國立政治大學應用數學系

碩士學位論文

暫現狀態下具長域隨機漫步在整數晶格點的格林函數與容積的漸近行為

Asymptotic Behaviors of the Green Function and Capacity for Transient State Random Walks with Long-Range Interactions

指導教授：陳隆奇　博士

研究生：陳濬程　撰

中 華 民 國 110 年 8 月

DOI:10.6814/NCCU202101556

致謝

歲月如歌，光陰似箭，兩年的研究生生活即將結束。回首這兩年的求學歷程，對那些引導我、幫助我、激勵我的人，我心中充滿了感激。

首先要感謝我的指導教授陳隆奇老師這一年來煞費苦心的指導，不論是論文定題到寫作定稿都傾注了老師大量的心血。在我攻讀碩士期間，深深受益於老師的關心、愛護和諄諄教導。他作為老師，點撥迷津，讓人如沐春風；作為長輩，關懷備至，讓人感念至深。

同時，我還要感謝所有教導過我、關心過我的同學和朋友們，你們為我的學業傾注了大量心血，你們為人師表的風範令我敬仰，嚴謹治學的態度令我敬佩。兩年來，我們朝夕相處，共同進步，感謝你們給予我的所有關心和幫助，同窗之誼，我將終生難忘！

這場既是惡夢也是美夢的兩年時光，終於在口試老師陳隆奇博士、洪芷漪博士、須上苑博士的口試提問喚聲中，即將清醒了。謝謝老師們撥空替我進行口試測驗，指導並更正論文之疑處與錯誤，助學生取得學位為之力。最後，祝福曾經幫助我的每一個朋友及師長及陪伴著我成長的家人，身體健康，快樂如意。

DOI:10.6814/NCCU202101556

中文摘要

在整數晶格 $\mathbb{Z}^d$ 上的隨機漫步 $S_n^x = x + \sum_{k=1}^{n} X_k$，其中每一個隨機向量 $X_i, i = 1, 2, \cdots, n$ 皆獨立且具有相同分佈 $D(x)$。此論文，我們假設 $D(x)$ 在 $\mathbb{Z}^d$ 空間情形下擁有對稱性且當 $|x| \to \infty$ 時，遞減速率為 $|x|^{-d-\alpha}$，其中 $\alpha \in (0, \infty) \setminus \{2\}$ 且 $d > \alpha \wedge 2$。本文主要是探討此具長域隨機漫步下的一些漸近行為。第一個主要結果在於獲得此模型之格林函數的漸近行為，此外我們還得到主要項係數及其收斂速度；第二個主要結果在討論容積的漸近行為，並且進一步得到在長域隨機漫步下的 Wiener's Criterion。

關鍵詞: 格林函數、容積、長域隨機漫步

DOI:10.6814/NCCU202101556

Abstract

Let $S_n^x = x + \sum_{k=1}^{n} X_k$ be the n-step random walk on $\mathbb{Z}^d$ starting at x, where $X_i's$ are independent identically distributed random vectors with distribution $D(x)$. In the thesis, we suppose that the distribution $D(x)$ is symmetric on $\mathbb{Z}^d$ and the rate of decay is of order $|x|^{-d-\alpha}$ as $|x| \to \infty$ with $\alpha \in (0, \infty) \setminus \{2\}$ and $d > \alpha \wedge 2$, where $a \wedge b = \min\{a, b\}$. The purpose of the thesis is to investigate asymptotic behaviors of the long-range random walk. First of all, we get the asymptotic behavior of the Green function. Moreover, we obtain the coefficient of the main term and its rates of convergence. Secondly, we discuss the asymptotic behavior of the capacity for the long-range random walk. Moreover, we derive the Wiener's Criterion for the long-range random walk.

Keywords : Green Function、Capacity、Long-Range Random Walk

DOI:10.6814/NCCU202101556

Contents

DOI:10.6814/NCCU202101556

Chapter 1

Introduction and The Main Results

1.1 Introduction

The term "random walk" was first proposed by Karl Pearson in 1905 (c.f. [6]). In mathematics, a random walk is a stochastic process whose formation is a successive summation of independent and identically distributed random variables, and it is one of the most extensively reasearched and interesting topics in probability. For example, the movement of the gas molecule during a small time inteval, tracking the feeding path of an animal, the price of a fluctuating stock or the financial difficulties of a gambler can all be approximated by random walk models. Other pertinent examples include, various diluted spin system, random copolymers (c.f. [8]), spin glasses (c.f. [2] [9]), random-graph models (c.f. [1]), etc.

The most interesting and important random walk model is the random walk on d-dimensional integer lattice $\mathbb{Z}^d$ whose probability distribution of going from a point to another point in each point is symmetry. In particular, a random walk is called a simple random walk if in the walk one can only jump to neighbours of the lattice $\mathbb{Z}^d$ with the equal probability. A d-dimensional simple random walk is usually assumed to be the Markov chain or Markov process, which is very closed to Brownian motion on $\mathbb{Z}^d$. Brownian motion was first discovered by the biologist Robert Brown in 1827 (c.f. [10]), when pollen grains suspended in water performed a continuous swarming motion. On the theoretical side, Brownian motion is a Guassian Markov process with stationary independent increments. It is well-known that the Brownian motion on $\mathbb{Z}^d$ is a scaled limit of a simple random walk on $\mathbb{Z}^d$ for all $d \geq 1$.

The simple random walk on $\mathbb{Z}^d$ lattice is studied sufficiently. In the thesis, we will consider

DOI:10.6814/NCCU202101556

the model of a long-range random walk on $\mathbb{Z}^d$, where the 1-step distribution $D(x)$ is symmetry and $D(x) \asymp |x|^{-d-\alpha}$, where the notation $f(x) \asymp g(x)$ means that there exist $0 < c < c' < \infty$ such that $cg(x) \le f(x) \le c'g(x)$ for all x. For example,

$$D(x) = \frac{|x|^{-d-\alpha}}{\sum\limits_{y \in \mathbb{Z}^d} |y|^{-d-\alpha}}, \tag{1.1.1}$$

where $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ and $\alpha \in (0, \infty) \backslash \{2\}$. It is easy to see that $D(x)$ is symmetry and the rate of decay is of order $\frac{1}{|x|^{d+\alpha}}$.

1.2 Main Results

First, we will investigate the behavior of the Green's function of the long-range random walk for $d > \alpha \wedge 2$. The Green function was introduced in a famous paper by Geoge Green in 1828 (c.f. [3]), and its definition for a random walk is as follows:

$$G^x(y) = \mathbb{E}\Big[\sum_{n=0}^{\infty} I_{\{S_n^x = y\}}\Big] = \sum_{n=0}^{\infty} \mathbb{E}[I_{\{S_n^x=y\}}] = \sum_{n=0}^{\infty} \mathbb{P}[S_n^x = y],$$

where $S_n^x = x + \sum\limits_{k=1}^{n} X_k$ is the n-step random walk on $\mathbb{Z}^d$ starting from x and $X_i's$ are i.i.d random vectors with the same distribution $D(x)$ for $x \in \mathbb{Z}^d$, $i.e,\ \mathbb{P}(X_k = x) = D(x),\ \forall x \in \mathbb{Z}^d$. That is, $G^x(y)$ is equal to the mean number of the visits to *y* starting from *x* and we let $S_n^0 = S_n$ and $G^0(x) = G(x)$ for convenience.

Let $\hat{D}$ and D^{*n} be the Fourier transform and the n-fold convolution of D, respectively:

$$\hat{D}(k) = \sum_{x \in \mathbb{Z}^d} e^{ik\cdot x} D(x), k \in [-\pi, \pi]^d$$

$$D^{*n}(x) = \begin{cases} \delta_{0,x}, & n = 0, \\ \sum\limits_{y \in \mathbb{Z}^d} D^{*(n-1)}(y) D(x-y), & n \ge 1, \end{cases}$$

where δ is the Kronecker delta function.

We assume that $D(x)$ satisfies the following assumptions and will show all the assumptions hold if D is as defined in (1.1.1) in Appendix A.

DOI:10.6814/NCCU202101556

Assumption 1.2.1. *We require the 1-step distribution $D(x)$ to be bounded as*

$$D(x) \asymp \frac{1}{|x|^{d+\alpha}}.$$

Assumption 1.2.2. *(Properties of $\hat{D}$)*

(a) Given $R > 0$ small and for $|k| \leq R$, there exist $v_\alpha \in (0, \infty)$ and $\epsilon \in (0, 1)$ such that

$$1 - \hat{D}(k) = v_\alpha |k|^{\alpha\wedge 2} + O(|k|^{\alpha\wedge 2+\epsilon})$$

In particular, if $\alpha > 2$, it is easy to see that $v_\alpha = \frac{\sigma^2}{2d}$, where $\sigma^2 = \sum_{x\in\mathbb{Z}^d} |x|^2 D(x) \in (0, \infty)$.

(b) For $|k| > R$ and $k \in [-\pi, \pi]^d$, there exists $\Delta \in (0, 1)$ such that

$$1 - \hat{D}(k) \begin{cases} < 2 - \Delta, \\ > \Delta. \end{cases}$$

Assumption 1.2.3. *(Bounds on D^{*n})*

For $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,

$$\|D^{*n}\|_\infty \leq O(1)n^{-\frac{d}{\alpha\wedge 2}}$$
$$D^{*n}(x) \leq \frac{O(1)n}{|x|^{d+\alpha\wedge 2}}.$$

Using the inverse formula for any dimension $d > (\alpha \wedge 2)$, $\alpha \neq 2$, we have

$$\begin{aligned} G(x) = \sum_{n=0}^{\infty} D^{*n}(x) =& \left(\frac{1}{2\pi}\right)^d \sum_{n=0}^{\infty} \int_{[-\pi,\pi]^d} e^{-ik\cdot x}[\hat{D}(k)]^n d^d k \\ =& \left(\frac{1}{2\pi}\right)^d \int_{[-\pi,\pi]^d} \sum_{n=0}^{\infty} e^{-ik\cdot x}[\hat{D}(k)]^n d^d k \\ =& \left(\frac{1}{2\pi}\right)^d \int_{[-\pi,\pi]^d} \frac{e^{-ik\cdot x}}{1-\hat{D}(k)} d^d k < \infty, \end{aligned}$$

since

$$\int_{[-\pi,\pi]^d} \frac{1}{1-\hat{D}(k)} d^d k \approx O(1) \int_0^1 \frac{1}{|k|^{\alpha\wedge 2}} d^d k \approx O(1) \int_0^1 \frac{r^{d-1}}{r^{\alpha\wedge 2}} dr,$$

DOI:10.6814/NCCU202101556

the Green function of long-range random walk does not exist if $d \leq (\alpha \wedge 2)$.
We have the first main result as following which will be proved in Section 2.2.

Theorem 1.2.4. *Let $d > \alpha \wedge 2$ and assume all assumptions of $D(x)$ as above, then there exists $\mu \in (0, \alpha \wedge 2)$ such that for $|x| \geq 1$,*

$$G(x) = \frac{C_{\alpha,d}}{|x|^{d-\alpha\wedge 2}}\left(1 + \frac{O(1)}{|x|^{\mu}}\right),$$

where

$$C_{\alpha,d} = \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{\frac{d}{2}}\Gamma(\frac{\alpha\wedge 2}{2})v_{\alpha}}.$$

Next, we investigate the *capacity* of the long-range random walk on $\mathbb{Z}^d$. For any intergers m and n, we define the range $R[m,n]$ to be the set of the visited sites in the time interval $[m,n]$, i.e.

$$R[m,n] = \{S_m, \cdots, S_n\}.$$

The *capacity* of a finite set $A \subset \mathbb{Z}^d$ is defined to be

$$cap(A) = \sum_{x \in A} Es_A(x) = \sum_{x \in A} \mathbb{P}_x(\tau_A^+ = \infty),$$

where $Es_A(x) = \mathbb{P}_x(\tau_A^+ = \infty)I_{\{x \in A\}}$ is the escape probability from a finite set $A \subset \mathbb{Z}^d$ and $\tau_A^+ = \min\{n \geq 1 : X_n \in A\}$ is the first return time to A. The study of the *capacity* of the range of a walk has a long history. For simple random walk, Jain and Orey proved (c.f. [4]), some fifty years ago, that $cap(R[0,n])$ satisfies a law of large numbers for all $d \geq 3$. i.e. almost surely

$$\lim_{n\to\infty} \frac{cap(R[0,n])}{n} = \alpha_d.$$

Moreover, they showed that $\alpha_d > 0$ if and only if $d \geq 5$. In the eighties, Lawler established estimates on intersection probabilities for random walks, which are relevant tools for estimating the expected capacity of the range (c.f. [5]). As for the capacity of the d-dimensional discrete ball centered at 0, $B(R) := B(0;R)$, the capacity in simple random walk is of order R^d (c.f. [7]); In the thesis, we are going to investigate the asymptotic of *capacity* in long-range random walk as the following theorem.

DOI:10.6814/NCCU202101556

Theorem 1.2.5. *Let $d > \alpha \wedge 2$,*

$$cap(B(R)) \asymp R^{d-\alpha\wedge 2} \text{ as } R \to \infty.$$

Lastly, we will investigate the *Wiener's Criterion*, the result for simple random walk was introduced by Serguei Popov (c.f. [7]). In the thesis, we say that a set $A \subset \mathbb{Z}^d$ is recurrent if $\mathbb{P}_x(\tau_A < \infty) = 1$ for all $x \in \mathbb{Z}^d$; Otherwise, we call the set A is transient. And we will derive an extension to the model of long-range random walk in which 1-step distribution $D(x)$ is defined in (1.1.1).

Theorem 1.2.6 (Wiener's Criterion)**.**
For $d > \alpha \wedge 2$, $A \subseteq \mathbb{Z}^d$ is recurrent if and only if

$$\sum_{k=1}^{\infty} \frac{cap(A_k)}{2^{(d-\alpha\wedge 2)k}} = \infty,$$

where $A_k = \{x \in A : 2^{k-1} < |x| \leq 2^k\}$ is the intersection of A with the annulus $B(2^k) \setminus B(2^{k-1})$.

The organization of this paper is as follows: in the next chapter, we will prove Theorem 1.2.4. In Chapter 3, we will show Theorem 1.2.5 for $d > \alpha \wedge 2$. In Chapter 4, we use the previous analysis to prove Theorem 1.2.6. In Appendix A, we will show the $D(x)$ defined in (1.1.1) satisfies all the assumptions.

DOI:10.6814/NCCU202101556

Chapter 2

Proof of Theorem 1.2.4

2.1 Propositions of Green function

It is important to note that some basic propositions of $G^x(y)$. First, in the case $x = y$ we do count this as "initial" visit (so, in particular, $G^x(x) > 1$). Moreover, by symmetry, it holds that $G^x(y) = G^y(x)$.

Proposition 2.1.1.

(a) $D^{*n}(x) = D^{*n}(-x)$

(b) $G^x(x) > 1$

(c) $G^x(y) = G^y(x)$

Proof.

(a) For $n = 1$, if $x \neq 0$, it is trivial by definition.

Suppose it holds for $n = k$ for some $k \in \mathbb{N}$.

Claim: $n = k + 1$, it holds.

$$\begin{aligned} D^{*(k+1)}(x) &= \sum_{y \in \mathbb{Z}^d} D^{*k}(y)D(x-y) = \sum_{y \in \mathbb{Z}^d} D^{*k}(-y)D(-x-(-y)) = \sum_{u \in \mathbb{Z}^d} D^{*k}(u)D(-x-u) \\ &= D^{*(k+1)}(-x). \end{aligned}$$

(b) $G^x(x) = \sum_{n=0}^{\infty} \mathbb{P}[S_n^x = x] = \mathbb{P}[S_0^x = x] + \sum_{n=1}^{\infty} \mathbb{P}[S_n^x = x] = 1 + \sum_{n=1}^{\infty} \mathbb{P}\left[\sum_{k=1}^{n} X_k = 0\right] > 1$,

since

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\sum_{k=1}^{n} X_k = 0\right] \quad > \quad 0.$$

DOI:10.6814/NCCU202101556

(c) To prove it, we first observe that $G^x(y) = G^0(y-x) := G(y-x)$ and using $G(y) = G(-y)$, since

$$\begin{aligned}
G^x(y) &= \sum_{n=0}^{\infty} \mathbb{P}[S_n^x = y] = \mathbb{P}[S_0^x = y] + \sum_{n=1}^{\infty} \mathbb{P}[S_n^x = y] = I_{\{x=y\}} + \sum_{n=1}^{\infty} \mathbb{P}\left[x + \sum_{k=1}^{n} X_k = y\right] \\
&= I_{\{x=y\}} + \sum_{n=1}^{\infty} \mathbb{P}\left[\sum_{k=1}^{n} X_k = y - x\right] = G(y-x) \\
G(y) &= \sum_{n=0}^{\infty} \mathbb{P}[S_n = y] = \mathbb{P}[S_0 = y] + \sum_{n=1}^{\infty} \mathbb{P}[S_n = y] = I_{\{y=0\}} + \sum_{n=1}^{\infty} \mathbb{P}\left[\sum_{k=1}^{n} X_k = y\right] \\
&= I_{\{y=0\}} + \sum_{n=1}^{\infty} D^{*n}(y) = I_{\{y=0\}} + \sum_{n=1}^{\infty} D^{*n}(-y) = G(-y),
\end{aligned}$$

where I is indicator function.

Hence, we get $G^x(y) = G(y-x) = G(x-y) = G^y(x)$. $\square$

DOI:10.6814/NCCU202101556

2.2 Proof of Theorem 1.2.4

By the inverse formula, we have

$$\begin{aligned} G(x) = \sum_{n=0}^{\infty} D^{*n}(x) =& \left(\frac{1}{2\pi}\right)^d \sum_{n=0}^{\infty} \int_{[-\pi,\pi]^d} e^{-ik\cdot x}[\hat{D}(k)]^n d^d k \\ =& \left(\frac{1}{2\pi}\right)^d \int_{[-\pi,\pi]^d} e^{-ik\cdot x} \frac{1}{1-\hat{D}(k)} d^d k. \end{aligned}$$

Using the following identity

$$\int_0^\infty e^{-ct} dt = \frac{1}{c}, \text{ for } c > 0,$$

and let $T = |x|^{(\alpha\wedge 2)(1-\mu)}$ with $\mu = \frac{\frac{\epsilon}{2}}{d-\alpha\wedge 2+\frac{\epsilon}{2}}$, we have

$$\begin{aligned} G(x) =& \left(\frac{1}{2\pi}\right)^d \int_0^\infty \int_{[-\pi,\pi]^d} e^{-[1-\hat{D}(k)]t} e^{-ik\cdot x} d^d k dt \\ =& I + I_1, \end{aligned}$$

where

$$\begin{aligned} I &= \left(\frac{1}{2\pi}\right)^d \int_T^\infty \int_{[-\pi,\pi]^d} e^{-[1-\hat{D}(k)]t} e^{-ik\cdot x} d^d k dt \\ I_1 &= \left(\frac{1}{2\pi}\right)^d \int_0^T \int_{[-\pi,\pi]^d} e^{-[1-\hat{D}(k)]t} e^{-ik\cdot x} d^d k dt. \end{aligned}$$

First, we consider I_1,

$$\begin{aligned} I_1 =& \left(\frac{1}{2\pi}\right)^d \int_0^T e^{-t} \int_{[-\pi,\pi]^d} \sum_{n=0}^{\infty} \frac{\hat{D}(k)^n t^n}{n!} e^{-ik\cdot x} d^d k dt \\ =& \int_0^T e^{-t} \sum_{n=0}^{\infty} \frac{t^n D^{*n}(x)}{n!} dt. \end{aligned}$$

DOI:10.6814/NCCU202101556

By Assumption 1.2.3, we get

$$I_1 \leq \int_0^T e^{-t} \sum_{n=0}^{\infty} \left(\frac{t^n}{n!} \frac{O(1)n}{|x|^{d+\alpha\wedge 2}} \right) dt = \frac{O(1)}{|x|^{d+\alpha\wedge 2}} \frac{T^2}{2}. \tag{2.2.1}$$

Next, we consider I,

$$I = \left(\frac{1}{2\pi}\right)^d \int_0^\infty \int_{\mathbb{R}^d} e^{-v_\alpha t|k|^{\alpha\wedge 2}} e^{-ik\cdot x} d^d k dt + \sum_{j=2}^{5} I_j,$$

for any $R \in (0, \pi)$, where

$$\begin{aligned}
I_2 &= -\left(\frac{1}{2\pi}\right)^d \int_0^T \int_{\mathbb{R}^d} e^{-v_\alpha t|k|^{\alpha\wedge 2}} e^{-ik\cdot x} d^d k dt \\
I_3 &= \left(\frac{1}{2\pi}\right)^d \int_T^\infty \int_{|k|\leq R} e^{-ik\cdot x} \left(e^{-[1-\hat{D}(k)]t} - e^{-v_\alpha t|k|^{\alpha\wedge 2}} \right) d^d k dt \\
I_4 &= \left(\frac{1}{2\pi}\right)^d \int_T^\infty \int_{[-\pi,\pi]^d} e^{-ik\cdot x} e^{-[1-\hat{D}(k)]t} I_{\{|k|>R\}} d^d k dt \\
I_5 &= -\left(\frac{1}{2\pi}\right)^d \int_T^\infty \int_{|k|>R} e^{-ik\cdot x - v_\alpha t|k|^{\alpha\wedge 2}} d^d k dt.
\end{aligned}$$

Using the identity

$$\int_0^\infty e^{-v_\alpha t|k|^{\alpha\wedge 2}} dt = \frac{1}{v_\alpha |k|^{\alpha\wedge 2}} = \frac{1}{v_\alpha \Gamma(\frac{\alpha\wedge 2}{2})} \int_0^\infty t^{\frac{\alpha\wedge 2}{2}-1} e^{-|k|^2 t} dt,$$

we obtain

$$\begin{aligned}
\left(\frac{1}{2\pi}\right)^d \int_0^\infty \int_{\mathbb{R}^d} e^{-v_\alpha t|k|^{\alpha\wedge 2}} e^{-ik\cdot x} d^d k dt &= \frac{1}{v_\alpha \Gamma(\frac{\alpha\wedge 2}{2})} \int_0^\infty t^{\frac{\alpha\wedge 2}{2}-1} \int_{\mathbb{R}^d} \left(\frac{1}{2\pi}\right)^d e^{-ik\cdot x - t|k|^2} d^d k dt \\
&= \frac{1}{(2\pi)^d v_\alpha \Gamma(\frac{\alpha\wedge 2}{2})} \int_0^\infty t^{\frac{\alpha\wedge 2}{2}-1} \left(\sqrt{\frac{\pi}{t}}\right)^d e^{-\frac{|x|^2}{4t}} dt,
\end{aligned}$$

DOI:10.6814/NCCU202101556

where the last equation holds by the inverse formula of Gaussian distribution. Then by the substituition with $u = \frac{|x|^2}{4t}$, we have

$$\left(\frac{1}{2\pi}\right)^d \int_0^\infty \int_{\mathbb{R}^d} e^{-v_\alpha t|k|^{\alpha\wedge 2}} e^{-ik\cdot x} d^d k dt = \frac{1}{2^d \pi^{\frac{d}{2}} v_\alpha \Gamma(\frac{\alpha\wedge 2}{2})} \int_0^\infty \left(\frac{|x|^2}{4u}\right)^{\frac{\alpha\wedge 2-d}{2}-1} e^{-u} \frac{|x|^2}{4u^2} du$$
$$= \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2} v_\alpha \pi^{\frac{d}{2}} \Gamma(\frac{\alpha\wedge 2}{2})} \frac{1}{|x|^{d-\alpha\wedge 2}}.$$

As a result, we arrive at

$$G(x) = \frac{C_{\alpha,d}}{|x|^{d-\alpha\wedge 2}} + \sum_{j=1}^{5} I_j,\ C_{\alpha,d} = \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2} v_\alpha \pi^{\frac{d}{2}} \Gamma(\frac{\alpha\wedge 2}{2})}.$$

Then it remains to estimate $I_j's$ which are error terms for $i = 2, 3, 4, 5$.

First, we estimate I_2 by substitution $u = \frac{|x|^2}{4t}$.

$$|I_2| = \left(\frac{1}{2\pi}\right)^d \int_0^T \int_{\mathbb{R}^d} e^{-v_\alpha t|k|^{\alpha\wedge 2}} e^{-ik\cdot x} d^d k dt$$
$$= \frac{1}{2^d \pi^{\frac{d}{2}} v_\alpha \Gamma(\frac{\alpha\wedge 2}{2})} \frac{2^{d-\alpha\wedge 2}}{|x|^{d-\alpha\wedge 2}} \int_{\frac{|x|^2}{4T}}^\infty u^{\frac{d-\alpha\wedge 2}{2}-1} e^{-u} du$$
$$\leq \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2} v_\alpha \pi^{\frac{d}{2}} \Gamma(\frac{\alpha\wedge 2}{2})} \frac{1}{|x|^{d-\alpha\wedge 2}} \left(\frac{|x|^2}{4T}\right)^{\frac{d-\alpha\wedge 2}{2}} e^{-\frac{|x|^2}{4T}}. \tag{2.2.2}$$

Next, we estimate I_3, for small R, whose value will be determined shortly, we use Assumption 1.2.2 and Taylor expansion to obtain

$$\left| e^{-[1-\hat{D}(k)]t} - e^{-v_\alpha t|k|^{\alpha\wedge 2}} \right| \leq O(1) t\, |k|^{\alpha\wedge 2+\epsilon} e^{-v_\alpha t|k|^{\alpha\wedge 2}}.$$

DOI:10.6814/NCCU202101556

Therefore,

$$
\begin{aligned}
|I_3| &= \left|\left(\frac{1}{2\pi}\right)^d \int_T^\infty \int_{|k|\le R} e^{-ik\cdot x}\left(e^{-[1-\hat{D}(k)]t} - e^{-v_\alpha t|k|^{\alpha\wedge 2}}\right) d^d k dt\right| \\
&\le O(1)\int_T^\infty t\int_{|k|\le R} |k|^{\alpha\wedge 2+\epsilon} e^{-v_\alpha t|k|^{\alpha\wedge 2}} d^d k dt \\
&= O(1)\int_T^\infty t\left(\frac{1}{v_\alpha t}\right)^{\frac{d+\alpha\wedge 2+\epsilon}{\alpha\wedge 2}} \frac{1}{\alpha\wedge 2}\int_0^{v_\alpha t R^{\alpha\wedge 2}} u^{\frac{d+\alpha\wedge 2+\epsilon}{\alpha\wedge 2}-1} e^{-u} du dt \\
&\le O(1)\int_T^\infty t v_\alpha t^{-\frac{d+\alpha\wedge 2+\epsilon}{\alpha\wedge 2}} dt \\
&= O(1) T^{-\frac{d-\alpha\wedge 2+\epsilon}{\alpha\wedge 2}}.
\end{aligned}
$$

Since $T = |x|^{(\alpha\wedge 2)(1-\mu)}$, $\mu = \frac{\frac{\epsilon}{2}}{d-\alpha\wedge 2+\frac{\epsilon}{2}}$, we have

$$
\begin{aligned}
|I_3| &\le O(1)\,|x|^{-(1-\mu)(d-\alpha\wedge 2+\epsilon)} \\
&= O(1)\,|x|^{-(d-\alpha\wedge 2)-\epsilon\frac{d-\alpha\wedge 2}{2(d-\alpha\wedge 2)+\epsilon}}.
\end{aligned} \tag{2.2.3}
$$

Note that

$$
\frac{d-\alpha\wedge 2}{2(d-\alpha\wedge 2)+\epsilon} > 0.
$$

And then, we keep going on estimating I_4. By Assumption 1.2.2, we have

$$
\begin{aligned}
|I_4| &= \left|\left(\frac{1}{2\pi}\right)^d \int_T^\infty \int_{[-\pi,\pi]^d} e^{-ik\cdot x} e^{-[1-\hat{D}(k)]t} I_{\{|k|>R\}} d^d k dt\right| \\
&\le \int_T^\infty \int_{[-\pi,\pi]^d} \left(\frac{1}{2\pi}\right)^d e^{-t\triangle} I_{\{|k|>R\}} d^d k dt \\
&\le O(1)\int_T^\infty e^{-t\triangle} dt = O(1) e^{-T\triangle}.
\end{aligned} \tag{2.2.4}
$$

Finally, we estimate I_5 by substitution $u = v_\alpha T\,|k|^{\alpha\wedge 2}$ and the following inequality

$$
\int_M^\infty t^{\beta-1} e^{-t} dt \le M^\beta e^{-M}.
$$

DOI:10.6814/NCCU202101556

Then we have

$$
\begin{aligned}
|I_5| &= \left| -\left(\frac{1}{2\pi}\right)^d \int_T^\infty \int_{|k|>R} e^{-ik\cdot x - v_\alpha t|k|^{\alpha\wedge 2}} d^d k dt \right| \\
&\le \frac{O(1)}{v_\alpha(\alpha\wedge 2)} \left(\frac{1}{v_\alpha T}\right)^{\frac{d-\alpha\wedge 2}{\alpha\wedge 2}} \int_{v_\alpha T R^{\alpha\wedge 2}}^\infty u^{\frac{d-\alpha\wedge 2}{\alpha\wedge 2}-1} e^{-u} du \\
&\le \frac{O(1)}{v_\alpha(\alpha\wedge 2)} \left(\frac{1}{v_\alpha T}\right)^{\frac{d-\alpha\wedge 2}{\alpha\wedge 2}} \left(v_\alpha T R^{\alpha\wedge 2}\right)^{\frac{d-\alpha\wedge 2}{\alpha\wedge 2}} e^{-v_\alpha T R^{\alpha\wedge 2}} \\
&= \frac{O(1)}{v_\alpha(\alpha\wedge 2)} R^{d-\alpha\wedge 2} e^{-v_\alpha T R^{\alpha\wedge 2}}. \qquad (2.2.5)
\end{aligned}
$$

Hence, we found that (2.2.1) to (2.2.5) are error terms.

Then we complete the proof of Theorem 1.2.4.

DOI:10.6814/NCCU202101556

Chapter 3

Proof of Theorem 1.2.5

For a finite set A, the notation τ_A is the first hitting time of A, which is defined as

$$\tau_A = \min\{n \geq 0 : X_n \in A\}.$$

3.1 Propositions of Capacity

Proposition 3.1.1.

$$\mathbb{P}_x(\tau_A < \infty) = \sum_{y \in A} G^x(y) Es_A(y) = \sum_{y \in \mathbb{Z}^d} G^x(y) Es_A(y).$$

Proof. For the proof, we use an important idea called *the last-visit decomposition* (c.f. [7]). On the event $\{\tau_A \leq \infty\}$, let $\sigma = max\{\ n : S_n \in A\}$ be the moment of last visit to A. The probability that the random walk visits $y \in A$ for exactly k times and then escapes to infinity

DOI:10.6814/NCCU202101556

is

$$\begin{aligned}
&\mathbb{P}_x\Bigg(\text{exactly k visits to y}, S_\sigma = y\Bigg)\\
=&\mathbb{P}_x\Bigg(\{\text{exactly k visits to y}\} \cap \{S_\sigma = y\}\Bigg)\\
=&\mathbb{P}_x\Bigg(\{\text{exactly k visits to y}\} \cap \bigcup_{m=0}^{\infty}\big(\{\text{m visits to y}\} \cap \{\tau_A^+ = \infty\} I_{\{y\in A\}}\big)\Bigg)\\
=&\mathbb{P}_x\Bigg(\{\text{at least k visits to y}\} \cap \{\tau_A^+ = \infty\} I_{\{y\in A\}}\Bigg)\\
=&\mathbb{P}_x\Bigg(\text{at least k visits to y}\Bigg)\,\mathbb{P}_x\Bigg(\{\tau_A^+ = \infty\} I_{\{y\in A\}}\Bigg)\\
=&\mathbb{P}_x\Bigg(\text{at least k visits to y}\Bigg)\, Es_A(y).
\end{aligned}$$

This means that for any $y \in A$ and $k \geq 1$, it holds that

$$\mathbb{P}_x\Bigg(\text{exactly k visits to y}, S_\sigma = y\Bigg) = \mathbb{P}_x\Bigg(\text{at least k visits to y}\Bigg)\, Es_A(y). \tag{3.1.1}$$

Then, summing (3.1.1) over k from 1 to ∞, we obtain

$$\mathbb{P}_x\big(\tau_A < \infty, S_\sigma = y\big) = \sum_{k=1}^{\infty} \mathbb{P}_x\big(N_y \geq k\big) Es_A(y), \text{where } N_y = \text{\# visit to y.}$$

Now, we are going to prove

$$G^x(y) = \sum_{k=1}^{\infty} \mathbb{P}_x\big(N_y \geq k\big).$$

$$\begin{aligned}
G^x(y) &= \mathbb{E}_x\Bigg[\sum_{k=1}^{\infty} I_{\{S_k=y\}}\Bigg] = \mathbb{E}_x(N_y) = \sum_{k=1}^{\infty} k\, \mathbb{P}_x\big(N_y = k\big)\\
&= \sum_{k=1}^{\infty}\Bigg(\sum_{m=1}^{k} 1\Bigg)\mathbb{P}_x\big(N_y = k\big) = \sum_{m=1}^{\infty}\sum_{k=m}^{k} \mathbb{P}_x\big(N_y = k\big) = \sum_{m=1}^{\infty} \mathbb{P}_x\big(N_y \geq m\big).
\end{aligned}$$

So we have

$$\mathbb{P}_x\big(\tau_A < \infty, S_\sigma = y\big) = \sum_{k=1}^{\infty} \mathbb{P}_x\big(N_y \geq k\big)\, Es_A(y) = G^x(y)\, Es_A(y). \tag{3.1.2}$$

DOI:10.6814/NCCU202101556

Then summing (3.1.2) over $y \in A$, we obtain

$$\mathbb{P}_x\big(\tau_A < \infty\big) = \sum_{y \in A} G^x(y)\ Es_A(y).$$

$\square$

Now, we are able to obtain the following useful proposition.

Proposition 3.1.2.

$$cap(A)\ \min_{z \in A} G^x(z)\ \le\ \mathbb{P}_x\big(\tau_A < \infty\big)\ \le\ cap(A)\ \max_{z \in A} G^x(z).$$

Proof. By Proposition 3.1.1, we have

$$\mathbb{P}_x\big(\tau_A < \infty\big) = \sum_{y \in \mathbb{Z}^d} G^x(y) Es_A(y).$$

Then it is easy to understand that

$$\sum_{y \in A} \min_{y \in A} G^x(y)\ Es_A(y)\ \le\ \mathbb{P}_x\big(\tau_A < \infty\big)\ \le\ \sum_{y \in A} \max_{y \in A} G^x(y)\ Es_A(y).$$

Then we get

$$\min_{y \in A} G^x(y)\ cap(A)\ \le\ \mathbb{P}_x\big(\tau_A < \infty\big)\ \le\ \max_{y \in A} G^x(y)\ cap(A).$$

$\square$

DOI:10.6814/NCCU202101556

3.2 Proof of Theorem 1.2.5

Since we know that the volume of a d-dimensional discrete ball $B\left(\frac{R}{2}\right)$ is of order R^d, we have

$$R^d \asymp \left|B\left(\frac{R}{2}\right)\right| = \sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \mathbb{P}_x\left(\tau^+_{B(R)} = \infty\right) + \sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \mathbb{P}_x\left(\tau^+_{B(R)} < \infty\right)$$

Calculating the first summation, we have

$$\begin{aligned}\sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \mathbb{P}_x\left(\tau^+_{B(R)} = \infty\right) &\asymp \sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \sum_{|y|>B\left(\frac{R}{2}\right)} D(y) \asymp \sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \sum_{|y|>B\left(\frac{R}{2}\right)} \frac{1}{|y|^{d+\alpha}} \\ &\asymp \sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \int_{\frac{R}{2}}^{\infty} \frac{r^{d-1}}{r^{d+\alpha}} dr \asymp \sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \frac{1}{R^\alpha} \asymp R^{d-\alpha}\end{aligned}$$

Therefore, we know that the second summation is of order R^d. i.e.

$$\sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \mathbb{P}_x\left(\tau^+_{B(R)} < \infty\right) \asymp R^d.$$

Considering the second summation and by Proposition 3.1.1, we obtain

$$\sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \mathbb{P}_x\left(\tau^+_{B(R)} < \infty\right) = \sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \sum_{y\in B(R)\setminus\{0\}} G^x(y)\, Es_{B(R)}(y)$$

We suppose that $Es_A(y) \asymp \dfrac{1}{|y|^\beta (\ln |y|)^\gamma}$ for some β and γ, and by Theorem 1.2.4, we know that $G(y) \asymp \dfrac{1}{|y|^{d-(\alpha\wedge 2)}}$, where $d > (\alpha \wedge 2)$. We get

$$\sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \mathbb{P}_x\left(\tau^+_{B(R)} < \infty\right) \asymp \sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \sum_{y\in B(R)\setminus\{0\}} \frac{1}{|y|^\beta (\ln |y|)^\gamma} \frac{1}{|x-y|^{d-(\alpha\wedge 2)}} = \sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \sum_{j=1}^{3} A_j,$$

DOI:10.6814/NCCU202101556

where

$$
\begin{aligned}
A_1 &= \sum_{\substack{y:\, |x-y| > \frac{3}{2}|x| \\ y\neq 0}} \frac{1}{|y|^{\beta}(\ln|y|)^{\gamma}} \frac{1}{|x-y|^{d-(\alpha\wedge 2)}} \\
A_2 &= \sum_{\substack{y:\, \frac{3}{2}|x| \geq |x-y| > \frac{5}{4}|x| \\ y\neq 0}} \frac{1}{|y|^{\beta}(\ln|y|)^{\gamma}} \frac{1}{|x-y|^{d-(\alpha\wedge 2)}} \\
A_3 &= \sum_{\substack{y:\, |x-y| \leq \frac{5}{4}|x| \\ y\neq 0}} \frac{1}{|y|^{\beta}(\ln|y|)^{\gamma}} \frac{1}{|x-y|^{d-(\alpha\wedge 2)}}.
\end{aligned}
$$

First, we are going to estimate A_1,

$$
\begin{aligned}
A_1 = \sum_{\substack{y:\, |x-y| > \frac{3}{2}|x| \\ y\neq 0}} \frac{1}{|y|^{\beta}(\ln|y|)^{\gamma}} \frac{1}{|x-y|^{d-(\alpha\wedge 2)}} &\asymp \frac{1}{|x|^{d-(\alpha\wedge 2)}} \int_{|x|}^{R} \frac{r^{d-1}}{r^{\beta}(\ln r)^{\gamma}} dr \\
&\asymp \frac{1}{|x|^{d-(\alpha\wedge 2)}} \begin{cases} \dfrac{R^{d-\beta}}{(\ln R)^{\gamma}}, & d > \beta \\ \dfrac{1}{(\ln|x|)^{\gamma-1}}, & d = \beta, \gamma > 1 \\ \ln\ln R, & d = \beta, \gamma = 1 \\ (\ln R)^{1-\gamma}, & d = \beta, \gamma < 1 \\ \dfrac{|x|^{d-\beta}}{(\ln|x|)^{\gamma}}, & d < \beta. \end{cases}
\end{aligned}
$$

Next, for A_2,

$$
\begin{aligned}
A_2 = \sum_{\substack{y:\, \frac{3}{2}|x| \geq |x-y| > \frac{5}{4}|x| \\ y\neq 0}} \frac{1}{|y|^{\beta}(\ln|y|)^{\gamma}} \frac{1}{|x-y|^{d-(\alpha\wedge 2)}} &\asymp \frac{1}{|x|^{d-(\alpha\wedge 2)}} \int_{\frac{1}{4}|x|}^{\frac{5}{2}|x|} \frac{r^{d-1}}{r^{\beta}(\ln r)^{\gamma}} dr \\
&\asymp \frac{1}{|x|^{d-(\alpha\wedge 2)}} \begin{cases} \dfrac{|x|^{d-\beta}}{(\ln|x|)^{\gamma}}, & d \neq \beta \\ (\ln|x|)^{1-\gamma}, & d = \beta, \gamma \neq 1 \\ \ln\ln|x|, & d = \beta, \gamma = 1. \end{cases}
\end{aligned}
$$

DOI:10.6814/NCCU202101556

Then, for A_3,

$$A_3 = \sum_{\substack{y:\ |x-y| \le \frac{5}{4}|x| \\ y \neq 0}} \frac{1}{|y|^\beta (\ln |y|)^\gamma} \frac{1}{|x-y|^{d-(\alpha\wedge 2)}} \asymp \frac{1}{|x|^{d-(\alpha\wedge 2)}} \int_1^{\frac{9}{4}|x|} \frac{r^{d-1}}{r^\beta (\ln r)^\gamma} dr$$

$$\asymp \frac{1}{|x|^{d-(\alpha\wedge 2)}} \begin{cases} \dfrac{|x|^{d-\beta}}{\ln |x|}, & d > \beta \\ 1, & d = \beta, \gamma > 1 \\ \ln \ln |x|, & d = \beta, \gamma = 1 \\ (\ln |x|)^{1-\gamma}, & d = \beta, \gamma < 1 \\ 1, & d < \beta. \end{cases}$$

Summarizing A_1 to A_3, we obtain that

$$\mathbb{P}_x\big(\tau^+_{B(R)} < \infty\big) \asymp \frac{1}{|x|^{d-(\alpha\wedge 2)}} \begin{cases} \dfrac{R^{d-\beta}}{(\ln R)^\gamma} + \dfrac{|x|^{d-\beta}}{(\ln |x|)^\gamma}, & d > \beta \\ (\ln |x|)^{1-\gamma} + 1, & d = \beta, \gamma > 1 \\ \ln \ln R + \ln \ln |x|, & d = \beta, \gamma = 1 \\ (\ln R)^{1-\gamma} + (\ln |x|)^{1-\gamma}, & d = \beta, \gamma < 1 \\ \dfrac{|x|^{d-\beta}}{(\ln |x|)^\gamma} + 1, & d < \beta. \end{cases} \tag{3.2.1}$$

Then we have

$$\sum_{x \in B\left(\frac{R}{2}\right)\setminus\{0\}} \mathbb{P}_x\big(\tau^+_{B(R)} < \infty\big) \asymp \sum_{x \in B\left(\frac{R}{2}\right)\setminus\{0\}} \frac{1}{|x|^{d-(\alpha\wedge 2)}} \left[\frac{R^{d-\beta}}{(\ln R)^\gamma} + \frac{|x|^{d-\beta}}{(\ln |x|)^\gamma}\right], \text{for } d > \beta,$$

since

$$\sum_{x \in B\left(\frac{R}{2}\right)\setminus\{0\}} \mathbb{P}_x\big(\tau^+_{B(R)} < \infty\big) \asymp R^d.$$

Now, consider that if $\frac{R^{d-\beta}}{(\ln R)^\gamma}$ of the first case is the main term, we get

$$\sum_{x \in B\left(\frac{R}{2}\right)\setminus\{0\}} \mathbb{P}_x\big(\tau^+_{B(R)} < \infty\big) \asymp \sum_{x \in B\left(\frac{R}{2}\right)\setminus\{0\}} \frac{1}{|x|^{d-\alpha\wedge 2}} \frac{R^{d-\beta}}{(\ln R)^\gamma} \asymp \frac{R^{d-\beta}}{(\ln R)^\gamma} \int_1^R \frac{r^{d-1}}{r^{d-\alpha\wedge 2}} dr \asymp \frac{R^{d-\beta+\alpha\wedge 2}}{(\ln R)^\gamma}.$$

DOI:10.6814/NCCU202101556

Then we obtain $\beta = \alpha \wedge 2, \gamma = 0$.
If $\frac{|x|^{d-\beta}}{(\ln|x|)^{\gamma}}$ of the first case is the main term, we get

$$\sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \mathbb{P}_x\big(\tau^+_{B(R)} < \infty\big) \asymp \sum_{x\in B\left(\frac{R}{2}\right)\setminus\{0\}} \frac{1}{|x|^{d-\alpha\wedge 2}} \frac{|x|^{d-\beta}}{(\ln x)^{\gamma}} \asymp \int_1^R \frac{r^{\alpha\wedge 2-\beta}}{(\ln r)^{\gamma}} r^{d-1} dr$$

$$\asymp \begin{cases} \frac{R^{d+\alpha\wedge 2-\beta}}{(\ln R)^{\gamma}}, & d+\alpha\wedge 2-\beta \neq 0 \\ \ln(\ln R), & \gamma = 1, d+\alpha\wedge 2-\beta = 0 \\ (\ln R)^{1-\gamma}, & \gamma < 1, d+\alpha\wedge 2-\beta = 0 \\ c, & \gamma > 1, d+\alpha\wedge 2-\beta = 0. \end{cases}$$

which can be true only in the first case.

Therefore, for both cases, we obtain that $\beta = \alpha \wedge 2, \gamma = 0$.

So,

$$Es_{B(R)}(y) \asymp \frac{1}{|y|^{\alpha\wedge 2}}$$

and

$$cap(B(R)) = \sum_{y\in B(R)\setminus\{0\}} Es_{B(R)}(y) \asymp \sum_{y\in B(R)\setminus\{0\}} \frac{1}{|y|^{\alpha\wedge 2}} \asymp \int_1^R \frac{r^{d-1}}{r^{\alpha\wedge 2}} dr \asymp R^{d-\alpha\wedge 2}.$$

DOI:10.6814/NCCU202101556

Chapter 4

Proof of Theorem 1.2.6

Assume that

$$\sum_{k=1}^{\infty} \frac{cap(A_k)}{2^{(d-\alpha\wedge 2)k}} < \infty,$$

and as we know before from Proposition 3.1.2

$$cap(A) \min_{z\in A} G^x(z) \le \mathbb{P}_x\big(\tau_A < \infty\big) \le cap(A) \max_{z\in A} G^x(z). \tag{4.0.1}$$

Then we have

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbb{P}_0\big(\tau_{A_k} < \infty\big) &\le \sum_{k=1}^{\infty} cap(A_k) \max_{z\in A_k} G(z) \le \sum_{k=1}^{\infty} cap(A_k) \max_{z\in A_k} \left(\frac{C_{\alpha,d}}{|z|^{d-\alpha\wedge 2}}\right) \\ &\le \sum_{k=1}^{\infty} cap(A_k) \left(\frac{2C_{\alpha,d}}{2^{(d-\alpha\wedge 2)k}}\right) < \infty.\end{aligned}$$

By Borel-Cantelli Lemma, we obtain $\mathbb{P}_0\big(\tau_{A_k} < \infty\ i.o.\big) = 0$.

So, $\mathbb{P}_0\big(\tau_A = \infty \text{ eventually}\big) = 1$. $i.e.\mathbb{P}_0\big(\tau_A < \infty\big) < 1$. Hence, A is transient.

This is contradicted to the assumption.

On the other hand, we can write

$$\sum_{k=1}^{\infty} \frac{cap(A_k)}{2^{(d-\alpha\wedge 2)k}} = \sum_{j=1}^{\infty}\sum_{i=0}^{3} \frac{cap(A_{4j+i})}{2^{(d-\alpha\wedge 2)(4j+i)}} = \infty.$$

DOI:10.6814/NCCU202101556

We may assume that $i=0$. Then we have

$$\sum_{j=1}^{\infty} \frac{cap(A_{4j})}{2^{(d-\alpha\wedge 2)(4j)}} = \infty.$$

First, for $j \geq n$, $y \in \partial B(2^{4j-2})$ and by (4.0.1), we have

$$\begin{aligned}\mathbb{P}_y\big(\tau_{A_{4j}} < \infty\big) &\geq cap(A_{4j}) \min_{z\in A_{4j}} G^y(z) \geq cap(A_{4j}) \min_{z\in A_{4j}} \left(\frac{C_{\alpha,d}}{|z-y|^{d-\alpha\wedge 2}}\right) \\ &\geq cap(A_{4j}) \frac{C_1}{2^{4j(d-\alpha\wedge 2)}},\ where\ C_1 = \frac{3}{4} C_{\alpha,d}.\end{aligned}$$

Next, for $z \in \partial B(2^{4j+2})$ and by (4.0.1), we have

$$\begin{aligned}\mathbb{P}_z\big(\tau_{A_{4j}} < \infty\big) &\leq cap(A_{4j}) \max_{y\in A_{4j}} G^y(z) \leq cap(A_{4j}) \max_{z\in A_{4j}} \left(\frac{C_{\alpha,d}}{|z-y|^{d-\alpha\wedge 2}}\right) \\ &\leq cap(A_{4j}) \frac{C_2}{2^{4j(d-\alpha\wedge 2)}},\ where\ C_2 = 3C_{\alpha,d}.\end{aligned}$$

Since we know that

$$\Big\{\tau_{A_{4j}} < \tau_{\partial B(2^{4j+2})}\Big\} \bigcup \Big\{\tau_{\partial B(2^{4j+2})} < \tau_{A_{4j}} < \infty\Big\} = \Big\{\tau_{A_{4j}} < \infty\Big\}.$$

Then for $y \in \partial B(2^{4j+2})$, we get

$$\mathbb{P}_y\big(\tau_{\partial B(2^{4j+2})} < \tau_{A_{4j}} < \infty\big) \leq \sup_{z\in\partial B(2^{4j+2})} \mathbb{P}_z\big(\tau_{A_{4j}} < \infty\big) \leq cap(A_{4j}) \frac{C_2}{2^{4j(d-\alpha\wedge 2)}}.$$

Therefore,

$$\begin{aligned}\mathbb{P}_y\big(\tau_{A_{4j}} < \tau_{\partial B(2^{4j+2})}\big) &\geq \mathbb{P}_y\big(\tau_{A_{4j}} < \infty\big) - \mathbb{P}_y\big(\tau_{\partial B(2^{4j+2})} < \tau_{A_{4j}} < \infty\big) \\ &\geq cap(A_{4j}) \frac{C_1 - C_2}{2^{4j(d-\alpha\wedge 2)}}.\end{aligned}$$

DOI:10.6814/NCCU202101556

Then we obtain

$$\begin{aligned}\mathbb{P}_0\big(\tau_{A_{4j}} > \tau_{\partial B(2^{4j+2})}, \forall j > n\big) &\leq \prod_{j\geq n} \mathbb{P}_0\big(\tau_{A_{4j}} > \tau_{\partial B(2^{4j+2})}\big) \\ &\leq \prod_{j\geq n}\left(1 - cap(A_{4j})\frac{C_1 - C_2}{2^{4j(d-\alpha\wedge 2)}}\right) \\ &\leq \exp\left(-\sum_{j\geq n} cap(A_{4j})\frac{C_1 - C_2}{2^{4j(d-\alpha\wedge 2)}}\right) = 0.\end{aligned}$$

Hence,

$$\mathbb{P}_0\big(\tau_{A_{4j}} < \tau_{\partial B(2^{4j+2})}\ i.o.\big) = 1.$$

Then we have

$$\mathbb{P}_0\big(\tau_A^+ < \tau_{A_{4j}} < \tau_{\partial B(2^{4j+2})}\ i.o.\big) = 1.$$

So, $A \subseteq \mathbb{Z}^d$ is recurrent. In other words, $\mathbb{P}_0\big(\tau_A^+ < \infty\big) = 1$.

DOI:10.6814/NCCU202101556

Appendix A

In this appendix, we prove that for $x \in \mathbb{Z}^d$,

$$D(x) = \frac{h(x)}{\sum_{y \in \mathbb{Z}^d} h(y)} = c_h h(x),$$

where

$$h(x) = \begin{cases} \frac{1 + O(|x|^{-\rho})}{|x|^{d+\alpha}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and

$$c_h = \left[\sum_{y \in \mathbb{Z}^d} h(y) \right]^{-1} < \infty,$$

which satisfies all the assumptions in Section 1.2. It is clearly that $D(x)$ satisfies Assumption 1.2.1.

Then, we prove $D(x)$ satisfies Assumption 1.2.2. First, we show that (a) in Assumption 1.2.2 holds. For $\alpha > 2$, by the Taylor expansion of $1 - \cos(k \cdot x)$ and using the $\mathbb{Z}^d$-symmetry of D, we obtain

$$1 - \hat{D}(k) = \sum_{x \in \mathbb{Z}^d} (1 - cos(k \cdot x)) D(x) = \frac{|k|^2}{2d} \sum_{x \in \mathbb{Z}^d} |x|^2 D(x) + O(1) \sum_{x \in \mathbb{Z}^d} |x|^{-(d+\alpha-\epsilon-2)}$$

holds provided that $\epsilon \in (0, \alpha - 2)$. This proves that $v_\alpha = \frac{\sigma^2}{2d}$, where $\sigma^2 \equiv \sum_{x \in \mathbb{Z}^d} |x|^2 D(x) \in (0, \infty)$.

Then it remains to prove $\alpha < 2$. By definition, we note that for $x \in \mathbb{Z}^d$,

$$D(x) = c_h h(x),$$

DOI:10.6814/NCCU202101556

where

$$c_h = \left[\sum_{y \in \mathbb{Z}^d} h(y) \right]^{-1} < \infty.$$

Taking the Fourier transform yields

$$\begin{aligned} 1 - \hat{D}(k) &= c_h \sum_{x \in \mathbb{Z}^d} (1 - \cos(k \cdot x)) h(x) \\ &= \frac{c_h}{|k|^d} \left[|k|^d \sum_{|y| > |k|} (1 - \cos(e_k \cdot y)) h\left(\frac{y}{|k|} \right) \right], \end{aligned}$$

where $e_k = \frac{k}{|k|}$. As $|k| \to 0$ and by the Riemann sum approximation, we obtain

$$\begin{aligned} 1 - \hat{D}(k) &= \frac{c_h(1 + O(|k|))}{|k|^d} \int_{|y| \geq |k|} (1 - \cos(e_k \cdot y)) h\left(\frac{y}{|k|} \right) d^d y \\ &= \frac{c_h(1 + O(|k|))}{|k|^d} \int_{|y| \geq |k|} (1 - \cos y_1) h\left(\frac{y}{|k|} \right) d^d y \\ &= c_h |k|^\alpha (1 + O(|k|)) \int_{|y| \geq |k|} (1 - \cos y_1) \left(\frac{1}{|y|^{d+\alpha}} + \frac{O(|k|^\rho)}{|y|^{d+\alpha+\rho}} \right) d^d y. \end{aligned}$$

where the second equation holds by symmetry of h.

We note that

$$\int_{|y| \geq |k|} \frac{1 - \cos y_1}{|y|^{d+\alpha}} d^d y = \int_{\mathbb{R}^d} \frac{1 - \cos y_1}{|y|^{d+\alpha}} d^d y - \int_{|y| \leq |k|} \frac{1 - \cos y_1}{|y|^{d+\alpha}} d^d y,$$

where we have used $|k| < 1$ to estimate the error terms. Moreover,

$$\int_{|y| \geq |k|} \frac{1 - \cos y_1}{|y|^{d+\alpha+\rho}} d^d y = \int_{|y| \geq 1} \frac{1 - \cos y_1}{|y|^{d+\alpha+\rho}} d^d y + \int_{|k| \leq |y| \leq 1} \frac{1 - \cos y_1}{|y|^{d+\alpha+\rho}} d^d y,$$

where

$$\int_{|k| \leq |y| \leq 1} \frac{1 - \cos y_1}{|y|^{d+\alpha+\rho}} d^d y = \begin{cases} O(1), & \rho < 2 - \alpha \\ O\left(\log \frac{1}{|k|} \right), & \rho = 2 - \alpha \\ O(|k|^{2-\alpha-\rho}), & \rho > 2 - \alpha. \end{cases}$$

DOI:10.6814/NCCU202101556

This proves (a) in Assumption 1.2.2 with $0 < \epsilon < 1 \wedge (2-\alpha) \wedge \rho$ and

$$v_\alpha = c_h \int_{\mathbb{R}^d} \frac{1-\cos y_1}{|y|^{d+\alpha}} d^d y.$$

Next, to prove the lower bound of $1-\hat{D}(x)$ in (b), we suppose that $\|k\|_\infty > R$ to obtain

$$\begin{aligned}
1-\hat{D}(x) &= \sum_{x\in\mathbb{Z}^d} D(x)(1-\cos(k\cdot x)) = 1 - \sum_{x\in\mathbb{Z}^d} D(x)\cos(k\cdot x)\\
&\geq 1 - \sum_{|x|\leq \frac{\pi}{2R}} D(x)\cos(k\cdot x)\\
&\geq 1 - \sum_{|x|\leq \frac{\pi}{2R}} D(x)\\
&= 1 - \Big[\sum_{x\in\mathbb{Z}^d} D(x) - \sum_{|x|>\frac{\pi}{2R}} D(x)\Big]\\
&= \sum_{|x|>\frac{\pi}{2R}} D(x)\\
&\geq \Delta.
\end{aligned}$$

Similarly, for the upper bound, we have

$$\begin{aligned}
1-\hat{D}(x) &= 1 - \sum_{x\in\mathbb{Z}^d} D(x)\cos(k\cdot x)\\
&\leq 1 - \sum_{|x|\geq \frac{\pi}{2R}} D(x)\cos(k\cdot x)\\
&\leq 1 + \sum_{|x|\geq \frac{\pi}{2R}} D(x)\\
&= 1 + \Big[\sum_{x\in\mathbb{Z}^d} D(x) - \sum_{|x|<\frac{\pi}{2R}} D(x)\Big]\\
&= 2 - \sum_{|x|<\frac{\pi}{2R}} D(x)\\
&\leq 2-\Delta.
\end{aligned}$$

Let

$$\Delta = \min\{\sum_{|x|>\frac{\pi}{2R}} D(x), \sum_{|x|<\frac{\pi}{2R}} D(x)\}.$$

DOI:10.6814/NCCU202101556

Then we have done the proof of Assumption 1.2.2.

Lastly, we show Assumption 1.2.3 as follows. We prove the bound of $\|D^{*n}\|_\infty$ first. By definition, it is trivial when $n = 1$.

For $n = 2$, we let

$$R = \{k \in [-\pi, \pi]^d : |k| \le 1, \hat{D}(k) \ge 0\},$$

so that $\left|\hat{D}(k)\right| = 1 - (1 - \hat{D}(k)) \le e^{-(1-\hat{D}(k))}$ for $k \in R$, and that $0 \le \left|\hat{D}(k)\right| < 1 - \Delta$ for $k \notin R$, due to the bound on $1 - \hat{D}(k)$ in Assumption 1.2.2.

Therefore, for any $x \in \mathbb{Z}^d$,

$$\begin{aligned} D^{*n}(x) &\le \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \left|\hat{D}(k)\right|^n \\ &\le \int_R \frac{d^d k}{(2\pi)^d} e^{-n(1-\hat{D}(k))} + (1-\Delta)^{n-2} \int_{R^c} \frac{d^d k}{(2\pi)^d} \hat{D}(k)^2, \end{aligned}$$

where the integral over $k \in R^c \equiv [-\pi, \pi]^d \setminus R$ is bounded by $\|D\|_\infty (1-\Delta)^{n-2} \le O(1) n^{-\frac{d}{\alpha \wedge 2}}$. For the integral over $k \in R$, we use the bounds on $1 - \hat{D}(k)$ in Assumption 1.2.2. If $\alpha \neq 2$, then

$$\int_R \frac{d^d k}{(2\pi)^d} e^{-n(1-\hat{D}(k))} \le c' \int_0^\infty \frac{dr}{r} r^d e^{-cnr^{\alpha\wedge 2}} = \frac{c' \Gamma(\frac{d}{\alpha \wedge 2})}{(\alpha \wedge 2)(cn)^{\frac{d}{\alpha\wedge 2}}},$$

for some $c, c' \in (0, \infty)$, where $r = |k|$. This completes the proof of the bound on $\|D^{*n}\|_\infty$ in Assumption 1.2.3.

Now, we prove the bound of $D^{*n}(x)$ and seperate it into the following two cases.

(i) the contribution from the walk that have at least one step which is longer than $c\,|x|$ for given $c > 0$ is bounded by $O(1)/|x|^{d+\alpha}$. In this case, $D^{*n}(x) \le \dfrac{O(1)n}{|x|^{d+\alpha}}$.

(ii) the contribution from the walks whose n steps are all shorter than $c\,|x|$ is bounded, due to the local CLT, by $O(\tilde{v}n)^{-\frac{d}{2}} \exp(-\frac{|x|^2}{O(\tilde{v}n)}) \le \frac{O(\tilde{v}n)}{|x|^{d+2}}$ (times an exponentially small normalization constant), where $\tilde{v}$ is the variance of the truncated 1-step distribution $\tilde{D}(y) \equiv D(y) 1_{\{|y| \le c|x|\}}$ and equals

$$\tilde{v} = \sum_{y \in \mathbb{Z}^d} |y|^2 \tilde{D}(y) \le O(1) \begin{cases} |x|^{2-\alpha}, & \alpha < 2 \\ 1, & \alpha > 2. \end{cases}$$

For $\alpha \neq 2$, the inequality is a discrete space-time version of the heat-kernal bound on the

DOI:10.6814/NCCU202101556

transition density $p_s(x)$ of an α-stable / Guassian process:

$$p_s(x) \equiv \int_{\mathbb{R}^d} \left(\frac{1}{2\pi}\right)^d e^{-ik\cdot x - s|k|^{\alpha\wedge 2}} d^d k \leq \frac{O(s)}{|x|^{d+\alpha\wedge 2}}.$$

DOI:10.6814/NCCU202101556

Bibliography

[1] Béla Bollobás. *Random Graphs*. Cambridge University Press, Cambridge, 2001.

[2] Maury Bramson, Ofer Zeitouni, and Martin P. W. Zerner. Shortest spanning trees and a counterexample for random walks in random environments. *The Annals of Probability*, 34(3):821 – 856, 2006.

[3] George Green. *An Essay on the Application of mathematical Analysis to the theories of Electricity and Magnetism*. Nottingham, July 1828.

[4] N. Jain and S. Orey. On the range of random walk. *Israel Journal of Mathematics*, 6(4): 373–380, 1968.

[5] Gregory F. Lawler. *Intersections of random walks / Gregory F. Lawler.* Probability and its applications. Birkhäuser, Boston, 1991.

[6] KARL PEARSON. The problem of the random walk. *Nature*, 72(1865):294–294, 1905.

[7] Serguei Popov. *Two-Dimensional Random Walk: From Path Counting to Random Interlacements*. Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2021.

[8] C E Soteros and S G Whittington. The statistical mechanics of random copolymers. *Journal of Physics A: Mathematical and General*, 37(41):R279–R325, sep 2004.

[9] Michel Talagrand. *Spin glasses : a challenge for mathematicians : cavity and mean field models / Michel Talagrand.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, v. 46. Springer, New York, 2003.

[10] Robert Brown F.R.S. Hon. M.R.S.E. & R.I. Acad. V.P.L.S. Xxvii. a brief account of microscopical observations made in the months of june, july and august 1827, on the

DOI:10.6814/NCCU202101556

particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies. *The Philosophical Magazine*, 4(21):161–173, 1828.

DOI:10.6814/NCCU202101556