as a fixed rate receiver at 6% and floating rate payer at LIBOR. Suppose that the counterparty has the option to terminate on the swap on payment dates between year 2 and year 5. The swap is a regular swap plus a short position in a Bermudan-style swap option, where the Bermudan-style swap option is an option to enter into a swap that matures in five years and involves a fixed payment at 6% being received and a floating payment at LIBOR being paid. The swap can be exercised on any payment date between year 2 and year 5.

In terms of the interest rate swap with variable change in principal, Longstaff and Schwartz (2001) used the LSM method to evaluate the Index Amortizing Interest Rate Swap. The variable change in principal is linked to the Constant Maturity Swap Rate. After decomposing the Callable Accreting Interest Rate, Bermudan swaptions were evaluated in Hippler (2008) by the LIBOR Market Model (LMM) along with LSM method. In the multi-factor interest rate, the LSM method is more feasible than the tree method. The results show that the parameters for volatility and correlation of the LMM model where there is a significant impact on the evaluated value using the LSM Method. Feng et al. (2016) used two models, the one-factor Hull–White model, as well as the two-factor G2++ model to calculate the value and risk of the Bermudan swaptions. The numerical methods used include LSM-all, LSM-bundle, Stochastic Grid Bundling method (SGBM), and COS method. Based on the empirical results, Feng et al. (2016) recommended using the SGBM method as the first choice in calculating the Bermudan commodities and risk for EE, CVA, PFE, with the LSM-bundle as the second choice. The difference between the four methods is small if only the commodity values are compared.

2.1. Numerical method for early exercise

The CAIRS have option that can be exercised at any time on or before its expiration date, the numerical method for early characteristics evaluation includes the tree method and simulation method. This paper will use the LSM method to evaluate commodities by the Hull and White short rate model. A good interest rate model satisfies two conditions. One is to accurately describe the current period structure of interest rates, and the other is to accurately describe the period structure of interest rate volatility and the structure of correlation coefficients that are different to its target. Only when the above two conditions are met can we accurately evaluate bonds and interest rate derivatives. One of the most widely used is the Hull and white model (Ball and Torous, 1983; Balck et al., 1990; Ho and Lee, 1986; Hull and White, 1990). After Hull and White (1990) proposed arbitrage free short rate model, Hull and White (1994) constructed the Hull and White short rate model by the interest rate trinomial tree method. First, the interest rate trinomial tree was developed by the initial hypothesis that the average level as zero. Under the boundary of interest rate range, trinomial tree analysis implies short rate distribution and Hull and White short rate distribution, to find the probability of the trinomial tree branch. Finally, the market price is used to correct the average level of each time point. The trinomial tree can fulfill the structure of the market interest rate period, to complete the construction of the tree method. In addition, Hull and White's trinomial tree method construction can be applied to other logarithmic normal distribution short rate models, for example those of Balck et al. (1990) and Black and Karasinski (1991). Andersen (1999) used the Monte-Carlo simulation method to simulate the interest rate path to evaluate the Bermudan options under an LMM interest rate model. Then, the maximized execution strategy function was used to find the execution time at each boundary point. According to this process, if the execution value is greater than the boundary at each boundary point, it represents the execution at the specific time point. This method is used to determine the execution strategy; thus, the value of the Bermudan options can be calculated.

Longstaff and Schwartz (2001) used American options as an example to illustrate the LSM method employed to evaluate the steps of the early exercise of commodities. The paper also uses several commodities with early exercise as the evidence, including the Cancellable Index Amortizing Interest Rate Swap, the US Interest Rate Swap, and the US stock options under the jump diffusion model. The LSM method proposed in this paper also employs the path of the underlying technology to simulate the uncertain factor, and then uses the expectation value implied by the regression estimate to find the conditions for determining the execution. The value of a commodity can be calculated by generating the execution strategy through the above method.

In the LSM method, the combined basis function is closer to the expectation value; it produces an error between the approximation value and the expectation value. Hence, Jain and Oosterlee (2015) proposed the Stochastic Grid Bundling Method (SGBM) to minimize this error in the recent year. During the SGBM process, the Monte-Carlo method and regression method are used as the foundation. However, in the process of regression estimation, the paths of the simulation are grouped, and each group is subjected to the regression estimation.

The group method includes the average split, K-means, etc. The objective of grouping is to divide the domain and use the region of approximation to reduce the error between approximation value and expectation value.

3. The callable accreting interest rate swap

This paper explores the Callable Accreting Interest Rate Commodity, which is an interest rate swap contract. The first section will introduce the standard interest rate swap contract, and the second will introduce the interest rate swap contact with special terms; the Callable Accreting Interest Rate Swap.

Interest rate swap allows both parties to swap the interest in accordance with the agreed terms under the contract, including principal, constant term, frequency of interest payment, and two interest-bearing indicators, after calculating both parties interest rate swap. The common interest rate swap pays fixed interest annually while the other party pays floating interest every quarter. The party who pays the fixed interest is commonly known as the payer while the other party who receives the fixed interested is known as the receiver. When an investor holds a long-term interest rate swap of the recipient, it indicates that the investor can receive fixed interest and pay the floating interest in the future. The types of the interest rate swaps have different agreed terms and conditions. According to the variation in the time of the principal, it can be divided into Accreting Interest Rate Swap and Amortizing Interest Rate Swap. The principal increases yearly according to the contractual schedule growth rate for the former; for the latter, the principal decreases yearly.

In addition to the terms mentioned above, if the contract gives the right (options) to one party, the party can decide whether to terminate the interest rate swap. This is called a "Cancellable Interest Rate Swap". When the receiver has the right to terminate the contract, this interest rate swap is called a "Puttable Interest Rate Swap". Conversely, if the payer has the right to terminate, this interest rate swap is referred to as "Callable Interest Rate Swap". This paper examines the CAIRS as an interest rate swap contract between the change of the principal and the termination rights. In greater details, the structure of the CAIRS is a 30-year payer interest rate swap with the following agreed terms:

1) The principal increases yearly according to the agreed growth rate.
2) Fixed interest rate and the annual payment frequency
3) Floating interest rate and quarterly receive frequency
4) The calculation indicator of the fixed interest as the agreed fixed interest rate
5) The calculation indicator of the floating interest is 3-months LIBOR.
6) The payer has the termination right of the contract; the frequency is annual.

Besides, in practice, the termination right of the contract is often accompanied by a non-call period. For example, a "5 × 1" callable right indicates that the payer is not allowed to call the contract in the first five years. The payer can consider whether to exercise the callable right annually from the sixth year.

3.1. Analysis of cash flow and commodity decompose

The previous section introduced the structure of the commodity, CAIRS. The first part of this section will analyze possible cash flow scenarios of the commodities, and introduce the decomposition of commodities through cash flow analysis. This section concludes that CAIRS can be decomposed as callable accrediting payer interest rate swap, plus callable accrediting Bermudan-style recipient interest rate swap.

First, CAIRS can be expressed as:

$$\mathrm{CAIRS}(\mathrm{N},\ \mathrm{A},\ \mathrm{K},\ \mathrm{T},\ \mathrm{Non\text{-}call}) \tag{3.1}$$

N is the principal, k is the fixed interest rate, T is the contract maturity date (year), A is the annual growth rate of the principal, and the non-call period is the lock-up period (year) of the callable right. According to the execution situation prior to the maturity of the commodity callable, the cash flow can be divided into the following two scenarios: the payer does not employ the callable feature until the maturity date shown in Table 1, while the payer calls at $t+1$ year ($t+1 >$ Non-call), as shown in Table 2.

In a non-callable situation, this contract is equivalent to the callable accreting payer interest rate swap at year T, while for the situation of callable right after the beginning of $t+1$, this contract is equivalent to the callable accrediting payer interest rate swap at year t. For the situation shown in Table 2, this contract can be regarded as the callable accrediting interest rate swap at year T, plus the callable accrediting receiver interest rate swap at year T-t shown in Fig. 1. Generally, since CAIRS can be considered callable at the beginning of $t+1$, and at the beginning of the lockout period to consider whether to exercise the callable right, the callable right is equivalent to the payer holding options of callable accrediting Bermudan-style interest rate swap. This option allows the payer of the

Table 1
The situation of cash flow when no callable upon maturity of CAIRS.

Time	1				2				…	t				…	T-1				T			
Paid Fixed Rate	K				K					K					K				K			
Received Floating Rate	L	L	L	L	L	L	L	L		L	L	L	L		L	L	L	L	L	L	L	L
Principal	$N(1+A)^0$				$N(1+A)^1$					$N(1+A)^{t-1}$					$N(1+A)^{T-2}$				$N(1+A)^{T-1}$			

Note: L is 3-month LIBOR; N is principal; K is fixed interest rate; A is the annual growth rate of principal; Fixed interest = Fixed interest rate × principal; Floating Interest = Floating Interest Rate × principal.

Table 2
The situation of cash flow when callable is done at t + 1 year upon maturity of CAIRS.

Time	1	2	…	t	…	T-1	T
Paid Fixed Rate	K	K		K		0	0
Received Floating Rate	L L L L	L L L L		L L L L		0 0 0 0	0 0 0 0
Principal	$N(1+A)^0$	$N(1+A)^1$		$N(1+A)^{t-1}$		$N(1+A)^{T-2}$	$N(1+A)^{T-1}$

Note: L is 3-month LIBOR, N is principal, K is fixed interest rate, A is the annual growth rate of principal; Fixed interest = Fixed interest rate × principal; Floating Interest = Floating Interest Rate × principal; If the contract is terminated at t + 1 year, and there is no cash flow swap from t + 1 year to the end year T, the interest rate is zero.

CAIRS at year T to exercise the CAIRS contract after the lock-up period. Obviously, it can be decomposed as callable accreting payer interest rate swap (CAIRS), plus the callable accreting Bermudan swap option (BSO). Hence, the evaluation of CAIRS commodities can be decomposed into the evaluation of the interest rate swap (IRS) and Bermudan options, as shown in 3.2.

$$\begin{aligned} &\text{CAIRS}(\text{N, A, K, T, Non - call}) \\ &= \text{IRS}(\text{N}, \text{A}, \text{K}, \text{T}) + \text{BSO}(\text{N, A, K, T, Non - call}) \end{aligned} \tag{3.2}$$

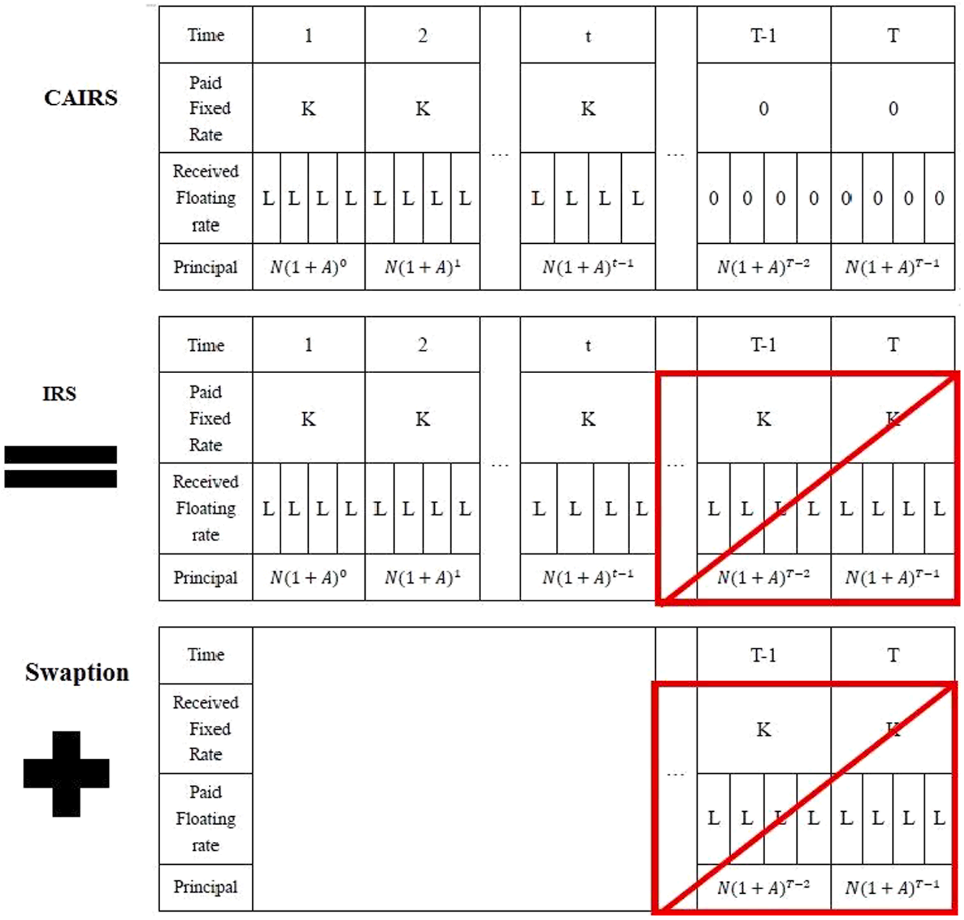

Fig. 1. Decompose CAIRS to a vanilla accreting IRS and a Bermudan swaption.

$$\begin{aligned} & \mathrm{IRS(N,A,K,T)} \\ & = \mathrm{floatleg} - \mathrm{fixleg} \\ & = E_t^Q\left\{\sum_{i=1}^{120}\left[\frac{\beta(t)}{\beta(T_{i/4})}\frac{L(T_{i/4+0.25},T_{i/4})}{4}N^{(i/4)}\right]-\sum_{j=1}^{30}\left[\frac{\beta(t)}{\beta(T_i)}KN^{(j)}\right]\right\} \\ & = \sum_{i=1}^{120}\left[P(t,T_{i/4+0.25})N^{(i/4)}\right]-\sum_{i=1}^{120}\left[P(t,T_{i/4})N^{(i/4)}\right]-\sum_{j=1}^{30}\left[P(t,T_j)KN^{(j)}\right] \end{aligned} \tag{3.3}$$

Note: P(t,s) is the bond from starting date, t to maturity date, s. $N^{(j)} = (1+A)^{j-1}$ is the calculation of principal with fixed interest annually. $N^{(i/4)} = (1+A)^{i/4}$ is the calculation of principal with floating interest quarterly. Beta is relative period discount factor.

For the evaluation of callable accreting payer interest rate swap, the CAIRS during exercising of commodities (t = 0) can be obtained as in 3.3, the equation of evaluation $IRS(N,\ A,\ K,\ T)$ if the current value of the floating interest amount received quarterly is deducted from the current value of fixed interest amount paid yearly.

In terms of the evaluation of Bermudan options, since there are multiple compliance points, there is no closed-form. This problem can be evaluated using simulation or tree method. In Section 4 of this paper will use the LSM method to evaluate the Bermudan options.

3.2. Suitability of the risk management tool

The interest rate swap contract is commonly used as the tool of risk management for the bond issuer. For the bond issuer with ZCB it has proven popular in recent years, the commonly used risk management tool is the CAIRS. After the bond issuer issues a ZCB, it implies that the issuer needs to calculate the internal return rate before the bond maturity date or expiration date. Then, the bond issuer exercises a contract of CAIRS with the swap back, which has the callable right, after taking consideration of the company risk management needs. This interest rate swap contract exercises the swap bank to pay a fixed interest to the bond issuer company annually, while the swap bank will receive the floating interest quarterly and has the contract termination right.

In the event of a falling market interest rate, the swap bank will pay the same fixed interest, but the floating interest received will gradually decrease. If the swap bank decides to exercise the termination right of interest rate swap contract, the bond issuer company will usually exercise the ZCB call options simultaneously. Based on the common transaction modes and risk management methods shared above, this section provides a detailed analysis and explains if the transaction modes to call simultaneously between bond issuer and swap bank are reasonable. This section includes three points:

A. The CAIRS can help the ZCB to pay the implied fixed interest at maturity date, distribute it every year and convert it into floating interest.
B. Time value is not considered in the cash flow calculation in the payment point A.
C. Comparing the Bermudan options and zero callable call options, the current value of cash flow is the same for each callable point in the future. This implies that the same value will be obtained with same strategy for two options.

To illustrate the three points, A, B, and C, if the bond issuer issues ZCB can be expressed as below:

$$\mathrm{ZCB(N,\ IRR,\ T,\ Non-call)} \tag{3.4}$$

Among them, N is the bond principal, IRR is the internal rate of return, T is the bond maturity date (annualization) and non-call is the callable lock-up period (annualization). This is based on the assumption that the bond issuer and swap bank have the contract of CAIRS on similar terms as $CAIRS(N,\ A,\ K,\ T,\ Non-call)$, shown in Eq. (3.1). CAIRS terms are related to the term of ZCB, with the lock-up period the same for both parties:

$$\mathrm{IRR(Internal\ Rate\ of\ Return) = A(Growth\ Rate\ of\ Initial\ Principal) = K(Fixed\ Interest\ Rate)} \tag{3.5}$$

Brennan and Schwartz (1977) shared that callable bonds, like ordinary bonds, deduct the callable right. Similarly, the ZCB can be decomposed into ZCB and callable right. The ZCB can be further subdivided into principal and internal interest. Therefore, the ZCB can be decomposed into principal, internal interest and callable portion. On the other hand, it is known that from the second section of section 3 that CAIRS can be decomposed into CAIRS plus Bermudan options. The CAIRS from the payer of CAIRS can be decomposed into fixed interest and floating interest. Hence, CAIRS can be decomposed into three parts: fixed interest (CAIRS-PFX); floating interest (CAIRS-PFL) and Bermudan options (BSO). Based on the decomposition above, and knowing that IRR = A = K, Eq. (3.6) is obtained. The first part of Eq. (3.6) represents the sum of fixed value of the fixed interest in CAIRS without consideration of the time value. The second part of Eq. (3.6) represents the fixed interest paid upon the maturity of ZCB. Two similar result are obtained under the design terms of Eq. (3.5).

$$\text{CAIRS - PFX} = \sum_{i=1}^{T} \text{KN}(1+\text{A})^{(i-1)} = \text{KN}\frac{(1+\text{A})^{\text{T}}-1}{\text{A}} = \text{N}(1+\text{IRR})^{\text{T}} - \text{N} \tag{3.6}$$

This explains that if the bond issuer can finance with ZCB, and accompanied simultaneously with CAIRS for management, it will not worry about the payment of the internal interest rate at the maturity of ZCB. This payment is supported by the fixed interest from CAIRS. The floating interest of the CAIRS is paid by the bond issuer to the swap bank on a quarterly basis, thus forming the characteristic of point A: "The CAIRS can help the ZCB to pay the implied fixed interest at the maturity date and convert it into floating interest." Eq. (3.6) mentioned above does not take into consideration the time value of CAIRS fixed interest. Hence, the characteristic of point B: "Time value is not considered in the cash flow calculation in point A" is proposed.

In order to illustrate the characteristic of point C, the following two options will be compared: the callable right of the Bermudan options and ZCB in CAIRS. First, reviewing the two options, the Bermudan options give the CAIRS payer the right to terminate the CAIRS contract at the time of expiration. For the Bermudan options, it is equivalent to having the right to exercise the callable accreting receiver interest rate swap. Therefore, the Bermudan options have the value equivalent to the callable accreting receiver interest rate.

On the other hand, the callable right for the zero-callable bond gives the bond issuer the right to call the bond at the appointed time before bond maturity. The value of the options for the callable right can be calculated by subtracting the principal and internal interest from the value of ZCB. To illustrate point C: "same value will be obtained with the same strategy for two options," for the calculation and consolidation of the two options above, this paper will divide the callable situation into options not to exercise till the maturity date and options to exercise at year t. If there is no execution from the two options, the cash flow produced post-execution must be equal to zero, as shown in Tables 3 and 4. If two options are chosen to be exercised at the beginning of year t (time = t − 1), the cash flow produced after the execution is shown as Tables 5 and 6. The current value of the cash flow in Tables 5 and 6 considered at the beginning of year t is shown in Table 7.

Bermudan options are the callable accreting receiver interest rate swap. Hence, if the options are to be exercised at the beginning of year t, it indicates that it exercises the callable accreting receiver interest rate swap; the cash flow exchange from year t to year T is shown in Table 5. For ZCB, the standards after execution of the contract are shown in Table 6, namely the cash flow from year t-1 (the beginning of year t) and year T. The comparison of Tables 5 and 6 shows that the cash flow produced are different based on the two options exercised.

For the two options mentioned above, since the two options are exercised at the same time, it can illustrate that: (1) the values of execution between two options are expected to be the same, (2) the values of execution between two options are expected to be the same, and this can explain the remaining characteristics of point C: "This infers that the same value will be obtained with same strategy for two options." At each point in time, if taking into consideration the execution value and expectation value in determining the strategy, and if both options can fulfill point (1), then point C: "same value will be obtained with same strategy for two options" can be realized. Besides, the value of options is determined by the best execution strategy, execution value and holding value. Therefore when points (1) and (2) are valid, it can illustrate point C: "same value will be obtained with same strategy for two options". The description above can be simplified as:

ZCB Callable Execution Value: X_i^{ZCB} , BSO Execution Value: $\mathrm{X_i^{BSO}}$

ZCB Callable Expectation value: H_i^{ZCB} , BSO Expectation value: $\mathrm{H_i^{BSO}}$

ZCB Callable Holding Value: h_i^{ZCB} , BSO Holding Value: h_i^{BSO}

$$h_i^{ZCB} = e^{-r_{i,t}}\left[X_{i+1}^{ZCB} I_{\left\{X_{i+1}^{ZCB} > H_{i+1}^{ZCB}\right\}} + h_{i+1}^{ZCB} I_{\left\{X_{i+1}^{ZCB} \leqslant H_{i+1}^{ZCB}\right\}}\right], H_i^{ZCB} = E_i\left[h_i^{ZCB}\right]$$

$$h_i^{BSO} = e^{-r_{i,t}}\left[X_{i+1}^{BSO} I_{\left\{X_{i+1}^{BSO} > H_{i+1}^{BSO}\right\}} + h_{i+1}^{BSO} I_{\left\{X_{i+1}^{BSO} \leqslant H_{i+1}^{BSO}\right\}}\right], H_i^{BSO} = E_i\left[h_i^{BSO}\right]$$

For any random time point $i = 1, 2, \cdots, T$, if the following conditions are valid:

$$X_i^{ZCB} = X_i^{BSO}$$

$$H_i^{ZCB} = H_i^{BSO}$$

$$h_i^{ZCB} = h_i^{BSO}$$

This is the callable options and Bermudan options in ZCB. The execution strategy and value are the same. Based on the symbol shown, based on the holding value h_i^{ZCB}, h_i^{BSO}, the value is determined by the later holding value h_{i+1}^{ZCB}, h_{i+1}^{BSO}, later execution value X_{i+1}^{ZCB},

Table 3
Cash flow situation of no-call CAIRS.

Time	first year	…	in year t	…	in year T
Fixed interest	0	…	0	…	0
Floating interest	0		0		0

Table 4
Cash flow situation of ZCB.

Time	first year	…	in year t	…	in year T
ZCB	0	…	0	…	$N(1+IRR)^T$
Principal	0		0		N
IRR	0		0		$N(1+IRR)^T - N$
Callable options	0		0		0

Note: ZCB = Principal + Internal Interest – Callable.

Table 5
Cash flow situation of CAIRS.

Time	first year	…	in year t				…	in year T			
Fixed interest	0	…	$KN(1+A)^{t-1}$				…	$KN(1+A)^{T-1}$			
Floating interest	0		F	F	F	F		F	F	F	F

Note: F represents floating interest of current period = $(3month - LIBOR)N(1+A)^{i-1}, i = t, \ldots T$. This tale shows the cash flow produced at the beginning of year t, year t-1.

Table 6
Cash flow situation of ZCB and callable options.

Time	first year	…	in year t-1	…	in year T
ZCB	0	…	$N(1+IRR)^{t-1}$	…	0
Principal	0		0		N
IRR	0		0		$N(1 + IRR)^T - N$
Callable options	0		$-N(1 + IRR)^{t-1}$		$N(1 + IRR)^T$

Note: ZCB = Principle + Internal Interest – Callable.

Table 7
Value of Callable between CAIRS and ZCB.

Part of the CAIRS	
Bermudan options	$E^Q_{t-1}\left\{\sum_{i=t}^T N(1+A)^{\ i-1}\left[\frac{\beta(t-1)}{\beta(i)}K - \sum_{j\in I_i}\frac{\beta(t-1)}{\beta(j)}\frac{L(j-0.25,j-0.25,j)}{4}\right]\right\}$ *where* $I_i = \{i-0.75, i-0.5, i-0.25, i\}$
Part of the ZCB	
ZCB	$N(1+IRR)^{t-1}$
Principal	$NP(t-1,T)$
IRR	$\left[N(1+IRR)^T - N\right]P(t-1,T)$
Callable options	$-N(1+IRR)^{t-1} + N(1+IRR)^T P(t-1,T)$

Note: CAIRS = Payment of Fixed Interest (CAIRS-PFX) + Receive Floating Interest (CAIRS-PFL) + Bermudan options. ZCB = Principle + Internal Interest – Callable. This table shows the current value of callable at year t.

X^{BSO}_{i+1}, and later expectation value H^{ZCB}_{i+1}, H^{BSO}_{i+1}. There is a relationship between the expected holding value and the later holding value. Therefore, this illustrates that “at any random time point t, the execution value of two options are similar.” The inductive method can then be used to infer that both options are equal with the holding value and expectation value. Based on the similar execution value and expectation value, this can infer the similar best execution strategy. In addition, if the holding value is similar, the two options are equal in value. In the end, the illustration of “at any random time point t, the execution value of two options is similar” is shown below:

Based on Table 7, the execution value of the two options can be obtained at the random time point at the beginning of year t (year t-1). Therefore, the explanation of “at any random time point t, the execution value of two options is similar” can be proved by the Eq. below. For details, please refer to the appendix.

Execution value of Bermudan options at the beginning of year t:

$$= E^Q_{t-1}\left\{\sum_{i=t}^T N(1+A)^{\ i-1}\left[\frac{\beta(t-1)}{\beta(i)}K - \sum_{j\in I_i}\frac{\beta(t-1)}{\beta(j)}\frac{L(j-0.25,j-0.25,j)}{4}\right]\right\}$$

$$= -N(1+IRR)^{t-1} + N(1+IRR)^T P(t-1,T) \tag{3.7}$$

4. Methodology

This section introduces the interest rate model and evaluation method for CAIRS. In terms of the evaluation of CAIRS, each time point of the cash flow and situation has been explained in Section 3. This section will illustrate further on the uncertainty factors.

The evaluation of CAIRS can be divided into callable accretions; the interest rate swap and Bermudan options. The uncertainty factor of CAIRS is the 3-month LIBOR. Since LIBOR can be regarded as a functional combination of bond, this part can be evaluated based on the bond value by using the IRS evaluation Eq. (3.3). In terms of the Bermudan options, the decision right is the receiver interest rate swap, while the uncertainty factor is still the 3-month LIBOR. This can be derived from the bonds in the corresponding period. Since the bond is a derivative of the interest rate, it can be regarded as a function of the interest rate. The discount factor in evaluating commodities is related to the interest rate. Therefore, once the interest rate model setting is completed, all the uncertainty factors in this commodity can be handled. The first part of this section introduces the Hull and White short rate model to address the interest rate uncertainty. The second and third sections describe how to use LSM method to build the term structure of the interest rate and evaluation of commodities, respectively.

4.1. Hull and White interest rate model

The interest rate model can be divided into short-term and long-term interest rate models based on the different variables. The common models shown in the former include the Vasicek model, the CIR model and the Hull and White model. The common models shown in the latter include the BGM model and HJM model. If the interest rate has good characteristics that can estimate the term structure of the market observation rate, it is known as a no-arbitrage model. The Hull and White single factor model, also known as the Extended Vasicek model used in this paper is a short rate no-arbitrage model, which was presented in 1990 by Hull and White. The model is illustrated below:

Given the probability $\{\Omega, F, Q\}$, where Ω is the sample space for all possible outcomes, F is the $\sigma-algebra$ that collects the Ω subset, and Q is a probability measurement that maps F to [0,1]. This probability space $\{F_t\}_{t=0}^{T}$ is used to collect time-related information and less than F. Let $W^Q(t)$ be the Brownian Movement Normal Distribution, the expectation value as 0, and the variable as t. Under the Hull and White model, the short rate r(t) fulfills the dynamic process below:

$$dr(t) = \kappa[\theta(t) - r(t)]dt + \sigma dW^Q(t) \tag{4.1}$$

Among them, $\theta(t)$ is the average of the short rate in time t. This parameter allows the Hull and White short rate to approach the structure of the market interest rate period. κ is the mean reversion speed parameter, representing the reversion strength of short-term deviates from the average level rate at time $\theta(t)$. σ is the instantaneous volatility parameter of the short rate. For instance, if the current short rate is lower than the average level of the period, the deviation value is calculated according to Eq. (4.1); the difference between the short rate and the previous period is the deviation multiplied by the reversion speed, plus a random term representing the volatility.

The dynamic process of Eq. (4.1) can be expressed by the 4.2

$$r(T) = r(t)e^{-\kappa(T-t)} + \int_t^T e^{-\kappa(T-s)}\kappa\theta(s)\, ds + \int_t^T e^{-\kappa(T-s)}\sigma\, dW^Q(s) \tag{4.2}$$

Since the Brownian Motion Normal Distribution is the last integral of the short rate integral model in Eq. (4.2), it can be regarded as a linear combination of Brownian motion, fulfilling the normal distribution. With the known information of F_t, the first two terms on the right side of Eq. (4.2) are known as the non-random terms. So, the r(T) short rate is the normal distribution translation. Since the ZCB price in Eq. (4.3) is similar to the short rate moment function, the short rate fulfills normal distribution. This deduces (Eq. (4.4)) the closed-form of the ZCB under the Hull and White model.

$$P(t,T) = E_t^Q[e^{-\int_t^T r(s)ds}] \tag{4.3}$$

$$= e^{E_t^Q\left[-\int_t^T r(s)ds\right] + \frac{1}{2}Var_t^Q\left[-\int_t^T r(s)ds\right]}$$

$$= e^{-r(t)B(t,T) - \kappa\int_t^T B(s,T)\theta(s)\, ds + \frac{\sigma^2}{4\kappa^2}\left[2(T-t) - 2B(t,T) - \kappa B(t,T)^2\right]} \tag{4.4}$$

$$Among\ this, B(t,T) = \frac{(1 - e^{-\kappa(T-t)})}{\kappa}$$

Then, the Hull and White model becomes a no-arbitrage model by the average level $\theta(t)$. The four steps below explain how the model parameters are set to meet the term structure of the market interest rate period. First, the overall process starts from the observed interest rate in the market. Since the market can observe a limited number of instant interest rates, the subsequent steps use the instant interest rate more than the observed quantity. If the market interest rate curve is a quadratic polynomial Eq. (4.5) of the time, the corresponding market instant rate can be found under this hypothesis. Therefore, step one assumes that the interest rate curve is quadratic and the interest rate curve observed by the market is a quadratic coefficient. In step two, the definition of Eq. (4.6) shows the long-term instantaneous interest rate, for example the extreme value of the future interest rate. This can infer the relationship between long-term instantaneous interest and bond, as shown in Eq. (4.7). Through the relationship between the instant interest rate and bond, shown in Eq. (4.8), the relationship between the long-term instantaneous interest rate and instant interest rate can be obtained. Then based on the relationship between instant interest rate to the time Eq. (4.5), the relationship between the long-term instantaneous interest rate and time can be obtained, as shown in Eq. (4.10). In Step three, based on the relationship between long-term instantaneous interest rate and bond, the Eq. of bond evaluation can be deduced, as shown in Eq. (4.4). This can also deduce the relationship between the long-term instantaneous interest rate and the Hull and White short rate average level $\theta(t)$, as shown in Eq. (4.11). Lastly, the fourth step is an independent long-term average level $\theta(t)$, which is the relationship between the long-term instantaneous interest and the short rate average level, with a partial differentiation to the time. The long-term instantaneous interest rate is replaced by the time relationship, as shown in Eq. (4.10). This can deduce the relationship between the final short rate average level and the instant interest rate quadratic coefficient, as shown in Eq. (4.12). Since the quadratic curve coefficient implies the term structural information during the market interest rate. Eq. (4.12) represents the short rate average level $\theta(t)$ that has the term structural information of the market interest rate, making the Hull and White model a no-arbitrage model.

Step 1: Correction of the instant rate curve coefficient by the market information

$$R(0,t) = \beta_0 + \beta_1 t + \beta_2 t^2 \tag{4.5}$$

Step 2: The relationship between long-term instantaneous interest rate, bond, and instant interest rate curve

$$\because F(0,t,t+\Delta t) = \frac{-1}{\Delta t} ln\left(\frac{P(0,t+\Delta t)}{P(0,t)}\right)$$

$$f(0,t) \equiv \lim_{\Delta t \to 0} F(0,t,t+\Delta t) \tag{4.6}$$

$$= \lim_{\Delta t \to 0} \frac{-1}{\Delta t} ln\left(\frac{P(0,t+\Delta t)}{P(0,t)}\right)$$

$$= -\lim_{\Delta t \to 0} \frac{lnP(0,t+\Delta t) - lnP(0,t)}{\Delta t}$$

$$= -\frac{\partial lnP(0,t)}{\partial t}$$

$$= \frac{\partial}{\partial t}\left[\frac{-lnP(0,t)}{t-0} t\right] \tag{4.7}$$

$$= \frac{\partial}{\partial t}[R(0,t)t] \tag{4.8}$$

$$= R(0,t) + \frac{\partial R(0,t)}{\partial t} t \tag{4.9}$$

$$= \beta_0 + 2\beta_1 t + 3\beta_2 t^2 \tag{4.10}$$

Step 3: The relationship between long-term instantaneous interest rate and long-term average level $\theta(t)$

$$f(0,t) = \frac{\partial}{\partial t}\left[\frac{-lnP(0,t)}{t-0} t\right]$$

$$= r(0)e^{-\kappa t} + \kappa \int_0^t \theta(s) e^{-\kappa(t-s)}\, ds - \frac{\sigma^2}{2} B(0,t)^2 \tag{4.11}$$

Step 4: The relationship between long-term average level $\theta(t)$ and instant rate curve

$$\theta(t) = f(0,t) + \frac{1}{\kappa}\frac{\partial f(0,t)}{t} + \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-2\kappa t}\right)$$

$$= \left(\beta_0 + \frac{2\beta_1}{\kappa}\right) + \left(2\beta_1 + \frac{6\beta_2}{\kappa}\right)t + 3\beta_2 t^2 + \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-2\kappa t}\right) \tag{4.12}$$

4.2. Least Squares Monte-Carlo method

This section follows the interest rate model set in the previous section and completes the evaluation of commodities by using a simulation method. Based on Section 3, the evaluation of commodities, CAIRS can be divided into callable accreting interest rate swap and Bermudan options. The CAIRS can be evaluated based on Eq. (3.3). The bond value in Eq. (3.3) can be deduced from the market interest rate curve information, or it can be obtained based on the simulation of the short rate according to Eq. (4.4). As for the Bermudan options, there is no closed-form or fixed evaluation method based on the early exercise term; this section introduces how to use the LSM method proposed by Longstaff and Schwartz (2001) to evaluate commodities.

This section introduces the LSM steps:

Step 1: for time t, the theoretical value of the commodity $V(t)$, expecting value of LSM, $V^{LSM}(t)$ are selected, and the uncertainty factor y(t), discounted factor r(t) and given the execution value of X(t, y), the expecting value $E[V(t+\Delta t)|y(t)]$.

Step 2: According to the hypothesis model of the uncertainty factor $y(t)$, $r(t)$, it generates N routes by the simulation method, and calculates the execution value $X_n(t,y)$ of the respective individual execution time point.

Step 3: for the path n, $n = 1,2,\ldots,N$, the commodity of the cash flow is discounted from the maturity date to the evaluation date; it can be divided into the following three situations according to the different time points: the evaluation of the time point, the early exercise time point, and non-early exercise time point. The following describes the three situations:

4.2.1. The early exercise time point

At this time point, the holder of the commodity will consider implementing the early exercise or continue to hold the commodity until the next time point. The holding value h(t) of the commodity is an uncertain random unknown in the current period. Based on the general assumption, the holder will consider and compare the discounted value $H(t)$ and execution value X(t, y) of the expected holding value. If the execution value is higher than the expected holding value, it will exercise the right. On the contrary, the holder will continue to hold the right of the commodity. Since the discount of expected holding value can be expressed as Eq. (4.13), it is a conditional expectation. Based on statistics, the least squares regression estimation method is the conditional expectation value. Therefore, it is possible to use the assumptions in Eqs. (4.14) and (4.15), to estimate the value of the discounted expected holding value based on the discounted holding value. Among these, $f_1, f_2, \cdots, f_M$ represents a set of basis function, commonly seen in Laguerre polynomials, combinations of Basis functions, etc. The estimation of basis function $\widehat{\beta}_1; \widehat{\beta}_2 \ldots \widehat{\beta}_M$ uses the least squares method.

$$H(t) = E\left[e^{-r_n(t)\Delta t} V(t+\Delta t)|y(t)\right] \equiv E[\,h(t)|y(t)] \tag{4.13}$$

$$H(t) = \beta_1 f_1(y(t)) + \beta_2 f_2(y(t)) + \cdots + \beta_M f_M(y(t)) \tag{4.14}$$

$$E[h(t)|y(t)] = \widehat{H}(t) = \widehat{\beta}_1 f_1(y(t)) + \widehat{\beta}_2 f_2(y(t)) + .. + \widehat{\beta}_M f_M(y(t)) \tag{4.15}$$

The method mentioned above determines the discount of the expected holding value, and it can determine at the time of comparison whether or not the route fulfills the execution value. It can determine the commodity value at this time.

$$V_n(t) = X_n(t,y) I_{(X_n > \widehat{H}_n)} + e^{-r_n(t)\Delta t} V_n(t+\Delta t) I_{(X_n \leqslant \widehat{H}_n)} \tag{4.16}$$

The Eq. above shows that if the decision consideration is more favorable$X_n > \widehat{H}_n$, then the execution value is similar to the value at the time. If the decision considerations continue to be more favorable$\widehat{H}_n \geqslant X_n$, then the current value of the commodity under the condition is equivalent to the holding value of the next discount time point commodity value $V_n(t+\Delta t)$.

In general, the maturity date of the early exercise is the last time point when the contract can be exercised. At the current time point there is no holding value. The holder does not need to consider the expectation value, and only needs to consider if the execution is favorable. Therefore, the value of the commodity maturity date is expressed as:

$$V_n(T) = max(X_n(T,\ y), 0)$$

4.2.2. Non-early exercise time point

Since there is only one options, to continue to hold the commodity at the time point, the current commodity value is the discounted value of the commodity value $V_n(t+1)$ until the next time point, as shown in Eq. (4.17).

$$V_n(t) = e^{-r_n(t)\Delta t} V_n(t+\Delta t) \tag{4.17}$$

4.2.3. Evaluation of the time point

During the processing time as the evaluation date, the current value of the individual route is first calculated as $V_n(t)$. When the contract is executable, the calculation can be calculated by Eqs. (4.16) and (4.17) can be calculated if it is a non-early exercise time point. Lastly, according to the Monte-Carlo method, the “Law of Large Numbers” $V^{LSM}(t)$ *is close to* $V(t)$; the evaluation value of the commodity is obtained based on Eq. (4.18).

$$V(t) \approx V^{LSM}(t) = \frac{1}{N}\sum_{n=1}^{N} V_n(t) \tag{4.18}$$

The above equation shows a generalized commodity evaluation step by using the LSM method. For the Bermudan options to be evaluated, the steps by this method in this paper are as follows:

Step 1: for the time t Bermudan options value $V(t)$, this commodity has stated its uncertainty factor with short rate at the beginning of this section, so $y(t) = r(t)$. In addition, the contract of Bermudan options is the callable accreting receiver interest rate swap. Therefore, the execution value is $\mathrm{X(t,r)} = \mathrm{Receiver\ IRS(N,A,K,T)}$; the expected holding value is $E[V(t+\Delta t)|r(t)]$.

Step 2: The hypothesis used in this short rate fulfills Hull and White model. Therefore, it can use the dynamic process shown in Eq. (4.1) to simulate the N-route production. According to Eq. (4.4), the bond price of each period is obtained. Since the execution contract is the callable accreting receiver interest rate swap, Eq. (4.4) can obtain the bond price at each period. Through the method mentioned above, the execution value $\mathrm{X_n(t,y)}$ of the individual path execution point can be calculated.

Step 3: According to the sample of commodity simulation shared by Longstaff and Schwartz (2001), when the interest rate is a two-factor model (Factor: $x,\ y;\ r = x + y$), the basis function is the constant function x, x^2, y, y^2 used for the swap contract of the special terms, and the cubic function that affects the subject matter. In addition, we use the basis function, short rate mono and quadratic function when dealing with ZCB. This paper uses the Hull and White single-factor model, considering the basis function as the constant function, r, r^2. Therefore, it is expected that the discounted value of the holding price will be shown in Eq. (4.19). When there are multiple callable points in the contract, the Eq. (4.19) regression estimation must be exercised at each callable point, and the expected holding value estimated by each time point regression is used to calculate the callable strategy.

$$E[h(t)|y(t)] = \widehat{H}(t) = \widehat{\theta}_0 + \widehat{\theta}_1 r(t) + \widehat{\theta}_2 r^2(t) \tag{4.19}$$

Then, according to the generalization method, the cash flow of the individual path is calculated from the maturity date, and the evaluation of result can be obtained by Bermudan options.

In addition, according to the short rate simulation shown in Step 2, evaluation Eq. (4.4) can be used to obtain the simulated price of the bond in each period. When this is included in Eq. (3.3), it can obtain the evaluation value of the CAIRS. Finally, the price of the CAIRS can be obtained by the sum of simulation related to the price of Bermudan options and the price of CAIRS.

5. Empirical results

This section implements the numerical method of the hypothesis model in the python programming language and the evaluation of the callable accreting callable interest rate. The evaluation of the commodity based on Eq. (3.1) can be expressed as:

$CAIRS(100,\ 4.1\%,\ 4.1\%,\ 30,\ 5)$

The callable accreting redeemable interest rate has the following terms:

1. Nominal Principal:100;
2. Principal Increase Rate (Internal rate of return): 4.1%;
3. Fixed Interest Rate (Payment frequency): 4.1% (Annually);
4. Floating Interest Rate (Payment frequency): 3month LIBOR + 34.3 bps (Quarterly);
5. Contract term: 30;
6. Callable lock-out structure (year): 5×1.

If the contract is assumed to start from the 27th of October 2017, the contract term is 30 years, so the maturity date is the 27th of October 2047. In practice, factors such as the future expected floating interest rates may adjust the fixed number of fixed points on the floating rate. In order to get closer to the actual contract form, the floating rate is assumed to be up to 34.3 bps, indicating that the interest rate of the payer can receive a floating interest rate equivalent to $100(1.04)^{t-1} \times \frac{(3\text{-}month\ LIBOR + 0.00343)}{4}$ quarterly.

The first section of this section explains how to use the market data to correct the parameters of the Hull and White model. The evaluation result is presented in the second section of this section. The third section will conduct a sensitivity analysis of the evaluation.

5.1. Parameter estimation

When using the Hull and White short rate model in Section 4, it is assumed that the mean reversion speed parameter and the volatility parameter are known. In this paper, the Hull and White dynamic process, as shown in Eq. (4.1), is discrete and obtains Eq. (5.1); $\sigma z(t)$ is the normal distribution with the expectation value of zero variability as $\sigma^2\Delta t$, so Eq. (5.1) is one of self-regression. The self-regression parameters can derive the self-regression coefficient estimated inversely.

When estimating the parameter in Eq. (5.1), the 3-month interest rate is used as the short rate. The data are based on the treasury yield curve published by the US Treasury Department. A total of 180 months of data are used for the evaluation. At the same time, from the treasury yield curve on the evaluation date, the interest rate curve can be introduced through Bootstrapping, and incorporated into Eq. (4.5), according to the four steps introduced in Section 4 to obtain the long term average level of short rate $\theta(t)$.

$$r(t+\Delta t) = \kappa\theta(t)\Delta t + [1-\kappa\Delta t]r(t) + \sigma z(t) \tag{5.1}$$

5.2. Evaluation result

For the evaluation date at the 31st of January 2019, the yield curve is shown in Figure 2, and the spot rate treasury curve is shown in Figure 3. Based on the parameter estimation result, the regression mean reversion speed parameter $\widehat{\kappa} = 0.0689$, volatility parameter $\widehat{\sigma} = 0.0068$, and long-term average level are shown in Figure 4.

After obtaining the Hull and White parameters, the commodity can be evaluated according to the method. Figure 5 uses the LSM method to evaluate Bermudan options simulation value and times relationship. Table 8 shows the corresponding numbers. According to the Central Limit Theorem, when the simulation number is enough, based on Eq. (4.18), the average number of the samples represented will approach the normal distribution. Therefore, according to the normal distribution, there is 95% confidence level. The theoretical price will include "simulated estimated $\pm 1.96\frac{Standard\ Deviation}{\sqrt{Times}}$". Therefore, when considering the tolerance between the simulated value and market value is for every one thousand dollars, the difference is one dollar. The minimum number of simulations required is $\left(\frac{1.96}{10^{-1}} \times s\right)^2$; s is the standard deviation of the sample. For example, when the number of simulations is 10,000 times, the standard error of the sample is 0.1587. The minimum required simulation number calculated by this standard deviation is $\left(\frac{1.96}{10^{-1}} \times 15.87\right)^2 = 96,753.3$ times. However, the number of simulations is small, so 10,000 times is not enough. When the number of simulations reaches 100,000 times, the standard error is 0.0504, and the standard deviation is 15.94. The actual number of simulations is greater than the minimum number of simulations required $\left(\frac{1.96}{10^{-1}} \times 15.94\right)^2 = 97,582.8$ times. Moreover, when the number of simulations is more than 100,000 times, the minimum number of simulations required is less than the actual number of simulations. The number of simulations over 100,000 times can produce a 95% confidence level. The error between the simulated commodity value and the theoretical value is a one-dollar difference in every thousand dollars. Figure 5 shows that after the number of simulations reach 100,000 times, the simulated value tends to be stable. Therefore, the LSM method mentioned in the subsequent analysis is based on the simulation of 100,000 times. Table 9 is the evaluation result.

5.3. Sensitivity analysis

In this section, three sensitivity analyses are performed; the mean reversion speed parameter, the short rate volatility parameter in the Hull and White model, and the spot rate curve on the evaluation date. After the CAIRS are decomposed, the callable accreting interest rate uses the evaluation equation. This paper analyzes only the Bermudan options without closed-form, and the influence of each factor under the simulation method. In Figure 6, the adjustment between up and down is made on the spot rate curve. The amount is adjusted by 1bps. Figure 6 also shows that the Bermudan options decrease as the curve rises. The spot rate curve lifts have two influence factors affecting discount and options value. When the spot rate curve rises, the discount factor is larger and the current value calculation is lower. On the other hand, due to the selection of the right of Bermudan options, to receive the fixed interest rate and pay the floating interest rate of the receiver interest rate swap the interest rate level increases, this implies that the cash flow of payment increases and cash flow received remains unchanged. Therefore, the value of the selection options drop, and the value of the section will also drop. Since the spot rate curve is shifted upwards, whether due to the influence of the discount factor or the options value, the result of the decline can be obtained, and the trend in Figure 6 can be understood.

Figure 6 shows the simulated value of Bermudan options after adjusting the mean reversion speed of the short rate model by 0.5 times to 1.5 times. The figure shows that the faster the mean reversion speed, the higher the options price. The changes mean reversion speed has two impacts: fluctuation of interest rate and long-term average level. First is the interest rate fluctuation. The high mean reversion speed indicates that the short rate has a stronger pullback to the average level. It is less likely to continue to have an overly large or small short rate, and can decrease the fluctuation. The same effect can be seen by considering the result from the variance. The

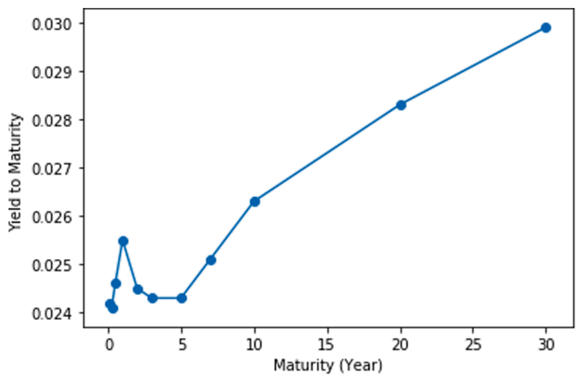

Fig. 2. Government bond yield curve.

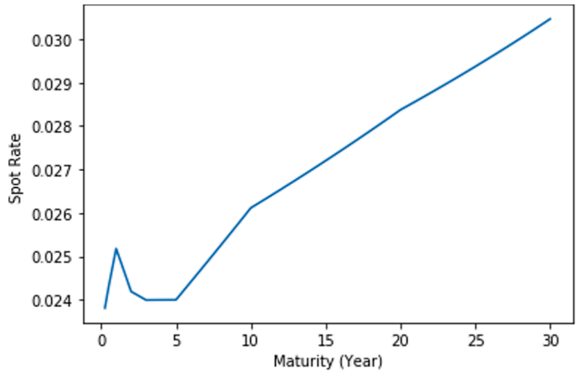

Fig. 3. Spot rate curve. Note: Evaluation on the 31st of January 2019.

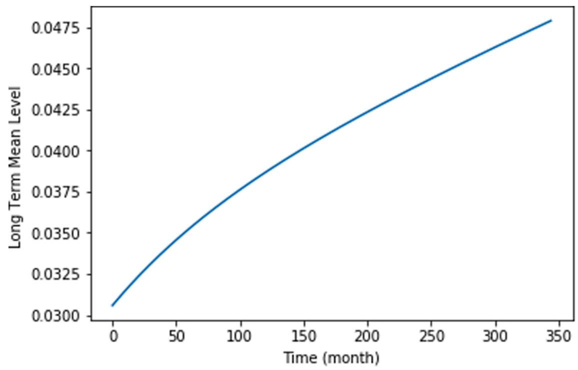

Fig. 4. Long-term average level.

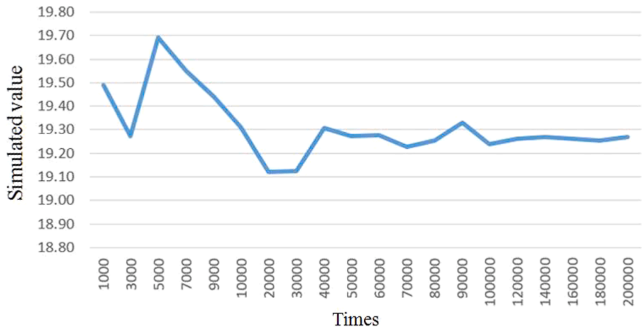

Fig. 5. Simulated value & times relationship.

Table 8
Simulated value & times.

Times	1,000	3,000	5,000	7,000	9,000	10,000	20,000	30,000	40,000
Simulated value	19.09	19.09	19.49	19.28	19.25	19.36	19.28	19.21	19.25
standard error	0.52	0.29	0.23	0.19	0.17	0.16	0.11	0.09	0.08
Times	50,000	60,000	70,000	80,000	90,000	100,000	120,000	140,000	160,000
Simulated value	19.29	19.25	19.19	19.22	19.24	19.24	19.26	19.27	19.26
standard error	0.07	0.07	0.06	0.06	0.05	0.05	0.05	0.04	0.04

Table 9
Evaluation result.

Assets	Simulated value
CAIRS	−2.80
IRS	−22.03
BSO	19.23

Note: Evaluation on the 31st of January 2019.

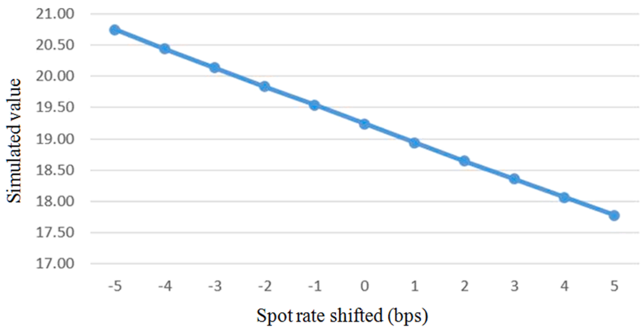

Fig. 6. Sensitivity analyses of the spot rate curve. Note: Evaluation on the 31st of January 2019.

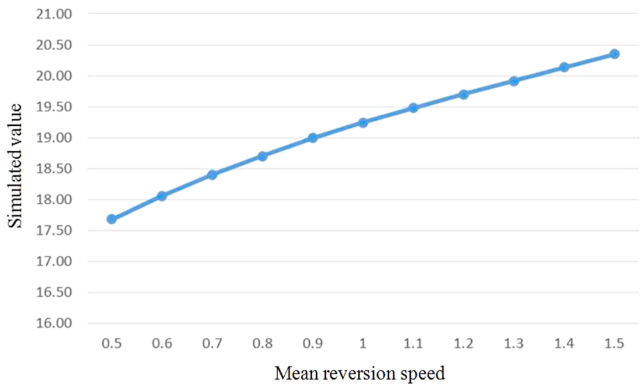

Fig. 7. Sensitivity analyses of the mean reversion speed. Note: Evaluation on the 31st of January 2019.

variance represents the short rate fluctuation at time point T and point t. The influence factors include the volatility parameter, the mean reversion speed parameter and days to the maturity date. If only the change of mean reversion speed is considered, the variance can be differentiated from the return rate, and Eq. (5.2) can be obtained. If the value in Eq. (5.2) is greater than zero (partial differential

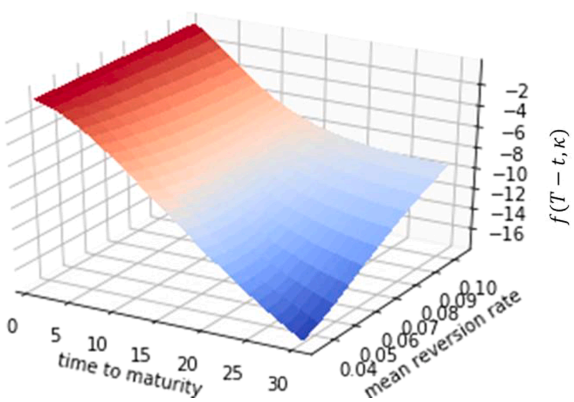

Fig. 8. Sensitivity analyses of the Eq. (5.2).

is greater than zero), the short rate fluctuation will increase with the increased mean reversion speed for a given volatility parameter from the maturity date. For the evaluation date of the 31st of October 2019, the recovery rate is 0.0689, and in the range of 0.5 times to 1.5 times, regardless of the value from zero to thirty-year period of the given maturity date. The corresponding partial deviation in Eq. (5.2) is obtained. The parentheses are all less than zero, as shown in Figure 8. Therefore, the increase in the representative mean reversion speed parameter will reduce the short rate fluctuation, which is the same as the inference that the mean reversion speed is intuitively interpreted as pulling the short rate back to the average level.

After analyzing the mean reversion speed for short rate fluctuations, the effect of the mean reversion speed on the long-term average level of interest rate $\theta(t)$ *is then analyzed*. When there is no rising or declining interest rate, the mean reversion speed will decrease, and the short rate fluctuation will increase. The chance of the early exercise of the options may increase and the options value will increase. However, the Hull and White short rate model plots the interest rate trend with a long-term average level. Eq. (4.12) shows that the mean reversion speed affects the long-term average level in the structure of this paper. In order to analyze the trend of influencing, Eq. (4.12) shows that the long-term average level can be obtained by sub-differentiating the mean reversion speed, obtaining Eq. (5.3). Since the mean reversion speed deviates from 0.5 to 1.5 times regardless of the time in the future, the long-term average level is less than zero for the mean reversion speed, as shown in Figure 9. When the represented mean reversion speed decreases, the long-term average level increases. As shown in Figure 10, when the mean reversion speed is changed from 1.5 times to the normal value of 0.5 times, the average value of the underlying calculation declines at each callable time point. Therefore, when the mean reversion speed decreases, the long-term average level of the interest rate increases; the selection value will decrease, eventually leading to a decline in the value of the options. Since the mean reversion speed affects the short rate fluctuation and the long-term average level, respectively, the final influence on the options are inconsistent. When the rate of mean reversion speed decreases, although it will also affect the increase of the short rate fluctuation, but the underlying fluctuation will increase, as shown in Figure 11. At the same time, the opportunity arises to increase the options to early exercise. However, the upper limit of the interest rate will have a great impact, and this will eventually decline the underlying value. The degree of decline in value is much higher and the chances of executing the options are reduced. Therefore, in Figure 7, the value tends to decrease as the mean reversion speed decreases.

Next, in Figure 12, after adjusting the volatility parameter of short rate model by 0.5 to 1.5, the Bermudan options are evaluated. The figure shows that the larger the short rate volatility parameter, the lower the options price. The change in the volatility parameter can be divided into two aspects: affecting short rate fluctuations and affecting the long-term average level of interest rates. Similarly, in analyzing the change of mean reversion speed, the short rate fluctuates as in the variance. It must be more than zero as shown in Eq. (5.4). Representing the short rate fluctuations, the volatility parameter rises regardless of the maturity date and mean reversion speed parameter status. In addition, in order to analyze the degree of long-term average level, Eq. (4.12) shows that the long-term average level is differentiated from the volatility parameter (as also shown in Eq. (5.5)). Eq. (5.5) must be greater than zero, which indicates that the increase in the volatility parameter will result in a long-term average interest rate increase. This implies that the payment floating interest of the selection will increase. Under the condition of floating interest rate increasing, the underlying value falls and the value of the options will fall. After consideration of the volatility parameters on the impact of short rate fluctuations and impact of long-term average level, the increase in volatility parameters leads to a rise in the long-term average level, which will reduce the underlying value; the probability of early exercise decreases, and the volatility parameters rises causing short rate fluctuations to rise, thereby increasing the probability of early exercise. Since the influence of the former is greater than the latter, the increased volatility parameter will result in more long-term average level and less possibility of the target being exercised early. Eventually the opportunity for the executable options is reduced and the options value is also reduced. Therefore, when the volatility parameter increases, the downward trend of the value can be explained by Figure 12.

$$\frac{\partial Var_t^Q[r(T)]}{\partial \kappa} = \frac{\partial \left[\frac{\sigma^2 \left(1-e^{-2\kappa(T-t)}\right)}{2\kappa} \right]}{\partial \kappa}$$

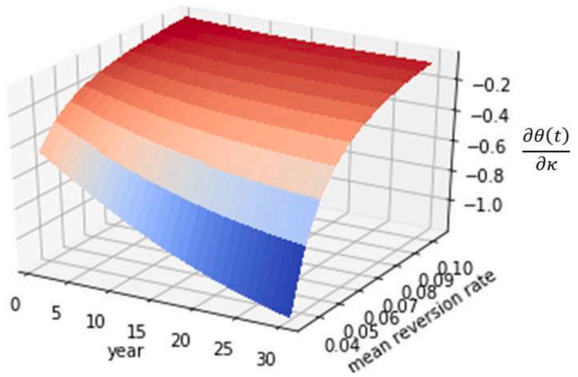

Fig. 9. Sensitivity analyses of the Eq. (5.3).

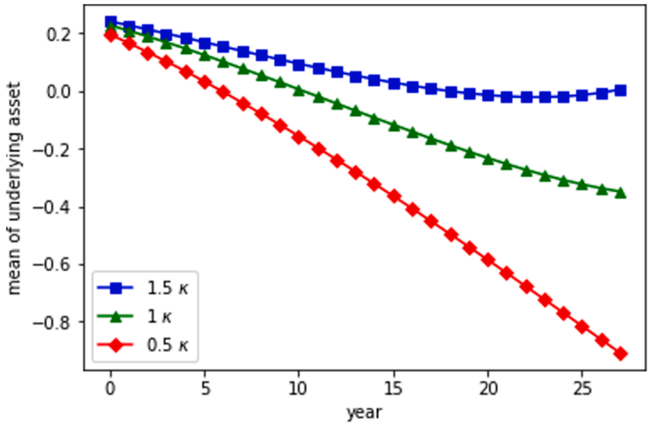

Fig. 10. Mean reversion speed of the underlying mean.

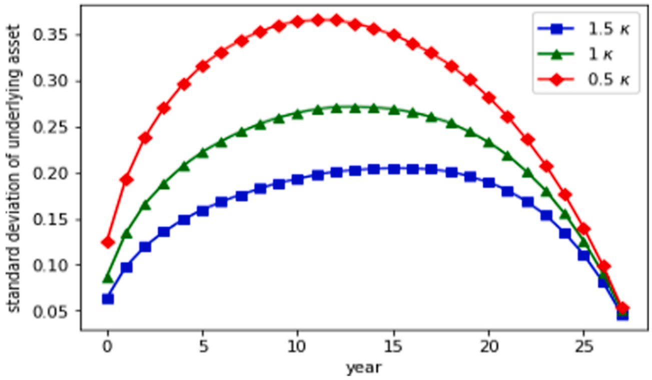

Fig. 11. Mean reversion of the underlying standard deviation.

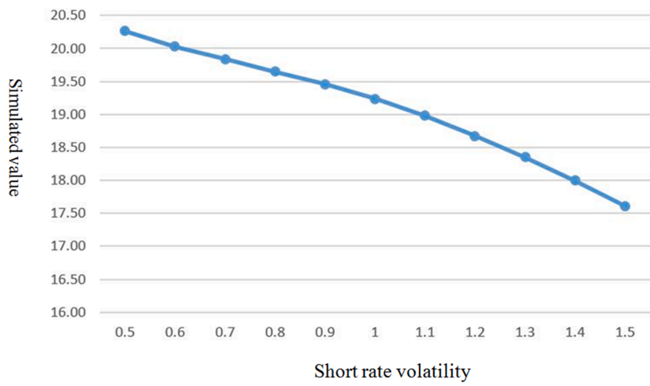

Fig. 12. Sensitivity analyses of the short rate volatility.

$$= \frac{\sigma^2}{2\kappa}\left[2(T-t)e^{-2\kappa(T-t)} - \frac{\left(e^{-2\kappa(T-t)}-1\right)}{\kappa}\right] \equiv \frac{\sigma^2}{2\kappa}f(T-t,\kappa) \tag{5.2}$$

$$\frac{\partial\theta(t)}{\partial\kappa} = \frac{-2\beta_1}{\kappa^2} - \frac{6\beta_2}{\kappa^2}t + \frac{\sigma^2}{2\kappa^2}\left(2te^{-2\kappa t}\right) - \frac{\sigma^2}{\kappa^3}\left(1-e^{-2\kappa t}\right) \tag{5.3}$$

$$\frac{\partial Var_t^Q[r(T)]}{\partial \sigma} = \frac{\partial\left[\frac{\sigma^2(1-e^{-2\kappa(T-t)})}{2\kappa}\right]}{\partial \sigma} = \frac{\sigma}{\kappa}\left[1-e^{-2\kappa(T-t)}\right] \tag{5.4}$$

$$\frac{\partial \theta(t)}{\partial \sigma} = \frac{\sigma}{\kappa^2}\left[1-e^{-2\kappa(T-t)}\right] \tag{5.5}$$

6. Conclusion

In most capital markets, the callable bond is a commonly used financing tool. In recent years, ZCBs have been particularly popular, and the issuing companies often use the CAIRS as the risk management tool. Therefore, this paper uses the Hull and White short rate model to evaluate the CAIRS under the LSM method. The CAIRS can be decomposed into the summation of callable accreting interest swap and Bermudan options. The former can be obtained by the corresponding bond price into the evaluation equation, while the latter must be obtained by numerical methods. This paper employs the LSM method to structure the Hull and White interest rate model evaluation process. In the LSM method, the regression estimation method is used to approximate the discount of the expected holding value. The selection of the basis function may result errors in evaluation. In addition, it is found that from the sensitivity analysis, the change rate and volatility of the interest rate model can affect the price of Bermudan options in two aspects. On one hand the influence of the short rate volatility parameters. On the other hand the influence of the long-term average levels parameters. In the result from the evaluation on the 31st of January 2019, it is found that the long-term average level caused by the parameter change has a greater effect on the options price. Therefore the mean reversion speed decreases and the volatility parameter increases, causing the price of the Bermudan options to decrease.

For the evaluation method, a different basis function can be considered in future research, to evaluate the error and result of CAIRS, or a more comprehensive analysis of the changes in the parameters, resulting in causal relationships of the price components between the long-term average level and short rate fluctuations.

Finally, the analysis of commodities in Section 3, in addition to the decomposition of the commodities by cash flow, also proposes three characteristics of CAIRS as the risk management tools: to help the issuing company spread the internal interest at every period, and convert to the floating interest. The time value is not considered in the internal interest and the Bermudan options strategy are equal to the callable options of the ZCB.

For the analysis in the real market, the research can further explore issues such as the risk management method of the counterparty when the swap bank or bond issuer does not use the best callable strategy. For example, when a swap bank exercises termination options on the CAIRS, if the bond issuer does not exercise the best strategy to call the ZCB, it swaps with the other swap bank. A new form of CAIRS will impact the bond issuers to have a more comprehensive practical value.

CRediT authorship contribution statement

Tang Kin-Boon: Conceptualization, Validation, Formal analysis. **Zheng Wen-Jie:** Writing - original draft, Investigation, Software. **Lin Chao-Yang:** Writing - review & editing, Data curation, Visualization. **Lin Shih-Kuei:** Project administration, Supervision, Methodology, Resources.

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Appendix

To prove the Bermudan options and ZCB at any random time point t, the execution value of both options is similar.

At the end of third part in Section 3, this paper proposes that at any random point of execution at time point t, the two implementation rights are considered to have the same execution value. From Table 7, that the execution value of the two options at the beginning of the year t (t-1 year) at any time of any execution is known. Therefore, the two sides will prove that the following equation is valid.

$$\text{Bermudan options the execution value in year t}$$

$$= E_{t-1}^Q\left\{\sum_{i=t}^{T} N(1+A)^{i-1}\left[\frac{\beta(t-1)}{\beta(i)}K - \sum_{j\in I_i}\frac{\beta(t-1)}{\beta(j)}\frac{L(j-0.25,j-0.25,j}{4}\right]\right\}$$

$$= -N(1+IRR)^{t-1} + N(1+IRR)^T P(t-1,T)$$

$= ZCB$ The execution value in year t

Proof:

$$E_{t-1}^{Q}\left\{\sum_{i=t}^{T}N(1+A)^{i-1}\left[\frac{\beta(t-1)}{\beta(i)}K-\sum_{j\in I_i}\frac{\beta(t-1)}{\beta(j)}\frac{L(j-0.25,j-0.25,j}{4}\right]\right\}$$

$=\sum_{i=t}^{T}N(1+A)^{i-1}\left\{KE_{t-1}^{Q}\left[\frac{\beta(t-1)}{\beta(i)}\right]-E_{t-1}^{Q}\left[\sum_{j\in I_i}\frac{\beta(t-1)}{\beta(j)}\frac{L(j-0.25,j-0.25,j}{4}\right]\right\}$

$=\sum_{i=t}^{T}N(1+A)^{i-1}\left\{KP(t-1,i)-E_{t-1}^{Q}\left[\sum_{j\in I_i}\frac{\beta(t-1)}{\beta(j)}\left(\frac{1}{P(j-0.25,j)}-1\right)\right]\right\}$

$=\sum_{i=t}^{T}N(1+A)^{i-1}\left\{KP(t-1,i)-\left(\sum_{j\in I_i}E_{t-1}^{Q}\left(\frac{\beta(t-1)}{\beta(j)}\frac{1}{P(j-0.25,j)}-1\right)-P(t-1,j)\right)\right\}$

$=(*)$

Since,

$$E_{t-1}^{Q}\left[\frac{\beta(t-1)}{\beta(j)}\frac{1}{P(j-0.25,j)}\right]$$

$=E_{t-1}^{Q}\left[\frac{\beta(t-1)}{\beta(j)}\left(\frac{P(j,j)}{P(t-1,j)}P(t-1,j)\right)\frac{1}{P(j-0.25,j)}\right]$

$=E_{t-1}^{Q_j}\left[P(t-1,j)\frac{1}{P(j-0.25,j)}\right]$

$=P(t-1,j)E_{t-1}^{Q_j}\left[\frac{P(j-0.25,j-0.25)}{P(j-0.25,j)}\right]$ since $P(t-1,j)$ is adapted to F_{t-1}

$=P(t-1,j)\frac{P(j-0.25)}{P(t-1,j)}$ since $\frac{P(j-0.25,j-0.25)}{P(j-0.25,j)}$ is martingale under Q_j

$=P(t-1,j-0.25)$

Then, (*)

$=\sum_{i=t}^{T}N(1+A)^{i-1}\left\{KP(t-1,i)-\left(\sum_{j\in I_i}E_{t-1}^{Q}\left(\frac{\beta(t-1)}{\beta(j)}\frac{1}{P(j-0.25,j)}\right)-P(t-1,j)\right)\right\}$

$=\sum_{i=t}^{T}N(1+A)^{i-1}\left\{KP(t-1,i)-\sum_{j\in I_i}[P(t-1,j-0.25)-P(t-1,j)]\right\}$

$=\sum_{i=t}^{T}N(1+A)^{i-1}\{KP(t-1,i)-[P(t-1,i-1)-P(t-1,j)]\}$

$=\sum_{i=t}^{T}N(1+A)^{i-1}\{(1+K)P(t-1,i)-P(t-1,i-1)\}$, since K=A

$=\sum_{i=t}^{T}N(1+A)^{i}P(t-1,i)-\sum_{i=t}^{T}N(1+A)^{i-1}P(t-1,i-1)$

$=\sum_{i=t}^{T}N(1+A)^{i}P(t-1,i)-\sum_{i=t}^{T-1}N(1+A)^{i}P(t-1,i)$

$=N(1+A)^{T}P(t-1,T)-N(1+A)^{t-1}P(t-1,t-1)$

$=N(1+A)^{T}P(t-1,T)-N(1+A)^{t-1}$, since A=IRR

$=N(1+IRR)^{T}P(t-1,T)-N(1+IRR)^{t-1}$ Q.E.D.

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