國立政治大學應用數學系

博士學位論文

一些具擴散項的霍林-坦納捕食者-被捕食者模型的行波解

Traveling Wave Solutions of Some Diffusive Holling-Tanner Predator-Prey Models

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中文摘要

在本文中，我們首先確立了一個具擴散項的廣義霍林-坦納 (Holling-Tanner) 捕食者-被捕食者模型的半行波解之存在，該模型的功能反應可能同時取決於捕食者和被捕食者的族群。接下來，利用建構利亞普諾夫 (Lyapunov) 函數和引用前面所獲得的半行波解，我們證明了此種模型在不同功能反應下行波解亦存在，這些功能反應包含洛特卡－沃爾泰拉 (Lotka-Volterra) 型、霍林二型 (Holling II) 以及貝丁頓-迪安傑利斯 (Beddington-DeAngelis) 型。最後，通過上下解方法，我們也證實了具有比率依賴功能反應的擴散霍林-坦納捕食者－被捕食者模型的半行波解存在。然後，藉由分析此半行波解在無限遠處的上、下極限，證明了行波解的存在。

關鍵詞：反應擴散系統；行波解；捕食者－被捕食者系統；霍林-坦納模型；貝丁頓-迪安傑利斯功能反應；比率相關功能反應。

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Abstract

In this thesis, we first establish the existence of semi-traveling wave solutions to a diffusive generalized Holling-Tanner predator-prey model in which the functional response may depend on both the predator and prey populations.

Next, by constructing the Lyapunov function, we apply the obtained result to show the existence of traveling wave solutions to the diffusive Holling-Tanner predator-prey models with various functional responses, including the Lotka-Volterra type functional response, the Holling type II functional response, and the Beddington-DeAngelis functional response.

Finally, we establish the existence of semi-traveling wave solutions of a diffusive Holling-Tanner predator-prey model with the Ratio-Dependent functional response by using the upper and lower solutions method. Then, by analyzing the limit superior and limit inferior of the semi-traveling wave solutions at infinity, we show the existence of traveling wave solutions.

Keywords: reaction-diffusion system; traveling wave solution; predator-prey system; Holling-Tanner model; Beddington-DeAngelis functional response; Ratio-Dependent functional response.

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Chapter 1

Introduction

Traveling wave solutions to diffusive predator-prey systems have been studied extensively (see [5, 8, 10], etc). In recent years, some researchers focused on the study of the diffusive Holling-Tanner predator-prey model. For example, Chen, Guo and Yao in [3] studied the diffusive Holling-Tanner model of Lotka-Volterra type functional response

$$\begin{aligned} u_t &= u_{xx} + ru(1-u) - rkuv, \\ v_t &= dv_{xx} + sv\left(1 - \frac{v}{u}\right), \end{aligned} \tag{1.1}$$

where d, r, s and k are positive constants. Under the condition $0 < k < 1$, they showed that system (1.1) admits a traveling wave solution with speed c iff $c \geq c^* := 2\sqrt{ds}$. Ai, Du and Peng in [1] first established the existence of semi-traveling wave solutions of a generalized Holling-Tanner model

$$u_t = u_{xx} + B(u) - f(u)v,$$

$$v_t = dv_{xx} + sv\left(1 - \frac{v}{u}\right),$$

and then applied it to show the existence of traveling wave solutions to the following diffusive Holling-Tanner model

$$\begin{aligned} u_t &= u_{xx} + u(1-u) - \frac{ku^m}{1+bu^m}v, \\ v_t &= dv_{xx} + sv\left(1 - \frac{v}{u}\right). \end{aligned} \tag{1.2}$$

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They showed that system (1.2) admits a traveling wave solution with speed $c \geq c^*$ for the two cases: (i) $m = 1$, $0 < k < 1$, and $b \geq 0$; (ii) $m = 2$, and either $k > 0$, $0 \leq b < 3$ or $0 < k < b^{2/3}(3 - b^{1/3})$, $0 < b < 27$. To the authors' knowledge, there are no results in the literature for traveling wave solutions of the diffusive Holling-Tanner system with the Beddington-DeAngelis functional response

$$\begin{aligned} u_t &= u_{xx} + ru(1-u) - \frac{rkuv}{1+bu+ev}, \\ v_t &= dv_{xx} + sv\left(1 - \frac{v}{u}\right), \end{aligned} \tag{1.3}$$

where $k,\ r,\ s,\ b$ and e are positive constants. Here $u(x,t)$ and $v(x,t)$ represent the density of prey and predators at position x and time t, respectively; d denotes the ratio of the diffusivity of the predator to that of the prey. Besides, let us consider two particular cases of (1.3). For $b = e = 0$, (1.3) is reduced to system (1.1) or system (1.2) with $m = 1$ and $b = 0$. For the case $b > 0$ and $e = 0$, (1.3) is reduced to system (1.2) with $m = 1$ and $b > 0$. Although the existence of traveling wave solutions to these two cases has been investigated in [3] and [1]. However, the results are under the restriction $0 < k < 1$. Based on the above reasons, we will first establish the existence of semi-traveling wave solutions to the following generalized diffusive Holling-Tanner model

$$\begin{aligned} u_t &= u_{xx} + B(u) - F(u,v)v, \\ v_t &= dv_{xx} + sv\left(1 - \frac{v}{u}\right), \end{aligned} \tag{1.4}$$

in Chapter 2, and then use the obtained result to show the existence of traveling wave solutions of system (1.3) with $b \geq 0$ and $e \geq 0$ in Chapter 3. In particular, for $b = e = 0$, we will relax the restriction on k.

For convenience, we rewrite (1.4) in the following form

$$\begin{aligned} u_t &= u_{xx} + h(u)[f(u) - g(u,v)v], \\ v_t &= dv_{xx} + sv\left(1 - \frac{v}{u}\right). \end{aligned} \tag{1.5}$$

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For functions h, f, and g, we impose the following hypotheses:

(H1) $h(u)$ and $f(u)$ are twice continuously differentiable functions on $[0,\infty)$, and $g(u,v)$ is a twice continuously differentiable function on $[0,\infty)\times[0,\infty)$.

(H2) $h(0)=0$ and $h(u)>0$ for all $u\in(0,1]$.

(H3) $f(0)>0,\ f(1)=0$, and $f(u)\geq 0$ for all $u\in(0,1)$.

(H4) $g(1,v)>0$ and $g(u,v)\geq 0$ for all $u\in[0,1)$ and $v\in[0,1]$.

(H5) There exists a unique positive number $\eta^*\in(0,1)$ such that $f(\eta^*)-g(\eta^*,\eta^*)\eta^*=0$.

Under the hypotheses **(H3)** and **(H5)**, system (1.5) has a boundary equilibrium point $(1,0)$ and a unique interior equilibrium point (i.e., the coexistence equilibrium point)(η^*,η^*). A solution (u,v) of system (1.5) is called a traveling wave solution if it is of the form

$$(u(x,t),v(x,t))=(U(z),V(z)),\ z=x+ct, \tag{1.6}$$

where c denotes the wave speed and $(U,V)\in C^2(\mathbb{R})\times C^2(\mathbb{R})$ is a pair of positive functions satisfying the boundary conditions $(U,V)(-\infty)=(1,0)$ and $(U,V)(+\infty)=(\eta^*,\eta^*)$. Upon substituting (1.6) into systems (1.5), we are led to the governing system for (U,V) as follows:

$$\begin{aligned} &U''-cU'+h(U)[f(U)-g(U,V)V]=0,\\ &dV''-cV'+sV\left(1-\frac{V}{U}\right)=0 \end{aligned} \tag{1.7}$$

on $\mathbb{R}$, together with the boundary conditions

$$(U,V)(-\infty)=(1,0) \text{ and } (U,V)(+\infty)=(\eta^*,\eta^*). \tag{1.8}$$

Therefore, to show the existence of traveling wave solutions of system (1.5) is equivalent to show the existence of positive solutions of system (1.7) satisfying (1.8). The main results for system (1.5) and (1.3) are stated in the following.

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Theorem 1.1. [9] *Suppose* **(H1)–(H5)** *hold. Then system (1.5) admits a semi-traveling wave solution* (u, v) *with speed* c *iff* $c \geq c^*$. *In addition,* $\delta < u < 1$ *and* $0 < v < 1$ *for some positive constant* δ.

Theorem 1.2. [9] *Suppose that* $0 \leq b \leq 1$, $e \geq 0$, *and*

$$0 < k < [-(1 + b + 2e) + \sqrt{(1 + b + 2e)^2 + 16(1 + b + e)^2}]/[2(1 + b + e)].$$

Then system (1.3) admits a traveling wave solution (u, v) *with speed* c *iff* $c \geq c^*$. *In addition,* $\delta < u < 1$ *and* $0 < v < 1$ *for some positive constant* δ.

In particular, when $b = e = 0$, the restriction on k can be further relaxed as follows.

Theorem 1.3. [9] *Suppose that* $b = e = 0$. *There exists a constant* $k_0 \in (4, 5)$ *such that if* $0 < k < k_0$, *then system (1.3) admits a traveling wave solution* (u, v) *with speed* c *iff* $c \geq c^*$. *In addition,* $\delta < u < 1$ *and* $0 < v < 1$ *for some positive constant* δ.

There are two parts in the proof of the existence of traveling wave solutions to system (1.3). In the first part, we show that system (1.5) admits a positive solution (u, v) of the form (1.6) with $(U, V)(-\infty) = (1, 0)$. Such a solution is so-called a semi-traveling wave solution. The argument of the proof for the existence of semi-traveling wave solutions of (1.5) is followed from that of [1] with a modification. In the second part, we apply the result in the first part to system (1.3) and show that the obtained semi-traveling wave solutions are actually traveling wave solutions under certain conditions by constructing the Lyapunov function and applying the LaSalle's invariance principle. Though this argument is standard, the main difficulty is the construction of the Lyapunov function. Motivated by [4], we construct a Lyapunov function different from that in [1] to improve the known result.

Next, we consider the following diffusive Holling-Tanner model with Ratio-Dependent type functional response

$$\begin{aligned} u_t &= u_{xx} + ru(1 - u) - \frac{rkuv}{u + qv}, \\ v_t &= dv_{xx} + sv\left(1 - \frac{v}{u}\right), \end{aligned} \tag{1.9}$$

where d, r, s, q and k are positive constants. Zuo and Shi in [11] proved that, under the condition

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$0 < k < (1+q)q/(2+q)$, system (1.9) admits a traveling wave solution with speed c iff $c \geq c^* := 2\sqrt{ds}$. In Chapter 4, we intend to release the condition.

Note that system (1.9) has a boundary equilibrium point $(1, 0)$ and a unique interior equilibrium point (i.e., the coexistence equilibrium point) (η^*, η^*) with $\eta^* := 1 - k/(1+q) \in (0, 1)$ iff $k < 1+q$. Therefore, throughout this thesis, we always assume that $k < 1+q$ for this system. Then it is easy to verify that system (1.9) satisfies the hypotheses **(H3)** and **(H5)** and the governing system for (U, V) is as follows:

$$U'' - cU' + rU(1-U) - \frac{rkUV}{U+qV} = 0, \tag{1.10a}$$

$$dV'' - cV' + sV\left(1 - \frac{V}{U}\right) = 0 \tag{1.10b}$$

on $\mathbb{R}$, together with the boundary conditions

$$(U, V)(-\infty) = (1, 0) \text{ and } (U, V)(+\infty) = (\eta^*, \eta^*). \tag{1.11}$$

Since system (1.9) does not satisfy the hypothesis **(H4)**, we cannot apply Theorem 1.1 directly. Instead, we will follow the arguments in [11] with a modification to get better results. More specifically, the structure of the proof is the same as that in [11]. But our proof is simpler and clearer. Besides, since system (1.10) has a singularity at $(U, V) = (0, 0)$, the authors in [11] derived a positive lower bound of U to overcome the difficulty. Here we get a better estimate of the lower bound of U to relax the restriction on k. The main result for system (1.9) are stated in the following.

Theorem 1.4. *Suppose that*

$$k < \sqrt{q^2 + 2q + 5} - 2. \tag{1.12}$$

Then system (1.9) admits a traveling wave solution with speed c iff $c \geq c^*$. In addition, $A < u < 1$ and $0 < v < 1$ for some positive number A.

We remark that $(1+q)q/(2+q) < \sqrt{q^2 + 2q + 5} - 2$ by a simple calculation. This shows that our range of k is wider than that in [11]. Besides, we note that $k < 1+q$ is a necessary condition to assure the coexistence equilibrium point exists. Since numerical simulation shows that our restriction on k is also a technical assumption (see the discussion in Section 4.5), we conjecture that the optimal result should be as follows: suppose that $k < q+1$, then system (1.9)

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admits a traveling wave solution with speed c iff $c \geq c^*$.

The rest of the thesis is organized as follows.

Chapter 2 is devoted to the study of the existence and non-existence of semi-traveling wave solutions to system (1.5). In Section 2.1, we analyze the trajectory of system (1.7) near the equilibrium point $(1, 0)$ to show that there exist no semi-traveling wave solutions with speed $c < c^*$. For the existence of traveling wave solutions, since system (1.7) has a singularity, we follow the idea of [1] to consider a modified system in which the reaction term $sV(1 - V/U)$ is replace by the function $sV(1 - V/\sigma_\epsilon(U))$, where σ_ϵ is a continuous function to be defined in Chapter 2 such that $\sigma_\epsilon(U) = U$ if $U \geq \epsilon$. We show that the modified system has a positive solution (U_ϵ, V_ϵ) and U_ϵ has a positive lower bound δ, which is independent of ϵ. Then (U_ϵ, V_ϵ) with $\epsilon \leq \delta$ is actually a semi-traveling wave solution of (1.5).

Chapter 3 is concerned with the existence and non-existence of traveling wave solutions to system (1.3). In Section 3.1, we apply the result obtained in Chapter 2 to get semi-traveling wave solutions to system (1.3). Then, by constructing the Lyapunov function and using the LaSalle's invariance principle, we prove that the obtained semi-traveling wave solutions to system (1.3) are actually traveling wave solutions under certain assumptions. Some numerical simulation results for system (1.3) are presented in Section 3.2.

Chapter 4 is focused on the existence and non-existence of traveling wave solutions to system (1.9). In Section 4.1, we review a general existence result in [11]. In Section 4.2, we construct a pair of upper and lower solutions to system (1.9) for $c \geq c^*$. In Section 4.3, by applying the general existence theorem together with the constructed upper and lower solutions, we get semi-traveling wave solutions to system (1.9). Then, in Section 4.4, we show that $(U, V)(+\infty) = (\eta^*, \eta^*)$ by analyzing limit superior and limit inferior of the semi-traveling wave solutions at $+\infty$. This confirms that the obtained semi-traveling wave solutions to system (1.9) are actually traveling wave solutions. Finally, some numerical simulation results of system (1.9) are also presented in Section 4.5.

We remark that the contents of Chapter 2 and Chapter 3 have been published in [9].

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Chapter 2

Semi-traveling wave solutions to system (1.5)

In this chapter, we prove Theorem 1.1 for the existence and non-existence of semi-traveling wave solutions to system (1.5). The proof is outlined as follows. First, the non-existence of semi-traveling wave solutions is followed by the standard phase plane analysis. Next, since system (1.7) has a singularity at $U = 0$, we follow the idea of [1] to consider a modified system in which the reaction term $sV(1 - V/U)$ is replace by the function $sV(1 - V/\sigma_\epsilon(U))$. Here $\sigma_\epsilon : [0, 1] \to (0, 1]$ is a continuous function defined by

$$\sigma_\epsilon(U) := \begin{cases} U, & \text{if } U \geq \epsilon, \\ U + \epsilon e^{\frac{1}{U-\epsilon}}, & \text{if } 0 \leq U < \epsilon, \end{cases} \tag{2.1}$$

where ϵ is a sufficiently small constant. Then we get a semi-traveling wave solution (U_ϵ, V_ϵ) to the modification system. Finally, we show that U_ϵ has a positive lower bound δ, which is independent of ϵ. It follows that (U_ϵ, V_ϵ) with $\epsilon \leq \delta$ is actually a semi-traveling wave solution of (1.5).

2.1 Non-existence of semi-traveling wave solutions

We show the non-existence of semi-traveling wave solutions of system (1.5) with speed $c < c^*$ in the following.

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Lemma 2.1. *Suppose* **(H1)–(H5)** *hold. For $c < c^*$, there exist no positive solutions of system (1.7) satisfying*

$$(U, V)(-\infty) = (1, 0).$$

Proof. First, we consider the linearized system of (1.7) around $(1, 0)$

$$U'' - cU' + h(1)f'(1)(U - 1) - h(1)g(1, 0)V = 0, \tag{2.2a}$$

$$dV'' - cV' + sV = 0. \tag{2.2b}$$

Note that (2.2b) has two eigenvalues λ_1 and λ_2, where

$$\lambda_1 := \frac{c - \sqrt{c^2 - 4ds}}{2}, \quad \lambda_2 := \frac{c + \sqrt{c^2 - 4ds}}{2}.$$

For contradiction, we assume (U, V) is a positive solution of system (1.7) with $c < 2\sqrt{ds}$ such that $(U, V)(-\infty) = (1, 0)$. Suppose that $c \leq -2\sqrt{ds}$. Then we have $\lambda_i < 0$, $i = 1, 2$, and so $V(z)$ is unbounded as $z \to -\infty$, a contradiction. Suppose $|c| < 2\sqrt{ds}$, then λ_1 and λ_2 form a complex conjugate pair. This would imply that $V(z)$ cannot be of the same sign for z near negative infinity, a contradiction again. Hence we complete the proof of this lemma. □

2.2 The modified system

We will establish the existence of semi-traveling wave solutions of (1.5) with speed $c \geq c^*$. Since system (1.7) has a singularity at $U = 0$, we first consider the following modified system

$$\begin{aligned} &U'' - cU' + h(U)[f(U) - g(U, V)V] = 0, \\ &dV'' - cV' + sV\left(1 - \frac{V}{\sigma_\epsilon(U)}\right) = 0 \end{aligned} \tag{2.3}$$

on $\mathbb{R}$, where σ_ϵ is the function defined by (2.1).

Following the arguments of [1, Lemma 2.3], we establish the following lemma for the existence of semi-traveling wave solutions to the modified system.

Lemma 2.2. *Suppose* **(H1)–(H5)** *hold. For $c \geq c^*$, system (2.3) admits a positive solution*

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(U_ϵ, V_ϵ) *on* $\mathbb{R}$ *satisfying*

$$0 < U_\epsilon(z) < 1, \quad 0 < V_\epsilon(z) < 1, \ \forall z \in \mathbb{R}, \tag{2.4}$$

and

$$(U_\epsilon, U_\epsilon', V_\epsilon, V_\epsilon')(-\infty) = (1, 0, 0, 0). \tag{2.5}$$

Furthermore, U_ϵ' *and* V_ϵ' *are bounded on* $\mathbb{R}$*, and there exist sufficiently small positive constants* δ *and* ϵ_0 *such that, for* $0 < \epsilon < \epsilon_0$*,*

$$U_\epsilon(z) > \delta, \forall z \in \mathbb{R}. \tag{2.6}$$

Proof. By applying [1, Theorem 2.1], system (2.3) admits at least one nonnegative solution (U_ϵ, V_ϵ) satisfying (2.5) if $c \geq c^*$. In addition, (U_ϵ, V_ϵ) satisfies

$$0 < U_\epsilon(z) < 1 \text{ and } 0 \leq V_\epsilon(z) \leq 1, \ \forall z \in \mathbb{R},$$

$$V_\epsilon(z) > 0, \ \forall z \leq 0, \tag{2.7}$$

and U_ϵ', V_ϵ' are bounded on $\mathbb{R}$. Furthermore, we claim that $0 < V_\epsilon(z) < 1$ for all $z \in \mathbb{R}$. For contradiction, we assume that there exists $z_1 > 0$ such that $V_\epsilon(z_1) = 0$. Then $V_\epsilon'(z_1) = 0$ and so the existence and unique theorem gives that $V_\epsilon = 0$ for all $z \in \mathbb{R}$, which contradicts (2.7). Assume that there exists $\hat{z}_1 \in \mathbb{R}$ such that $V_\epsilon(\hat{z}_1) = 1$. Since V_ϵ attains the maximum at the point $z = \hat{z}_1$, it follows that $V_\epsilon'(\hat{z}_1) = 0$ and $V_\epsilon''(\hat{z}_1) \leq 0$. On the other hand, by (2.3), we have

$$dV_\epsilon''(\hat{z}_1) = cV_\epsilon'(\hat{z}_1) - sV_\epsilon(\hat{z}_1)\left(1 - \frac{V_\epsilon(\hat{z}_1)}{\sigma_\epsilon(U_\epsilon(\hat{z}_1))}\right) > 0,$$

a contradiction. Hence $V_\epsilon(z) < 1$ for all $z \in \mathbb{R}$.

Now it suffices to claim that there exist sufficiently small positive constants δ and ϵ_0 such that $U_\epsilon(z) > \delta$ for all $z \in \mathbb{R}$ and $0 < \epsilon < \epsilon_0$. We divide the proof of this claim into several steps. For convenience, we denote $U := U_\epsilon$, $V := V_\epsilon$, and $\sigma(U) := \sigma_\epsilon(U)$ in the remaining proof.

Step 1. *Show that*

$$|U'(z)|/U(z) \leq \gamma_+, \ \forall z \in \mathbb{R}. \tag{2.8}$$

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Here $\gamma_+ := (c + \sqrt{c^2 + 4M_1})/2$ *is a positive solution of* $c\gamma + M_1 - \gamma^2 = 0$, *where*

$$M_1 := \sup_{(U,V)\in(0,1]\times[0,1]} \frac{h(U)[g(U,V)V - f(U)]}{U} > 0.$$

By (2.3), the function $\phi_1 := U'/U$ satisfies

$$\phi_1' = \frac{U''}{U} - \phi_1^2 = c\phi_1 - \frac{h(U)[f(U) - g(U,V)V]}{U} - \phi_1^2 \leq c\phi_1 + M_1 - \phi_1^2.$$

Since $\psi(z) := \gamma_+$ satisfies $\psi' = c\psi + M_1 - \psi^2$ and $\phi_1(-\infty) = 0 < \psi(-\infty)$, one can easily verify that $\phi_1(z) \leq \psi(z)$ for all $z \in \mathbb{R}$. Hence $\phi_1 \leq \gamma_+$ on $\mathbb{R}$.

Now we claim that $\phi_1 \geq -\gamma_+$. For contradiction, suppose that $\phi_1(z_2) < -\gamma_+$ at some point $z_2 \in \mathbb{R}$. Let $\phi(z)$ be the unique solution of the equation

$$\phi' = c\phi + M_1 - \phi^2 \tag{2.9}$$

such that $\phi(z_2) = \phi_1(z_2)$. By the comparison principle, we have

$$\phi_1(z) \leq \phi(z),\ \forall z \geq z_2. \tag{2.10}$$

Let $\gamma_- := (c - \sqrt{c^2 + 4M_1})/2$ be another root of $c\gamma + M_1 - \gamma^2 = 0$. By solving (2.9), we have

$$\phi(z) = \frac{\gamma_+ - \gamma_- e^{-\sqrt{c^2+4M_1}(z-z_3)}}{1 - e^{-\sqrt{c^2+4M_1}(z-z_3)}},$$

where

$$z_3 := z_2 + \frac{1}{\sqrt{c^2 + 4M_1}} \ln\left(\frac{\gamma_+ - \phi(z_2)}{\gamma_- - \phi(z_2)}\right) > z_2.$$

This yields $\phi \to -\infty$ as $z \to z_3-$. Together with (2.10), we find that $\phi_1 \to -\infty$ as $z \to z_4-$ for some point $z_4 \in (z_2, z_3]$, which contradicts the fact that ϕ_1 is defined for all $z \in \mathbb{R}$. Hence (2.8) holds.

Step 2. *Show that*

$$V'(z)/V(z) \leq \lambda,\ \forall z \in \mathbb{R}, \tag{2.11}$$

where $\lambda := (c - \sqrt{c^2 - 4ds})/(2d)$.

For contradiction, we assume that $(V'/V)(\hat{z}_1) > \lambda$ at some point $\hat{z}_1$. Then $(V'/V)(z) > \lambda$

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for all $z \geq \hat{z}_1$. To see this, we also use a contradictory argument and assume that there exists $\hat{z_2} > \hat{z}_1$ such that $(V'/V)(\hat{z}_2) = \lambda$ and $(V'/V)'(\hat{z}_2) \leq 0$. Then, by using $d\lambda^2 - c\lambda + s = 0$, we have

$$\left(\frac{V''}{V'}\right)(\hat{z}_2) = \frac{cV'(\hat{z}_2) - sV(\hat{z}_2)(1 - V(\hat{z}_2)/\sigma(U(\hat{z}_2)))}{d\lambda V(\hat{z}_2)} > \frac{c\lambda V(\hat{z}_2) - sV(\hat{z}_2)}{d\lambda V(\hat{z}_2)} = \frac{c\lambda - s}{d\lambda} = \lambda,$$

which, together with $(V'/V)(\hat{z}_2) = \lambda$, yields that

$$\left(\frac{V'}{V}\right)'(\hat{z}_2) = \left(\frac{V''}{V}\right)(\hat{z}_2) - \left(\frac{V'}{V}\right)^2(\hat{z}_2) > 0,$$

a contradiction. Hence $V'(z)/V(z) > \lambda$ for all $z \geq \hat{z}_1$ and so $V(z) \geq V(\hat{z}_1)e^{\lambda(z-\hat{z}_1)} \to \infty$ as $z \to \infty$, which contradicts the fact that $V \leq 1$. Therefore, (2.11) holds.

Step 3. *Show that there exist a positive constant ϵ_1 such that, for $0 < \epsilon < \epsilon_1$,*

$$V(z)/\sigma(U(z)) < dM_2/s, \ \forall z \in \mathbb{R}, \tag{2.12}$$

where $M_2 := s/d + c|1/d - 1|\gamma_+ + M_1 + \gamma_+^2$.

Let $\rho := V/\sigma(U)$. By computation, we obtain that

$$\rho' = \left(\frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)}\right)\rho \tag{2.13}$$

and

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$$\rho'' = \left(\frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)}\right)\rho' + \left[\frac{V''}{V} - \left(\frac{V'}{V}\right)^2 - \frac{\sigma''(U)(U')^2 + \sigma'(U)U''}{\sigma(U)} + \left(\frac{(\sigma(U))'}{\sigma(U)}\right)^2\right]\rho$$
$$= \left(\frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)}\right)\rho' + \{\frac{c}{d}\frac{V'}{V} - \frac{s}{d}(1-\rho) - \left(\frac{V'}{V}\right)^2 - \frac{\sigma''(U)(U')^2}{\sigma(U)}$$
$$- c\frac{\sigma'(U)U'}{\sigma(U)} + \frac{\sigma'(U)}{\sigma(U)} \cdot h(U)[f(U) - g(U,V)V] + \left(\frac{(\sigma(U))'}{\sigma(U)}\right)^2\}\rho$$
$$= \left(\frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)}\right)\rho' + \{\frac{c}{d}\left(\frac{\rho'}{\rho} + \frac{(\sigma(U))'}{\sigma(U)}\right) - \frac{s}{d}(1-\rho) - \frac{(\sigma''(U)(U')^2)}{\sigma(U)}$$
$$- c\frac{\sigma'(U)U'}{\sigma(U)} + \frac{\sigma'(U)}{\sigma(U)} \cdot h(U)[f(U) - g(U,V)V]\}\rho - \left[\left(\frac{V'}{V}\right)^2 - \left(\frac{(\sigma(U))'}{\sigma(U)}\right)^2\right]\rho$$
$$= \left(\frac{c}{d} - \frac{2(\sigma(U))'}{\sigma(U)}\right)\rho'$$
$$+ \left[\frac{s}{d}\rho - \frac{s}{d} + c\left(\frac{1}{d} - 1\right)\frac{\sigma'(U)U'}{\sigma(U)} + \frac{\sigma'(U)}{\sigma(U)} \cdot h(U)[f(U) - g(U,V)V] - \frac{\sigma''(U)(U')^2}{\sigma(U)}\right]\rho. \tag{2.14}$$

From step 3 in [1, Lemma 2.3], we can find $\epsilon_1 > 0$ such that, for $0 < \epsilon < \epsilon_1$ and $U > 0$,

$$\max\{U, \epsilon e^{-1/\epsilon}\} \le \sigma(U) \le \min\{1, U + \epsilon\},\ \ 0 < \sigma'(0) \le \sigma'(U) \le 1,\ \ 0 \le \sigma''(U) \le \sigma''(0) < 1, \tag{2.15}$$

so that

$$\frac{\sigma'(U)}{\sigma(U)} \le \frac{1}{U} \ \text{ and } \ \frac{\sigma''(U)}{\sigma(U)} \le \frac{1}{U} \le \frac{1}{U^2}. \tag{2.16}$$

Using (2.16), (2.8), (2.11), and the definition of M_1, we get from (2.14) that

$$\rho'' > \left(\frac{c}{d} - \frac{2(\sigma(U))'}{\sigma(U)}\right)\rho' + \left(\frac{s}{d}\rho - M_2\right)\rho. \tag{2.17}$$

Multiplying (2.17) by the integrating factor $Q(z) := \sigma^2(U)e^{-cz/d}$, we obtain that

$$[Q(z)\rho'(z)]' > Q(z)\left[\frac{s}{d}\rho(z) - M_2\right]\rho(z),\ \ \forall z \in \mathbb{R}. \tag{2.18}$$

Suppose that $\rho(z) < dM_2/s$ for all $z \in \mathbb{R}$ is false. Due to $\rho(-\infty) = 0$, there exists a smallest z_5 such that $\rho(z_5) = dM_2/s$ and $\rho'(z_5) \ge 0$. Together with (2.18), we get $[Q(z_5)\rho'(z_5)]' > 0$. So there exists a positive constant η_2 such that $Q(z)\rho'(z) > Q(z_5)\rho'(z_5) \ge 0$ for all $z \in (z_5, z_5+\eta_2)$.

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Since $Q(z) > 0$, it implies that $\rho'(z) > 0$ and $\rho(z) > dM_2/s$ for all $z \in (z_5, z_5 + \eta_2)$. Further, we claim that $\rho'(z) > 0$ for all $z > z_5$. For contradiction, we assume that there exists a smallest $z_6 > z_5$ such that $\rho'(z_6) \leq 0$. By integrating (2.18) from z_5 to z_6 and using the fact that $\rho'(z_5) \geq 0$, $\rho'(z_6) \leq 0$, and $\rho' > 0$ in (z_5, z_6), we have

$$0 \geq Q(z_6)\rho'(z_6) - Q(z_5)\rho'(z_5) \geq \int_{z_5}^{z_6} Q(z) \left[\frac{s}{d}\rho(z) - M_2\right] \rho(z) dz > 0,$$

a contradiction. So $\rho'(z) > 0$ and $\rho(z) > dM_2/s$ for all $z > z_5$, which gives

$$[Q(z)\rho'(z)]' > 0,\ \forall z > z_5 \ \text{ and } \ Q(z)\rho'(z) > Q(z_5+\eta_2)\rho'(z_5+\eta_2),\ \forall z > z_5 + \eta_2,$$

and therefore

$$\begin{aligned} \rho'(z) &> \frac{Q(z_5+\eta_2)}{Q(z)} \cdot \rho'(z_5+\eta_2) \\ &= \frac{\sigma^2(U(z_5+\eta_2))}{\sigma^2(U(z))} \cdot e^{c(z-(z_5+\eta_2))/d} \rho'(z_5+\eta_2) \\ &\geq \sigma^2(U(z_5+\eta_2)) e^{c(z-(z_5+\eta_2))/d} \rho'(z_5+\eta_2),\ \forall z > z_5+\eta_2. \end{aligned}$$

Thus $\rho'(z) \to \infty$ as $z \to \infty$ and so

$$\rho(z) \to \infty \text{ as } z \to \infty. \tag{2.19}$$

By (2.16) and (2.8), we have

$$\left|\frac{(\sigma(U(z)))'}{\sigma(U(z))}\right| = \left|\frac{\sigma'(U(z))U'(z)}{\sigma(U(z))}\right| \leq \gamma_+. \tag{2.20}$$

For all $z > z_5$, we use (2.13) and $\rho'(z) > 0$ to deduce that

$$\frac{V'(z)}{V(z)} - \frac{(\sigma(U(z)))'}{\sigma(U(z))} > 0,$$

which, together with (2.20), yields

$$\frac{V'(z)}{V(z)} > -\gamma_+. \tag{2.21}$$

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Combining (2.11) and (2.21), we get

$$\frac{|V'(z)|}{V(z)} \leq M_3 \tag{2.22}$$

for all $z > z_5$, where $M_3 := \max\{c/(2d), \gamma_+\}$. By (2.13), (2.22), and (2.20), we deduce that

$$\left|\left(\frac{c}{d} - \frac{2(\sigma(U))'}{\sigma(U)}\right)\rho'\right| = \left|\left(\frac{c}{d} - \frac{2(\sigma(U))'}{\sigma(U)}\right)\left(\frac{V'}{V} - \frac{(\sigma(U))'}{\sigma(U)}\right)\rho\right| \leq M_4\rho,$$

where $M_4 := (c/d + 2\gamma_+)(M_3 + \gamma_+)$. Then, by (2.17), we get

$$\rho''(z) > \left(\frac{s}{d}\rho(z) - M_2 - M_4\right)\rho(z)$$

for all $z > z_5$. Due to $\rho(\infty) = \infty$, there exists $\hat{z_6} > z_5$ such that, for $z > \hat{z_6}$,

$$\rho(z) > 2d(M_2 + M_4)/s,$$

which follows that

$$\rho''(z) > \frac{s}{2d}\rho(z)^2,$$

and so, by multiplying the above inequality by ρ' and integrating the resulting inequality, we obtain

$$(\rho')^2(z) > (\rho')^2(\hat{z_6}) + \frac{s}{3d}\left[\rho(z)^3 - \rho(\hat{z_6})^3\right].$$

Now we take $z_7 > \hat{z_6}$ such that, for $z > z_7$,

$$\rho(z)^3 > 2\rho(\hat{z_6})^3,$$

which yields that

$$(\rho')^2(z) > \frac{s}{6d}\rho(z)^3,$$

and therefore,

$$\rho'(z) > \sqrt{\frac{s}{6d}}\rho(z)^{3/2}.$$

This implies that ρ blows up in finite time, a contradiction. Thus we finish the proof of step 3.

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Step 4. *Show that there exist positive constants ϵ_0 and δ such that, for $0 < \epsilon < \epsilon_0$,*

$$U(z) > \delta,\ \forall z \in \mathbb{R}. \tag{2.23}$$

Set

$$M_0 := \max_{(u,v)\in[0,1]\times[0,1]} g(u,v).$$

We consider the function $\Theta(\tau) := f(\tau) - (dM_0M_2/s)(\tau+\alpha)$, where $\alpha := f(0)s/(2dM_0M_2)$. Due to $\Theta(0) > 0$, there exist a constant $\delta \in (0,1)$ such that

$$\Theta(\tau) > 0,\ \forall \tau \in [0,\delta]. \tag{2.24}$$

Recall that $U(-\infty) = 1$. For contradiction, we assume that there exists a smallest z_8 such that $U(z_8) = \delta$ and $U'(z_8) \leq 0$. For $0 < \epsilon < \epsilon_0 := \min\{\epsilon_1, \alpha\}$, it follows from (2.3), (2.12), (2.15), and the definitions of M_0 that, as long as $U \leq \delta$ and $U' \leq 0$,

$$\begin{aligned} U'' \quad &= cU' - h(U)[f(U) - g(U,V)V] \\ &\leq cU' - h(U)[f(U) - \tfrac{dM_0M_2}{s}(U+\alpha)] < 0. \end{aligned}$$

Hence one can easily verify that $U''(z) < 0$ and $U'(z) < 0$ for all $z > z_8$ and therefore $U(z) \to -\infty$ as $z \to \infty$, which contradicts the boundedness of U. Thus we finish the proof of (2.23) and the lemma. $\square$

2.3 Proof of Theorem 1.1

With the help of Lemma 2.2, we establish the following result for the existence of semi-traveling wave solutions of system (1.5), which, together with Lemma 2.1, gives Theorem 1.1.

Lemma 2.3. *Suppose* **(H1)–(H5)** *hold. For any $c \geq c^*$, there is a sufficiently small constant $\delta > 0$ such that the system (1.7) has at least one positive solution (U, V) on $\mathbb{R}$ satisfying*

$$\delta < U(z) < 1,\ \ 0 < V(z) < 1,\ \ \forall z \in \mathbb{R};\ V(z) > \delta,\ \ \forall z \geq z_0 \tag{2.25}$$

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for some $z_0 \in \mathbb{R}$, and

$$(U, U', V, V')(-\infty) = (1, 0, 0, 0). \tag{2.26}$$

Furthermore, U', V' are bounded on $\mathbb{R}$.

Proof. Pick ϵ sufficiently small such that $0 < \epsilon < \min\{\epsilon_0, \delta\}$, where ϵ_0 and δ are constants defined in Lemma 2.2. By Lemma 2.2, system (2.3) admits a solution (U_ϵ, V_ϵ) satisfying (2.4), (2.5), and (2.6). Since $U_\epsilon > \delta > \epsilon$, it follows from definition of σ_ϵ that $\sigma_\epsilon(U_\epsilon) = U_\epsilon$ and so $(U, V) := (U_\epsilon, V_\epsilon)$ is a solution of (1.7) satisfying

$$\delta < U(z) < 1, \quad 0 < V(z) < 1, \ \forall z \in \mathbb{R},$$

and

$$(U, U', V, V')(-\infty) = (1, 0, 0, 0).$$

Now we claim that there is a $z_0 \in \mathbb{R}$ such that $V(z_0) > \delta$. For contradiction, we assume that

$$V(z) \le \delta, \ \forall z \in \mathbb{R}. \tag{2.27}$$

Since $V(-\infty) = 0$ and $V(z) > 0$ for all $z \in \mathbb{R}$, there are two possibilities: (i) there is a smallest $\hat{z}$ such that $V'(\hat{z}) \le 0$; (ii) $V'(z) > 0$ for all $z \in \mathbb{R}$. For the case (i), we note that, as long as $V \le \delta$ and $V' \le 0$, it follows from (2.23) that

$$dV'' = cV' - sV\left(1 - \frac{V}{U}\right) < 0,$$

which implies that $V''(z) < 0$ and $V'(z) < 0$ for all $z > \hat{z}$. This leads to $V(z) \to -\infty$ as $z \to \infty$, that contradicts the boundedness of $V(z)$. For the case (ii), $V(\infty)$ exists and is positive. Since V and $sV(1 - V/U)$ are bounded on $\mathbb{R}$, it follows from Lemma A.3 that V' and V'' are bounded on $\mathbb{R}$. Then, differentiating the second equation of (1.7) and using Lemma A.3 again, we also have V''' is bounded on $\mathbb{R}$. Thus, by Lemma A.1, we have $V'(\infty) = V''(\infty) = 0$. This, together with (1.7) and the fact that $V(\infty) > 0$, yields that $U(\infty)$ also exists and $U(\infty) = V(\infty)$. From (2.23) and (2.27), we get $U(\infty) = V(\infty) = \delta$. Arguing as above, we have $U'(\infty) = 0$ and $U''(\infty) = 0$. Therefore, using the first equation of (1.7), we get that $f(\delta) - g(\delta, \delta)\delta = 0$. On the other hand, by the definitions of M_2 in step 3, we know that

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$dM_2/s > 1$. Together with definition of M_0 and (2.24), we get that

$$f(\delta) - g(\delta, \delta)\delta > f(\delta) - dM_0 M_2/s \cdot \delta > 0,$$

a contradiction. Hence we conclude that there is a $z_0 \in \mathbb{R}$ such that $V(z_0) > \delta$.

Furthermore, we claim that $V(z) > \delta$ for all $z \geq z_0$. For contradiction, we assume that there exists $\tilde{z} > z_0$ such that $V(\tilde{z}) = \delta$ and $V'(\tilde{z}) \leq 0$. Recall we have shown in the case (i) that $V'' < 0$ as long as $V \leq \delta$ and $V' \leq 0$. It follows that $V''(z) < 0$ and $V'(z) < 0$ for all $z > \tilde{z}$ and therefore $V(z) \to -\infty$ as $z \to \infty$, which contradicts the boundedness of V. So we finish the proof of this lemma. $\square$

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Chapter 3

Traveling wave solution to system (1.3)

In this section, we apply the obtained result to system (1.3) and show that, under certain conditions, the semi-traveling wave solutions are indeed traveling wave solutions by constructing the Lyapunov function and applying the LaSalle's invariance principle.

3.1 Proof of Theorem 1.2 and Theorem 1.3

In this section, we prove Theorem 1.2 and Theorem 1.3 for the existence of traveling wave solutions of system (1.3). To begin with, we set $h(u) := ru$, $f(u) := 1 - u$, and $g(u,v) := k/(1 + bu + ev)$, where $r > 0$, $k > 0$, $b \geq 0$ and $e \geq 0$. Then (1.5) and (1.7) become (1.3) and

$$\begin{aligned} &U'' - cU' + rU(1-U) - \frac{rkUV}{1+bU+eV} = 0, \\ &dV'' - cV' + sV(1 - \frac{V}{U}) = 0, \end{aligned} \tag{3.1}$$

respectively. Besides, it is easy to check that the functions f, g, and h satisfy the assumptions **(H1)–(H5)** with

$$\eta^* = \begin{cases} \dfrac{1}{k+1}, & \text{if } b = e = 0, \\ \dfrac{b+e-1-k+\sqrt{(b+e-1-k)^2 + 4(b+e)}}{2(b+e)}, & \text{otherwise.} \end{cases} \tag{3.2}$$

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So it follows from Lemma 2.3 that for any $c \geq c^*$, there is a sufficiently small constant $\delta > 0$ such that system (3.1) has at least one positive solution (U, V) on $\mathbb{R}$ satisfying

$$\delta < U(z) < 1, \quad 0 < V(z) < 1, \ \ \forall z \in \mathbb{R}; \ V(z) > \delta, \ \ \forall z \geq z_0 \tag{3.3}$$

for some $z_0 \in \mathbb{R}$, and

$$(U, U', V, V')(-\infty) = (1, 0, 0, 0). \tag{3.4}$$

In addition, there exists a positive constant M such that

$$|U'| < M, \ |V'| < M, \ \ \forall z \in \mathbb{R}. \tag{3.5}$$

Moreover, we will use the LaSalle's invariance principle to prove that

$$(U, U', V, V')(+\infty) = (\eta^*, 0, \eta^*, 0). \tag{3.6}$$

For this, we first rewrite system (3.1) as a first-order ODEs system:

$$\begin{aligned} &U' = Y, \\ &Y' = cY - rU\left[1 - U - \frac{kV}{1 + bU + eV}\right], \\ &V' = Z, \\ &dZ' = cZ - sV\left(1 - \frac{V}{U}\right), \end{aligned} \tag{3.7}$$

and set

$$\Sigma := (\delta, 1) \times (-M, M) \times (\delta, 1) \times (-M, M).$$

Next, motivated by [4], we define the Lyapunov function $L : \Sigma \to \mathbb{R}$ as follows:

$$L(U, Y, V, Z) := \frac{cU}{r} - \frac{Y}{r} + \frac{(\eta^*)^2 Y}{rU^2} + \frac{c(\eta^*)^2}{rU} + \kappa\left(cV - dZ + \frac{d\eta^* Z}{V} - c\eta^* \ln V\right),$$

where κ is a positive constant to be determined later. To proceed, we need the following lemma.

Lemma 3.1. *The equation* $-k^3 - k^2 + 16k + 32 = 0$ *has a unique solution, saying* k_0*. In*

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addition, $k_0 \in (4,5)$ *and*

$$-k^3 - k^2 + 16k + 32 > 0, \forall k < k_0.$$

Proof. Note that $P_0(k) := -k^3 - k^2 + 16k + 32$ has a positive local minimum at $k = -8/3$ and a positive local maximum at $k = 2$. In addition, the graph of P_0 is concave up on $(-\infty, -1/3)$ and concave down on $(-1/3, \infty)$. Since $P_0(4) > 0$ and $P_0(5) < 0$, we can find a number $k_0 \in (4,5)$ such that $P_0(k_0) = 0$, and $P_0 > 0$ in $(-\infty, k_0)$ and $P_0 < 0$ in (k_0, ∞). Hence we complete the proof of this lemma. □

Lemma 3.2. *Suppose one of the following conditions holds:*

- **(i)** $b = e = 0$ *and* $0 < k < k_0$*, where* k_0 *is defined in Lemma 3.1;*
- **(ii)** $0 \leq b \leq 1$*,* $e \geq 0$*, and*
  $0 < k < [-(1+b+2e) + \sqrt{(1+b+2e)^2 + 16(1+b+e)^2}]/[2(1+b+e)] < 2.$

If $c \geq c^*$*, then the orbital derivative of* L *along any trajectory* $X(z) := (U(z), Y(z), V(z), Z(z))$ *of (3.7) lying in* Σ *is non-positive; that is,*

$$\frac{d}{dz} L(X(z)) \leq 0,$$

and " $=$ " *holds only at* $(\eta^*, 0, \eta^*, 0)$.

Proof. By (3.7), we see that

$$1 - \eta^* = \frac{k\eta^*}{1 + (b+e)\eta^*}. \tag{3.8}$$

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Together with (3.7), we deduce that

$$
\begin{aligned}
\frac{d}{dz}L(X(z)) \quad & = \nabla L(X(z)) \cdot X'(z) \\
& = \frac{cY}{r} - \left[\frac{cY}{r} - U\left(1 - U - \frac{kV}{1+bU+eV}\right)\right] \\
& \quad + \frac{(\eta^*)^2}{U^2} \cdot \left[\frac{cY}{r} - U\left(1 - U - \frac{kV}{1+bU+eV}\right)\right] \\
& \quad - \frac{2(\eta^*)^2 Y^2}{rU^3} - \frac{c(\eta^*)^2}{rU^2} Y + c\kappa Z - \kappa\left[cZ - sV(1 - \frac{V}{U})\right] \\
& \quad + \frac{\kappa\eta^*}{V} \cdot \left[cZ - sV(1 - \frac{V}{U})\right] - \frac{\kappa d\eta^* Z^2}{V^2} - \frac{\kappa c\eta^* Z}{V} \\
& \leq \frac{U^2 - (\eta^*)^2}{U} \cdot \left[1 - U - \frac{kV}{1+bU+eV}\right] + \frac{\kappa s}{U}(V - \eta^*)(U - V) \\
& = \frac{1}{U}(U + \eta^*)(U - \eta^*)\left[-(U - \eta^*) + \frac{k(\eta^* - V) + kb\eta^*(U - V)}{(1 + (b+e)\eta^*)(1 + bU + eV)}\right] \\
& \quad + \frac{1}{U}\left[\kappa s(U - \eta^*)(V - \eta^*) - \kappa s(V - \eta^*)^2\right] \\
& = \frac{1}{U}\{-A(U - \eta^*)^2 + B(U - \eta^*)(V - \eta^*) - C(V - \eta^*)^2\},
\end{aligned}
$$

where

$$
A := \left[1 - \frac{kb\eta^*}{(1 + (b+e)\eta^*)(1 + bU + eV)}\right](U + \eta^*),
$$

$$
B := \kappa s - \frac{k(1 + b\eta^*)(U + \eta^*)}{(1 + (b+e)\eta^*)(1 + bU + eV)},
$$

and

$$
C := \kappa s.
$$

By $U, V > 0$ and using (3.8), and (i) and (ii), we get

$$
\frac{kb\eta^*}{[1 + (b+e)\eta^*](1 + bU + eV)} < \frac{kb\eta^*}{1 + (b+e)\eta^*} = b(1 - \eta^*) < 1, \tag{3.9}
$$

and so $A > 0$. Thus, if $B^2 - 4AC < 0$, that is,

$$
\left[\kappa s - \frac{k(1 + b\eta^*)(U + \eta^*)}{(1 + (b+e)\eta^*)(1 + bU + eV)}\right]^2 - 4\kappa s\left[1 - \frac{kb\eta^*}{(1 + (b+e)\eta^*)(1 + bU + eV)}\right](U + \eta^*) < 0, \tag{3.10}
$$

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then the quadratic form

$$-A(U-\eta^*)^2 + B(U-\eta^*)(V-\eta^*) - C(V-\eta^*)^2$$

is negative unless $U = \eta^*$ and $V = \eta^*$. So we get $dL(X(z))/dz \le 0$.

Now we must find $\kappa > 0$ such that (3.10) holds. For this, we rewrite (3.10) as the form

$$\begin{aligned}&(\kappa s)^2 - 2(U+\eta^*)\left[\frac{k(1-b\eta^*)}{(1+(b+e)\eta^*)(1+bU+eV)} + 2\right](\kappa s)\\ &+\frac{k^2(1+b\eta^*)^2(U+\eta^*)^2}{(1+(b+e)\eta^*)^2(1+bU+eV)^2} < 0.\end{aligned} \tag{3.11}$$

We claim that

$$D := \left[\frac{k(1-b\eta^*)}{(1+(b+e)\eta^*)(1+bU+eV)} + 2\right]^2 - \frac{k^2(1+b\eta^*)^2}{(1+(b+e)\eta^*)^2(1+bU+eV)^2} > 0.$$

After computation, we use $1 + bU + eV > 1$ and (3.8) to get

$$\begin{aligned} D &= 4\left[1 + \frac{k(1-b\eta^*)}{(1+(b+e)\eta^*)(1+bU+eV)} - \frac{k^2 b\eta^*}{(1+(b+e)\eta^*)^2(1+bU+eV)^2}\right]\\ &> 4\left[1 + \frac{k(1-b\eta^*)}{(1+(b+e)\eta^*)(1+bU+eV)} - \frac{kb(1-\eta^*)}{(1+(b+e)\eta^*)(1+bU+eV)}\right]\\ &= 4\left[1 + \frac{k(1-b)}{(1+(b+e)\eta^*)(1+bU+eV)}\right]\\ &> 4\left[1 + \frac{1-b}{1+b+e}\left(\frac{1}{\eta^*} - 1\right)\right] \end{aligned} \tag{3.12}$$

Since $b \le 1$, (3.12) implies $D > 0$. So (3.11) holds if and only if

$$(U+\eta^*)(H-\sqrt{D}) < \kappa s < (U+\eta^*)(H+\sqrt{D}), \tag{3.13}$$

where

$$H := 2 + \frac{k(1-b\eta^*)}{(1+(b+e)\eta^*)(1+bU+eV)}.$$

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By (i), (ii), $0 < U \leq 1$ and $0 < V \leq 1$, we have

$$2 + \frac{k(1-b\eta^*)}{(1+(b+e)\eta^*)(1+b+e)} < H < 2 + \frac{k(1-b\eta^*)}{1+(b+e)\eta^*}.$$

This, together with (3.8), yields

$$2 + \frac{1-b\eta^*}{1+b+e}\left(\frac{1}{\eta^*}-1\right) < H < 2 + (1-b\eta^*)\left(\frac{1}{\eta^*}-1\right). \tag{3.14}$$

Note that $H - \sqrt{D}$ is positive. By (3.12), (3.14), and $0 < U \leq 1$, we can find a suitable $\kappa > 0$ such that (3.13) holds if

$$\begin{aligned} & (1+\eta^*)\left[2 + (1-b\eta^*)\left(\frac{1}{\eta^*}-1\right) - 2\sqrt{1 + \frac{1-b}{1+b+e}\left(\frac{1}{\eta^*}-1\right)}\right] \\ < \quad & \eta^*\left[2 + \frac{1-b\eta^*}{1+b+e}\left(\frac{1}{\eta^*}-1\right) + 2\sqrt{1 + \frac{1-b}{1+b+e}\left(\frac{1}{\eta^*}-1\right)}\right]. \end{aligned} \tag{3.15}$$

Rearranging (3.15), we get

$$\begin{aligned} & (1+\eta^*)\left[2 + (1-b\eta^*)\left(\frac{1}{\eta^*}-1\right)\right] - \eta^*\left[2 + \frac{1-b\eta^*}{1+b+e}\left(\frac{1}{\eta^*}-1\right)\right] \\ < \quad & (2+4\eta^*)\sqrt{1 + \frac{1-b}{1+b+e}\left(\frac{1}{\eta^*}-1\right)}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & 1 - b + \frac{b+e}{1+b+e} + \frac{1}{\eta^*} - \frac{e\eta^*}{1+b+e} + \frac{b(b+e)(\eta^*)^2}{1+b+e} \\ < \quad & (2+4\eta^*)\sqrt{\frac{2b+e}{1+b+e} + \left(\frac{1-b}{1+b+e}\right)\frac{1}{\eta^*}}. \end{aligned} \tag{3.16}$$

For (i), we use $b = e = 0$ and $\eta^* = 1/(k+1)$ to simplify (3.16) in the form

$$-k^3 - k^2 + 16k + 32 > 0. \tag{3.17}$$

Since $0 < k < k_0$, it follows from Lemma 3.1 that (3.17) holds, and so does (3.15). For (ii), we

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first claim that $\eta^* > 1/(1+k)$. Using (3.8) and the fact $1+(b+e)\eta^* > 1$, we have $1-\eta^* < k\eta^*$, which implies $\eta^* > 1/(1+k)$. By $1/(1+k) < \eta^* < 1$, (3.16) holds if

$$\begin{aligned} & 1-b+\frac{b+e}{1+b+e}+1+k-\frac{e}{(1+b+e)(1+k)}+\frac{b(b+e)}{1+b+e} \\ < \quad & \frac{2k+6}{1+k}\sqrt{\frac{2b+e}{1+b+e}+\frac{1-b}{1+b+e}}. \end{aligned} \tag{3.18}$$

Simplifying (3.18), we get

$$2+k+\frac{e}{1+b+e}-\frac{e}{(1+b+e)(1+k)} < 2+\frac{4}{1+k}. \tag{3.19}$$

After some simple calculations, we get (3.19) holds if

$$(1+b+e)k^2+(1+b+2e)k-4(1+b+e) < 0. \tag{3.20}$$

From the assumption on k, (3.20) holds, and so does (3.15). Hence the proof is complete. $\square$

Finally, since L is continuous and bounded below in Σ and $X(z) := (U(z), Y(z), V(z), Z(z))$ is a solution of (3.7) obtained in Lemma 2.3 and is positively invariant in Σ for all $z \geq z_0$, it follows from Lemma 3.2 and the LaSalle's invariance principle that $(U, Y, V, Z)(\infty) = (\eta^*, 0, \eta^*, 0)$. So we establish the existence of traveling wave solutions of system (1.3) in the following lemma, which, together with Lemma 2.1, gives Theorem 1.2 and Theorem 1.3.

Lemma 3.3. *Suppose one of the following conditions holds:*

(i) $b = e = 0$ *and* $0 < k < k_0$, *where* k_0 *is defined in Lemma 3.1;*

(ii) $0 \leq b \leq 1$, $e \geq 0$, *and*
$0 < k < [-(1+b+2e)+\sqrt{(1+b+2e)^2+16(1+b+e)^2}]/[2(1+b+e)]$.

For $c \geq c^*$, system (3.1) has a positive solution (U, V) satisfying (1.8). In addition, there exists a positive constant δ such that

$$\delta < U(z) < 1, \quad 0 < V(z) < 1, \ \forall z \in \mathbb{R}.$$

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3.2 Numerical simulation results

Now we present some numerical results for system (1.3). In Figure 3.1 and Figure 3.2, we see that the large time behaviours of the solutions of the initial value problem of system (1.3) are two traveling waves propagating outwards in opposite directions, where the initial data (u_0, v_0) is chosen so that $u_0 = 1$ and $v_0 = 0.05(1 + \text{sign}(51 - x))(1 + \text{sign}(x - 49))/4$. From Figure 3.3 and Figure 3.4, we find that the restrictions $b \leq 1$ in Theorem 1.2 and on k in Theorem 1.2 and Theorem 1.3 are technical assumptions since the values of the parameters b and k in Figure 3.3 and Figure 3.4 do not satisfy the restrictions.

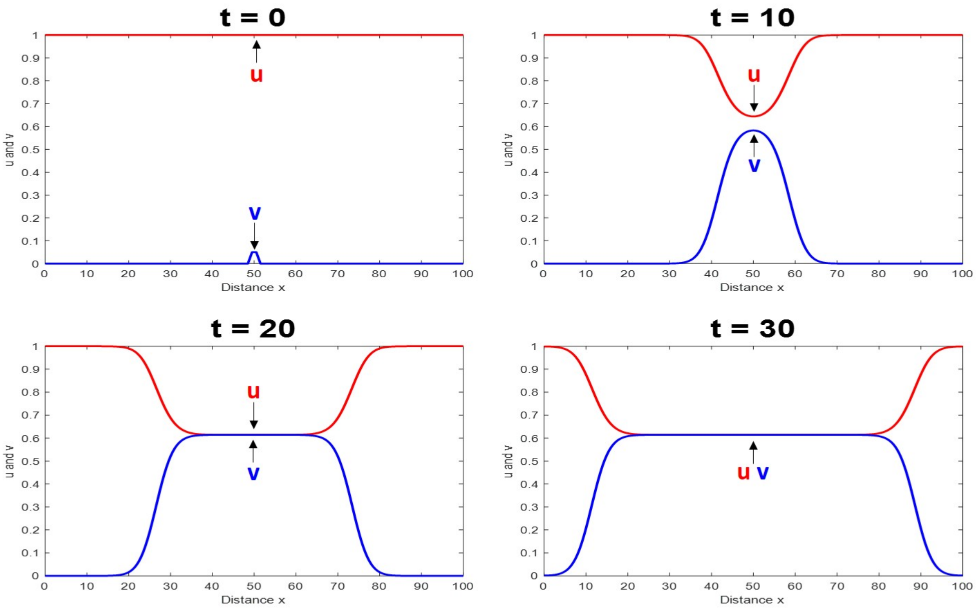

Figure 3.1: The solution of system (1.3) as a function of the spatial variable x is plotted at t=0, t=10, t=20 and t=30. The initial data (u_0, v_0) is chosen so that $u_0 = 1$ and $v_0 = 0.05 * (1 + sign(51 - x)) * (1 + sign(x - 49))/4$. The parameter values are $k = 1.4$, $b = e = 1$, $d = 1$, $r = 4$ and $s = 0.6$.

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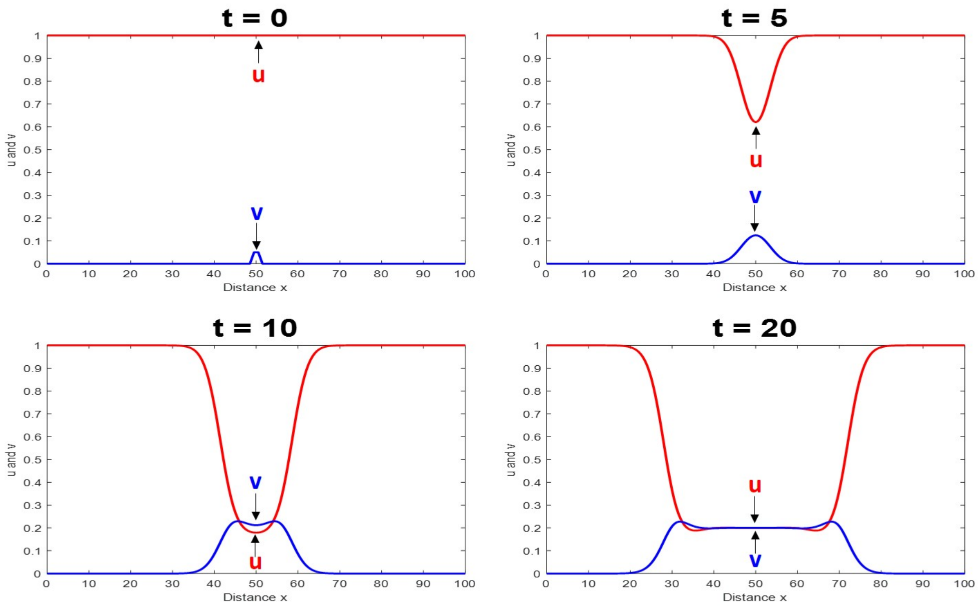

Figure 3.2: The solution of system (1.3) as a function of the spatial variable x is plotted at t=0, t=5, t=10 and t=20. The initial data (u_0, v_0) is chosen so that $u_0 = 1$ and $v_0 = 0.05 * (1 + sign(51 - x)) * (1 + sign(x - 49))/4$. The parameter values are $k = 4$, $b = e = 0$, $d = 1$, $r = 2$ and $s = 0.5$.

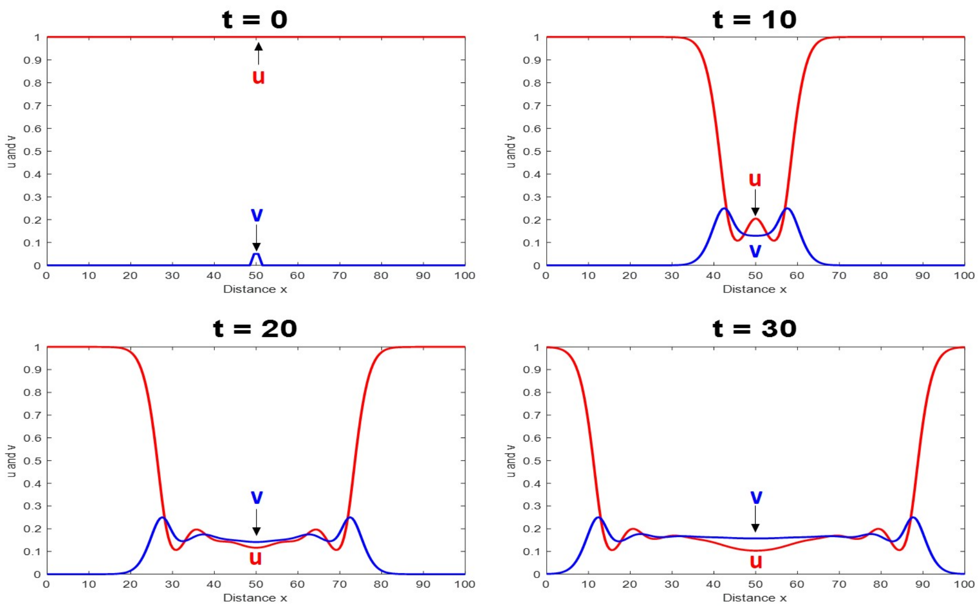

Figure 3.3: The solution of system (1.3) as a function of the spatial variable x is plotted at t=0, t=10, t=20 and t=30. The initial data (u_0, v_0) is chosen so that $u_0 = 1$ and $v_0 = 0.05 * (1 + sign(51 - x)) * (1 + sign(x - 49))/4$. The parameter values are $k = 10$, $b = 5$, $e = 1$, $d = 1$, $r = 4$ and $s = 0.6$.

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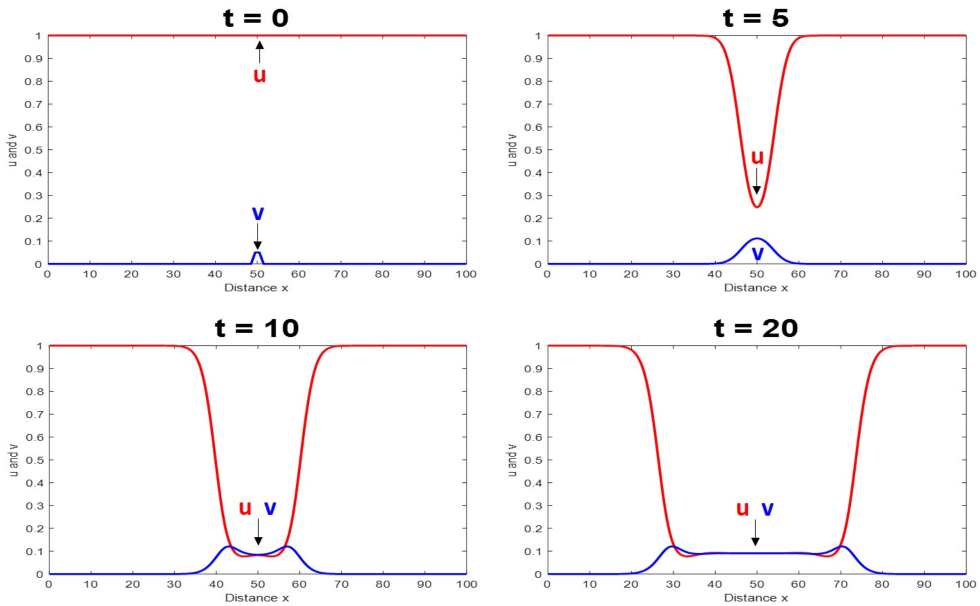

Figure 3.4: The solution of system (1.3) as a function of the spatial variable x is plotted at t=0, t=5, t=10 and t=20. The initial data (u_0, v_0) is chosen so that $u_0 = 1$ and $v_0 = 0.05 * (1 + sign(51 - x)) * (1 + sign(x - 49))/4$. The parameter values are $k = 10$, $b = e = 0$, $d = 1$, $r = 2$ and $s = 0.5$.

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Chapter 4

Traveling wave solutions to system (1.9)

4.1 A general system

In this section, we review a general existence result obtained in [11]. Consider the general system

$$\begin{aligned} &U'' - cU' + f_1(U,V) = 0, \\ &dV'' - cV' + f_2(U,V) = 0 \end{aligned} \tag{4.1}$$

on $\mathbb{R}$. Here $c, d > 0$ and f_i, $i = 1, 2$, are continuously differentiable functions.

Definition 4.1. *(U^+, V^+) and (U^-, V^-) are called a pair of upper and lower solutions of (4.1) if $U^+, V^+, U^-, V^- \in C(\mathbb{R})$ satisfy*

$$(U^+)''(z) - c(U^+)'(z) + f_1(U^+(z), V^-(z)) \le 0,$$

$$(U^-)''(z) - c(U^-)'(z) + f_1(U^-(z), V^+(z)) \ge 0,$$

$$d(V^+)''(z) - c(V^+)'(z) + f_2(U^+(z), V^+(z)) \le 0,$$

$$d(V^-)''(z) - c(V^-)'(z) + f_2(U^-(z), V^-(z)) \ge 0$$

except for finitely many points of z in $\mathbb{R}$.

Using the Schauder's fixed point theorem, the authors in [11] established the following theorem for the existence of solutions to system (4.1).

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Theorem 4.2. *Let a_*, a^*, b_*, b^* be real numbers such that $a_* < a^*$ and $b_* < b^*$. Suppose that the functions f_1 and f_2 satisfy the following conditions:*

(A1) *(Lipschitz condition) There exists a positive constant L such that*

$$|f_i(u_1, v_1) - f_i(u_2, v_2)| \leq L(|u_1 - u_2| + |v_1 - v_2|), i = 1, 2$$

for all $a_* \leq u_1, u_2 \leq a^*$ and $b_* \leq v_1, v_2 \leq b^*$;

(A2) *(Mixed quasi-monotonicity) There exist two positive constants β_1 and β_2 such that*

$$f_1(u_1, v_1) - f_1(u_2, v_1) + \beta_1(u_1 - u_2) \geq 0,$$

$$f_1(u_1, v_1) - f_1(u_1, v_2) \leq 0,$$

$$f_2(u_1, v_1) - f_2(u_1, v_2) + \beta_2(v_1 - v_2) \geq 0,$$

$$f_2(u_1, v_1) - f_2(u_2, v_1) \geq 0$$

for all $a_* \leq u_2 \leq u_1 \leq a^*$ and $b_* \leq v_2 \leq v_1 \leq b^*$.

If (4.1) has a pair of upper and lower solutions (U^+, V^+) and (U^-, V^-) which satisfy

$$a_* \leq U^- \leq U^+ \leq a^*,\ b_* \leq V^- \leq V^+ \leq b^*,$$

on $\mathbb{R}$, and

$$(U^+)'(z+) \leq (U^+)'(z-),\ (V^+)'(z+) \leq (V^+)'(z-),$$

$$(U^-)'(z-) \leq (U^-)'(z+), (V^-)'(z-) \leq (V^-)'(z+)$$

for all $z \in \mathbb{R}$, then system (4.1) has a solution (U, V) satisfying

$$U^- \leq U \leq U^+ \text{ and } V^- \leq V \leq V^+$$

on $\mathbb{R}$.

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4.2 Upper and lower solutions to system (1.10)

In this section, we will construct upper and lower solutions of (1.10) for $c \geq c^* = 2\sqrt{ds}$. By linearizing the second equation of (1.10) around the equilibrium point $(1, 0)$, we get two positive eigenvalues

$$\lambda = \frac{c - \sqrt{c^2 - 4ds}}{2d} \text{ and } \hat{\lambda} = \frac{c + \sqrt{c^2 - 4ds}}{2d}, \tag{4.2}$$

which are two roots of the characteristic equation

$$P(y) := dy^2 - cy + s = 0.$$

Now we assume that (1.12) holds. Then one can easily verify that $(q-1)^2 > k^2 + 4(k-q)$ and so the equation

$$\Psi(y) := y^2 + (q-1)y + k - q = 0 \tag{4.3}$$

has two real roots, saying

$$A^- := \frac{1 - q - \sqrt{(q-1)^2 - 4(k-q)}}{2} \text{ and } A^+ := \frac{1 - q + \sqrt{(q-1)^2 - 4(k-q)}}{2}.$$

In addition, $A^+ > (1 - q + k)/2$. Moreover, we claim that A^+ is positive. For $q \leq 1$, it is obviously true. For $q > 1$, it can be done if one can show that $q > k$. For this, we consider two cases: (i) $k \leq 1$; (ii) $k > 1$. For the case (i), we have $q > 1 \geq k$. For the case (ii), we have $\sqrt{k^2 + 4k} - 1 > k$. Also, by (1.12), we have $q > \sqrt{k^2 + 4k} - 1$. Taken together, we get $q > k$. Therefore there exists a positive constant A satisfying

$$\max\{0, A^-, \frac{1 - q + k}{2}\} < A < A^+. \tag{4.4}$$

Clearly, $\Psi(A) < 0$ and so $A^2 + (q-1)A + k - q < 0$, which follows that

$$1 - A - \frac{k}{A + q} > 0. \tag{4.5}$$

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The case $c > c^*$

For $c > c^*$, we define four nonnegative continuous functions as follows:

$$U^+(z) := 1,\ \ \forall z \in \mathbb{R}, \qquad U^-(z) := \begin{cases} 1 - Me^{\nu z}, & \text{if } z \leq z_1, \\ A, & \text{if } z > z_1, \end{cases} \tag{4.6}$$

$$V^+(z) := \begin{cases} e^{\lambda z}, & \text{if } z \leq 0, \\ 1, & \text{if } z > 0, \end{cases} \qquad V^-(z) := \begin{cases} e^{\lambda z}(1 - \alpha e^{\eta z}), & \text{if } z \leq z_2, \\ 0, & \text{if } z > z_2, \end{cases} \tag{4.7}$$

where λ is defined in (4.2), A, ν, η, M, α are positive constants such that (4.4) holds and

$$\nu < \min\{c, \lambda\}, \tag{4.8}$$

$$\eta < \min\{\lambda, \hat{\lambda} - \lambda\}, \tag{4.9}$$

$$M > \max\{1 - A, k/A\}, \tag{4.10}$$

$$\alpha > \max\{[M/(1 - A)]^{\eta/\nu}, -s/[AP(\lambda + \eta)]\}, \tag{4.11}$$

and

$$z_1 := 1/\nu \ln[(1 - A)/M] < 0 \text{ and } z_2 := -1/\eta \ln \alpha < z_1. \tag{4.12}$$

The graphs of U^+, U^-, V^+, V^- for $c > c^*$ are given in Figure 4.1.

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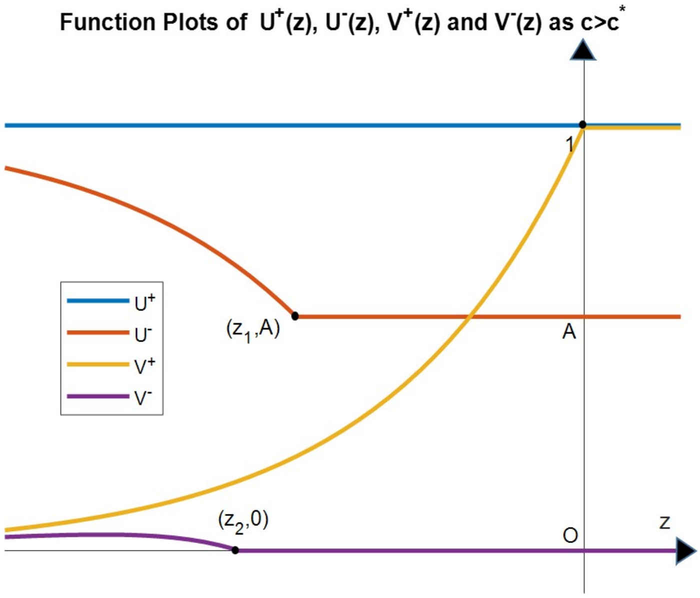

Figure 4.1: Upper and lower solutions to system (1.10) : U^+, U^-, V^+, V^- for $c > c^*$.

Lemma 4.3. *Assume that (1.12) holds, then* (U^+, V^+) *and* (U^-, V^-)*, where* U^+*,* V^+*,* U^-*, and* V^- *are defined in (4.6) and (4.7), are a pair of upper and lower solutions to system (1.10) for* $c > c^*$.

Proof. Firstly, it is obvious that the function $U^+(z) \equiv 1$ satisfies the inequality

$$(U^+)''(z) - c(U^+)'(z) + rU^+(z)(1 - U^+(z)) - \frac{rkU^+(z)V^-(z)}{U^+(z) + qV^-(z)} \leq 0 \tag{4.13}$$

for all $z \in \mathbb{R}$.

Secondly, we claim that the function $V^+(z)$ satisfies the inequality

$$d(V^+)''(z) - c(V^+)'(z) + sV^+(z)\left(1 - \frac{V^+(z)}{U^+(z)}\right) \leq 0 \tag{4.14}$$

for all $z \neq 0$. For $z > 0$, since $V^+(z) = U^+(z) = 1$, (4.14) immediately holds. For $z < 0$, $V^+(z) = e^{\lambda z}$ and $U^+(z) = 1$. Since $P(\lambda) = 0$, it follow that

$$d(V^+)''(z) - c(V^+)'(z) + sV^+(z)\left(1 - \frac{V^+(z)}{U^+(z)}\right) = P(\lambda)e^{\lambda z} - se^{2\lambda z} \leq 0,$$

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and thus (4.14) holds.

Thirdly, we claim that the function $U^-(z)$ satisfies the inequality

$$(U^-)''(z) - c(U^-)'(z) + rU^-(z)(1 - U^-(z)) - \frac{rkU^-(z)V^+(z)}{U^-(z) + qV^+(z)} \geq 0, \tag{4.15}$$

for all $z \neq z_1$. For $z > z_1$, $U^-(z) = A$ and $V^+(z) \leq 1$. By (4.5), we have

$$\begin{aligned} & (U^-)''(z) - c(U^-)'(z) + rU^-(z)(1 - U^-(z)) - \frac{rkU^-(z)V^+(z)}{U^-(z) + qV^+(z)} \\ \geq \quad & rA(1 - A - \frac{k}{A+q}) > 0. \end{aligned}$$

So (4.15) holds. For $z \leq z_1$, we have that $U^-(z) = 1 - Me^{\nu z}$. By (4.8), (4.10), $V^+(z) = e^{\lambda z}$ and $U^-(z) \geq A$, we deduce that

$$\begin{aligned} & (U^-)''(z) - c(U^-)'(z) + rU^-(z)(1 - U^-(z)) - \frac{rkU^-(z)V^+(z)}{U^-(z) + qV^+(z)} \\ \geq \quad & -\nu^2 Me^{\nu z} + c\nu Me^{\nu z} + r(1 - Me^{\nu z})Me^{\nu z} - \frac{rk(1 - Me^{\nu z})e^{\lambda z}}{A} \\ \geq \quad & r(1 - Me^{\nu z}) \left(Me^{\nu z} - \frac{ke^{\nu z}}{A} \right) \\ \geq \quad & r(1 - Me^{\nu z})e^{\nu z}(M - \tfrac{k}{A}) \geq 0. \end{aligned}$$

Hence (4.15) holds.

Finally, we claim that the function $V^-(z)$ satisfies the inequality

$$d(V^-)''(z) - c(V^-)'(z) + sV^-(z) \left(1 - \frac{V^-(z)}{U^-(z)} \right) \geq 0 \tag{4.16}$$

for all $z \neq z_2$. For $z > z_2$, since $V^-(z) = 0$, it follows that (4.16) holds. For $z < z_2$, we have that $V^-(z) = e^{\lambda z}(1 - \alpha e^{\eta z})$ and $U^-(z) = 1 - Me^{\nu z}$. By (4.9), (4.11), (4.12), $P(\lambda) = 0$,

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$P(\lambda+\eta) < 0$ and $U^-(z) \geq A$, we deduce that

$$
\begin{aligned}
& d(V^-)''(z) - c(V^-)'(z) + sV^-(z)\left(1 - \frac{V^-(z)}{U^-(z)}\right) \\
\geq \quad & P(\lambda)e^{\lambda z} - P(\lambda+\eta)\alpha e^{(\lambda+\eta)z} - \frac{se^{2\lambda z}(1-\alpha e^{\eta z})^2}{A} \\
\geq \quad & -P(\lambda+\eta)\alpha e^{(\lambda+\eta)z} - \frac{se^{2\lambda z}}{A} \\
\geq \quad & e^{(\lambda+\eta)z}\left(-\alpha P(\lambda+\eta) - \frac{se^{(\lambda-\eta)z}}{A}\right) \\
\geq \quad & e^{(\lambda+\eta)z}\left(-\alpha P(\lambda+\eta) - s/A\right) \geq 0.
\end{aligned}
$$

So (4.16) holds. This completes the proof of this lemma. □

The case $c = c^*$

In this case, $\lambda = \hat{\lambda}$ and $c = 2d\lambda$. Let ν and κ be positive constants such that

$$\nu < \min\{c, \lambda\} \tag{4.17}$$

and

$$\kappa > \lambda e. \tag{4.18}$$

Then there exists a negative number z_0 such that

$$-\kappa z_0 e^{\lambda z_0} = 1 \;\text{ and }\; z_0 < -1/\lambda. \tag{4.19}$$

Let M be a positive constant such that

$$M > \max\{1, (1-A)e^{-\nu \hat{z_1}}\}, \tag{4.20}$$

where $\hat{z_1}$ is a negative constant such that $\hat{z_1} < z_0$ and

$$-ze^{(\lambda-\nu)z} < \frac{A}{k\kappa}, \;\; \forall z < \hat{z_1}. \tag{4.21}$$

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Then

$$z_1 := -1/\nu \ln[M/(1-A)] < \hat{z}_1 < z_0. \tag{4.22}$$

Select

$$l > \max\{1, \kappa\sqrt{-\hat{z}_2}\}, \tag{4.23}$$

where $\hat{z}_2$ is a negative constant such that $\hat{z}_2 < z_1$ and

$$(-z)^{7/2}e^{\lambda z} < \frac{Ad}{4s\kappa^2}, \ \ \forall z < \hat{z}_2. \tag{4.24}$$

Then

$$z_2 := -(l/\kappa)^2 < \hat{z}_2 < z_1. \tag{4.25}$$

For $c = c^*$, we define four nonnegative continuous functions as follows:

$$U^+(z) := 1, \ \ \forall z \in \mathbb{R}, \qquad U^-(z) := \begin{cases} 1 - Me^{\nu z}, & \text{if } z \leq z_1, \\ A, & \text{if } z > z_1, \end{cases} \tag{4.26}$$

$$V^+(z) := \begin{cases} -\kappa z e^{\lambda z}, & \text{if } z \leq z_0, \\ 1, & \text{if } z > z_0, \end{cases} \qquad V^-(z) := \begin{cases} (-\kappa z - l\sqrt{-z})e^{\lambda z}, & \text{if } z \leq z_2, \\ 0, & \text{if } z > z_2, \end{cases} \tag{4.27}$$

where A, ν, κ, M, l, z_0, z_1, z_2 are constants such that (4.5), (4.17)–(4.25) hold. The graphs of U^+, U^-, V^+, V^- for $c = c^*$ are given in Figure 4.2.

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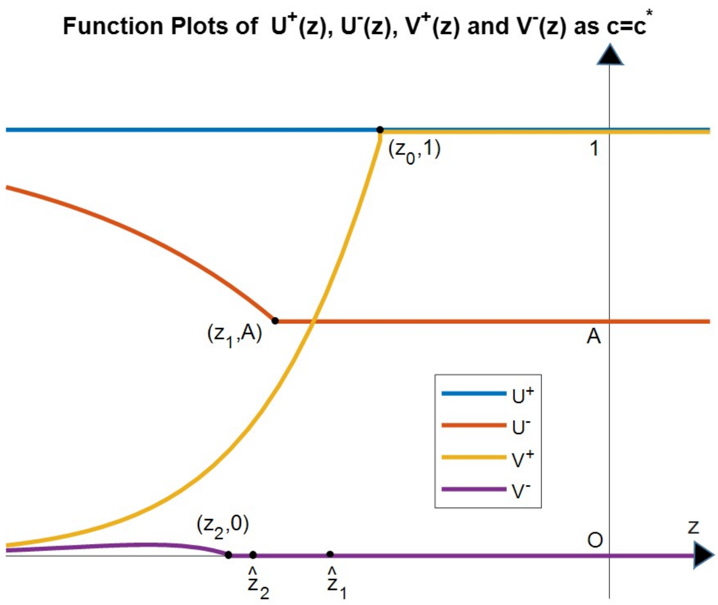

Figure 4.2: Upper and lower solutions to system (1.10) : U^+, U^-, V^+, V^- for $c = c^*$.

Lemma 4.4. *Assume that (1.12) holds, then* (U^+, V^+) *and* (U^-, V^-)*, where* U^+*,* V^+*,* U^-*, and* V^- *are defined in (4.26) and (4.27), are a pair of upper and lower solutions to system (1.10) for* $c = c^*$.

Proof. Firstly, it is obvious that the function $U^+(z) \equiv 1$ satisfies the inequality

$$(U^+)''(z) - c(U^+)'(z) + rU^+(z)(1 - U^+(z)) - \frac{rkU^+(z)V^-(z)}{U^+(z) + qV^-(z)} \leq 0 \tag{4.28}$$

for all $z \in \mathbb{R}$.

Secondly, we claim that the function $V^+(z)$ satisfies the inequality

$$d(V^+)''(z) - c(V^+)'(z) + sV^+(z)\left(1 - \frac{V^+(z)}{U^+(z)}\right) \leq 0 \tag{4.29}$$

for all $z \neq z_0$. For $z > z_0$, since $V^+(z) = U^+(z) = 1$, (4.29) holds. For $z \leq z_0$, $V^+(z) =$

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$-\kappa z e^{\lambda z}$ and $U^+(z) = 1$. From the fact that $P(\lambda) = 0$ and $c = 2d\lambda$, it follows that

$$
\begin{aligned}
& d(V^+)''(z) - c(V^+)'(z) + sV^+(z)\left(1 - \frac{V^+(z)}{U^+(z)}\right) \\
= \quad & -P(\lambda)\kappa z e^{\lambda z} - (2d\lambda - c)\kappa e^{\lambda z} - s(V^+)^2 \leq 0,
\end{aligned}
$$

and therefore (4.29) holds.

Thirdly, we claim that the function $U^-(z)$ satisfies the inequality

$$
(U^-)''(z) - c(U^-)'(z) + rU^-(z)(1 - U^-(z)) - \frac{rkU^-(z)V^+(z)}{U^-(z) + qV^+(z)} \geq 0 \tag{4.30}
$$

for all $z \neq z_1$. For $z > z_0$, since $U^-(z) = A$ and $V^+(z) = 1$, from (4.5), we obtain that

$$
\begin{aligned}
& (U^-)''(z) - c(U^-)'(z) + rU^-(z)(1 - U^-(z)) - \frac{rkU^-(z)V^+(z)}{U^-(z) + qV^+(z)} \\
= \quad & rA(1 - A) - \frac{rkA}{A + q} = rA\left(1 - A - \frac{k}{A + q}\right) > 0,
\end{aligned}
$$

and so (4.30) holds. For $z_1 < z \leq z_0$, $U^-(z) = A$ and $V^+(z) = -\kappa z e^{\lambda z}$. By (4.5) and $V^+(z) \leq 1$, we get

$$
\begin{aligned}
& (U^-)''(z) - c(U^-)'(z) + rU^-(z)(1 - U^-(z)) - \frac{rkU^-(z)V^+(z)}{U^-(z) + qV^+(z)} \\
& \geq rA(1 - A - \frac{k}{A + q}) > 0,
\end{aligned}
$$

and so (4.30) holds. For $z \leq z_1$, we have that $U^-(z) = 1 - Me^{\nu z}$ and $V^+(z) = -\kappa z e^{\lambda z}$. By (4.17)-(4.22), $U^- \geq A$ and $V^+ \geq 0$, we deduce that

$$
\begin{aligned}
& (U^-)''(z) - c(U^-)'(z) + rU^-(z)(1 - U^-(z)) - \frac{rkU^-(z)V^+(z)}{U^-(z) + qV^+(z)} \\
\geq \quad & -\nu^2 Me^{\nu z} + c\nu Me^{\nu z} + r(1 - Me^{\nu z})Me^{\nu z} - \frac{rk(1 - Me^{\nu z})(-\kappa z e^{\lambda z})}{A} \\
\geq \quad & r(1 - Me^{\nu z})\left(Me^{\nu z} - \frac{k\kappa(-z)e^{\lambda z}}{A}\right) \\
\geq \quad & r(1 - Me^{\nu z})e^{\nu z}(M - \frac{k\kappa(-z)e^{(\lambda - \nu)z}}{A}) > 0.
\end{aligned}
$$

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Hennce (4.30) holds.

Finally, we claim that the function $V^-(z)$ satisfies the inequality

$$d(V^-)''(z) - c(V^-)'(z) + sV^-(z)\left(1 - \frac{V^-(z)}{U^-(z)}\right) \geq 0 \tag{4.31}$$

for all $z \neq z_2$. For $z > z_2$, since $V^-(z) = 0$, it follows that (4.31) holds. For $z \leq z_2$, we have that $V^-(z) = (-\kappa z - l\sqrt{-z})e^{\lambda z}$ and $U^-(z) \geq A$. By (4.24)-(4.25), $P(\lambda) = 0$ and $c = 2d\lambda$, we deduce that

$$\begin{aligned}
& d(V^-)''(z) - c(V^-)'(z) + sV^-(z)\left(1 - \frac{V^-(z)}{U^-(z)}\right) \\
\geq\ & P(\lambda)(-\kappa z - l\sqrt{-z})e^{\lambda z} + (2d\lambda - c)\left(-\kappa + \frac{l}{2\sqrt{-z}}\right)e^{\lambda z} \\
& + \frac{dle^{\lambda z}}{4(-z)^{3/2}} - \frac{se^{2\lambda z}[\kappa(-z) - l\sqrt{-z}]^2}{A} \\
\geq\ & \frac{dle^{\lambda z}}{4(-z)^{3/2}} - \frac{s\kappa^2 z^2 e^{2\lambda z}}{A} + \frac{2s\kappa l(-z)^{3/2}e^{2\lambda z}}{A} - \frac{sl^2(-z)e^{2\lambda z}}{A} \\
=\ & \frac{e^{\lambda z}}{4A(-z)^{3/2}}(Adl - 4s\kappa^2(-z)^{7/2}e^{\lambda z}) + \frac{sl(-z)e^{2\lambda z}}{A}(2\kappa(-z)^{1/2} - l) > 0.
\end{aligned}$$

Thus (4.31) holds. □

4.3 Semi-traveling wave solutions to system (1.9)

Following the proof of Lemma 2.1 with a slight modification, we have the following lemma. Since its proof is almost the same as that in Lemma 2.1, we omit it here.

Lemma 4.5. *If $c < c^* := 2\sqrt{ds}$, then system (1.10) has no positive solution (U, V) satisfying $(U, V)(-\infty) = (1, 0)$.*

Lemma 4.6. *Suppose that (1.12) holds. If $c \geq c^*$, then system (1.10) admits a positive solution (U, V) satisfying $(U, V)(-\infty) = (1, 0)$. In addition,*

$$A < U < 1 \text{ and } 0 < V < 1$$

on $\mathbb{R}$, where A is a positive constant satisfying (4.4).

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Proof. Define

$$\begin{aligned} f_1(u,v) &= ru(1-u) - \frac{rkuv}{u+qv}, \\ f_2(u,v) &= sv\left(1 - \frac{v}{u}\right). \end{aligned} \tag{4.32}$$

One can easily verify that f_i satisfies (A1)-(A2) and $f_i(1,0) = f_i(\eta^*, \eta^*) = 0$ for $i = 1, 2$. So it follows from Theorem 4.2, Lemma 4.3, Lemma 4.4, and definition of (U^+, V^+) and (U^-, V^-) that system (1.10) has a solution (U, V) satisfying

$$A \le U^-(z) \le U(z) \le U^+(z) \le 1 \text{ and } 0 \le V^-(z) \le V(z) \le V^+(z) \le 1,\ \forall z \in \mathbb{R}. \tag{4.33}$$

Due to $U^-(-\infty) = U^+(-\infty) = 1$ and $V^-(-\infty) = V^+(-\infty) = 0$, we have

$$(U, V)(-\infty) = (1, 0). \tag{4.34}$$

We now just need to prove that $A < U < 1$ and $0 < V < 1$. First, we claim $V(z) > 0$ for all $z \in \mathbb{R}$. For contradiction, we assume that there exists a smallest $z_3 \ge z_2$ such that $V(z_3) = 0$. By the definition of V^-, we have that $V(z) > 0$ for all $z < z_3$ that implies $V'(z_3) = 0$ and so the existence and unique theorem gives that $V = 0$ for all $z \in \mathbb{R}$ which makes a contradiction. Second, we claim $U(z) < 1$ for all $z \in \mathbb{R}$. For contradiction, we assume that there exists $z_4 \in \mathbb{R}$ such that $U(z_4) = 1$, then $U'(z_4) = 0$ and $U''(z_4) \le 0$. But, by (1.10) and $V > 0$, we have that

$$U''(z_4) = -rU(z_4)(1 - U(z_4)) + \frac{rkU(z_4)V(z_4)}{U(z_4) + qV(z_4)} > 0,$$

yielding a contradiction. Using a similar argument, we can show that $V < 1$ for all $z \in \mathbb{R}$. Finally, we show that $U(z) > A$ for all $z \in \mathbb{R}$. For contradiction, we assume that there exists a smallest $z_5 \ge z_1$ such that $U(z_5) = A$. By the definition of U^-, we have that $U(z) > A$ for all

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$z < z_5$ that implies $U'(z_5) = 0$ and $U''(z_5) \geq 0$. But, by (1.10), (4.5), and $V < 1$, we have that

$$\begin{aligned} 0 &= U''(z_5) - cU'(z_5) + rU(z_5)(1 - U(z_5)) - \frac{rkU(z_5)V(z_5)}{U(z_5) + qV(z_5)} \\ &\geq rA\left(1 - A - \frac{kV(z_5)}{A + qV(z_5)}\right) \\ &> rA\left(1 - A - \frac{k}{A + q}\right) > 0, \end{aligned}$$

yielding a contradiction. So we have that $U(z) > A$ for all $z \in \mathbb{R}$. Now we finish the proof of Lemma 4.6. □

In the remaining, we will show that the semi-traveling wave solutions established in Lemma 4.6 to system (1.9) are actually traveling wave solutions. To this end, we need the following lemma.

Lemma 4.7. *Suppose that (1.12) holds. Let $c \geq c^*$ and let (U, V) be a positive solution obtained in Lemma 4.6 to system (1.10) on $\mathbb{R}$. Then there exists a constant $z_6 \in \mathbb{R}$ such that*

$$V(z) > A \tag{4.35}$$

for all $z \geq z_6$, where A is a positive constant satisfying (4.4).

Proof. First, we claim that there exists $z_6 \in \mathbb{R}$ such that $V(z_6) > A$. For contradiction, we assume that

$$V(z) \leq A, \ \forall z \in \mathbb{R}. \tag{4.36}$$

Since $V(-\infty) = 0$ and $V(z) > 0$ for all $z \in \mathbb{R}$, there are two possibilites: (i) there is a smallest z_7 such that $V'(z_7) \leq 0$; (ii) $V'(z) > 0$ for all $z \in \mathbb{R}$. For the case (i), we note that as long as $V \leq A$ and $V' \leq 0$, it follows from (1.10) and Lemma 4.6 that

$$dV'' = cV' - sV(1 - \frac{V}{U}) < 0,$$

which implies that $V''(z) < 0$ and $V'(z) < 0$ for all $z > z_6$. This leads to $V(z) \to -\infty$ as $z \to \infty$, which contradicts the boundedness of $V(z)$. For the case (ii), $V(\infty)$ exists and is positive. Since V and $sV(1 - V/U)$ are bounded on $\mathbb{R}$, it follows from Lemma A.3 that V' and

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V'' are bounded on $\mathbb{R}$. Then differentiating the second equation of (1.10) and using Lemma A.3 again, we also have V''' is bounded on $\mathbb{R}$. Thus, by Lemma A.1, we have $V'(\infty) = V''(\infty) = 0$. This, together with (1.10) and the fact that $V(\infty) > 0$, yields that $U(\infty)$ also exists and $U(\infty) = V(\infty)$. From (1.10b), $U > A$ and (4.36), we get $U(\infty) = V(\infty) = A$. Arguing as above, we have $U'(\infty) = 0$ and $U''(\infty) = 0$. Therefore, by (1.10a), we get that $1 - A - k/(1+q) = 0$. On the other hand, by (4.5) and $A < 1$, we get $1 - A - k/(1+q) > 0$, a contradiction. Hence we conclude that there is a $z_6 \in \mathbb{R}$ such that $V(z_6) > A$. Furthermore, we claim that $V(z) > A$ for all $z \geq z_6$. For contradiction, we assume that there exists $z_8 > z_6$ such that $V(z_8) = A$ and $V'(z_8) \leq 0$. Recall we have shown in the case (i) that $V'' < 0$ as long as $V \leq A$ and $V' \leq 0$. It follows that $V''(z) < 0$ and $V'(z) < 0$ for all $z > z_8$ and therefore $V(z) \to -\infty$ as $z \to \infty$, which contradicts the boundedness of V. So we finish the proof of this lemma. □

4.4 Proof of Theorem 1.4

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Lemma 4.6, system (1.10) has a positive solution (U, V) satisfying $(U, V)(-\infty) = (1, 0)$. In addition, $A < U < 1$ and $0 < V < 1$ on $\mathbb{R}$, where A is a positive constant satisfying (4.4). To complete the proof, we just need to show that

$$(U, V)(\infty) = (\eta^*, \eta^*). \tag{4.37}$$

For this, we define

$$\bar{u} := \limsup_{z\to+\infty} U(z), \underline{u} := \liminf_{z\to+\infty} U(z), \bar{v} := \limsup_{z\to+\infty} V(z), \underline{v} := \liminf_{z\to+\infty} V(z), \tag{4.38}$$

and claim that

$$\bar{u}(1-\bar{u}) - \frac{k\bar{u}\underline{v}}{\bar{u} + q\underline{v}} \geq 0. \tag{4.39}$$

Note that there are two possibilities for the behaviour of U at ∞: (i) U is eventually monotone; (ii) U is oscillatory. For the case (i), since U is bounded, it follows that $U(\infty)$ exists and

$$U(\infty) = \bar{u}. \tag{4.40}$$

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Referring to the proof of Lemma 4.7, we also get

$$U'(\infty) = U''(\infty) = 0. \tag{4.41}$$

For any given $\epsilon > 0$, by definition of limit inferior, there exists $z^* > 0$ such that

$$V(z) > \underline{v} - \epsilon \tag{4.42}$$

for all $z > z^*$. So it follows from (1.10a) and (4.42) that

$$U''(z) - cU'(z) + rU(z)(1 - U(z)) - \frac{rkU(z)(\underline{v} - \epsilon)}{U(z) + q(\underline{v} - \epsilon)} > 0$$

for all $z > z^*$. Letting $z \to \infty$ and using (4.40) and (4.41), we get

$$\bar{u}(1 - \bar{u}) - \frac{k\bar{u}(\underline{v} - \epsilon)}{\bar{u} + q(\underline{v} - \epsilon)} \geq 0.$$

Finally, letting $\epsilon \to 0$, we get (4.39). For the case (ii), we select a sequence $\{z_n\}$ of the maximum points of U such that $z_n \geq z^*$ for all $n \in \mathbb{N}$, $z_n \to \infty$ as $n \to \infty$, and $\lim_{n\to+\infty} U(z_n) = \bar{u}$. Then $U'(z_n) = 0$ and $U''(z_n) \leq 0$ for all $n \in \mathbb{N}$. Together with (1.10a) and (4.42), we get

$$\begin{aligned} 0 \quad &= \quad U''(z_n) - cU'(z_n) + rU(z_n)(1 - U(z_n)) - \frac{rkU(z_n)V(z_n)}{U(z_n) + qV(z_n)} \\ &\leq \quad rU(z_n)(1 - U(z_n)) - \frac{rkU(z_n)(\underline{v} - \epsilon)}{U(z_n) + q(\underline{v} - \epsilon)} \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ and then letting $\epsilon \to 0$, we get (4.39). By similar arguments, we also get the following inequalities

$$\underline{u}(1 - \underline{u}) - \frac{k\underline{u}\bar{v}}{\underline{u} + q\bar{v}} \leq 0, \tag{4.43}$$

$$\bar{v}\left(1 - \frac{\bar{v}}{\bar{u}}\right) \geq 0, \tag{4.44}$$

and

$$\underline{v}\left(1 - \frac{\underline{v}}{\underline{u}}\right) \leq 0. \tag{4.45}$$

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Note that $\underline{v} > 0$ by Lemma 4.7. Together with (4.45), we have

$$\underline{v} \geq \underline{u}. \tag{4.46}$$

Substituting (4.46) into (4.39), we deduce that

$$(1 - \bar{u}) - \frac{k\underline{u}}{\bar{u} + q\underline{u}} \geq 0,$$

which gives

$$\bar{u} - \bar{u}^2 + q\underline{u} - q\bar{u}\underline{u} - k\underline{u} \geq 0. \tag{4.47}$$

Similarly, using (4.43) and (4.44), we also have

$$\bar{u} \geq \bar{v} \tag{4.48}$$

and

$$-\underline{u} + \underline{u}^2 - q\bar{u} + q\bar{u}\underline{u} + k\bar{u} \geq 0. \tag{4.49}$$

Combining (4.47) and (4.49), we obtain that

$$(\bar{u} - \underline{u})(1 - q + k - \bar{u} - \underline{u}) \geq 0.$$

By (4.4), we have

$$\bar{u} + \underline{u} \geq 2\underline{u} \geq 2A > 1 - q + k.$$

Taken together, we get $\bar{u} = \underline{u}$. This, together with (4.46) and (4.48), gives that

$$\underline{u} \leq \underline{v} \leq \bar{v} \leq \bar{u} = \underline{u}, \tag{4.50}$$

and so

$$\underline{v} = \bar{v} = \underline{u} = \bar{u}. \tag{4.51}$$

Hence both $U(\infty)$ and $V(\infty)$ exist, and $U(\infty) = V(\infty)$. Substituting (4.51) into (4.39) and (4.43), we obtain that $U(\infty) = V(\infty) = 1 - k/(1+q) = \eta^*$. Then (4.37) holds and we finish the proof of Theorem 1.4.

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4.5 Numerical simulation results

Now we present some numerical results for system (1.9). In Figure 4.3 and Figure 4.4, we see that the large time behaviors of the solutions of the initial value problem of system (1.9) are two traveling waves propagating outwards in opposite directions, where the initial data (u_0, v_0) is chosen so that $u_0 = 1$ and $v_0 = 0.05(1 + \text{sign}(51 - x))(1 + \text{sign}(x - 49))/4$. From Figure 4.5, we see that the restriction $k < \sqrt{q^2 + 2q + 5} - 2$ in Theorem 1.4 is a technical assumption since the values of the parameters q and k in Figure 4.5 do not satisfy the restriction.

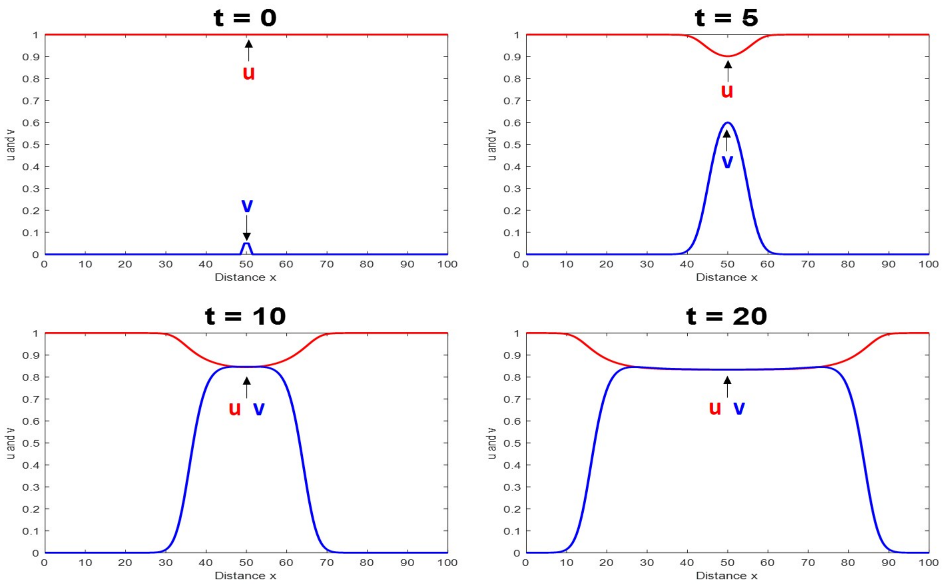

Figure 4.3: The solution of system (1.9) as a function of the spatial variable x is plotted at t=0, t=5, t=10 and t=20. The initial data (u_0, v_0) is chosen so that $u_0 = 1$ and $v_0 = 0.05 * (1 + sign(51 - x)) * (1 + sign(x - 49))/4$. The parameter values are $k = 1$, $q = 5$, $d = 1$, $r = 0.5$ and $s = 1$.

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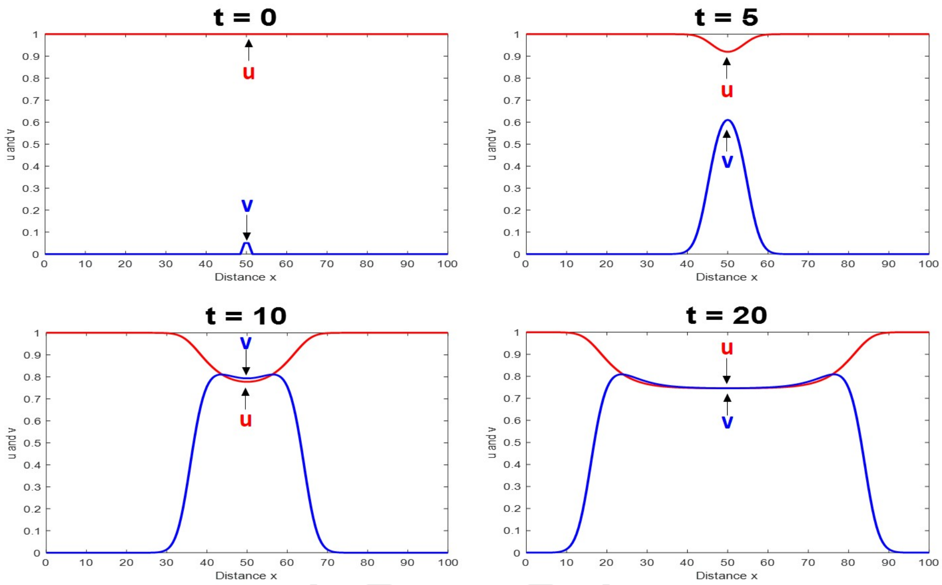

Figure 4.4: The solution of system (1.9) as a function of the spatial variable x is plotted at t=0, t=5, t=10 and t=20. The initial data (u_0, v_0) is chosen so that $u_0 = 1$ and $v_0 = 0.05 * (1 + sign(51 - x)) * (1 + sign(x - 49))/4$. The parameter values are $k = 0.28$, $q = 0.1$, $d = 1$, $r = 0.5$ and $s = 1$.

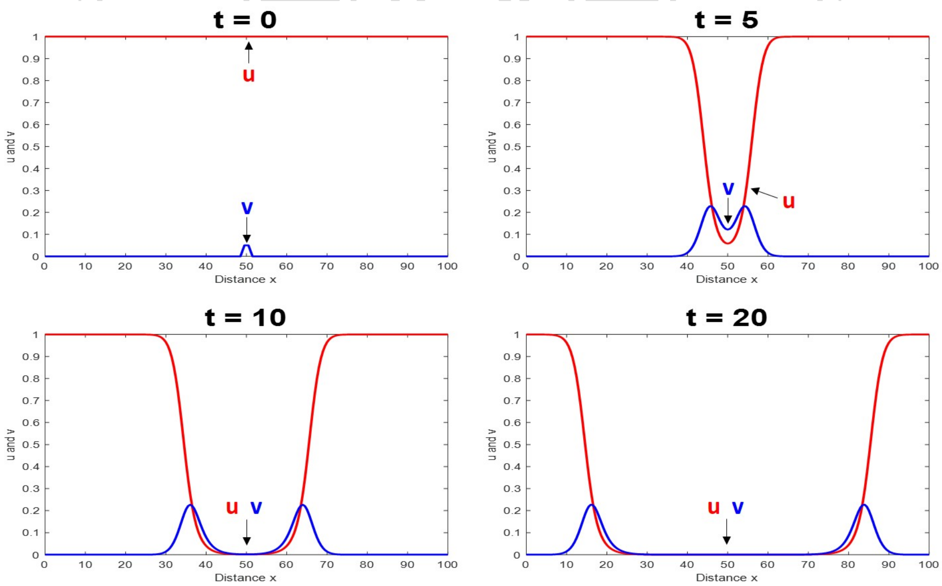

Figure 4.5: The solution of system (1.9) as a function of the spatial variable x is plotted at t=0, t=5, t=10 and t=20. The initial data (u_0, v_0) is chosen so that $u_0 = 1$ and $v_0 = 0.05 * (1 + sign(51 - x)) * (1 + sign(x - 49))/4$. The parameter values are $k = 2.99$, $q = 2$, $d = 1$, $r = 5$ and $s = 1$.

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Appendix

In this chapter, we collect some useful lemmas which are used in the proof.

Lemma A.1. (Barbălat's Lemma [2]) *Suppose* $w \in C^1(b,\infty)$ *and* $\lim_{t\to\infty} w(t)$ *exists. If* w' *is uniformly continuous, then* $\lim_{t\to\infty} w'(t) = 0$.

Lemma A.2. (LaSalle's Invariance Principle [7]) *Consider the following initial value problem:*

$$X' = f(X),\ X \in \mathbb{R}^n. \tag{A.1}$$

Let $\Sigma \subseteq \mathbb{R}^n$ *be an open set in* $\mathbb{R}^n$*. Suppose* $X(z)$ *is a solution of (A.1) which is positive invariant in* Σ*. If there is a continuous and bounded below function* $V : \Sigma \to \mathbb{R}$ *such that the orbital derivative of* V *along* $X(z)$ *is non-positive, i.e.,*

$$\frac{d}{dz}V(X(z)) = \nabla V(X(z)) \cdot X'(z) \le 0,$$

then the ω*-limit set of* $X(z)$ *is contained in* $\mathscr{I}$*, where* $\mathscr{I}$ *is the largest invariant set in* $\{X \in \Sigma : dV/dz = 0\}$.

The following *a priori* estimates for the second-order differential equations can be found in [6].

Lemma A.3. (Fu [6, Lemma 3.2]) *Let* B *be a positive number and* $G \in C(\mathbb{R})$*. Suppose that* $w \in C^2(\mathbb{R})$ *is a solution of*

$$w'' - Bw' = G(z)$$

in $\mathbb{R}$*. If* w *and* G *are bounded in* $\mathbb{R}$*, then so are* w' *and* w''*. Moreover,*

$$\|w'\|_{L^\infty(\mathbb{R})} \le \frac{\|G\|_{L^\infty(\mathbb{R})}}{B}$$

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and

$$\|w''\|_{L^\infty(\mathbb{R})} \leq 2\|G\|_{L^\infty(\mathbb{R})}.$$

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Bibliography

[1] S. AI, Y. DU, AND R. PENG, *Traveling waves for a generalized Holling–Tanner predator–prey model*, J. Differ. Eqn., 263 (2017), pp. 7782–7814.

[2] I. BARBALAT, *Systemes déquations différentielles dóscillations non linéaires*, Rev. Math. Pures Appl., 4 (1959), pp. 267–270.

[3] Y.-Y. CHEN, J.-S. GUO, AND C.-H. YAO, *Traveling wave solutions for a continuous and discrete diffusive predator–prey model*, J. Math. Anal. Appl., 445 (2017), pp. 212–239.

[4] Y. DU AND S.-B. HSU, *A diffusive predator–prey model in heterogeneous environment*, J. Differ. Eqn., 203 (2004), pp. 331–364.

[5] S.-C. FU, *Traveling waves for a diffusive SIR model with delay*, J. Math. Anal. Appl., 435 (2016), pp. 20–37.

[6] S.-C. FU, M. MIMURA, AND J.-C. TSAI, *Traveling waves in a hybrid model of demic and cultural diffusions in Neolithic transition*, J. Math. Biol., 82 (2021), p. article 26.

[7] J. K. HALE, *Ordinary Differential Equations*, R.E. Krieger Publ., (1980).

[8] W.-T. LI, G. LIN, AND S. RUAN, *Existence of travelling wave solutions in delayed reaction–diffusion systems with applications to diffusion–competition systems*, Nonlinearity, 19 (2006), pp. 1253–1273.

[9] C.-H. WANG AND S.-C. FU, *Traveling wave solutions to diffusive Holling-Tanner predator-prey models*, Discrete Cont. Dyn.-B, 26 (2021), pp. 2239–2255.

[10] X.-S. WANG, H. WANG, AND J. WU, *Traveling waves of diffusive predator-prey systems: disease outbreak propagation*, Discrete Cont. Dyn. S., 32 (2012), pp. 3303–3324.

DOI:10.6814/NCCU202100501

[11] W. ZUO AND J. SHI, *Traveling wave solutions of a diffusive ratio-dependent Holling-Tanner system with distributed delay*, Commun. Pur. Appl. Anal., 17 (2018), pp. 1179–1200.

DOI:10.6814/NCCU202100501