國立政治大學應用數學系

碩士學位論文

非對稱分支隨機漫步的範圍

The Range of Asymmetric Branching Random Walk

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中文摘要

考慮一個分支過程且族群中的每個個體在出生時皆在實數線上移動, 作一非對稱的隨機漫步, 並記錄每一個個體的位置。在本篇論文中, 我們證明了當時間趨近於無限大時, 實數線上有個體佔據的位置將會是一個區間。

關鍵字：分支隨機過程, 分支過程, 隨機漫步

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Abstract

We consider a Galton-Watson branching process in which each individual performs an asymmetric random walk on the real line and record the positions of all individuals in each generation. In this thesis, we show that the set of occupied positions is eventually an interval.

keywords: branching random walk, Galton-Watson process, random walk

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Chapter 1

Preliminary

1.1 Introduction

The concept of branching processes was formulated by Francis Galton in late nineteenth century. The motivation of his research came from the observation of the extinction of certain European nobility compared to the rapid growth of the overall population. He Posted this extinction problem in the Educational Times in 1874 and Henry William Watson replied with a solution (see Harris [7]). Seneta and Heyde pointed out that the French mathematician Bienayme had proposed basically the same model several years ago.

Since the creation of Galton and Watson's model (called the Galton-Watson branching process), it had been neglected for many years. After 1940, people's interest in this model increased because of the analogy between the population growth and nuclear chain reactions and also because of people being more and more interested in the application of probability theory. Since then, branching processes have been considered to be suitable probability models used to describe the behavior of systems whose individuals reproduce, are transformed, and die. Now, this theory is an active and interesting research field.

There are many studies on branching process and one of them is the study of branching random walks, which combines the concepts of branching process and random walk. (see Biggins [1] [2] [3], Bramson [4], Dekking [5]). Grill [6] raised the question of the range of branching random walk. He considered the symmetric simple branching random walk in which each individual, as it is created, may move a unit to right or remain in the same place with probability of one-half and showed that the set of occupied points is eventually an interval. In

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addition, he gave a limit law of the number of individuals in the position close to the border of the set of occupied positions. Later on Johnson [8] worked on the similar problem with asymmetric movement. But, in his paper, he considered the case that each individual only had a deterministic constant number of children.

In this thesis, we are asking the same question for Galton-Watson branching processes when offspring distribution is no longer deterministic equipped with asymmetric random walks.

In the next sections of this chapter, we review the definitions and basic results of Galton-Watson branching process (Section 1.2) and branching random walk (Section 1.3)

In Chapter 2, we investigate the local population and study the probability of extinction at the position k and the properties about the population at two extreme positions, 0 and n, in the n-th generation.

In Chapter 3, we present our main results on the occupied positions and provide the proofs for our main results.

1.2 Galton-Watson branching process

A Galton-Watson branching process is a stochastic process often interpreted as the population size. Usually we assume that this population starts with one ancestor. Each individual lives a unit of time and produces its offsprings when it dies according to the probability distribution$\{\pi_k\}_{k\geq 0}$. The reproduction of each individual is assumed to be independent of that of others.

1.2.1 Model setting

Let $\{Z_n\}_{n\geq 0}$ be a Galton-Watson branching process, where Z_n is the number of individuals in the n-th generation and $Z_0 = 1$.

Let $\xi_i^{(n)}$ be the random variable denoting the number of children of the i-th individual in n-th generation.

Assume that $\{\xi_i^{(n)} : n \geq 0, i \geq 1\}$ are i.i.d. random variables with the probability distribution $\{\pi_k\}$, where $\{ \pi_k\}$ is called the offspring distribution and

$$\pi_k = P(Z_1 = k)$$

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means the probability of having k offsprings.

Then the total number of individuals in the $(n+1)$-th generation is the sum of $\xi_i^{(n)}$'s, which means

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{(n)}.$$

Clearly, $Z_1 = \xi_1^{(0)}$.

Moreover, if **T** denotes the family tree generated by the above way, each individual in **T** can be labeled by $\vec{i} = (i_0, i_1, ..., i_n)$, which means that it is in the n-th generation and is the i_n-th child of the individual $\vec{i'} = (i_0, i_1, ..., i_{n-1})$ of the $(n-1)$-generation. Here we denote the initial ancestor by (i_0), where $i_0 = 1$.

If $\vec{i} = (i_0, i_1, ..., i_n)$, define $|\vec{i}| = n$, i.e the generation number of $\vec{i}$.

In order to analyze the process, we introduce the probability generating function of the offspring distribution. Let

$$F(z) = E(z^{Z_1}) = \sum_{k=0}^{\infty} \pi_k z^k$$

be the probability generating function of Z_1.

Let

$$m = F'(1) = \sum_{k=0}^{\infty} k\pi_k$$

be the offspring mean.

1.2.2 Classial results

First, from the definition of F as a power series with nonnegtiove coefficients $\{\pi_k\}$ adding to 1 we have the following properties.

1. F is strictly convex and increasing in $[0, 1]$.
2. $F(0) = \pi_0$; $F(1) = 1$.
3. If $m \leq 1$, then $F(t) > t$ for $t \in [0, 1)$.
4. If $m > 1$, then $F(t) = t$ has a unique solution in $[0, 1)$.

Let q_{ex} be the smallest solution of $F(t) = t$ for $t \in [0, 1]$ (Figure 1.1). Then we have the following lemmas.

Lemma 1.2.1. *If $m \leq 1$ then $q_{ex} = 1$; if $m > 1$ then $q_{ex} < 1$. Moreover, q_{ex} is the extinction*

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probability of the branching process $\{Z_n\}$, *i.e.* $q_{ex} = P(Z_n = 0 \text{ for some } n \geq 1)$.

Lemma 1.2.2. *Let* F^n *be the* n*-th iteration of* F, *i.e.,*

$$F^0(z) = z, \qquad F^1(z) = F(z), \qquad F^{n+1}(z) = F(F^n(z)).$$

If $t \in [0, q_{ex})$, *then* $F^n(t)$ *is increasing to* q_{ex} *as* $n \to \infty$.

If $t \in (q_{ex}, 1)$, *then* $F^n(t)$ *is decreasing to* q_{ex} *as* $n \to \infty$.

If $t = q_{ex}$ *or* 1, $F^n(t) = t$ *for all* n.

We have further discussions on the upper and lower bounds of F^k in Section 2.2 .

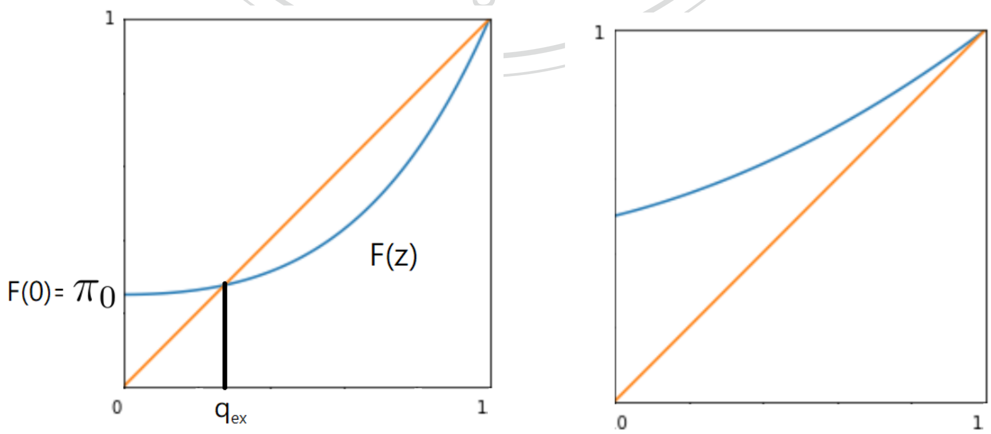
1.3 Branching random walk

The one dimentional branching random walk is a branching process equipped a movement structure.

1.3.1 Model setting

Let $\{Z_n\}$ be a Galton-Watson branching process with offspring distribution $\{\pi_k\}$ and $Z_0 = 1$, then we impose a random walk to the Galton-Watson branching process $\{Z_n\}$.

Assume the initial ancestor (i_0) is placed at 0 in the real line, and upon its death, it gives birth to a random number Z_1 children according to $\{\pi_k\}$, then the j-th child $(1, j)$ among those



(a) $m > 1$ (b) $m \leq 1$

Figure 1.1: The probability generating function

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Z_1 children moves away from the position of (i_0) with movement $X_{(1,j)}$, where $\{X_{(1,j)} : 1 \leq j \leq Z_1\}$ are i.i.d random variables in **R**.

Similarly, for each individual $\vec{i}'$, it gives birth according to $\{\pi_k\}$ when it dies, and then its child $\vec{i}$ moves away from the position of $\vec{i}'$ with movement $X_{\vec{i}}$.

We assume that $\{X_{\vec{i}}\}_{\vec{i}}$ are i.i.d., i.e., the movements of all individuals in the branching random walk are i.i.d., and the movements are independent of the reproduction of the branching process.

Let $S_{\vec{i}}$ be the position of individual $\vec{i}$, and $\mathbf{S}_n = \{S_{\vec{i}} : |\vec{i}| = n\}$ be the position chart of the individuals in the n-th generation, i.e. it is the set of occupied sites on the real line. Moreover, $S_{\vec{i}} = S_{\vec{i}'} + X_{\vec{i}'}$, where $\vec{i}'$ is the parent of $\vec{i}$.

The sequence of random vectors $\{(Z_n, \mathbf{S}_n)\}_{n\geq 0}$ is called a branching random walk with offspring distribution $\{\pi_k\}_{k\geq 0}$ and the movement distribution $\{r_k\}_{k\in\mathbf{Z}}$, where $r_k \equiv P(X_{\vec{i}} = k)$, $k \in \mathbf{Z}$.

In this thesis, we assume that, for each individual $\vec{i}$, the movement $X_{\vec{i}}$ has distribution $\{r_k\}_{k\geq 0}$

$$r_k \equiv P(X_{\vec{i}} = k) = \begin{cases} p, & \text{if } k = 1 \\ q, & \text{if } k = 0 \end{cases}$$

where $0 < p, q < 1$ and $p + q = 1$.

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Chapter 2

Properties on local population

Throughout this thesis, we shall always assume that

$$E(Z_1) = m > 1$$

to avoid the trivial result. Otherwise the process must be extinct with probability 1.

2.1 Local extinction probabilities

First of all, we concentrate on the number of individuals at each posotion, and the probability that no individual lies at the positions which are less than or equal to k.

Let $\lambda(n,k)$ be the number of individuals in the position k at time n (i.e. in the n-th generation).

Let $G_{n,k}(z) = E(z^{\lambda(n,k)})$ denote the generating function of $\lambda(n,k)$.

Let

$$p_{n,k} = P(\lambda(n,k) = 0)$$

be the probability that no one in the n-th generation is in the position k.

Let

$$q_{n,k} = P(\lambda(n,j) = 0,\ \forall j \leq k)$$

be the probability that no one in the n-th generation is in the position less than or equal to k.

Note that $p_{n,-1} = 1$ for all $n = 1, 2, ...$, we obtain the following recursion for $p_{n,k}$.

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Lemma 2.1.1. *For all $n \geq 1$, $k = 0, 1, ..., n$, we have*

$$p_{n,k} = F(qp_{n-1,k} + pp_{n-1,k-1}).$$

Proof. Let $\tilde{\lambda}_j(n,k)$ be the number of offsprings in the n-th generation of the j-th individual in the 1st generation whose position is k. Then

$$\lambda(n,k) = \sum_{j=1}^{Z_1} \tilde{\lambda}_j(n,k)$$

and hence

$$G_{n,k}(z) = E(z^{\lambda(n,k)}) = E(z^{\sum_{j=1}^{Z_1} \tilde{\lambda}_j(n,k)}).$$

Due to the reproductive structure of the Golton-Watson branching process, we have that, conditioned on Z_1, the random variables $\{\tilde{\lambda}_j(n,k) : j = 1, ..., Z_1\}$ are independent and independent of Z_1. So, by the law of total expectation, we have

$$\begin{aligned} G_{n,k}(z) \quad &= E(E(z^{\sum\limits_{j=1}^{Z_1} \tilde{\lambda}_j(n,k)} | Z_1)) \\ &= E(\prod_{j=1}^{Z_1} E(z^{\tilde{\lambda}_j(n,k)} | Z_1)) \\ &= E(\prod_{j=1}^{Z_1} E(z^{\tilde{\lambda}_j(n,k)})). \end{aligned}$$

Also, we know that, conditioned on Z_1, $\{\tilde{\lambda}_j(n,k) : j = 1, ..., Z_1\}$ are identically distributed, so we have

$$G_{n,k}(z) = E(\prod_{j=1}^{Z_1} E(z^{\tilde{\lambda}_1(n,k)})).$$

Now, note that the 1st individual in the 1st generation only can move to 1 or stay at 0. Therefore, conditioned on its movement $X_{(1,1)}$, it follows that

$$\begin{aligned} G_{n,k}(z) = E(\prod_{j=1}^{Z_1} \quad & [E(z^{\tilde{\lambda}_1(n,k)} | X_{(1,1)} = 0) P(X_{(1,1)} = 0) \\ & + E(z^{\tilde{\lambda}_1(n,k)} | X_{(1,1)} = 1) P(X_{(1,1)} = 1)]). \end{aligned}$$

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Since, conditioned on $X_{(1,1)} = 0$, $\tilde{\lambda_1}(n,k)$ has the same distribution as $\lambda(n-1,k)$, and, conditioned on $X_{(1,1)} = 1$, $\tilde{\lambda_1}(n,k)$ has the same distribution as $\lambda(n-1,k-1)$, we have

$$\begin{aligned} G_{n,k}(z) &= E(\prod_{j=1}^{Z_1}[E(z^{\lambda(n-1,k)})q + E(z^{\lambda(n-1,k-1)})p]) \\ &= E(\prod_{j=1}^{Z_1}[G_{n-1,k}(z)q + G_{n-1,k-1}(z)p]) \\ &= E([G_{n-1,k}(z)q + G_{n-1,k-1}(z)p]^{Z_1}) \\ &= F([qG_{n-1,k}(z) + pG_{n-1,k-1}(z)]), \end{aligned}$$

where F is the probability generating function of Z_1.

Finally, we obtain

$$\begin{aligned} p_{n,k} &= P(\lambda(n,k) = 0) = G_{n,k}(0) \\ &= F([qG_{n-1,k}(0) + pG_{n-1,k-1}(0)]) \\ &= F(qp_{n-1,k} + pp_{n-1,k-1}). \end{aligned}$$

□

The next lemma is a recursive property for $q_{n,k}$ with $q_{n,-1} = 1$, for all n.

Lemma 2.1.2. *For all $n \geq 1$, $k = 0, 1, ..., n$, we have*

$$q_{n,k} = F(qq_{n-1,k} + pq_{n-1,k-1}).$$

Proof. Let $Y(n,k)$ be the number of individuals in the n-th generation whose position is less than or equal to k.

Let $\tilde{\lambda_j}(n,k)$ be defined as given in the proof of Lemma 2.1.1. Then

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$$
\begin{aligned}
Y(n,k) &= \sum_{j=0}^{k} \lambda(n,j) \\
&= \sum_{j=0}^{k} \sum_{i=0}^{Z_1} \tilde{\lambda}_i(n,j) \\
&= \sum_{i=0}^{Z_1} \sum_{j=0}^{k} \tilde{\lambda}_i(n,j).
\end{aligned}
$$

Let $H_{n,k} = E(z^{Y(n,k)})$ be the probability generating function of $Y(n,k)$, then

$$
\begin{aligned}
H_{n,k}(z) &= E(z^{Y(n,k)}) \\
&= E(z^{\sum_{i=1}^{Z_1} \sum_{j=0}^{k} \tilde{\lambda}_i(n,j)}).
\end{aligned}
$$

By the law of total expectation, we have that

$$
\begin{aligned}
H_{n,k}(z) &= E(E(z^{\sum_{i=1}^{Z_1} \sum_{j=0}^{k} \tilde{\lambda}_i(n,j)} | Z_1)) \\
&= E(\prod_{i=1}^{Z_1} E(z^{\sum_{j=0}^{k} \tilde{\lambda}_i(n,j)} | Z_1)), \\
&\quad \text{because } Z_1 \text{ is } \sigma(Z_1)\text{-measurable and } \tilde{\lambda}_i \text{ are independent.} \\
&= E(\prod_{i=1}^{Z_1} E(z^{\sum_{j=0}^{k} \tilde{\lambda}_i(n,j)})), \\
&\quad \text{by the independence between } \tilde{\lambda}_j(n,k) \text{ and } Z_1.
\end{aligned}
$$

Also, we know that $\tilde{\lambda}_i(n,k)$ has the same distribution as $\tilde{\lambda}_1(n,k)$, so we have

$$
H_{n,k}(z) = E(\prod_{i=1}^{Z_1} E(z^{\sum_{j=0}^{k} \tilde{\lambda}_1(n,j)})).
$$

Conditioned on the movement $X_{(1,1)}$, by the law of total probability, it follows that

$$
\begin{aligned}
H_{n,k}(z) = E(\prod_{i=1}^{Z_1} \quad & [E(z^{\sum_{j=0}^{k} \tilde{\lambda}_1(n,j)} | X_{(1,1)} = 0) P(X_{(1,1)} = 0) \\
& + E(z^{\sum_{j=0}^{k} \tilde{\lambda}_1(n,j)} | X_{(1,1)} = 1) P(X_{(1,1)} = 1)]).
\end{aligned}
$$

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Since, conditioned on $X_{(1,1)} = 0$, $\sum_{j=0}^{k} \tilde{\lambda_1}(n,j)$ has the same distrinbution as $\sum_{j=0}^{k} \lambda(n-1,j)$, and conditioned on $X_{(1,1)} = 1$, $\sum_{j=0}^{k} \tilde{\lambda_1}(n,j)$ has the same distribution as $\sum_{j=0}^{k-1} \lambda(n-1,j)$, we have

$$\begin{aligned}
H_{n,k}(z) &= E(\prod_{i=1}^{Z_1}[E(z^{\sum_{j=0}^{k}\lambda(n-1,j)})q + E(z^{\sum_{j=0}^{k-1}\lambda(n-1,j)})p]) \\
&= E(\prod_{j=1}^{Z_1}[E(z^{Y(n-1,k)})q + E(z^{Y(n-1,k-1)})p]) \\
&= E(\prod_{j=1}^{Z_1}[H_{n-1,k}(z)q + H_{n-1,k-1}(z)p]) \\
&= E(H_{n-1,k}(z)q + H_{n-1,k-1}(z)p]^{Z_1}) \\
&= F(qH_{n-1,k}(z) + pH_{n-1,k-1}(z)).
\end{aligned}$$

Then, we have that

$$\begin{aligned}
q_{n,k} &= P(Y(n,k) = 0) = H_{n,k}(0) \\
&= F([qH_{n-1,k}(0) + pH_{n-1,k-1}(0)]) \\
&= F(qq_{n-1,k} + pq_{n-1,k-1}).
\end{aligned}$$

□

Now we discuss the monotonicity of $q_{n,k}$ and $p_{n,k}$. Because each individual can only stay or move to the right, it is clear that if no individual is located at position less than or eqaul to k, then no one can go back to there. So, for a fixed k, that $\lambda(n,j) = 0$, for all $j \leq k$, implies $\lambda(n+1,j) = 0$, for all $j \leq k$, and it means

$$q_{n,k} = P(\lambda(n,j) = 0\ \forall j \leq k) \leq P(\lambda(n+1,j) = 0\ \forall j \leq k) = q_{n+1,k}.$$

Moreover, for a fixed n, that $\lambda(n,j) = 0$, for all $j \leq k+1$, implies $\lambda(n,j) = 0$, for all $j \leq k$, so

$$q_{n,k} = P(\lambda(n,j) = 0\ \forall j \leq k) \geq P(\lambda(n,j) = 0\ \forall j \leq k+1) = q_{n,k+1}.$$

Therefore, $q_{n,k}$ is decreasing in k and increasing in n.

On the other hand, Lemma 2.1.3 tells the behavior of $p_{n,k}$ as a function of n.

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Lemma 2.1.3. *For each k, $\exists n_0(k) \in \mathbf{N}_0$ s.t. for $n > n_0(k)$, $p_{n,k}$ is monotone as a function of n.*

Proof. We prove it by induction on k. First, for $k = 0$,

$$p_{n,0} = P(\lambda(n, 0) = 0) = q_{n,0}$$

and hence it is increasing as n increases. So, for $k = 0$, $n_0(0) = 0$.

Suppose that it holds for $k = m$, i.e. $\exists n_0(m) \in \mathbf{N}_0$ s.t. for $n > n_0(m)$, $p_{n,m}$ is monotone as a function of n.

For $k = m + 1$, since $p_{n,m}$ is monotone for $n > n_0(m)$, we assume that $p_{n,m}$ is increasing as a function of n, $\forall n > n_0(m)$.

There are two cases:

Case1: If there exists $n_0(m + 1) \geq n_0(m)$ s.t. $p_{n_0(m+1)+1,m+1} \geq p_{n_0(m+1),m+1}$, then by the recursion of $p_{n,k}$, we have the following:

$$\begin{aligned} p_{n_0(m+1)+1,m+1} &= F(qp_{n_0(m+1),m+1} + pp_{n_0(m+1),m}) \\ p_{n_0(m+1)+2,m+1} &= F(qp_{n_0(m+1)+1,m+1} + pp_{n_0(m+1)+1,m}) \\ &\cdots \end{aligned}$$

Since $n_0(m + 1) \geq n_0(m)$, by the induction hypothesis, $p_{n_0(m+1)+1,m} \geq p_{n_0(m+1),m}$. So,

$$qp_{n_0(m+1)+1,m+1} + pp_{n_0(m+1)+1,m} \geq qp_{n_0(m+1),m+1} + pp_{n_0(m+1),m}$$

and, since F is increasing, we get

$$p_{n_0(m+1)+2,m+1} \geq p_{n_0(m+1)+1,m+1}.$$

Repeated the same procedure, we can have

$$\begin{aligned} p_{n_0(m+1)+3,m+1} &= F(qp_{n_0(m+1)+2,m+1} + pp_{n_0(m+1)+2,m}) \\ &\geq F(qp_{n_0(m+1)+1,m+1} + pp_{n_0(m+1)+1,m}) \\ &= p_{n_0(m+1)+2,m+1} \end{aligned}$$

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and so on.

Therefore, we can conclude that

$$p_{n+1,m+1} \geq p_{n,m+1}, \quad \forall n \geq n_0(m+1).$$

Case2: If there is no such an $n_0(m+1)$ as required in Case 1, then it means that $p_{n,m+1}$ is decreasing for all $n \geq n_0(m)$. In this case, we set $n_0(m+1) = n_0(m)$.

In either case, we prove that $p_{n,m+1}$ is monotone for all $n \geq n_0(m+1)$.

By the similarly arguments, we can also obtain the result that there exists $n_0(m+1)$ s.t. $p_{n,m+1}$ is monotone for $n \geq n_0(m+1)$ when $p_{n,m}$ is decreasing as a function of n, $\forall n > n_0(m)$.

Therefore, by induction, we proved that for each k, $\exists n_0(k) \in \mathbf{N}_0$ s.t. for $n > n_0(k)$, $p_{n,k}$ is monotone as a function of n. □

From the lemma above, we have that, for each fixed k, $p_{n,k}$ and $q_{n,k}$ are eventually monotone as function of n, so the limits

$$p_k = \lim_{n\to\infty} p_{n,k}$$

and

$$q_k = \lim_{n\to\infty} q_{n,k}$$

both exsit. Moreover, because F is continuous, they also satisfy the recursions

$$p_k = F(qp_k + pp_{k-1})$$

$$q_k = F(qq_k + pq_{k-1}),$$

where $p_{-1} = q_{-1} = 1$. In addition, due to the fact that $p_{n,0} = q_{n,0}$ for all n, we have that $p_0 = q_0$ and hence $p_k = q_k$ for all k.

Next, we consider the leftmost and rightmost occupied position in n-th generation. Let

$$K_n = \inf\{k : \lambda(n,k) > 0\}$$

$$L_n = \inf\{j : \lambda(n, n-j) > 0\},$$

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then the distribution of K_n can be found as

$$\begin{aligned} P(K_n > k) &= P(\inf\{k : \lambda(n,k) > 0\} > k) \\ &= P(\lambda(n,j) = 0,\ \forall j \le k) \\ &= q_{n,k}. \end{aligned}$$

Furthermore, by monotonicity of K_n,

$$K = \lim_{n\to\infty} K_n$$

exists and

$$P(K > k) = q_k.$$

2.2 Population at extreme points

In this section, we present the results on the limit behavior of the occupied positions. First of all, we would like to consider the population at two extreme points, 0 and n, in the n-th generation.

Recall that $\lambda(n,0)$ is the number of individuals in the n-th generation whose positions are 0, and $\lambda(n,n)$ is the number of individuals in the n-th generation whose positions are n. Also, note that the extinction probability of a Galton-Watson branching process $\{Z_n\}$ is the probability of the event $\{Z_n = 0$, for some $n\}$.

The following lemma gives the distributions of the processes, $\{\lambda(n,0)\}_{n\ge 0}$ and $\{\lambda(n,n)\}_{n\ge 0}$.

Lemma 2.2.1. *The processes $\{\lambda(n,0)\}_{n\ge 0}$ and $\{\lambda(n,n)\}_{n\ge 0}$ are two Galton-Watson branching processes, with the offspring probability generating functions $g_1(z) = F(qz+p)$ and $g_2(z) = F(pz+q)$, respectively. In addition, $p_0 = \lim_{n\to\infty} p_{n,0}$ is the extinction probability of $\{\lambda(n,0)\}$ and $p_0' := \lim_{n\to\infty} p_{n,n}$ is the extinction probability of $\{\lambda(n,n)\}$.*

Proof. Because each individual can only go to the right by one step or stay at the same position, only parents at the position 0 can reproduce the children at the position 0. Note that each child stays at 0 with probability q.

Let $\eta_i^{(n)}$ be the number of children of the i-th individual in the n-th generation located at 0.

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Let $\tilde{\eta}_i^{(n)}$ be the number of children of the i-th individual in the n-th generation located at 0 whose position is 0.

Then, for all $j = 0, 1, 2, ...$

$$\begin{aligned}\tilde{\pi}_j &= P(\tilde{\eta}_i^{(n)} = j) \\ &= \sum_{k=j}^{\infty} P(\tilde{\eta}_i^{(n)} = j,\ \eta_i^{(n)} = k) \\ &= \sum_{k=j}^{\infty} P(\tilde{\eta}_i^{(n)} = j | \eta_i^{(n)} = k) P(\eta_i^{(n)} = k) \\ &= \sum_{k=j}^{\infty} \binom{k}{j} q^j p^{k-j} \pi_k\end{aligned}$$

which is independent of n and i.

So, $\{\tilde{\eta}_i^{(n)}\}_{n,i}$ are i.i.d. and

$$\begin{aligned}\lambda(n+1,0) &= \text{the number of individuals in the } (n+1)\text{-th generation} \\ &\quad\ \text{whose position is } 0 \\ &= \text{the number of children of individuals in the } n\text{-th generation} \\ &\quad\ \text{located at } 0 \text{ whose position is } 0 \\ &= \sum_{i=1}^{\lambda(n,0)} \tilde{\eta}_i^{(n)}.\end{aligned}$$

So, $\{\lambda(n,0)\}_{n\geq 0}$ is a Galton-Watson branching process with $\lambda(0,0) = 1$ and offspring distribution $\{\tilde{\pi}_j\}_{j\geq 0}$.

Also, $\lambda(1,0) = \sum_{i=1}^{\lambda(0,0)} \tilde{\eta}_i^{(0)} = \tilde{\eta}_1^{(0)}$ has distribution$\{\tilde{\pi}_j\}_{j\geq 0}$.

Now, let $g_1(z) = E(z^{\lambda(1,0)})$ be the probability generating function of the process

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$\{\lambda(n,0)\}_{n\geq 0}$. Then

$$\begin{aligned} g_1(z) &= E(z^{\lambda(1,0)}) = \sum_{j=0}^{\infty} P(\lambda(1,0)=j)z^j \\ &= \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} \pi_k \tbinom{k}{j} q^j p^{k-j} z^j \\ &= \sum_{k=0}^{\infty} \pi_k \sum_{j=0}^{k} \tbinom{k}{j} q^j p^{k-j} z^j, \text{ by Tonelli's theorem} \\ &= \sum_{k=0}^{\infty} \pi_k (qz+p)^k \\ &= F(qz+p). \end{aligned}$$

So, we can treat $\{\lambda(n,0)\}_{n\geq 0}$, as a branching process with probability generating function $F(qz+p)$.

Similarly, we can show that $\{\lambda(n,n)\}_{n\geq 0}$ is a Galton-Watson branching process. To determine the offspring distribution for $\lambda(n,n)$, the construction tells us that, if individual $\vec{i}$, $|\vec{i}| = n$, is in the position n, then its parent must be in the position $n-1$ and it goes to the position n with probability p. So,

$$P(\lambda(1,1)=j) = \sum_{k=j}^{\infty} \pi_k \tbinom{k}{j} p^j q^{k-j}.$$

Let $g_2(z) = E(z^{\lambda(n,n)})$. Similarly to the argument of g_1, we have $g_2(z) = F(pz+q)$ is the generatin function of the branching process $\{\lambda(n,n)\}_{n\geq 0}$.

And, since each individual can only move to the right or stay in the same position, $\lambda(n,0) = 0$ implies $\lambda(n+1,0) = 0$. So, $\{\lambda(n,0)=0\}_{n\geq 0}$ is increasing as n is increasing.

Then the extinction probability of the branching process $\{\lambda(n,0)=0\}_{n\geq 0}$ is

$$\begin{aligned} P(\lambda(n,0)=0 \text{ for some n}) &= P\Big(\bigcup_{n=0}^{\infty}\{\lambda(n,0)=0\}\Big) \\ &= \lim_{n\to\infty} P(\lambda(n,0)=0), \text{ by the contiuity from above} \\ &= \lim_{n\to\infty} p_{n,0} \\ &= p_0. \end{aligned}$$

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Simlarly, we can prove that $p'_0 := \lim_{n\to\infty} p_{n,n}$ is the extinction probability of the branching process $\{\lambda(n,n)\}_{n\geq 0}$.

Note that $Z_n = 0$ implies $\lambda(n,k) = 0$ for all k, so $P(Z_n = 0) < P(\lambda(n,0) = 0)$ for all n, and hence $p_0 = q_0 > q_{ex}$. □

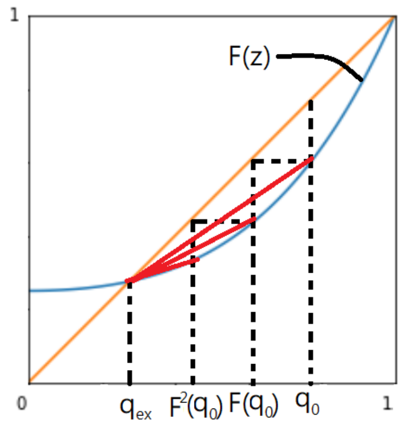
Lemma 2.2.2. *For all k, $F^k(q_0) \leq q_{ex} + \gamma_3^k$, where $\gamma_3 = (F(q_0) - q_{ex})/(q_0 - q_{ex})$, and $\gamma_3 < 1$.*

Proof. Because $F^k(q_0)$ is decreasing in k, and F is convex (Figure 2.1), the slopes of the secant lines of F satisfy the inequality

$$\frac{F(F^{k-1}(q_0)) - F(q_{ex})}{F^{k-1}(q_0) - q_{ex}} \leq \frac{F(F^{k-2}(q_0)) - F(q_{ex})}{F^{k-2}(q_0) - q_{ex}},$$

where $q_{ex} = F(q_{ex})$ and hence

$$\frac{F^k(q_0) - F^k(q_{ex})}{F^{k-1}(q_0) - F^{k-1}(q_{ex})} = \frac{F^k(q_0) - q_{ex}}{F^{k-1}(q_0) - q_{ex}} \leq \frac{F(q_0) - q_{ex}}{q_0 - q_{ex}}.$$



(a)

Figure 2.1: $F^k(q_0)$

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So,

$$\begin{aligned}
F^k(q_0) - q_{ex} &\leq \frac{F(q_0) - F(q_{ex})}{q_0 - q_{ex}}(F^{k-1}(q_0) - q_{ex}) \\
&= \gamma_3(F^{k-1}(q_0) - q_{ex}) \\
&\leq \gamma_3^2(F^{k-2}(q_0) - q_{ex}) \\
&\leq \ldots \\
&\leq \gamma_3^k(q_0 - q_{ex}) \\
&\leq \gamma_3^k.
\end{aligned}$$

And as shown in Figure 2.1, because $F(z)$ is convex and increasing to 1 as z is increasing to 1, the slope of secant line $(F(q_0) - q_{ex})/(q_0 - q_{ex}) = \gamma_3$ is smaller than 1. □

Lemma 2.2.3. *If* $0 < q_{ex}$*, then* $F^n(0) \geq q_{ex} - \gamma_4^n$*, for all* n*, where* $\gamma_4 = (q_{ex} - F(0))/q_{ex}$ *and* $\gamma_4 < 1$.

Proof. $\because F^n(0)$ is increasing in n, and F is convex.

$\therefore$

$$\frac{F^n(q_{ex}) - F^n(0)}{F^{n-1}(q_{ex}) - F^{n-1}(0)} = \frac{F^n(0) - q_{ex}}{F^{n-1}(0) - q_{ex}} \geq \frac{q_{ex} - F(0)}{q_{ex}}.$$

So,

$$\begin{aligned}
F^n(0) - q_{ex} &\geq \frac{q_{ex} - F(0)}{q_{ex}}(F^{n-1}(0) - q_{ex}) \\
&= \gamma_4(F^{n-1}(0) - q_{ex}) \\
&\geq \gamma_4^2(F^{n-2}(0) - q_{ex}) \\
&\geq \ldots \\
&\geq \gamma_4^n(0 - q_{ex}) \\
&\geq -\gamma_4^n.
\end{aligned}$$

Then , we have

$$F^n(0) \geq q_{ex} - \gamma_4^n.$$

□

Remark 2.2.4. *For the Galton-Watson branching process* $\{Z_n\}_{n\geq 0}$*, if* $\pi_0 > 0$*, then* $q_{ex} > 0$.

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Chapter 3

Main results on occupied positions

3.1 Main theorems

In this section, we present the main results in this thesis. That is, under some proper assumptions on the probability generating functions g_1 and g_2 of $\{\lambda(n,0)\}_{n\geq 0}$ and $\{\lambda(n,n)\}_{n\geq 0}$, respectively, we can prove that the occupied positions eventually form an interval on the real line.

Recall the following notations:

$$K_n = \inf\{k : \lambda(n,k) > 0\}, \quad L_n = \inf\{j : \lambda(n,n-j) > 0\},$$

$$K = \lim_{n\to\infty} K_n, \text{ and } L = \lim_{n\to\infty} L_n.$$

Theorem 3.1.1. *Suppose $g_1'(p_0) < q/p$ and $g_2'(p_0') < p/q$ Then, almost surely, eventually, $\{k : \lambda(n,k) > 0\} = [K_n, n - L_n]$.*

Remark 3.1.2. *Because K_n and K take only integer values, we have $K_n = K$ eventually with probability 1. Similarly, $L_n = L$ eventually with probability 1.*
So, our main result can be written as $[K, n-L]$ is eventully occupied a.s..

3.2 Proofs of main theorems

In order to prove Theorem 3.1.1, we need the following lemmas.

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Lemma 3.2.1. *$\forall p \in (0,1)$, if $p \geq q \geq pF(p)$ or $q > p \geq qF(q)$, then $\forall n \geq 2$, $\exists k_n$, $0 < k_n < n$ such that $p_{n,0} \geq p_{n,1} \geq ... \geq p_{n,k_n} \leq ... \leq p_{n,n-1} \leq p_{n,n}$, and k_n is increasing as n is increasing.*

Proof. We prove it by induction on n. First of all, for $n = 2$ by Lemma 2.1.1, we have

$$\begin{aligned}
p_{0,0} &= P(\lambda(0,0) = 0) = 0, \text{because } \lambda(0,0) = Z_0 = 1 \\
p_{1,0} &= F(qp_{0,0} + pp_{0,-1}) = F(p) \\
p_{1,1} &= F(qp_{0,1} + pp_{0,0}) = F(q) \\
p_{2,0} &= F(qp_{1,0} + pp_{1,-1}) = F(qF(p) + p) \\
p_{2,1} &= F(qp_{1,1} + pp_{1,0}) = F(qF(q) + pF(p)) \\
p_{2,2} &= F(qp_{1,2} + pp_{1,1}) = F(q + pF(q)).
\end{aligned}$$

Suppose $p \geq q \geq pF(p)$. Since $F(z) \leq 1$ for all $z \in [0,1]$ and F is increasing function, $qF(p) + p \geq qF(q) + pF(p)$, then $p_{2,0} = F(qF(p) + p) \geq F(qF(q) + pF(p)) = p_{2,1}$. And by assumption $q \geq pF(p)$, we have $q + pF(q) \geq pF(p) + qF(q)$, so $p_{2,2} = F(q + pF(q)) \geq F(qF(q) + pF(p)) = p_{2,1}$. Combining the above two inequalities together gives us that

$$p_{2,0} \geq p_{2,1} \leq p_{2,2}$$

which means it holds for $n = 2$, and $k_2 = 1$.

Similarly, if $q > p \geq qF(q)$, we have $q > qF(p)$ and $F(q) > F(p)$. So $p_{2,2} = F(q + pF(q)) \geq F(qF(q) + pF(p)) = p_{2,1}$. And by assumption $q > p \geq qF(q)$, we have $p_{2,0} = F(qF(p) + p) \geq F(pF(p) + qF(q)) = p_{2,1}$. Combining the above two inequalities together also gives us that $p_{2,0} \geq p_{2,1} \leq p_{2,2}$.

Suppose that it holds for $n \geq 2$.

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Then, for $n+1$,

$$\begin{aligned} p_{n+1,0} &= F(qp_{n,0}+pp_{n,-1}) \\ p_{n+1,1} &= F(qp_{n,1}+pp_{n,0}) \\ &\vdots \\ p_{n+1,k_n-1} &= F(qp_{n,k_n-1}+pp_{n,k_n-2}) \\ p_{n+1,k_n} &= F(qp_{n,k_n}+pp_{n,k_n-1}). \end{aligned}$$

By induction hypothesis, we get $qp_{n,0}+pp_{n,-1} = qp_{n,0}+p \geq qp_{n,1}+pp_{n,0} \geq ... \geq qp_{n,k_n-1}+pp_{n,k_n-2} \geq qp_{n,k_n}+pp_{n,k_n-1}$. Since F is increasing, we have

$$p_{n+1,0} \geq p_{n+1,1} \geq ... \geq p_{n+1,k_n} \geq p_{n+1,k_n}.$$

On the other hand,

$$\begin{aligned} p_{n+1,n+1} &= F(qp_{n,n+1}+pp_{n,n}) \\ p_{n+1,n} &= F(qp_{n,n}+pp_{n,n-1}) \\ &... \\ p_{n+1,k_n+2} &= F(qp_{n,k_n+2}+pp_{n,k_n+1}) \\ p_{n+1,k_n+1} &= F(qp_{n,k_n+1}+pp_{n,k_n}) \end{aligned}$$

Similarly, by induction hypothesis, we get $qp_{n,n+1}+pp_{n,n} = q+pp_{n,n} \geq qp_{n,n}+pp_{n,n-1} \geq ... \geq qp_{n,k_n+2}+pp_{n,k_n+1}+pp_{n,k_n-2} \geq qp_{n,k_n+1}+pp_{n,k_n}$, hence

$$p_{n+1,n+1} \geq p_{n+1,n} \geq ... \geq p_{n+1,k_n+2} \geq p_{n+1,k_n+1}.$$

Now, we compare p_{n+1,k_n} and p_{n+1,k_n+1}.

If $p_{n+1,k_n} \geq p_{n+1,k_n+1}$, then

$$p_{n+1,0} \geq ... \geq p_{n+1,k_n} \geq p_{n+1,k_n+1} \leq p_{n+1,k_n+2} \leq ... \leq p_{n+1,n+1}.$$

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So, we set $k_{n+1} = k_n + 1$.

Otherwise, if $p_{n+1,k_n} \leq p_{n+1,k_n+1}$, then

$$p_{n+1,0} \geq \dots \geq p_{n+1,k_n-1} \geq p_{n+1,k_n} \leq p_{n+1,k_n+1} \leq \dots \leq p_{n+1,n+1}$$

and set $k_{n+1} = k_n$.

In either case, we can conclude that there exists k_n such that

$$p_{n+1,0} \geq p_{n+1,1} \geq \dots \geq p_{n+1,k_{n+1}} \leq \dots \leq p_{n+1,n} \leq p_{n+1,n+1}$$

and clearly $k_{n+1} \geq k_n$.

Therefore, by induction, we get $\forall n \geq 2$, $p_{n,0} \geq p_{n,1} \geq \dots \geq p_{n,k_n} \leq \dots \leq p_{n,n-1} \leq p_{n,n}$, and k_n is increasing as n is increasing. □

Lemma 3.2.2. *For any $\theta \in (0,1)$, $\exists \alpha \in (0, \frac{1}{2})$, and $\gamma_2 \in (0,1)$ such that, for large n, and $0 \leq k \leq \alpha n$, we have $\binom{n}{k}\theta^{n-k} \leq \gamma_2^n$.*

Proof. First, we know, for $0 \leq k < \alpha n < \frac{1}{2}n$,

$$\binom{n}{k}\theta^{n-k} \leq \binom{n}{k}\theta^{n-\alpha n} \leq \binom{n}{\lceil \alpha n \rceil}\theta^{n-\alpha n}.$$

Also, for large n, by Stirling formula, we have

$$\begin{aligned}\binom{n}{\lceil \alpha n \rceil} &= \frac{n!}{\lceil \alpha n \rceil!(n-\lceil \alpha n \rceil)!} \\ &\approx \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{\sqrt{2\pi \alpha n}(\frac{\alpha n}{e})^{\alpha n}\sqrt{2\pi n - \alpha n}(\frac{n-\alpha n}{e})^{n-\alpha n}} \\ &= \frac{1}{\sqrt{2\pi(1-\alpha)\alpha n}}\frac{1}{(\alpha^\alpha(1-\alpha)^{(1-\alpha)})^n}.\end{aligned}$$

Claim: $\frac{1}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}} \to 1 \; as \; \alpha \to 0+$.

By the facts that $\log x$ is a continuous function at 1 and

$$\lim_{\alpha \to 0+} \log \alpha^\alpha = \lim_{\alpha \to 0+} \alpha \log \alpha = 0,$$

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we have

$$\alpha^\alpha \to 1 \; as \; \alpha \to 0+$$

and hense

$$\alpha^\alpha (1-\alpha)^{(1-\alpha)} \to 1 \quad as \quad \alpha \to 0+.$$

.

Then we get $\frac{1}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}} \to 1 \; as \; \alpha \to 0+$.

Since $0 < \theta < 1$, $\theta^{-\frac{1}{2}} > 1$ and hence $\theta^{-\frac{1}{2}} - 1 > 0$. So, for any $\epsilon \in (0, \theta^{-\frac{1}{2}} - 1)$, we have that

$$0 < (1+\epsilon)\theta^{\frac{1}{2}} < 1$$

and $\exists \delta > 0$ s.t. $\forall \alpha \in (0, \delta)$, $\frac{1}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}} < 1 + \epsilon$.

Therefore, for any given $\theta \in (0,1)$, for any $\epsilon \in (0, \theta^{-\frac{1}{2}} - 1)$, $\exists \delta > 0$ and $\exists \gamma_2 \in ((1+\epsilon)\theta^{\frac{1}{2}}, 1)$ s.t. $\forall \alpha \in (0, min\{\frac{1}{2}, \delta\})$

$$\frac{\gamma_2}{\theta^{1-\alpha}} > \frac{\gamma_2}{\theta^{\frac{1}{2}}} > 1 + \epsilon > \frac{1}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}}.$$

Therefore we can find $\gamma_2 < 1$ s.t. $(\frac{\gamma_2}{\theta^{(1-\alpha)}}) \geq (\frac{1}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}})$.

Then, for large n

$$(\frac{\gamma_2}{\theta^{(1-\alpha)}})^n \geq (\frac{1}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}})^n \frac{1}{\sqrt{2\pi(1-\alpha)\alpha n}}.$$

So, for large n

$$\gamma_2^n \geq \binom{n}{\lceil \alpha n \rceil} \theta^{n-\alpha n} \geq \binom{n}{k} \theta^{n-k}.$$

□

Note that g_1 and g_2 are probability generating functions and p_0 and p_0' are extinction probabilities of $\{\lambda(n,0)\}$ and $\{\lambda(n,n)\}$, respectively.

Lemma 3.2.3. *Suppose $g_1'(p_0) < q/p$ and $g_2'(p_0') < p/q$, then $\exists 0 < \rho < 1$ and a constant C s.t. for $n \geq 2$, $0 \leq k \leq k_n$*

$$p_{n,k} - q_{n,k} \leq C\rho^n,$$

where k_n is defined as in Lemma 3.2.1.

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Proof. Let

$$d_{n,k} = p_{n,k} - q_{n,k}.$$

For $k = 0$, $d_{n,0} = p_{n,0} - q_{n,0} = 0$, and by the basic recursion,

$$\begin{aligned} d_{n+1,k} &= p_{n+1,k} - q_{n+1,k} \\ &= F(qp_{n,k} + pp_{n,k-1}) - F(qq_{n,k} + pq_{n,k-1}) \\ &= F'(z)(qd_{n,k} + pd_{n,k-1}), \text{ by the mean value theorem,} \end{aligned}$$

where $qq_{n,k} + pq_{n,k-1} \leq z \leq qp_{n,k} + pp_{n,k-1}$.

Because $p_{n,k}$ is decreasing in k, $\forall 0 \leq k \leq k_n$ and $p_{n,0}$ is increasing in n, we have, for any $0 \leq k \leq k_n$,

$$z \leq qp_{n,k} + pp_{n,k-1} \leq qp_{n,0} + p \leq q \lim_{n\to\infty} p_{n,0} + p \leq qp_0 + p.$$

Also, $F'(z)$ is increasing function for positive z, so we get that

$$d_{n+1,k} \leq F'(qp_0 + p)(qd_{n,k} + pd_{n,k-1}) \leq \max\{p, q\}F'(qp_0 + p)(d_{n,k} - d_{n,k-1}).$$

Similarly, for $k_n + 1 \leq k \leq n$,

$$z \leq qp_{n,k} + pp_{n,k-1} \leq q + pp_{n,n} \leq q + p \lim_{n\to\infty} p_{n,n} \leq q + pp'_0$$

and hence

$$d_{n+1,k} \leq F'(q + pp'_0)(qd_{n,k} + pd_{n,k-1}) \leq \max\{p, q\}F'(q + pp'_0)(d_{n,k} - d_{n,k-1}).$$

Note that, since p_0 and p'_0 are the extionction probabilities and g_1 and g_1 are the probability generating functions for $\{\lambda(n, 0)\}$ and $\{\lambda(n, n)\}$, respectively, we know that $g'_1(p_0) < 1$ and $g'_2(p'_0) < 1$.

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By assumptions and the fact that $g_1'(z) = qF'(qz+p)$ and $g_2'(z) = pF'(pz+q)$, we have

$$
\begin{aligned}
0 < \gamma_1 &:= \max\{p,q\}\max\{F'(qp_0+p), F'(pp_0'+q)\} \\
&= \max\{p,q\}\max\{\frac{g_1'(p_0)}{q}, \frac{g_2'(p_0')}{p}\} < 1.
\end{aligned}
$$

Then we get that, for all $k \le n$,

$$d_{n+1,k} \le \gamma_1(qd_{n,k} + pd_{n,k-1}).$$

Next, we show the following:

$$d_{n,k} \le \binom{n}{k}\gamma_1^{n-k}, \quad \forall k \le n, \quad \forall n.$$

We prove this inequality by induction on n.

For $n = 0$, we know that $d_{0,k} = p_{0,k} - q_{0,k} = 0$.

Suppose $d_{n,k} \le \binom{n}{k}\gamma_1^{n-k}$.

For $n+1$, by the above argument, we have, for $0 \le k \le n$,

$$
\begin{aligned}
d_{n+1,k} &\le \gamma_1(qd_{n,k} + pd_{n,k-1}) \\
&\le \gamma_1(q\binom{n}{k}\gamma_1^{n-k} + p\binom{n}{k-1}\gamma_1^{n-k+1}) \\
&\le \gamma_1(q\binom{n}{k}\gamma_1^{n-k} + p\binom{n}{k-1}\gamma_1^{n-k}) \\
&\le \gamma_1^{n-k+1}(q\binom{n}{k} + p\binom{n}{k-1}) \\
&\le \gamma_1^{n-k+1}(\binom{n}{k} + \binom{n}{k-1}) \\
&= \gamma_1^{n-k+1}\binom{n+1}{k}, \text{ by double counting.}
\end{aligned}
$$

For $k = n+1$, it is clear that $d_{n+1,k+1} = p_{n+1,k+1} - q_{n+1,k+1} \le 1 = \binom{n+1}{n+1}\gamma_1^{(n+1)-(n+1)}$.

So, by induction, we have $d_{n,k} \le \binom{n}{k}\gamma_1^{n-k}, \quad \forall k \le n, \quad \forall n$.

Then, by Lemma 3.2.2, we can find $0 < \alpha_1 < 1/2$, $\gamma_2 < 1$, and $n_0 \ge 1$, for all $n > n_0$ and $k \le \alpha_1 n$,

$$d_{n,k} \le \binom{n}{k}\gamma_1^{n-k} \le \gamma_2^n.$$

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Now, consider for small n with $n \leq n_0$, and take $\alpha_2 = \frac{1}{n_0}$, then $k < \frac{n}{n_0} < 1$, i.e., $k = 0$, and,

$$d_{n,0} \leq \binom{n}{0} \gamma_1^{n-0} = \gamma_1^n,$$

i.e., in this case, we can take $\gamma_2 = \gamma_1$.

Let $\alpha = \min\{\frac{1}{n_0}, \alpha_1\}$ and $\gamma = \max\{\gamma_1, \gamma_2\}$. Thus, we find $\alpha > 0$ and $\gamma < 1$ s.t. for all n and for $k \leq \alpha n$, we have

$$d_{n,k} \leq \gamma^n.$$

We also need to take care the case when $\alpha n < k_n$. For $\alpha n < k < k_n$, we first compute the upper bound of $p_{n,k}$, and lower bound of $q_{n,k}$.

$$\begin{aligned} p_{n,k} &= F(qp_{n-1,k} + pp_{n-1,k-1}) \\ &\leq F(p_{n-1,k-1}), \text{because } p_{n,k} \text{ is decreasing in } k \text{ for } k < k_n \\ &\leq F^k(p_{n-k,0}) \\ &\leq F^k(q_0), \text{because } p_{n,0} = q_{n,0} \text{ is increasing in } n \text{ and } q_0 = \lim_{n\to\infty} q_{n,0}. \end{aligned}$$

So, by Lemma 2.2.2, we have, for $k > \alpha n$,

$$p_{n,k} \leq q_{ex} + \gamma_3^k \leq q_{ex} + \gamma_3^{\alpha n}.$$

By the similar way, we can get lower bound of $q_{n,k}$.

$$\begin{aligned} q_{n,k} &= F(qq_{n-1,k} + pq_{n-1,k-1}) \\ &\geq F(q_{n-1,k}), \text{because } q_{n,k} \text{ is decreasing in } k \\ &\geq F^n(q_{0,k}) \\ &= F^n(0). \end{aligned}$$

So, according whether the value of $F(0)$ is 0 or not, we have two cases.

If $F(0) > 0$, then $q_{ex} > 0$ and by Lemma 2.2.3, we have that $q_{n,k} \geq q_{ex} - \gamma_4^n$. Because F is convex, the slope of secant line $(q_{ex} - F(0))/q_{ex} = \gamma_4$ is smaller than 1, then $p_{n,k} - q_{n,k} \leq \gamma_3^{\alpha n} + \gamma_4^n \leq \gamma_3^{\alpha} + \gamma_4$. Let $\rho = \max\{\gamma, \gamma_3^{\alpha}, \gamma_4\}$, and $C = 2$.

On the other hand, if $F(0) = \pi_0 = 0$, then $q_{ex} = 0$ and the lemma is also true when

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$\rho = \max\{\gamma, \gamma_3^{\alpha}\}$, that is, in this case, $q_{n,k} \geq F^n(0) = 0$ with $\gamma_4 = 0$.

Summarizing the above, we get, for all $k \leq k_n$, there are constant C and $0 < \rho < 1$, s.t.

$$p_{n,k} - q_{n,k} \leq C\rho^n.$$

□

Now, we are ready to prove our main result, Theorem 3.1.1.

Recall that $K_n = \inf\{k : \lambda(n,k) > 0\}$.

First, we prove the claim that $\exists K_n \leq k \leq k_n$ s.t. $\lambda(n,k) = 0$ if and only if $\exists 0 \leq k \leq k_n$ s.t. $\lambda(n,k) = 0, \lambda(n,k-1) > 0$.

($\Rightarrow$) Suppose $\exists K_n \leq k \leq k_n : \lambda(n,k) = 0$. Because $K_n = \inf\{k : \lambda(n,k) > 0\}$, by the well-ordering principle, there exists a smallest number between K_n and k_n, denote it by k, then $0 \leq k \leq k_n$, $\lambda(n,k) = 0$, $\lambda(n,k-1) > 0$.

($\Leftarrow$)We prove it by contradiction. Suppose that $\forall K_n \leq k \leq k_n : \lambda(n,k) > 0$, then, by definition of K_n, $\lambda(n,k) = 0$ for all $k < K_n$. So, it is clear that there is no such $0 \leq k \leq k_n$ s.t. $\lambda(n,k) = 0, \lambda(n,k-1) > 0$.

So, the claim holds.

Now, let $A_n = \{\exists K_n \leq k \leq k_n : \lambda(n,k) = 0\}$.

Then

$$\begin{aligned}
P(A_n) &= P(\exists K_n \leq k \leq k_n : \lambda(n,k) = 0) \\
&= P(\exists 0 \leq k \leq k_n : \lambda(n,k) = 0,\ \lambda(n,k-1) > 0) \\
&\leq P(\bigcup_{k=0}^{k_n}(\{\lambda(n,k) = 0)\} - \{\lambda(n,j) = 0, \forall j \leq k\})) \\
&\leq \sum_{k=0}^{k_n}(p_{n,k} - q_{n,k}) \\
&\leq Cn\rho^n, \text{ by Lemma 3.2.3.}
\end{aligned}$$

Then

$$\sum_{n=0}^{\infty} P(A_n) \leq \sum_{n=0}^{\infty} Cn\rho^n < \infty.$$

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Thus, by Borel-Cantelli lemma, $P(A_n\ i.o.) = 0$, i.e., $[K_n, k_n]$ is a.s. eventully occupied.

Finally, we will show that the interval $[k_n, n - L_n]$ is also a.s. eventually occupied.

Consider another branching random walk $\{(Z_n^*, \mathbf{S}_n^*)\}_{n\geq 0}$, where $Z_n^* = Z_n$ for the branching structure and for each individual $\vec{i}$ in $\{(Z_n^*, \mathbf{S}_n^*)\}_{n\geq 0}$, the movement $X_{\vec{i}}^* = 1 - X_{\vec{i}}$. So, if $|\vec{i}| = n$, then we have

$$S_{\vec{i}}^* = n - S_{\vec{i}}.$$

Therefore, for each $n \geq 0$ and $k = 0, 1, 2, ..., n$,

$$\begin{aligned} \lambda^*(n,k) &= \text{the number of individuals in the } n\text{-th generation of } \{(Z_n^*, \mathbf{S}_n^*)\}_{n\geq 0} \text{ at position } k \\ &= \lambda(n, n-k). \end{aligned}$$

Hence,

$$\begin{aligned} L_n &\equiv \text{inf}\{j : \lambda(n, n-j) > 0\} \\ &= \text{inf}\{j : \lambda^*(n, j) > 0\} \equiv K_n^* \end{aligned}$$

and

$$\begin{aligned} p_{n,k}^* &\equiv P(\lambda^*(n,k) = 0) \\ &= P(\lambda(n, n-k) = 0) \\ &= p_{n,n-k}. \end{aligned}$$

So, by reversing the role of p and q and by the similar lines of proofs, we have that there exists a increasing sequence $\{k_n^*\}_{n\geq 0}$ s.t. the interval $[K_n^*, k_n^*]$ is a.s. eventually occupied by individuals in $\{(Z_n^*, \mathbf{S}_n^*)\}_{n\geq 0}$ and $k_n^* = n - k_n$.

Since

$$\begin{aligned} B_n &\equiv \{\exists k \in [k_n, n - L_n], \lambda(n,k) = 0\} \\ &= \{\exists k \ \ s.t.\ \ n - k \in [L_n, n - k_n], \lambda(n,k) = 0\} \\ &= \{\exists j \in [K_n^*, k_n^*], \lambda(n,j) = 0\} \\ &\equiv A_n^*. \end{aligned}$$

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It is clear, by the previous arguments, that $P(A_n^* \quad i.o.) = 0$.

So, $P(B_n \quad i.o.) = 0$, i.e., $[k_n, n - L_n]$ is a.s. eventually occupied.

Therefore, we conclude that $[K_n, n - L_n]$ is a.s. eventually occupied and the proof of Theorem 3.1.1 is complete.

Remark 3.2.4. *If we consider the branching random walk in which each individual has probability p to go the right by one step and, instead of staying in the same position, it has probability q to go to the left by one step, then we can get a similar result. In this case, we let*

$$Y_{\vec{i}} = \begin{cases} 1, & \text{if } X_{\vec{i}} = 1 \\ -1, & \text{if } X_{\vec{i}} = 0 \end{cases}$$

and recall that $X_{\vec{i}}$ is the movement of $\vec{i}$ and $S_{\vec{i}}$ is the position of $\vec{i}$ in the branching random walk $\{(Z_n^*, \mathbf{S}_n^*)\}_{n\geq 0}$ defined for Theorem 3.1.1. Let $\lambda'(n,k)$ be the number of individuals in position k at time n, and $S'_{\vec{i}}$ be the position of $\vec{i}$ in the branching random walk with movement $\{Y_{\vec{i}}\}$. Then $S'_{\vec{i}} = 2k - n$ if and only if $S_{\vec{i}} = k$. So,

$$\lambda'(n, 2k - n) = \lambda(n, k).$$

Thus, Theorem3.1.1 implies that, in this branching random walk, $\lambda'(n, 2k - n) > 0$ for all $k \in [K_n, n - L_n]$.

Note that, it is impossible for people to go to even position with odd steps, so the occupied positions do not form an interval. In this case, we only can say that all occupied positions are a.s. eventually contained in the interval $[2K_n - n, n - 2L_n]$.

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