國立政治大學應用數學系

碩士學位論文

# 關於黃金平均樹上子平移的條型熵研究
# On the Strip Entropy of the Golden-Mean Tree Shift



指導教授：班榮超　博士

研究生：林韋霖　撰

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# 中文摘要

在 2019 年，彼得森跟莎拉曼 [11] 證明在樹平移上拓樸熵的存在性。之後他們取最左邊的樹枝當作基底，用條型法 [12] 去估計黃金平均樹上子平移的熵。以 $\{0,1\}$ 為字母，並且不接受連續兩個 1，他們證明在 $k$ 維樹上，$h_n^{(k)}$ 會收斂到 $h^{(k)}$。

本篇文章會考慮在黃金平均樹上的週期路徑，並且定義條型熵在這些週期路徑上，稱作 $h_n(\mathcal{T})$。我們證明 $h_n(\mathcal{T})$ 會收斂到到在黃金平均樹上的熵 $h(\mathcal{T})$。

關鍵字：樹平移，拓樸熵，黃金平均樹，條型熵



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# Abstract

In 2019, Petersen and Salama [11] demonstrated the existence of topological entropy for tree shifts. Later, they took the most left branch as a fixed base, and use the *strip method* [12] to evaluate the entropy of the golden mean tree shift. By alphabet $\{0, 1\}$ with no adjacent 1's, they proved that $h_n^{(k)}$ converges to $h^{(k)}$ on the k-tree shift.

In this paper, the periodic paths on the golden mean tree are considered, and the *strip entropy*, said $h_n(\mathcal{T})$, is defined in these periodic paths. We prove that $h_n(\mathcal{T})$ converges to the topological entropy $h(\mathcal{T})$ on the golden mean tree.

Keywords: tree shifts, topological entropy, golden mean tree, strip entropy



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# Contents





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# Chapter 1

# Introduction

A set $X$ of infinite words over $\mathcal{A}$ is a *shift space* if it is a closed set of a full shift $\mathcal{A}^{\mathbb{Z}}$ and invariant under the shift operator, it can be described by a forbidden set $\mathcal{F}$, where the forbidden set collects the elements which would be rejected, such shift space is denoted by $X = X_{\mathcal{F}}$. If the set $\mathcal{F}$ is finite, then we called it the *shift of finite type* (SFT).

In 2012, Aubrun and Béal $[1, 2]$ studied the shifts of finite type defined on infinite ranked trees by a natural structure of symbolic dynamical system. A tree shift only attaches to child nodes and parent node, so it is more complicated than one-dimensional shifts, but it is not complicated as multidimensional shifts. In other words, tree shifts are the case between one-dimensional shifts and multidimensional shifts. Because multidimensional shifts have only few results than one-dimensional shifts, researchers are interested in tree shifts.

The *topological entropy* is a quantity to describe the complexity of a given system. In [10], Lind and Marcus provided the $1$-*dimensional topological entropy* as

$$h(X) = \lim_{n\to\infty} \frac{\log|B_n(X)|}{n},$$

where $X$ is a $1$-*dimensional* shift space and $B_n(X)$ is the set of all $n$-length words occur in $X$, and $|\cdot|$ denotes the number of cardinality.

In [5], Ban and Chang defined the $entropy$ of tree shifts by

$$h(X^{\mathcal{B}}) = \lim_{n\to\infty} \frac{\ln^2|B_n(X^{\mathcal{B}})|}{n},$$

where $\mathcal{B}$ is a basic set of allowed 2-block patterns. And let $X^{\mathcal{B}} = X_{\mathcal{F}}$, where $\mathcal{F} = B_2(\mathcal{T}) \setminus \mathcal{B}$,



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$\mathcal{T}$ is the set of all infinite trees on a finite alphabet, then $X^{\mathcal{B}}$ is a tree shift of finite type. They proved that the *entropy* is either $0$ or $\ln 2$, and the mixing and chaotic behavior for tree shifts are also studied [3, 4, 6, 8].

In [11], Peterson and Salama defined the *topological entropy* on $k$-tree $\tau$ in a different way by

$$h(\tau) = \lim_{n\to\infty} \frac{\log p_\tau(n)}{1 + k + \cdots + k^n} = \inf\left\{ \frac{\log p_\tau(n)}{1 + k + \cdots + k^n} : n \in \mathbb{N} \right\},$$

where $p_\tau(n)$ is the number of $n$-blocks that appear in the tree $\tau$. They demonstrated the existence of the topological entropy, and proved that $h(\mathcal{T}X) \geq h(X)$, where $\mathcal{T}X$ is the hom tree-shift derived from X, hom tree-shift means the rule for every generator is the same. In [7], Ban et al. characterized a necessary and sufficient condition when the preceding equality holds.

In [12], Peterson and Salama raised the strip method. In $k$-tree by alphabet $\{0, 1\}$ with no adjacent 1's, they took the most left branch as a fixed base (figure1) to draw the strip tree (figure2). The width of strip tree is the length of the most left branch in $k$-tree, and the height of strip tree is the longest length that the node in the most left branch attaches to other nodes. That is, if the height in strip tree is $n$ means the node in the most left branch would attach to $k-1$ $\Delta_{n-1}^{(k)}$, where $\Delta_{n-1}^{(k)}$ is the initial height $n-1$ subtree of the $k$-tree. Fix width and height, a recurrence relation is obtained

$$\begin{bmatrix} a_{i+1}(0) \\ a_{i+1}(1) \end{bmatrix} = \begin{bmatrix} b_{n-1}^{k-1} & b_{n-1}^{k-1} \\ b_{n-1}(0)^k & 0 \end{bmatrix} \begin{bmatrix} a_i(0) \\ a_i(1) \end{bmatrix},$$

where $a_i(0)$ denotes the number of ways of the node with width $i$ height $0$ labeled $0$, and define $a_i(1)$ similarly, $b_n$ is the number of different labelings of $\Delta_n$. Thus, the entropy is obtained by

$$\lim_{n\to\infty} {h_n}^{(k)} = \lim_{n\to\infty} \frac{(k-1)\log b_{n-1}}{k^n - 1} = h^{(k)}.$$



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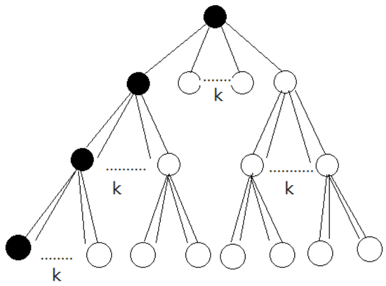
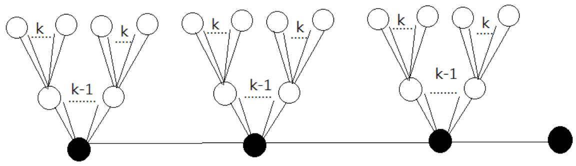



figure(1)




figure(2)

In $k$-tree, every node has the same out-degree $k$, so if we use another path rather than the most left path, we will get the same strip tree. Thus, we consider the golden mean tree whose node has out-degree either $1$ or $2$, such if we choose a different path, we will obtain different strip tree. We want to know if the strip method can still work.

In this paper, we first find the general $m$-step recurrence matrix $M$ (**Theorem 3.1.1**) corresponds the strip tree with height $n$. Second, we find two irreducible matrix (**Lemma 3.1.2**) $C, D$ such that

$$B_{n-2}^{\alpha} B_{n-1}^{\beta} D \leq M \leq B_{n-2}^{\alpha} B_{n-1}^{\beta} C,$$

where $B_{n-1}, B_{n-2}$ is the number of all different labelings of the subtree of the golden mean tree with level $n-1, n-2$. Finally, we use the equation to prove the strip entropy converges to the entropy (**Theorem 3.1.4**) on the golden mean tree.





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# Chapter 2

# Preliminaries

In this section, we provide necessary materials and some known results for the study of the strip entropy.

A *binary tree* is a tree in which each node has at most two children, the offspring of the nodes are referred to the left child and the right child. So the paths of the tree correspond to the set of all finite words with alphabet $\Sigma = \{0, 1\}$, the left child corresponds to $0$, the right child corresponds $1$. Consider a special kind of binary trees that no two consequent right children are allowed, called it the *golden mean tree* $\mathcal{T}$. The *empty word* $\epsilon$ is the root of the tree. For $n \in \mathbb{N}$ all words of length $n$ are denoted by

$$\Sigma^n = \{w_1 w_2 w_3 \cdots w_n \mid w_i \in \Sigma, \forall 1 \leq i \leq n \text{ and } w_i w_{i+1} \neq 11, 1 \leq i \leq n-1\}.$$

All finite words is the set

$$\Sigma^* = \{w \in \Sigma^n | n \in \mathbb{N} \cup \{0\}\}.$$

A word $w \in \Sigma^*$ corresponds a unique path in the tree from the root. So the set $\Delta_n = \bigcup_{i=0}^{n} \Sigma^i$ is the initial height $n$ subtree. Now, consider the tree from a different perspective.

In order to draw the strip tree, choose an *infinite path*

$$G = \{g_1, g_1 g_2, g_1 g_2 g_3, \cdots\}$$

of the tree $\mathcal{T}$, and set $G$ to be the subtree of the strip tree with height $0$. For all $g \in G$, the set of



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all words of length $k$ attaches to $g$ is defined by

$$\Sigma_g^k = \{gg' \in \Sigma^* \mid |g'| = k\}.$$

Let the $i$th branch above $g$ with height $n$ defined by

$$T_i^n = \left\{ \bigcup_{i=0}^{n} \Sigma_g^k \mid |g| = i \text{ the first path of } g' \neq g_{i+1} \right\},$$

and defined the number of different paths in $T_i^n$ to be $c_i$. So the tree with width $m$ and height $n$ is the set

$$B_{n,m} = \bigcup_{i=0}^{m} T_i^n \cup \{\epsilon\}.$$

Thus, each word appears in $\Sigma^*$ will also appear in some $T_k$, the paths in the golden mean tree and the paths in the strip tree are one-one correspondence. A tree can be express by these two ways.

Let $f_i$ be the $i$th Fibonacci number, defined by $f_1 = 1$, $f_2 = 1$, and the recursive

$$f_{n+2} = f_{n+1} + f_n, \forall\, n \in \mathbb{N},$$

and

$$f_{n,m} := f_n + f_{n+1} + f_{n+2} + \cdots + f_m,\ \forall\, n, m \in \mathbb{N}.$$

If $g_i g_{i+1} = 00$, the structure of $T_i^n$ is illustrated as G1. Thus, the number of the branch is $f_1 + f_2 + \cdots + f_n + 1 = f_{1,n} + 1$. If $g_i g_{i+1} = 01$, we draw the structure of $T_i^n$ as G2 in the same fashion, and the number of the branch is equal to $f_2 + f_3 + \cdots + f_{n+1} + 1 = f_{2,n+1} + 1$. Finally, if $g_i g_{i+1} = 10$, the structure of $T_i^n$ is as G3, and the number of the branch is $1$.



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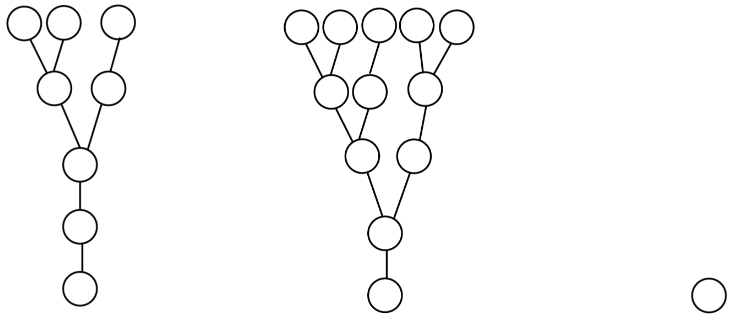


figure(3)

Let $\mathcal{A} = \{0, 1\}$ be an alphabet. A *labeled tree* is a function $t : \Sigma^* \to \mathcal{A}$, where $t(w)$ is the label attached to $w \in \Sigma^*$. A *tree shift* is the set of all labeled tree rejects some forbidden words. In this paper, we consider $\{11\}$ as the forbidden word, that is, the pattern of consecutive 1's is not allowed. For each $n = 0, 1, \cdots$, denote by $b_n$ the number of all different labelings of $\Delta_n$, and denote by $b_n(i)$ the number of such labelings have $i \in \mathcal{A}$ at the root. Finally, define $r_n(i) := \frac{b_n(i)}{b_n}$. In another way to see the labeling tree, we set $a_{n,m}$ to be the number of all different labelings of $B_{n,m}$. If $g$ is of length $k$, $a_{n,m}(g)$ is the number of all different labelings of $B_{n,m}$ with $T^0_{m-k+1}$ to $T^0_m$ draws $g$.

**Strip method** we use the the most left path $\{0, 00, 000, \cdots\}$, we obtained a recurrence relation below

$$\begin{bmatrix} a_{n,i+1}(0) \\ a_{n,i+1}(1) \end{bmatrix} = \begin{bmatrix} b_{n-2}(0) + b_{n+2} & b_{n-2}(0) + b_{n+2} \\ b_{n-2} & 0 \end{bmatrix} \begin{bmatrix} a_{n,i}(0) \\ a_i(n,1) \end{bmatrix}.$$

So the *entropy* of the tree is defined as follows.

$$h(\mathcal{T}) = \lim_{n\to\infty} \lim_{m\to\infty} \frac{\log a_{n,m}}{c_1 + c_2 + \cdots + c_m} = \lim_{n\to\infty} \lim_{m\to\infty} \frac{\log b^m_{n-2}}{m(f_{1,n} + 1)}.$$

Thus, we obtain

$$h(\mathcal{T}) = \lim_{n\to\infty} \frac{\log b_{n-2}}{f_{1,n} + 1}.$$

Choose a periodic path $g$, and defined

$$h_n(\mathcal{T}) = \lim_{m\to\infty} \frac{\log a_{n,m}}{c_1 + c_2 + \cdots + c_m},$$



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the existence of $h$ and $h_n$ is proved in [9].

**Problem.** Suppose $h(\mathcal{T})$ is the *entropy* of the golden mean tree evaluated by strip method, $h_n(\mathcal{T})$ is the *strip entropy* evaluated along a periodic path. Does $h_n(\mathcal{T})$ converge to $h(\mathcal{T})$ as $n$ tends to infinity ?





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# Chapter 3

# Results and examples

## 3.1 Main results and their proves

In this section, we are going to construct the strip entropy along a periodic path on the golden mean tree.

Let $S_i = \{g' \mid gg' \in T_i^n\}$ be the set which remove the common part $g$ from $T_i^n$, so it has the same cardinality as $T_i^n$. Give two words of the same length, say

$$x_1x_2x_3\cdots x_n \leq y_1y_2y_3\cdots y_n$$

if exists $1 \leq i \leq n$ such that $x_k = y_k$ for all $k \leq i$ and $x_i \leq y_i$.

The labeling alphabet is $\mathcal{A}$, and $\{11\}$ is the forbidden word for the labeling tree. The number of distinct word of length $m$ is equal to $f_{m+2}$, we give it an order by it size, that is, all the $m$-length golden mean word be the set

$$\ell(m) = \{\ell_{m,1}, \ell_{m,2}, \ell_{m,3}\cdots \ell_{m,f_{m+2}}\} \text{ where } \ell_i \leq \ell_{i+1}.$$

If two matrices $C, D \in M_{n\times n}$, and we defined $C \leq D$ if every entry of $C$ is less than or equal to the corresponded entry in $D$.

We defined $\{S_0, S_1, S_2, \cdots\}$ be a periodic $m$ sequence when $S_i = S_{i+m}\ \forall\ i \in \mathbb{N}\cup\{0\}$. **Theorem 3.1.1** yields the $m$-step matrix corresponds the strip tree with height $n$.

**Theorem 3.1.1.** *Let $\{S_0, S_1, S_2, \cdots\}$ be a periodic m sequence. Then there exists a matrix $M$,*



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*such that*

$$\begin{bmatrix} a_{n,(i+1)m}(\ell_{m,1}) \\ a_{n,(i+1)m}(\ell_{m,2}) \\ \vdots \\ a_{n,(i+1)m}(\ell_{m,f_{m+2}}) \end{bmatrix} = M \begin{bmatrix} a_{n,im}(\ell_{m,1}) \\ a_{n,im}(\ell_{m,2}) \\ \vdots \\ a_{n,im}(\ell_{m,f_{m+2}}) \end{bmatrix},$$

*and $M = A^m \circ P$, $A^m$ is the adjacency matrix of $m$th higher block code of golden mean, and $P = [P_{ij}]$ where $P_{ij}$ is some product of*

$$\{(b_{n-2} + b_{n-2}(0)), b_{n-2}, b_{n-1}, b_{n-1}(0), 1\}.$$

*Furthermore,*

$$M_{1,1} = (b_{n-2} + b_{n-2}(0))^{\alpha} \times b_{n-1}^{\beta} \times 1^{\gamma}$$

*where $\alpha$,$\beta$,$\gamma$ are the numbers that G1,G2,G3 appears in $\{S_{i+1}, S_{i+1}, S_{i+2}, \cdots, S_{i+m}\}$ respectively, and $\alpha + \beta + \gamma = m$.*

*Proof.* Note that $A^m_{ij}$ unveils whether $\ell_{m,j}$ can attach to $\ell_{m,i}$ or not. That is, if $\ell_{m,i}$ can attach to $\ell_{m,j}$, then $A^m_{ij} = 1$, if it can't, $A^m_{ij} = 0$, $1 \leq i, j \leq f_{m+2}$. In G1, if the root label $0$, then the label number above is $b_{n-2} + b_{n-2}(0)$, if the root label $1$, then the label number above is $b_{n-2}$. In G2, if the root label $0$, then the label number above is $b_{n-1}$, if the root label $1$, then the label number above is $b_{n-1}(0)$. In G3, whether the root is $0$ or $1$, there is no branch above, so it have only one choice.

Because every word in $\ell_{m,i}$ is either $0$ or $1$, and corresponds the branch either structure as G1 or G2 or G3, so $P_{i,j}$ is some product of

$$\{(b_{n-2} + b_{n-2}(0)), b_{n-2}, b_{n-1}, b_{n-1}(0), 1\}.$$

Furthermore, in golden mean, $0$ can attach to $0$, so $0^m$ can attach to $0^m$, so $A^m_{11} = 1$, and all root are $0$, so it only need to consider the branch, if there is a branch structure as G1, there has one $b_{n-2} + b_{n-2}(0)$, so $\alpha$ equal the number G1 appears in $\{S_i, S_{i+1}, S_{i+2}, S_{i+m-1}\}$, same argument for $\beta$ and $\gamma$. The proof is completed. □



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**Lemma 3.1.2.** *Under the same assumption of* **Theorem 3.1.1***, if $M$ is irreducible, then $\exists\ C, D \in \mathcal{M}_{f_{m+2} \times f_{m+2}}$ are irreducible such that*

$$b_{n-2}^{\alpha} b_{n-1}^{\beta} D \leq M \leq b_{n-2}^{\alpha} b_{n-1}^{\beta} C.$$

*Proof.* Choose $C = [C_{ij}] \in \mathcal{M}_{f_{m+2} \times f_{m+2}}$ with $C_{ij} = 0$ if $M_{ij} = 0$ otherwise $C_{ij} = 2^{\alpha}$. Since when root label $0$ the labeling number of branch above is more than root label $1$. So $M_{11}$ is the biggest entry of $M$. Divide $M_{11}$ by $b_{n-2}^{\alpha} b_{n-1}^{\beta}$, we obtain $(1 + r_{n-1}(0))^{\alpha}$. Since

$$r_{n-1}(0) \leq 1, (1 + r_{n-1}(0))^{\alpha} \leq 2^{\alpha},$$

this implies that $M \leq b_{n-2}^{\alpha} b_{n-1}^{\beta} C$. Choose $D = [D_{ij}] \in \mathcal{M}_{f_{m+2} \times f_{m+2}}$ with $D_{ij} = 0$ if $M_{ij} = 0$ otherwise $D_{ij} = 2^{-\beta}$. We note that $1^m$ can not be a word in golden mean. The golden mean word's labeling number is more than $1^m$. Divide $b_{n-2}^{\alpha} b_{n-1}^{\beta}$ by $b_{n-2}^{\alpha} b_{n-1}(0)$, we obtan $r_{n-1}(0)^{\beta}$. Since

$$r_{n-1}(0) \geq 2^{-1}, r_{n-1}(0)^{\beta} \geq 2^{-\beta},$$

this implies that $b_{n-2}^{\alpha} b_{n-1}^{\beta} D \leq M$. The proof is completed. □

**Remark 3.1.3.** *Note that $b_n$ is the number of all different labelings of $\Delta_n$, $f_i$ is the $i$th term of Fibonacci number, then we have*

$$\begin{aligned}\lim_{n\to\infty} \frac{\log b_{n-1}}{f_{1,n+1} + 1} &= \lim_{n\to\infty} \frac{\log b_n}{f_{2,n+2} + 1} \\ &= \lim_{n\to\infty} \frac{\log b_{n-1} + \log b_n}{(f_{1,n+1} + 1) + (f_{2,n+2} + 1)}\end{aligned}.$$

**Theorem 3.1.4.** *Let $G = \{g_1, g_1 g_2, g_1 g_2 g_3, \cdots\}$ be an infinite path in the golden mean tree. We set $G$ to be the subtree of strip tree with height $0$. If $\{S_1, S_2, S_3, \cdots\}$ be a periodic $m$ sequence, then*

$$h_n(\mathcal{T}) \to h(\mathcal{T}) \text{ as } n \to \infty,$$

*where $h(\mathcal{T})$ is the topological entropy of the golden mean tree, and $h_n(\mathcal{T})$ is the strip entropy evaluated along the path.*



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*Proof.* By the **Lemma 3.1.2**, we have

$$b_{n-2}^{\alpha} b_{n-1}^{\beta} D \leq M \leq b_{n-2}^{\alpha} b_{n-1}^{\beta} C.$$

Thus,

$$b_{n-2}^{\alpha} b_{n-1}^{\beta} \lambda_D \leq \lambda_M \leq b_{n-2}^{\alpha} b_{n-1}^{\beta} \lambda_C,$$

where $\lambda_k$ be the maximal eigenvalue for $k = M, C, D$,so we can get the result

$$\begin{aligned} \frac{\log b_{n-2}^{\alpha} b_{n-1}^{\beta} \lambda_D}{\alpha(f_{1,n}+1)+\beta(f_{2,n+1}+1)} &\leq \frac{log \lambda_M}{\alpha(f_{1,n}+1)+\beta(f_{2,n+1}+1)} \\ &\leq \frac{\log b_{n-2}^{\alpha} b_{n-1}^{\beta} \lambda_C}{\alpha(f_{1,n}+1)+\beta(f_{2,n+1}+1)}. \end{aligned}$$

The limit of left side of inequality

$$\begin{aligned} \lim_{n\to\infty} \frac{\log b_{n-2}^{\alpha} b_{n-1}^{\beta} \lambda_D}{\alpha(f_{1,n}+1)+\beta(f_{2,n+1}+1)} &= \lim_{n\to\infty} \frac{\log b_{n-2}^{\alpha} b_{n-1}^{\beta} - \beta}{\alpha(f_{1,n}+1)+\beta(f_{2,n+1}+1)} \\ &= h(\mathcal{T}). \end{aligned}$$

The limit of right side of inequality

$$\begin{aligned} \lim_{n\to\infty} \frac{\log b_{n-2}^{\alpha} b_{n-1}^{\beta} \lambda_C}{\alpha(f_{1,n}+1)+\beta(f_{2,n+1}+1)} &= \lim_{n\to\infty} \frac{\log b_{n-2}^{\alpha} b_{n-1}^{\beta} + \alpha}{\alpha(f_{1,n}+1)+\beta(f_{2,n+1}+1)} \\ &= h(\mathcal{T}). \end{aligned}$$

Hence,

$$\begin{aligned} h(\mathcal{T}) &\leq \lim_{n\to\infty} h_n(\mathcal{T}) \\ &= \lim_{n\to\infty} \lim_{m\to\infty} \frac{\log a_{n,m}}{\alpha(f_{1,n}+1)+\beta(f_{2,n+1}+1)} \\ &\leq h(\mathcal{T}). \end{aligned}$$

This complete the proof. □

## 3.2 Examples

**Example 3.2.1.** *Let $g_1 g_2 g_3 \cdots = 0\overline{100}$, then $G = \{g_1, g_1 g_2, g_1 g_2 g_3\}$ is a periodic $3$ path whose structures are as figure(4) and figure(5) respectively. The corresponding matrix $M$ is defined*



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*as follows.*

$$\begin{bmatrix} a_{n,3i+3}(000) \\ a_{n,3i+3}(001) \\ a_{n,3i+3}(010) \\ a_{n,3i+3}(100) \\ a_{n,3i+3}(101) \end{bmatrix} = M \begin{bmatrix} a_{n,3i}(000) \\ a_{n,3i}(001) \\ a_{n,3i}(010) \\ a_{n,3i}(100) \\ a_{n,3i}(101) \end{bmatrix}.$$

*And* $M = [M_{ij}]_{i,j=1}^{5}$*, where the entries* $M_{ij}$ *is set below.*

$$M_{11} = (b_{n-2} + b_{n-2}(0))b_{n-1} \qquad M_{12} = (b_{n-2} + b_{n-2}(0))b_{n-1}$$

$$M_{13} = (b_{n-2} + b_{n-2}(0))b_{n-1} \qquad M_{14} = (b_{n-2} + b_{n-2}(0))b_{n-1}$$

$$M_{15} = (b_{n-2} + b_{n-2}(0))b_{n-1}$$

$$M_{21} = (b_{n-2} + b_{n-2}(0))b_{n-1} \qquad M_{22} = (b_{n-2} + b_{n-2}(0))b_{n-1}$$

$$M_{23} = (b_{n-2} + b_{n-2}(0))b_{n-1} \qquad M_{24} = (b_{n-2} + b_{n-2}(0))b_{n-1}$$

$$M_{25} = (b_{n-2} + b_{n-2}(0))b_{n-1}$$

$$M_{31} = (b_{n-2} + b_{n-2}(0))b_{n-1}(0) \qquad M_{32} = (b_{n-2} + b_{n-2}(0))b_{n-1}(0)$$

$$M_{33} = (b_{n-2} + b_{n-2}(0))b_{n-1}(0) \qquad M_{34} = (b_{n-2} + b_{n-2}(0))b_{n-1}(0)$$

$$M_{35} = (b_{n-2} + b_{n-2}(0))b_{n-1}(0)$$

$$M_{41} = b_{n-2}b_{n-1} \qquad M_{42} = 0$$

$$M_{43} = b_{n-2}b_{n-1} \qquad M_{44} = b_{n-2}b_{n-1} \qquad M_{45} = 0$$

$$M_{51} = b_{n-2}b_{n-1} \qquad M_{52} = 0$$

$$M_{53} = b_{n-2}b_{n-1} \qquad M_{54} = b_{n-2}b_{n-1} \qquad M_{55} = 0$$



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*Choose*

$$D = \begin{bmatrix} 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} \\ 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} \\ 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} \\ 2^{-1} & 0 & 2^{-1} & 2^{-1} & 0 \\ 2^{-1} & 0 & 2^{-1} & 2^{-1} & 0 \end{bmatrix}, \; C = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 2 & 0 \end{bmatrix}.$$

*Then we have*

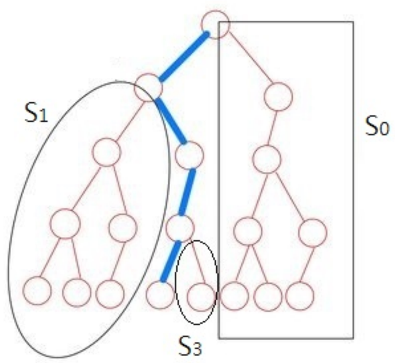
$$b_{n-2}b_{n-1}D \leq M \leq b_{n-2}b_{n-1}C,$$

$$\lim_{n\to\infty} \frac{\log b_{n-2}b_{n-1}\lambda_D}{(f_{1,n}+1)+(f_{2,n+1}+1)} \leq \lim_{n\to\infty} h_n(\mathcal{T}) \leq \lim_{n\to\infty} \frac{\log b_{n-2}b_{n-1}\lambda_C}{(f_{1,n}+1)+(f_{2,n+1}+1)}.$$

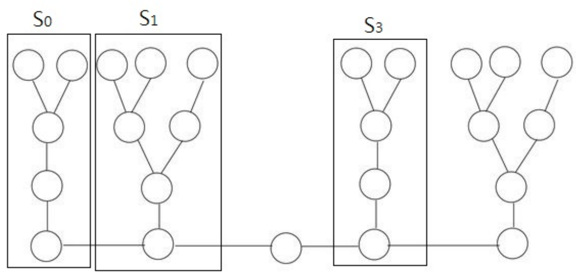
*That is*

$$\lim_{n\to\infty} \frac{\log b_{n-2}b_{n-1}-1}{(f_{1,n}+1)+(f_{2,n+1}+1)} \leq \lim_{n\to\infty} h_n(\mathcal{T}) \leq \lim_{n\to\infty} \frac{\log b_{n-2}b_{n-1}+1}{(f_{1,n}+1)+(f_{2,n+1}+1)}.$$

*So $h_n(\mathcal{T})$ converges to $h(\mathcal{T})$ as $n$ tends to infinity.*




*figure(4)*




*figure(5)*

**Example 3.2.2.** *Let $g_1g_2g_3\cdots = \overline{000}$, then $G = \{g_1, g_1g_2, g_1g_2g_3\}$ is a periodic $3$ path. The*



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*corresponding matrix $M$ is defined as follows.*

$$\begin{bmatrix} a_{n,3i+3}(000) \\ a_{n,3i+3}(001) \\ a_{n,3i+3}(010) \\ a_{n,3i+3}(100) \\ a_{n,3i+3}(101) \end{bmatrix} = M \begin{bmatrix} a_{n,3i}(000) \\ a_{n,3i}(001) \\ a_{n,3i}(010) \\ a_{n,3i}(100) \\ a_{n,3i}(101) \end{bmatrix}.$$

*And $M = [M_{ij}]_{i,j=1}^{5}$, where the entries $M_{ij}$ is set below.*

$$M_{11} = (b_{n-2} + b_{n-2}(0))^3 \qquad M_{12} = (b_{n-2} + b_{n-2}(0))^3$$

$$M_{13} = (b_{n-2} + b_{n-2}(0))^3 \qquad M_{14} = (b_{n-2} + b_{n-2}(0))^3$$

$$M_{15} = (b_{n-2} + b_{n-2}(0))^3$$

$$M_{21} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2} \qquad M_{22} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2}$$

$$M_{23} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2} \qquad M_{24} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2}$$

$$M_{25} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2}$$

$$M_{31} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2} \qquad M_{32} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2}$$

$$M_{33} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2} \qquad M_{34} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2}$$

$$M_{35} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2}$$

$$M_{41} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2} \qquad M_{42} = 0$$

$$M_{43} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2} \qquad M_{44} = (b_{n-2} + b_{n-2}(0))^2 b_{n-2} \qquad M_{45} = 0$$

$$M_{51} = (B_{n-2} + B_{n-2}(0)) B_{n-2}^2 \qquad M_{52} = 0$$

$$M_{53} = (B_{n-2} + B_{n-2}(0)) B_{n-2}^2 \qquad M_{54} = (B_{n-2} + B_{n-2}(0)) B_{n-2}^2 \qquad M_{55} = 0$$





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*Choose*

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 2^3 & 2^3 & 2^3 & 2^3 & 2^3 \\ 2^3 & 2^3 & 2^3 & 2^3 & 2^3 \\ 2^3 & 2^3 & 2^3 & 2^3 & 2^3 \\ 2^3 & 0 & 2^3 & 2^3 & 0 \\ 2^3 & 0 & 2^3 & 2^3 & 0 \end{bmatrix}.$$

*Then we have*

$$b_{n-2}b_{n-1}D \leq M \leq b_{n-2}b_{n-1}C,$$

$$\lim_{n\to\infty} \frac{\log b_{n-2}b_{n-1}\lambda_D}{3(f_{1,n}+1)} \leq \lim_{n\to\infty} h_n(\mathcal{T}) \leq \lim_{n\to\infty} \frac{\log b_{n-2}b_{n-1}\lambda_C}{3(f_{1,n}+1)}.$$

*That is*

$$\lim_{n\to\infty} \frac{\log b_{n-2}b_{n-1}}{3(f_{1,n}+1)} \leq \lim_{n\to\infty} h_n(\mathcal{T}) \leq \lim_{n\to\infty} \frac{\log b_{n-2}b_{n-1}+3}{3(f_{1,n}+1)}.$$

*So $h_n(\mathcal{T})$ converges to $h(\mathcal{T})$ as $n$ tends to infinity.*





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# Chapter 4

# Conclusion and discussion

*In this paper, we prove the strip entropy along the periodic paths of the golden mean tree converges to the entropy $h(\mathcal{T})$ on the golden mean tree (***Theorem 3.1.4***). Some problems remain in the future study.*

*(1) Whether the strip entropy along the aperiodic paths will also converge to the entropy $h(\mathcal{T})$ on the golden mean tree?*

*(2) Whether the strip entropy along arbitrary path will also converge to the entropy $h(\mathcal{T})$ on the golden mean tree?*

*(3) In this paper, we consider the golden mean tree. Does the strip method still work on general tree?*

*(4) In this paper, the labeling alphabet $\mathcal{A} = \{0, 1\}$ have only two element, what if the labeling alphabet have element more than two? Does the entropy $h_n(\mathcal{T})$ which defined on strip method still converge to the entropy $h(\mathcal{T})$ on the golden mean tree?*

*In Tsai's paper [13], it proves on the golden mean tree with labeling alphabet $\mathcal{A} = \{1, \cdots, d\}$ with irreducible adjacency matrix $A = (\alpha_{ij})$, the entropy $h_n(\mathcal{T})$ which defined on strip method converges to the entropy $h(\mathcal{T})$ on golden mean tree.*



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# Appendix A

# Computation of the $m$-step matrix

*In* **Theorem 3.1.1***, we know the entries* $[M_{ij}]$ *is either* $0$ *or some product of*

$$\{(b_{n-2}+b_{n-2}(0)), b_{n-2}, b_{n-1}, b_{n-1}(0), 1\},$$

*and*

$$M_{1,1} = (b_{n-2}+b_{n-2}(0))^{\alpha} \times b_{n-1}^{\beta} \times 1^{\gamma}$$

*where* $\alpha$*,*$\beta$*,*$\gamma$ *are the number that G1,G2,G3 appear in* $\{S_{i+1}, S_{i+1}, S_{i+2}, \cdots, S_{i+m}\}$ *respectively. The other entries is decided by the periodic path, here is an algorithm below, input the periodic path and* $i$ $j$*, output would be the entries* $[M_{ij}]$ *under the periodic path.*

*https://github.com/wekewei/paper*



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