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# On the Entropy of Multidimensional Multiplicative Integer Subshifts

**Jung-Chao Ban**[1,2] **· Wen-Guei Hu**[3] **· Guan-Yu Lai**[4]

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**Abstract**
A multiplicative integer subshift $X_\Omega$ derived from the subshift $\Omega$ is invariant under multiplicative integer action, which is closely related to the level set of multiple ergodic average. The complexity of $X_\Omega$ is usually measured by entropy (or box dimension). This work concerns on two types of multi-dimensional multiplicative integer subshifts (MMIS) with different coupling constraints, and then obtains their entropy formulae.

**Keywords** Entropy · Multiplicative integer subshift · Multiple ergodic average · Box dimension

## 1 Introduction

### 1.1 Motivations

Before stating our main theorem, let us now briefly explain the connection between the studies of MMIS and multifractal analysis. Multifractal analysis, which was introduced by

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✉ Wen-Guei Hu
wghu@scu.edu.cn

Jung-Chao Ban
jcban@nccu.edu.tw

Guan-Yu Lai
guanyu.am04g@g2.nctu.edu.tw

1 Department of Mathematical Sciences, National Chengchi University, Taipei 11605, Taiwan, ROC

2 Math. Division, National Center for Theoretical Science, National Taiwan University, Taipei 10617, Taiwan, ROC

3 College of Mathematics, Sichuan University, Chengdu 610064, China

4 Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan, ROC

Mandelbrot's works on multiplicative chaos in 1970's [25], plays a crucial role in physics and mathematics. The multifractal analysis has now become a set of tools applicable in physics, analysis, ergodic theory, fractal geometry, and other sciences.

Given a dynamical system, it is of interest to study the multifractal analysis of the (Birkhoff) ergodic averages. The aim of multifractal analysis of ergodic averages is to calculate the Hausdorff dimensions or topological entropy of the level sets consisting of the points whose limits of ergodic averages are given as levels. We will not attempt to review the extensive literature here, referring only to [5,13,29,30] for background and references.

Motivated by his famous ergodic proof of Szemerédi's Theorem, Furstenberg [20] studied the multiple ergodic averages. Later, many mathematicians including Conze and Lesigne [13], Bourgain [6], Host and Kra [22] and many others have studied the convergence of the multiple ergodic averages. The multifractal analysis of the multiple ergodic averages was initiated by Fan, Liao and Ma [16], which can be described as follows. Let $(X, \rho)$ be a metric space, $\{T_i : X \to X\}_{i=1}^d$ be maps from $X$ to $X$ and $\{f_i : X \to \mathbb{R}\}_{i=1}^d$ be maps from $X$ into $\mathbb{R}$. Let $\mathbb{F} = (f_1, \ldots, f_d)$, the *multiple Birkhoff ergodic averages of* $\mathbb{F}$ are defined by

$$A_n\mathbb{F}(x) = \frac{1}{n}\sum_{k=0}^{n-1} f_1(T_1^k x) f_2(T_2^k x)\cdots f_d(T_d^k x).$$

The *Hausdorff dimension multifractal spectrum of the multiple ergodic averages* with respect to $\mathbb{F}$ is defined by

$$d_H(\alpha) = \dim_H E_{\mathbb{F}}(\alpha),\ \alpha \in \mathbb{R},$$

where

$$E_{\mathbb{F}}(\alpha) = \left\{x \in X : \lim_{n\to\infty} A_n\mathbb{F}(x) = \alpha\right\}. \tag{1}$$

The set $X_2^{(1,2)}$ (defined in next subsection), considered in [16], is a special subset of $E_{\mathbb{F}}(0)$ and of the same Hausdorff dimension as $E_{\mathbb{F}}(0)$. The authors of [16] compute the box dimension of $X_2^{(1,2)}$ and since then, the research on the subject has been very active. Many works focus on the dimension formula of the generalizations of $X_2^{(1,2)}$ (called multiplicative subshifts, see Subsect. 1.2) [4,23,24], the multifractal spectrum $\alpha \mapsto d_H(\alpha)$ [17,18,28] and its relation to the non-linear transfer operators and thermodynamic formalism [31], and the multiplicative Ising model in statistical physics [9,10]. The results related to our works will be presented rigorously in next subsection. However, the problem of multifractal analysis on multiple Birkhoff averages is at present far from being solved.

The *topological entropy* of a dynamical system was introduced by Adler, Konheim and McAndrew [1] in 1965. The topological entropy of any subset $K$ in a topological dynamical system $(X, T)$, denoted by $h_{top}^B(K)$, was firstly introduced by Bowen [7] using the method of spanning sets. Note that the set $K$ is not necessary compact or invariant (see also Section 7.2 of [32] for more details of the definition $h_{top}^B(K)$). In [19], Feng and Huang introduced the *upper capacity topological entropy of $T$ restricted on $K$*, say $h_{top}^{UC}(K)$, which is a generalization of the Adler-Konheim-McAndrew and Bowen topological entropy to arbitrary subset $K$. In the same spirit of the entropy $h_{top}^{UC}(K)$, the concept of entropy for MMIS is also defined by the formula (12) in Subsect. 1.2 and our goal is to establish the entropy formula of the multidimensional version of the multiplicative subshifts, namely, the MMIS. We emphasize that in the symbolic dynamical system $(X, \sigma)$, where X is the space $\{0, 1, \ldots, m-1\}^{\mathbb{N}}$ endowed with the classical metric (see (6) in Subsect. 1.2), and $\sigma$ is the usual shift map, the topological entropy of any subset is exactly its box dimension multiplied by $\log m$. In fact, let $d_n(x, y) := \max\{d(T^k(x), T^k(y)) : k = 0, 1, \ldots, n-1\}$. Then the $(n, \epsilon)$-Bowen balls $B_n(x, \epsilon) := \{y \in X : d_n(x, y) < \epsilon\}$ when $\epsilon > 0$ is small enough. Thus we have $h_{top}^B(K) = \overline{\dim}_B(K) \log m$.

There is a close connection between MMIS and statistical physics. In [9,10], the authors study the multiple ergodic averages (also called nonconventional averages) in the context of lattice spin systems and construct the multiplicative Ising model. More precisely, they define the *multiplicative Ising model* as the lattice spin system on $\{-1, 1\}^{\mathbb{N}}$ (i.e., the lattice spin system with Ising $\pm 1$ spins on $\mathbb{N}$) with Hamiltonian

$$H(\sigma) = -\beta \left( \sum_{i \in \mathbb{N}} J \sigma_i \sigma_{2i} + \sum_{i \in \mathbb{N}} h \sigma_i \right), \tag{2}$$

where $\sigma \in \{-1, 1\}^{\mathbb{N}}$ is a spin configuration. The parameters $\beta$, $J$ and $h$ stand for the inverse temperature, coupling strength and magnetic field, respectively. Meanwhile, the Hamiltonian of the classical Ising model on the lattice $[0, N]$ with boundary condition $\pm 1$ on the right and free on the left on $\{-1, 1\}^N$ is defined by

$$H_N(\sigma_{[0,N]}) = -\beta \left( \sum_{i=0}^{N-1} J \sigma_i \sigma_{i+1} + \sum_{i=0}^{N} h \sigma_i + \sigma_N(\pm 1) \right). \tag{3}$$

In [9], the authors study the thermodynamic limit of the free energy function $\mathcal{F}_p(\lambda) = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_p(e^{\lambda S_N})$ associated to the sum $S_N = \sum_{i=1}^{N} \sigma_i \sigma_{2i}$ by choosing $\mathbb{P}_p$ to be a product of Bernoulli with the parameter $p$ on two symbols $\{+, -\}$. They prove the large deviation principle and central limit theorem therein. Later, the thermodynamic formalism such as existence of pressure and entropies of the multiplicative Ising model are established in [10]. In contrast with the Hamiltonian defined in (3), the multiplicative Ising model is to study the Hamiltonian $\sum_{i=1}^{N} \sigma_i \sigma_{2i}$. It is described in [9]:

> The sum $\sum_{i=1}^{N} \sigma_i \sigma_{i+1}$ is simply a nearest neighbor translation-invariant interaction, whereas the sum $\sum_{i=1}^{N} \sigma_i \sigma_{2i}$ is a long-range non-translation invariant interaction. Therefore, from the point of view of computing partition functions, the Hamiltonian $\sum_{i=1}^{N} \sigma_i \sigma_{2i}$ will be much harder to deal with.

They also mentioned that when the sum is of the type $\sum_{i=1}^{N} \sigma_i \sigma_{2i} \cdots \sigma_{ki}$ for $k > 2$, the dimension of the corresponding lattice spin system is related to the number of primes in $\{2, \ldots, k\}$, e.g., $k = 3$ ($k = 5$) corresponds to a 2-dimensional (3 -dimensional) nearest-neighbor spin system (Sect. 5, [9]). It is seen that the MMIS studied in this paper is a kind of multiplicative Ising model defined in $\mathbb{N}^d$ for $d \geq 1$. Therefore, the present paper could be a starting point for the research on multidimensional multiplicative Ising models in statistic physics.

### 1.2 Main Results

Let $\Sigma_m = \{0, 1, \ldots, m-1\}$ be a finite alphabet, $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{N}_{\geq k} = \{n \in \mathbb{N} : n \geq k\}$. Let $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{k-1} \in \mathbb{N}^d$ and for any two vectors $\mathbf{i}, \mathbf{j} \in \mathbb{N}^d$, we denote the coordinate-wise product vector of $\mathbf{i}$ and $\mathbf{j}$ as $\mathbf{i} \cdot \mathbf{j}$, i.e., $\mathbf{i} \cdot \mathbf{j} = (i_1 j_1, i_2 j_2, \ldots, i_d j_d)$ for $\mathbf{i} = (i_1, i_2, \ldots, i_d)$ and $\mathbf{j} = (j_1, j_2, \ldots, j_d)$. The $(d, k)$-*multidimensional multiplicative integer subshift* $X_{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{k-1}}^{(d,k)} \subseteq \Sigma_m^{\mathbb{N}^d}$ ($(d, k)$-MMIS) is defined as

$$X^{(d,k)}_{\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{k-1}} = \{(x_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d} \in \Sigma_m^{\mathbb{N}^d} : x_{\mathbf{i}} x_{\mathbf{i}\bullet\mathbf{p}_1} \cdots x_{\mathbf{i}\bullet\mathbf{p}_{k-1}} = 0 \text{ for all } \mathbf{i} \in \mathbb{N}^d\}. \tag{4}$$

The motivation for studying the $(d, k)$-MMIS comes from Fan et al. [16], where the authors investigated the box dimension of (4) for $d = 1$, $k = m = 2$ and $p_1 = 2$ (since $d = 1$, $\mathbf{p}_1 = p_1$ is scalar), i.e., they considered

$$X_2^{(1,2)} = \{(x_k)_{k=1}^{\infty} \in \{0, 1\}^{\mathbb{N}} : x_k x_{2k} = 0 \text{ for all } k \in \mathbb{N}\} \tag{5}$$

with the metric

$$\rho(x, y) = 2^{-\min\{k\geq 1: x_k \neq y_k\}} \tag{6}$$

for $x, y \in \{0, 1\}^{\mathbb{N}}$ and obtained the box dimension for $X_2^{(1,2)}$ as

$$\dim_B X_2^{(1,2)} = \frac{1}{2\log 2} \sum_{n=1}^{\infty} \frac{\log F_n}{2^n},$$

where $F_n$ is the Fibonacci sequence with $F_1 = 2$, $F_2 = 3$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{N}$. Subsequently, Kenyon et al. [23,24] noted that (5) can be expressed as

$$X_{\Omega}^{(q)} = \{(x_k)_{k=1}^{\infty} \in \Sigma_m^{\mathbb{N}} : (x_{iq^{\ell}})_{\ell=1}^{\infty} \in \Omega \text{ for all } i, q \nmid i\}, \tag{7}$$

where $q = m = 2$; and $\Omega = \Sigma_G$ is the one-dimensional *golden mean shift*, i.e., the subshift of finite type in $\Sigma_2^{\mathbb{N}_0}$ with forbidden set $\mathcal{F} = \{11\}$. Equation (7) has the same form as (4) for $d = 1$, $k = m = 2$, and $p_1 = q$ when $\Omega = \Sigma_G$. Kenyon et al. also developed the box dimension formula and extended it to general cases. Let $q, m \geq 2$ and $\Omega \subseteq \Sigma_m^{\mathbb{N}_0}$ be a subshift, then

$$\dim_B X_{\Omega}^{(q)} = \frac{(q-1)^2}{q\log m} \sum_{n=1}^{\infty} \frac{\log |\mathrm{Pref}_n(\Omega)|}{q^n}, \tag{8}$$

where $\mathrm{Pref}_n(\Omega) = \{u \in \{0, 1, \ldots, m-1\}^n : \Omega \cap [u] \neq \emptyset\}$. In particular, if $\Omega$ is a shift of finite type with transition matrix $A$, then (8) can also be expressed as

$$\dim_B X_{\Omega}^{(q)} = \frac{(q-1)^2}{q\log m} \sum_{n=1}^{\infty} \frac{\log |A^{n-1}|}{q^n}.$$

Kenyon et al. called (7) a *multiplicative subshift*, since (7) is invariant under multiplicative integer action, i.e., if $(x_k)_{k=1}^{\infty} \in X_{\Omega}^{(q)}$, then $(x_{rk})_{k=1}^{\infty} \in X_{\Omega}^{(q)}$ for all $r \in \mathbb{N}$.

Peres et al. [28] considered the more general case,

$$X_{\Omega}^{(S)} = \{(x_k)_{k=1}^{\infty} \in \Sigma_m^{\mathbb{N}} : x|_{iS} \in \Omega \text{ for all } i \in \mathbb{N}, \gcd(i, S) = 1\}, \tag{9}$$

where $S$ is semigroup generated by primes $p_1, p_2, \ldots, p_k$, and $\gcd(i, S) = 1$ means $\gcd(i, s) = 1$ for all $s \in S$. A typical example of (9) is

$$X_{2,3}^{(1,3)} = \{(x_k)_{k=1}^{\infty} \in \Sigma_2^{\mathbb{N}} : x_k x_{2k} x_{3k} = 0 \text{ for all } k \in \mathbb{N}\}, \tag{10}$$

where $S$ is the semigroup generated by 2 and 3. Equation (10) also has the form of (4) for $d = 1$, $k = 3$, $m = 2$ and $p_1 = 2$, $p_2 = 3$. The box dimension formula for (9) is

$$\dim_B X_{\Omega}^{(S)} = \frac{1}{\log m} \left[\prod_{i=1}^{k} (1 - \frac{1}{p_i})\right] \sum_{n=1}^{\infty} (\frac{1}{\ell_n} - \frac{1}{\ell_{n+1}}) \log |\mathrm{Pref}_n(\Omega)|,$$

where $S = \{1 = \ell_1 < \ell_2 < \ldots\}$. As we have explained in Subsect. 1.1, in the case of symbolic dynamical systems, the entropy is just the box dimension multiplied by $\log m$. Ban et al. [4] obtained entropy for general cases as

$$X^{(1,d+1)}_{p_1,p_2,\ldots,p_d} = \{(x_k)_{k=1}^{\infty} \in \Sigma_2^{\mathbb{N}} : x_k x_{p_1 k} x_{p_2 k} \cdots x_{p_d k} = 0 \text{ for all } k \in \mathbb{N}\},$$

which have the form of (4) for $d = 1$, $k = d + 1$, $m = 2$ and $p_i \geq 2$ for all $i = 1, 2, \ldots, d$. Since all results presented concern about the $(d, k)$-MMIS for $d = 1$, the question arises: what is the entropy (box dimension) of $(d, k)$-MMIS for $d \geq 2$? This paper considers two types of $(d, k)$-MMISs : one with the form (4) for $d \geq 2$; and the other is

$$\bar{X}^{(d,k)}_{\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{k-1}} = \{(x_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d} \in \Sigma_m^{\mathbb{N}^d} : x_{\mathbf{i}} x_{\mathbf{i}\bullet\mathbf{p}_1^j} \cdots x_{\mathbf{i}\bullet\mathbf{p}_{k-1}^j} = 0 \text{ for all } 1 \leq j \leq d \text{ and } \mathbf{i} \in \mathbb{N}^d\}, \tag{11}$$

where $\mathbf{p}_i^j \in \mathbb{N}^d$ is the vector that the $j$-th component is the $j$-th component $(\mathbf{p}_i)_j$ of $\mathbf{p}_i$ and all other components are 1. It is clear that

$$\bar{X}^{(d,k)}_{\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{k-1}} = \bigcap_{j=1}^{d} X^{(d,k)}_{\mathbf{p}_1^j,\mathbf{p}_2^j,\ldots,\mathbf{p}_{k-1}^j},$$

where the multiplicative constraint of $X^{(d,k)}_{\mathbf{p}_1^j,\mathbf{p}_2^j,\ldots,\mathbf{p}_{k-1}^j}$ is only built on the $j$-th components of $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{k-1}$.

***Remark 1.1*** For $d = m = 2$, $k = 3$, $\mathbf{p}_1 = (2, 3)$ and $\mathbf{p}_2 = (5, 7)$,

1. Type I is

$$X^{(d,k)}_{\mathbf{p}_1,\mathbf{p}_2} = \{(x_{(i,j)})_{i,j=1}^{\infty} \in \{0,1\}^{\mathbb{N}^2} : x_{(i,j)} x_{(2i,3j)} x_{(5i,7j)} = 0 \text{ for all } (i, j) \in \mathbb{N}^2\}.$$

2. Type II is

$$\begin{aligned}\bar{X}^{(d,k)}_{\mathbf{p}_1,\mathbf{p}_2} = &\{(x_{(i,j)})_{i,j=1}^{\infty} \in \{0,1\}^{\mathbb{N}^2} : x_{(i,j)} x_{(2i,j)} x_{(5i,j)} = 0 \text{ for all } (i, j) \in \mathbb{N}^2\} \\ &\cap\{(x_{(i,j)})_{i,j=1}^{\infty} \in \{0,1\}^{\mathbb{N}^2} : x_{(i,j)} x_{(i,3j)} x_{(i,7j)} = 0 \text{ for all } (i, j) \in \mathbb{N}^2\}.\end{aligned}$$

3. We consider these two types because their coupling constraints are different, e.g. for type I, the positions in $\mathbb{N}^2$ that will be affected by (1,1) are $\{(2^{\alpha}5^{\beta}, 3^{\alpha}7^{\beta}) : \alpha, \beta \geq 0\}$, whereas $\{(2^{\alpha_1}5^{\beta_1}, 3^{\alpha_2}7^{\beta_2}) : \alpha_1, \beta_1, \alpha_2, \beta_2 \geq 0\}$ are affected for type II (see Fig. 1). Thus the constraints of the Type I MMIS are tighter than those of the Type II MMIS, and hence produce different expressions for entropy. We use these two types to show that different coupling constraints cause different entropy formulae for higher dimensional cases.
4. Both Type I and II have the same form when $d = 1$.

This paper extends the entropy formula of $(d, k)$-MMIS for $d \geq 2$. For simplicity, we only consider $m = 2$.

The *entropy* of a $(d, k)$-MMIS $X$ is defined as

$$h(X) = \limsup_{k,\ell\to\infty} \frac{\log \Gamma_{k\times\ell}(X)}{k\ell}, \tag{12}$$

where $\Gamma_{k\times\ell}(X)$ is the number of admissible patterns on $k \times \ell$ lattice. The results of our study are summarized in the following theorems.

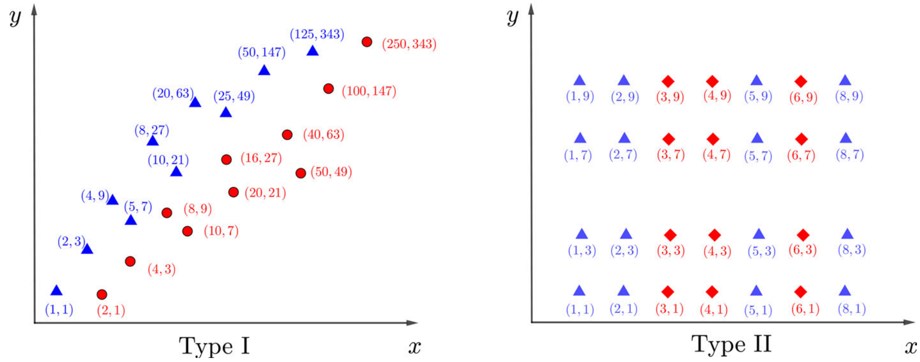


**Fig. 1** The points with symbol triangle are affected by (1, 1); the points with symbol circle are affected by (2, 1); the points with symbol diamond are affected by (3, 1)

**Theorem 1.2** *Let* $\mathbf{p} = (p, q) \in \mathbb{N}^2_{\geq 2}$*, then the entropy of* $X^{(2,2)}_{\mathbf{p}}$ *is*

$$h(X^{(2,2)}_{\mathbf{p}}) = (pq-1)\left(1 - \frac{1}{pq}\right)\sum_{m=1}^{\infty} \frac{\log F_m}{(pq)^m},$$

*where* $F_m$ *is the Fibonacci sequence with* $F_1 = 2$*,* $F_2 = 3$*, and* $F_{m+2} = F_{m+1} + F_m$ *for all* $m \in \mathbb{N}$*.*

When $\mathbf{p} = (2, 3)$, $h(X^{(2,2)}_{\mathbf{p}}) \approx 0.6479$. The following theorem covers the general case for $d \geq 3$.

**Theorem 1.3** *Let* $\mathbf{p} = (p_1, p_2, \ldots, p_d) \in \mathbb{N}^d_{\geq 2}$*, then the entropy of* $X^{(d,2)}_{\mathbf{p}}$ *is*

$$h(X^{(d,2)}_{\mathbf{p}}) = (p_1 p_2 \ldots p_d - 1)\left(1 - \frac{1}{p_1 p_2 \ldots p_d}\right)\sum_{m=1}^{\infty} \frac{\log F_m}{(p_1 p_2 \ldots p_d)^m}.$$

We define $\{\mathbf{e}_i : 1 \leq i \leq d\}$ as the standard basis for $\mathbb{R}^d$, for example, $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ for $d = 2$.

**Theorem 1.4** *Let* $\mathbf{p}_1 = (p_1, p_2)$ *and* $\mathbf{p}_2 = (q_1, q_2) \in \mathbb{N}^2_{\geq 2}$ *with* $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$*, then the entropy of* $X^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}$ *is*

$$\begin{aligned} h(X^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}) = &\left(1 - \frac{1}{p_1 p_2}\right)\left(1 - \frac{1}{q_1 q_2}\right) \times \\ &\sum_{M,N=1}^{\infty} \left(\frac{1}{r_M} - \frac{1}{r_{M+1}}\right)\left(\frac{1}{r'_N} - \frac{1}{r'_{N+1}}\right)\log b_{M,N}, \end{aligned}$$

*where* $b_{M,N}$ *is the number of admissible patterns on* $\mathbb{L}_{M,N}$ *lattice in* $\mathbb{N}^2_0$ *with forbidden set* $\mathcal{F} = \{x_{\mathbf{0}} = x_{\mathbf{e}_1} = x_{\mathbf{e}_2} = 1\}$ *(see Definition 2.5 for definitions of* $\mathbb{L}_{M,N}$ *and* $r_M, r'_M$*).*

When $\mathbf{p}_1 = (2, 3)$ and $\mathbf{p}_2 = (5, 7)$, $h(X^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}) \approx 0.5353$. The following theorem covers the general case for $d \geq 2$ and $k \geq 2$.

**Theorem 1.5** *For $k \geq 2$, let $\mathbf{p}_i = (p_{i,1}, p_{i,2}, \ldots, p_{i,d}) \in \mathbb{N}^d_{\geq 2}$, $1 \leq i \leq k-1$. Assume $\gcd(p_{i,\ell}, p_{j,\ell}) = 1$ for all $1 \leq i < j \leq k-1$ and $1 \leq \ell \leq d$, then the entropy of $X^{(d,k)}_{\mathbf{p}_1,\mathbf{p}_2,\cdots,\mathbf{p}_{k-1}}$ is*

$$h(X^{(d,k)}_{\mathbf{p}_1,\mathbf{p}_2}) = \left[\prod_{i=1}^{k-1}\left(1 - \frac{1}{p_{i,1}p_{i,2}\cdots p_{i,d}}\right)\right] \times \sum_{M_1,M_2,\ldots,M_d=1}^{\infty}\left[\prod_{i=1}^{d}\left(\frac{1}{r^{(i)}_{M_i}} - \frac{1}{r^{(i)}_{M_i+1}}\right)\right]\log b_{M_1,M_2,\ldots,M_d},$$

*where $b_{M_1,M_2,\ldots,M_d}$ is the number of admissible patterns on the lattice $\mathbb{L}_{M_1,M_2,\ldots,M_d}$ in $\mathbb{N}^{k-1}_0$ with forbidden set $\mathcal{F} = \{x_{\mathbf{0}} = x_{\mathbf{e}_1} = x_{\mathbf{e}_2} = \cdots = x_{\mathbf{e}_{k-1}} = 1\}$ (see Definition 2.6 for definitions of $\mathbb{L}_{M_1,M_2,\ldots,M_d}$ and $r^{(i)}_{M_i+1}$).*

We develop the entropy formula for (11) as follows.

**Theorem 1.6** *Let $\mathbf{p} = (p, q) \in \mathbb{N}^2_{\geq 2}$, then the entropy of $\bar{X}^{(2,2)}_{\mathbf{p}}$ is*

$$h(\bar{X}^{(2,2)}_{\mathbf{p}}) = (p-1)(q-1)(1-\frac{1}{p})(1-\frac{1}{q})\sum_{m,n=1}^{\infty}\frac{\log a_{m,n}}{p^m q^n},$$

*where $a_{m,n}$ is the number of admissible patterns on $m \times n$ lattice in $\mathbb{N}^2_0$ with forbidden set $\mathcal{F} = \{x_{\mathbf{0}} = x_{\mathbf{e}_1} = 1, x_{\mathbf{0}} = x_{\mathbf{e}_2} = 1\}$.*

When $\mathbf{p} = (2, 3)$, $h(\bar{X}^{(2,2)}_{\mathbf{p}}) \approx 0.5212$. The following theorem covers the general case for $d \geq 3$ and $k = 2$.

**Theorem 1.7** *Let $\mathbf{p} = (p_1, p_2, \ldots, p_d) \in \mathbb{N}^d_{\geq 2}$. The entropy of $\bar{X}^{(d,2)}_{\mathbf{p}_1}$ is*

$$h(\bar{X}^{(d,2)}_{\mathbf{p}_1}) = \left[\prod_{i=1}^{d}(p_i - 1)(1 - \frac{1}{p_i})\right]\sum_{m_1,m_2,\ldots,m_d=1}^{\infty}\frac{\log a_{m_1,m_2,\ldots,m_d}}{{p_1}^{m_1}{p_2}^{m_2}\cdots{p_d}^{m_d}},$$

*where $a_{m_1,m_2,\ldots,m_d}$ is the number of admissible patterns on $m_1 \times m_2 \times \cdots \times m_d$ lattice in $\mathbb{N}^d_0$ with forbidden set $\mathcal{F} = \{x_{\mathbf{0}} = x_{\mathbf{e}_1} = 1, x_{\mathbf{0}} = x_{\mathbf{e}_2} = 1, \ldots, x_{\mathbf{0}} = x_{\mathbf{e}_d} = 1\}$.*

For $d = 2$ and $k = 3$, we have the following theorem.

**Theorem 1.8** *Let $\mathbf{p}_1 = (p_1, p_2)$ and $\mathbf{p}_2 = (q_1, q_2) \in \mathbb{N}^d_{\geq 2}$ with $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$. The entropy of $X^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}$ is*

$$h(X^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}) = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{q_1}\right)\left(1 - \frac{1}{q_2}\right) \times \sum_{M,N=1}^{\infty}\left(\frac{1}{r_M} - \frac{1}{r_{M+1}}\right)\left(\frac{1}{r'_N} - \frac{1}{r'_{N+1}}\right)\log c_{M,N},$$

*where $c_{M,N}$ is the number of admissible patterns on $\bar{\mathbb{L}}_{M,N}$ in $\mathbb{N}^4_0$ with forbidden set $\mathcal{F} = \{x_{\mathbf{0}} = x_{\mathbf{e}_1} = x_{\mathbf{e}_2} = 1, x_{\mathbf{0}} = x_{\mathbf{e}_3} = x_{\mathbf{e}_4} = 1\}$ (see Definition 3.7 for definitions of $\bar{\mathbb{L}}_{M,N}$ and $r_M, r'_M$).*

When $\mathbf{p}_1 = (2,3)$ and $\mathbf{p}_2 = (5,7)$, $h(\bar{X}^{(2,2)}_{\mathbf{p}_1}) \approx 0.4077$. The following theorem covers the general case for $d \geq 2$ and $k \geq 2$.

**Theorem 1.9** *For $k \geq 2$, let $\mathbf{p}_i = (p_{i,1}, p_{i,2}, \ldots, p_{i,d}) \in \mathbb{N}^d_{\geq 2}$, $1 \leq i \leq k-1$. Assume $\gcd(p_{i,\ell}, p_{j,\ell}) = 1$ for all $1 \leq i < j \leq k-1$ and $1 \leq \ell \leq d$, then the entropy of $\bar{X}^{(d,k)}_{\mathbf{p}_1,\mathbf{p}_2,\cdots,\mathbf{p}_{k-1}}$ is*

$$h(\bar{X}^{(d,k)}_{\mathbf{p}_1,\mathbf{p}_2}) = \left[\prod_{i=1}^{k-1}\prod_{\ell=1}^{d}(1-\frac{1}{p_{i,\ell}})\right] \times \sum_{M_1,M_2,\ldots,M_d=1}^{\infty}\left[\prod_{i=1}^{d}(\frac{1}{r^{(i)}_{M_i}} - \frac{1}{r^{(i)}_{M_i+1}})\right]\log c_{M_1,M_2,\ldots,M_d},$$

*where $c_{M_1,M_2,\ldots,M_d}$ is the number of admissible patterns on $\bar{\mathbb{L}}_{M_1,M_2,\ldots,M_d}$ in $\mathbb{N}_0^{(k-1)d}$ with forbidden set*

$$\mathcal{F} = \{x_{\mathbf{0}} = x_{\mathbf{e}_j} = x_{\mathbf{e}_{j+1}} = \cdots x_{\mathbf{e}_{j+k-2}} = 1, j = m(k-1)+1, 0 \leq m \leq d-1\}$$

*(see Definition 3.9 for definitions of $\bar{\mathbb{L}}_{M_1,M_2,\ldots,M_d}$ and $r^{(i)}_{M_i}$).*

Although Theorems 1.2 to 1.9 provide some entropy formulae for $(d,k)$-MMISs, exact values are extremely difficult to calculate because we need to calculate the number of admissible patterns on their underlying lattices for a multidimensional SFT $\Omega$ [2,3,8,11,12,15,21,26,27,33].

***Remark 1.10*** 1. The entropy formula in Theorem 1.5 for $d = 2$ and $k = 2$ is

$$h(X^{(2,2)}_{\mathbf{p}_1}) = \left(1 - \frac{1}{p_{1,1}p_{1,2}}\right)\sum_{M_1,M_2=1}^{\infty}\left[\prod_{i=1}^{2}\left(\frac{1}{p_{1,i}^{M_i-1}} - \frac{1}{p_{1,i}^{M_i}}\right)\right]\log F_{\min\{M_1,M_2\}}.$$

By taking $m = \min\{M_1, M_2\}$, it can be shown that

$$h(X^{(2,2)}_{\mathbf{p}_1}) = \left(1 - \frac{1}{p_{1,1}p_{1,2}}\right)\sum_{m=1}^{\infty}\left(\frac{1}{(p_{1,1}p_{1,2})^{m-1}} - \frac{1}{(p_{1,1}p_{1,2})^{m}}\right)\log F_m,$$

which coincides with the entropy formula in Theorem 1.2. Similarly, the entropy formula in Theorem 1.5 (or Theorem 1.9) for $k = 2$ can be simplified to that in Theorem 1.3 (or Theorem 1.7), respectively.

2. In Theorem 1.4, we have the condition $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$. Our method is highly dependent on a 'nice' partition of $\mathbb{N}^2$ as in Lemma 2.4, whose elements $\mathcal{M}_{\mathbf{p},\mathbf{q}}(i,j)$, $(i,j) \in \mathcal{I}_{\mathbf{p},\mathbf{q}}$, (see Definition 2.3) are not affected by each other under the coupling constraint $x_{(i,j)}x_{(ip_1,jq_1)}x_{(ip_2,jq_2)} = 0$ and have some structural similarities inside. Without the condition, the partition in Lemma 2.4 fails, for example, when $\mathbf{p}_1 = (2,3)$ and $\mathbf{p}_2 = (2,5)$, it can be checked that $\mathcal{M}_{\mathbf{p},\mathbf{q}}(i,j) \cap \mathcal{M}_{\mathbf{p},\mathbf{q}}(i',j') \neq \emptyset$ for $(i,j) = (2m-1, 3^n)$ and $(i',j') = (2m-1, 5^n)$ in $\mathcal{I}_{\mathbf{p},\mathbf{q}}$, $m,n \geq 1$. Therefore, our approach cannot work in this circumstances, and further study is needed for obtaining a nice partition and entropy formula. The situation for Theorem 1.8 is similar.

The remainder of this paper is organized as follows. Sections 2 and 3 collect the necessary materials to calculate the entropies of (4) and (11). Section 2 also provides proofs for Theorems 1.2 and 1.4. Since the proofs for the general cases (Theorems 1.3 and 1.5) are similar to those for Theorems 1.2 and 1.4, respectively, we provide the necessary definitions rather than their detail proofs. Section 3 provides detailed proofs for Theorems 1.6 and 1.8, and we similarly omit the proofs for Theorems 1.7 and 1.9.

## 2 Proofs for Theorems 1.2 and 1.4

The following definitions and lemmas are required to prove Theorem 1.2. Given $p, q \geq 2$, let $\mathcal{M}_{p,q} = \{(p^m, q^m) : m \geq 0\}$ be a subset of $\mathbb{N}^2$, denoting $\mathcal{M}_{p,q}(i, j) = \{(ip^m, jq^m) : m \geq 0\}$ as the lattice $\mathcal{M}_{p,q}$ starts at $(i, j)$, and let $\mathcal{I}_{p,q} = \{(i, j) : p \nmid i \text{ or } q \nmid j\}$ be the complementary index set of $X_{\mathbf{p}}^{(2,2)}$.

The idea of proofs of Theorems 1.2 and 1.4 can be divided into three parts.

(I) Identify the partition of $\mathbb{N}^2$ by index set $\mathcal{I}_{p,q}$ and lattices $\mathcal{M}_{p,q}$ (Lemmas 2.1, 2.4, 3.2 and 3.6). Then for an admissible global pattern $U \in X_{\mathbf{p}}^{(2,2)}$ the restriction of $U$ on $\mathcal{M}_{p,q}(i, j)$ (denoted by $U|_{\mathcal{M}_{p,q}(i,j)}$) and the restriction of $U$ on $\mathcal{M}_{p,q}(i', j')$ (denoted by $U|_{\mathcal{M}_{p,q}(i',j')}$) are independent for all $(i, j) \neq (i', j') \in \mathcal{I}_{p,q}$.

(II) Compute the density limit for independent lattices $\mathcal{M}_{p,q}(i, j)$ with the same size in the $k \times \ell$ lattice (Lemmas 2.2, 2.7, 3.4 and 3.8).

(III) Determine the set of all admissible patterns on $\mathcal{M}_{p,q}(i, j)$ with size $m \geq 1$ and compute their numbers e.g. $F_m$ in Theorem 1.2.

The following lemma shows that $\mathcal{M}_{p,q}(i, j)$ forms a partition of $\mathbb{N}^2$.

**Lemma 2.1** *For $p, q \geq 2$,*

$$\mathbb{N}^2 = \coprod_{(i,j)\in\mathcal{I}_{p,q}} \mathcal{M}_{p,q}(i, j).$$

***Proof*** We first claim that for all $(i, j) \neq (i', j') \in \mathcal{I}_{p,q}$, $\mathcal{M}_{p,q}(i, j) \cap \mathcal{M}_{p,q}(i', j') = \emptyset$. Suppose this did not hold. Then there exist $(i, j) \neq (i', j') \in \mathcal{I}_{p,q}$ such that $\mathcal{M}_{p,q}(i, j) \cap \mathcal{M}_{p,q}(i', j') \neq \emptyset$. Since $(i, j) \neq (i', j')$, then there exist $m_1 \neq m_2 \geq 0$ such that $(ip^{m_1}, jq^{m_1}) = (i'p^{m_2}, j'q^{m_2})$. Without loss of generality we assume $m_1 > m_2$, then $ip^{m_1-m_2} = i'$ and $jq^{m_1-m_2} = j'$ gives $p|i'$ and $q|j'$, which contradict $(i', j') \in \mathcal{I}_{p,q}$. It remains to show that the equality holds. For $(i, j) \in \mathbb{N}^2$, then $i = i'p^{\alpha}$ and $j = j'q^{\beta}$ for some $p \nmid i', q \nmid j'$ and $\alpha, \beta \geq 0$. Take $\gamma = \min\{\alpha, \beta\}$, then $(\frac{i}{p^\gamma}, \frac{j}{q^\gamma}) \in \mathcal{I}_{p,q}$, $(i, j) \in \mathcal{M}_{p,q}(\frac{i}{p^\gamma}, \frac{j}{q^\gamma})$, and converse is clear. □

For part (II), we need more definitions to characterize the partition in $k \times \ell$ lattice. For $k, \ell \geq 1$, let $\mathcal{N}_{k\times\ell} = \{(i, j) : 1 \leq i \leq k, 1 \leq j \leq \ell\}$ be a $k \times \ell$ lattice and $\mathcal{L}_{k\times\ell}(i, j) = \mathcal{M}_{p,q}(i, j) \cap \mathcal{N}_{k\times\ell}$ be the subset of $\mathcal{M}_{p,q}(i, j)$ in $k \times \ell$ lattice. For $m \geq 1$, define $\mathcal{J}_{k\times\ell;m} = \{(i, j) \in \mathcal{N}_{k\times\ell} : |\mathcal{L}_{k\times\ell}(i, j)| = m\}$ as the set of points such that $\mathcal{M}_{p,q}(i, j)$ in $k \times \ell$ lattice with length exactly $m$, where $|\cdot|$ denotes cardinal numbers. Let $\mathcal{K}_{k\times\ell;m} = \{(i, j) \in \mathcal{I}_{p,q} \cap \mathcal{N}_{k\times\ell} : |\mathcal{L}_{k\times\ell}(i, j)| = m\}$ denote the set of points in $\mathcal{I}_{p,q}$ that $\mathcal{M}_{p,q}(i, j)$ in $k \times \ell$ lattice with length exactly $m$. In the following lemma, we compute the limit of density of $\mathcal{K}_{k\times\ell;m}$.

**Lemma 2.2** *For $k, \ell$, and $m \geq 1$, we have the following assertions.*

$$1. |\mathcal{J}_{k\times\ell;m}| = \left\lfloor \frac{k}{p^{m-1}} \right\rfloor \left\lfloor \frac{\ell}{q^{m-1}} \right\rfloor - \left\lfloor \frac{k}{p^m} \right\rfloor \left\lfloor \frac{\ell}{q^m} \right\rfloor.$$

2. $\lim_{k,\ell\to\infty} \dfrac{|\mathcal{K}_{k\times\ell;m}|}{|\mathcal{J}_{k\times\ell;m}|} = 1 - \dfrac{1}{pq}$.

3. $\lim_{k,\ell\to\infty} \dfrac{1}{k\ell} \sum_{m=1}^{k\ell} |\mathcal{K}_{k\times\ell;m}| \log F_m = \sum_{m=1}^{\infty} \lim_{k,\ell\to\infty} \dfrac{1}{k\ell} |\mathcal{K}_{k\times\ell;m}| \log F_m$.

***Proof*** **1.** Since $|\mathcal{L}_{k\times\ell}(i,j)| = m$, then by definition we have $\mathcal{J}_{k\times\ell;m} = \{(i,j) : ip^{m-1} \le k \text{ and } jq^{m-1} \le \ell\} \cap \{(i,j) : ip^m > k \text{ or } jq^m > \ell\}$.

Thus, there are three disjoint cases to consider:

I. $\{(i,j) : ip^{m-1} \le k \text{ and } jq^{m-1} \le \ell\} \cap \{(i,j) : ip^m > k \text{ and } jq^m > \ell\}$;
II. $\{(i,j) : ip^{m-1} \le k \text{ and } jq^{m-1} \le \ell\} \cap \{(i,j) : ip^m > k \text{ and } jq^m \le \ell\}$;
III. $\{(i,j) : ip^{m-1} \le k \text{ and } jq^{m-1} \le \ell\} \cap \{(i.j) : ip^m \le k \text{ and } jq^m > \ell\}$.

Then,

$$
\begin{aligned}
|\mathcal{J}_{k\times\ell;m}| &= |\{(i,j) \in \mathcal{N}_{k\times\ell} : |\mathcal{L}_{k\times\ell}(i,j)| = m\}| \\
&= \underbrace{|\{(i,j) : (\frac{k}{p^m} < i \le \frac{k}{p^{m-1}}) \text{ and } (\frac{\ell}{q^m} < j \le \frac{\ell}{q^{m-1}})\}|}_{\text{I}} \\
&+ \underbrace{|\{(i,j) : (\frac{k}{p^m} < i \le \frac{k}{p^{m-1}}) \text{ and } (j \le \frac{\ell}{q^m})\}|}_{\text{II}} \\
&+ \underbrace{|\{(i,j) : (i \le \frac{k}{p^m}) \text{ and } (\frac{\ell}{q^m} < j \le \frac{\ell}{q^{m-1}})\}|}_{\text{III}} \\
&= \left(\left\lfloor \frac{k}{p^{m-1}} \right\rfloor - \left\lfloor \frac{k}{p^m} \right\rfloor\right)\left(\left\lfloor \frac{\ell}{q^{m-1}} \right\rfloor - \left\lfloor \frac{\ell}{q^m} \right\rfloor\right) \\
&+ \left(\left\lfloor \frac{k}{p^{m-1}} \right\rfloor - \left\lfloor \frac{k}{p^m} \right\rfloor\right)\left\lfloor \frac{\ell}{q^m} \right\rfloor + \left\lfloor \frac{k}{p^m} \right\rfloor\left(\left\lfloor \frac{\ell}{q^{m-1}} \right\rfloor - \left\lfloor \frac{\ell}{q^m} \right\rfloor\right) \\
&= \left\lfloor \frac{k}{p^{m-1}} \right\rfloor \left\lfloor \frac{\ell}{q^{m-1}} \right\rfloor - \left\lfloor \frac{k}{p^m} \right\rfloor \left\lfloor \frac{\ell}{q^m} \right\rfloor.
\end{aligned}
$$

**2.** For $m_2 > m_1 \ge 1$ and $n_2 > n_1 \ge 1$, let

$$
\mathcal{R}_{m_1,m_2;n_1,n_2} = \left\{(i,j) \in \mathbb{N}^2 : m_1 \le i \le m_2 \text{ and } n_1 \le j \le n_2\right\}
$$

be a rectangle lattice. Clearly, the complement of $\mathcal{I}_{p,q}$ is $\mathcal{I}_{p,q}^c = \{(i,j) : p \mid i \text{ and } q \mid j\}$ and

$$
|\mathcal{R}_{m_1,m_2;n_1,n_2} \cap \mathcal{I}_{p,q}| = |\mathcal{R}_{m_1,m_2;n_1,n_2}| - |\mathcal{R}_{m_1,m_2;n_1,n_2} \cap \mathcal{I}_{p,q}^c|.
$$

Thus,

$$
|\mathcal{R}_{m_1,m_2;n_1,n_2} \cap \mathcal{I}_{p,q}| \ge |\mathcal{R}_{m_1,m_2;n_1,n_2}| - \frac{1}{pq} |\mathcal{R}_{m_1,m_2+2p;n_1,n_2+2q}|
$$

and

$$
|\mathcal{R}_{m_1,m_2;n_1,n_2} \cap \mathcal{I}_{p,q}| \le |\mathcal{R}_{m_1,m_2;n_1,n_2}| - \frac{1}{pq} |\mathcal{R}_{m_1,m_2-2p;n_1,n_2-2q}|.
$$

Then, by the Squeeze theorem,

$$\lim_{\substack{m_2-m_1\to\infty\\ n_2-n_1\to\infty}} \frac{|\mathcal{R}_{m_1,m_2;n_1,n_2}\cap\mathcal{I}_{p,q}|}{|\mathcal{R}_{m_1,m_2;n_1,n_2}|} = 1-\frac{1}{pq}.$$

Since the regions I, II, and III are rectangular, and both the length and width of each of the three rectangles approach $\infty$ as $k, l \to \infty$. Therefore,

$$\begin{aligned}\lim_{k,\ell\to\infty}\frac{|\mathcal{K}_{k\times\ell;m}|}{|\mathcal{J}_{k\times\ell;m}|} &= \lim_{k,\ell\to\infty}\frac{|\mathcal{J}_{k\times\ell;m}\cap\mathcal{I}_{p,q}|}{|\mathcal{J}_{k\times\ell;m}|}\\ &= 1-\frac{1}{pq}.\end{aligned}$$

**3.** Define

$$\bar{K}_{k\times\ell;m} = \begin{cases} |\mathcal{K}_{k\times\ell;m}| & \text{if } m \le k\ell,\\ 0 & \text{if } m > k\ell.\end{cases}$$

Then

$$\lim_{k,\ell\to\infty}\frac{1}{k\ell}\sum_{m=1}^{k\ell}|\mathcal{K}_{k\times\ell;m}|\log F_m = \lim_{k,\ell\to\infty}\frac{1}{k\ell}\sum_{m=1}^{\infty}\bar{K}_{k\times\ell;m}\log F_m.$$

Hence from Weierstrass M-test with

$$\begin{aligned}\frac{1}{k\ell}\left|\bar{K}_{k\times\ell;m}\log F_m\right| &\le \tfrac{1}{k\ell}|\mathcal{J}_{k\times\ell;m}|\log F_m\\ &= \tfrac{1}{kl}\left(\left\lfloor\tfrac{k}{p^{m-1}}\right\rfloor\left\lfloor\tfrac{\ell}{q^{m-1}}\right\rfloor - \left\lfloor\tfrac{k}{p^{m}}\right\rfloor\left\lfloor\tfrac{\ell}{q^{m}}\right\rfloor\right)\log F_m\\ &\le \tfrac{1}{k\ell}\left(\tfrac{k\ell}{p^{m-1}q^{m-1}}\right)\log F_m\\ &= \tfrac{1}{(pq)^{m-1}}\log F_m\end{aligned}$$

for all $k, \ell \in \mathbb{N}$ and $\displaystyle\sum_{m=1}^{\infty}\frac{\log F_m}{(pq)^{m-1}} < \infty$, we deduce that $\displaystyle\sum_{m=1}^{\infty}\frac{\bar{K}_{k\times\ell;m}\log F_m}{k\ell}$ converges uniformly in $k, \ell$. Then,

$$\begin{aligned}\lim_{k,\ell\to\infty}\frac{1}{k\ell}\sum_{m=1}^{k\ell}|\mathcal{K}_{k\times\ell;m}|\log F_m &= \lim_{k,\ell\to\infty}\frac{1}{k\ell}\sum_{m=1}^{\infty}\bar{K}_{k\times\ell;m}\log F_m\\ &= \sum_{m=1}^{\infty}\lim_{k,\ell\to\infty}\frac{1}{k\ell}\bar{K}_{k\times\ell;m}\log F_m\\ &= \sum_{m=1}^{\infty}\lim_{k,\ell\to\infty}\frac{1}{k\ell}|\mathcal{K}_{k\times\ell;m}|\log F_m.\end{aligned}$$

The proof is complete.

□

For part (III), from Lemma 2.1, $(i,j) \neq (i',j') \in \mathcal{I}_{p,q}$, $X_{\mathbf{p}}^{(2,2)}|_{\mathcal{M}_{p,q}(i,j)}$ and $X_{\mathbf{p}}^{(2,2)}|_{\mathcal{M}_{p,q}(i',j')}$ are independent, which means there are no restrictions on each other. Then,

$$\Gamma_{k\times\ell}(X_{\mathbf{p}}^{(2,2)}) = |X_{\mathbf{p}}^{(2,2)}|_{\mathcal{N}_{k\times\ell}}| = \prod_{(i,j)\in\mathcal{I}_{p,q}\cap\mathcal{N}_{k\times\ell}} |X_{\mathbf{p}}^{(2,2)}|_{\mathcal{M}_{p,q}(i,j)\cap\mathcal{N}_{k\times\ell}}|.$$

Clearly, $X_{\mathbf{p}}^{(2,2)}|_{\mathcal{M}_{p,q}(i,j)\cap\mathcal{N}_{k\times\ell}}$ are independent and $|\mathcal{M}_{p,q}(i,j)\cap\mathcal{N}_{k\times\ell}| = m$ for all $(i,j) \in \mathcal{K}_{k\times\ell;m}$. Therefore, we have

$$\Gamma_{k\times\ell}(X_{\mathbf{p}}^{(2,2)}) = \prod_{m=1}^{k\ell} F_m^{|\mathcal{K}_{k\times\ell;m}|},$$

which is crucial in proving Theorem 1.2.

***Proof for Theorem 1.2*** Since

$$\begin{aligned} h(X_{\mathbf{p}}^{(2,2)}) &= \limsup_{k,\ell\to\infty} \frac{1}{k\ell} \log \Gamma_{k\times\ell}(X_{\mathbf{p}}^{(2,2)}) \\ &= \limsup_{k,\ell\to\infty} \frac{1}{k\ell} \log \prod_{m=1}^{k\ell} F_m^{|\mathcal{K}_{k\times\ell;m}|} \\ &= \limsup_{k,\ell\to\infty} \frac{1}{k\ell} \sum_{m=1}^{k\ell} |\mathcal{K}_{k\times\ell;m}| \log F_m. \end{aligned} \tag{13}$$

Combining Lemma 2.2 and (13), the limit for (13) exists, and

$$\begin{aligned} h(X_{\mathbf{p}}^{(2,2)}) &= \sum_{m=1}^{\infty} \lim_{k,\ell\to\infty} \frac{1}{k\ell} |\mathcal{K}_{k\times\ell;m}| \log F_m \\ &= \sum_{m=1}^{\infty} \lim_{k,\ell\to\infty} \frac{1}{k\ell} |\mathcal{J}_{k\times\ell;m}| (1 - \frac{1}{pq}) \log F_m \\ &= \sum_{m=1}^{\infty} \frac{1}{p^m} \frac{1}{q^m} (pq-1)(1-\frac{1}{pq}) \log F_m \\ &= (pq-1)(1-\frac{1}{pq}) \sum_{m=1}^{\infty} \frac{\log F_m}{(pq)^m}. \end{aligned} \tag{14}$$

The proof is complete. □

The proof for Theorem 1.4 proceeds similarly, as follows.

**Definition 2.3** Let $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2) \in \mathbb{N}^2_{\geq 2}$, then

1. $\mathcal{I}_{\mathbf{p},\mathbf{q}} = \{(i,j) : (p_1 \nmid i \text{ or } p_2 \nmid j) \text{ and } (q_1 \nmid i \text{ or } q_2 \nmid j)\}$.
2. $\mathcal{M}_{\mathbf{p},\mathbf{q}} = \{(p_1^\alpha q_1^\beta, p_2^\alpha q_2^\beta) : \alpha, \beta \geq 0\}$.
3. $\mathcal{M}_{\mathbf{p},\mathbf{q}}(i,j) = \{(ip_1^\alpha q_1^\beta, jp_2^\alpha q_2^\beta) : \alpha, \beta \geq 0\}$ denote the lattice $\mathcal{M}_{\mathbf{p},\mathbf{q}}$ starts at $(i,j)$.

The following lemma gives the partition of $\mathbb{N}^2$.

**Lemma 2.4** *For* $\mathbf{p}, \mathbf{q} \in \mathbb{N}^2_{\geq 2}$ *with* $\gcd(p_1, q_1) = 1$ *and* $\gcd(p_2, q_2) = 1$,

$$\mathbb{N}^2 = \coprod_{(i,j)\in\mathcal{I}_{\mathbf{p},\mathbf{q}}} \mathcal{M}_{\mathbf{p},\mathbf{q}}(i,j).$$

***Proof*** We first claim that for all $(i, j) \neq (i', j') \in \mathcal{I}_{\mathbf{p},\mathbf{q}}$, $\mathcal{M}_{\mathbf{p},\mathbf{q}}(i, j) \cap \mathcal{M}_{\mathbf{p},\mathbf{q}}(i', j') = \emptyset$.

Suppose not, then there exist $(i, j) \neq (i', j') \in \mathcal{I}_{\mathbf{p},\mathbf{q}}$ such that $\mathcal{M}_{\mathbf{p},\mathbf{q}}(i, j) \cap \mathcal{M}_{\mathbf{p},\mathbf{q}}(i', j') \neq \emptyset$. Since $(i, j) \neq (i', j')$,

$$(ip_1^{\alpha_1} q_1^{\beta_1}, jp_2^{\alpha_1} q_2^{\beta_1}) = (i' p_1^{\alpha_2} q_1^{\beta_2}, j' p_2^{\alpha_2} q_2^{\beta_2})$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ with $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$. Without loss of generality, we may assume $\alpha_1 > \alpha_2$. Then since $ip_1^{\alpha_1-\alpha_2} q_1^{\beta_1} = i' q_1^{\beta_2}$, $jp_2^{\alpha_1-\alpha_2} q_2^{\beta_1} = j' q_2^{\beta_2}$ and $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$; $p_1 | i'$, and $p_2 | j'$, which contradicts $(i', j') \in \mathcal{I}_{\mathbf{p},\mathbf{q}}$.

It remains to show that the equality holds. For $(i, j) \in \mathbb{N}^2$, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ such that $(i, j) = (i' p_1^{\alpha_1} q_1^{\beta_1}, j' p_2^{\alpha_2} q_2^{\beta_2})$, where $p_1, q_1 \nmid i'$, and $p_2, q_2 \nmid j'$. Take $\alpha = \min\{\alpha_1, \alpha_2\}$ and $\beta = \min\{\beta_1, \beta_2\}$. Then, we have $(i, j) \in \mathcal{M}_{\mathbf{p},\mathbf{q}}(\frac{i}{p_1^\alpha q_1^\beta}, \frac{j}{p_2^\alpha q_2^\beta})$ and $(\frac{i}{p_1^\alpha q_1^\beta}, \frac{j}{p_2^\alpha q_2^\beta}) \in \mathcal{I}_{\mathbf{p},\mathbf{q}}$. The converse is then clear. □

Similar to the proof of Theorem 1.2, the following definitions and lemmas are needed.

**Definition 2.5** For $k, \ell, M$, and $N \geq 1$, let

1. $\mathcal{M}_{p_1,q_1} = \{p_1^\alpha q_1^\beta : \alpha, \beta \geq 0\} = \{r_1 < r_2 < \cdots\}$ and
   $\mathcal{M}_{p_2,q_2} = \{p_2^\alpha q_2^\beta : \alpha, \beta \geq 0\} = \{r'_1 < r'_2 < \cdots\}$.
2. $\mathcal{J}_{k\times\ell;M,N} = \{(i, j) \in \mathcal{N}_{k\times\ell} : ir_M \leq k < ir_{M+1}$ and $jr'_N \leq \ell < jr'_{N+1}\}$ be the subset of $\mathcal{N}_{k\times\ell}$ which satisfies $ir_M \leq k < ir_{M+1}$ and $jr'_N \leq \ell < jr'_{N+1}$.
3. $\mathcal{K}_{k\times\ell;M,N} = \{(i, j) \in \mathcal{I}_{\mathbf{p},\mathbf{q}} \cap \mathcal{N}_{k\times\ell} : ir_M \leq k < ir_{M+1}$ and $jr'_N \leq \ell < jr'_{N+1}\}$ be the subset of $\mathcal{J}_{k\times\ell;M,N}$, which belongs to $\mathcal{I}_{\mathbf{p},\mathbf{q}}$.
4. For $M, N \geq 1$, we define $\mathbb{L}_{M,N}$ as
   $$\mathbb{L}_{M,N} := \{(\alpha, \beta) : p_1^\alpha q_1^\beta \leq r_M\} \cap \{(\alpha, \beta) : p_2^\alpha q_2^\beta \leq r'_N\}.$$

The lattice $\mathbb{L}_{M_1,M_2,\ldots,M_d}$, which is used in Theorem 1.5, for $d, k \geq 2$ is defined as follows.

**Definition 2.6** For $d, k \geq 2$ and $M_1, M_2, \ldots, M_d \geq 1$, the lattice

$$\mathbb{L}_{M_1,M_2,\ldots,M_d} := \bigcap_{i=1}^{d} \{(\alpha_1, \alpha_2, \cdots, \alpha_{k-1}) : p_{1,i}^{\alpha_1} p_{2,i}^{\alpha_2} p_{k-1,i}^{\alpha_{k-1}} \cdots \leq r_{M_i}^{(i)}\},$$

where

$\mathcal{M}_{p_{1,j},p_{2,j},\cdots,p_{k-1,j}} = \{p_{1,j}^{\alpha_1} p_{2,j}^{\alpha_2} \cdots p_{k-1,j}^{\alpha_{k-1}} : \alpha_1, \alpha_2, \cdots, \alpha_{k-1} \geq 0\} = \{r_1^{(j)} < r_2^{(j)} < \cdots\}$

for all $j = 1, 2, \ldots, d$.

**Lemma 2.7** *For $k, \ell, M$, and $N \geq 1$, we have the following assertions.*

1. $|\mathcal{J}_{k\times\ell;M,N}| = \left(\left\lfloor \frac{k}{r_M} \right\rfloor - \left\lfloor \frac{k}{r_{M+1}} \right\rfloor\right)\left(\left\lfloor \frac{\ell}{r'_N} \right\rfloor - \left\lfloor \frac{\ell}{r'_{N+1}} \right\rfloor\right)$.
2. $\lim\limits_{k,\ell\to\infty} \dfrac{|\mathcal{K}_{k\times\ell;M,N}|}{|\mathcal{J}_{k\times\ell;M,N}|} = 1 - (\dfrac{1}{p_1p_2} + \dfrac{1}{q_1q_2} - \dfrac{1}{p_1p_2q_1q_2})$.
3. $\lim\limits_{k,\ell\to\infty} \dfrac{1}{k\ell} \sum\limits_{M=1}^{k} \sum\limits_{N=1}^{\ell} |\mathcal{K}_{k\times\ell;M,N}| \log b_{M,N} = \sum\limits_{M,N=1}^{\infty} \lim\limits_{k,\ell\to\infty} \dfrac{1}{k\ell} |\mathcal{K}_{k\times\ell;M,N}| \log b_{M,N}$,

*where* $b_{M,N}$ is the number of admissible patterns on the lattice $\mathbb{L}_{M,N}$ *of two-dimensional subshift with forbidden set* $\mathcal{F} = \{x_{\mathbf{0}} = x_{\mathbf{e}_1} = x_{\mathbf{e}_2} = 1\}$.

***Proof*** **1.** Since $(i, j) \in \mathcal{J}_{k\times\ell;M,N}$ if and only if $ir_M \le k < ir_{M+1}$ and $jr'_N \le \ell < jr'_{N+1}$. It follows that $\frac{k}{r_{M+1}} < i \le \frac{k}{r_M}$ and $\frac{\ell}{r'_{M+1}} < j \le \frac{\ell}{r'_M}$.

Therefore

$$|\mathcal{J}_{k\times\ell;M,N}| = \left(\left\lfloor\frac{k}{r_M}\right\rfloor - \left\lfloor\frac{k}{r_{M+1}}\right\rfloor\right)\left(\left\lfloor\frac{\ell}{r'_N}\right\rfloor - \left\lfloor\frac{\ell}{r'_{N+1}}\right\rfloor\right).$$

**2.** Let the complement of $\mathcal{I}_{\mathbf{p},\mathbf{q}}$ be

$$\mathcal{I}^c_{\mathbf{p},\mathbf{q}} = \mathcal{S}_1 \cup \mathcal{S}_2 = \{(i,j) : p_1|i \text{ and } p_2|j\} \cup \{(i,j) : q_1|i \text{ and } q_2|j)\}.$$

Since $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$, we have

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \{(i,j) : p_1q_1|i \text{ and } p_2q_2|j\}.$$

Then, as in the proof of Lemma 2.2,

$$\begin{aligned}
\lim_{k,\ell\to\infty} \frac{|\mathcal{K}_{k\times\ell;M,N}|}{|\mathcal{J}_{k\times\ell;M,N}|} &= \lim_{k,\ell\to\infty} \frac{|\mathcal{J}_{k\times\ell;M,N} \cap \mathcal{I}_{\mathbf{p},\mathbf{q}}|}{|\mathcal{J}_{k\times\ell;M,N}|} \\
&= \lim_{k,\ell\to\infty} 1 - \frac{|\mathcal{J}_{k\times\ell;M,N} \cap \mathcal{I}^c_{\mathbf{p},\mathbf{q}}|}{|\mathcal{J}_{k\times\ell;M,N}|} \\
&= \lim_{k,\ell\to\infty} 1 - \left(\frac{|\mathcal{J}_{k\times\ell;M,N} \cap \mathcal{S}_1|}{|\mathcal{J}_{k\times\ell;M,N}|} + \frac{|\mathcal{J}_{k\times\ell;M,N} \cap \mathcal{S}_2|}{|\mathcal{J}_{k\times\ell;M,N}|} - \frac{|\mathcal{J}_{k\times\ell;M,N} \cap \mathcal{S}_1 \cap \mathcal{S}_2|}{|\mathcal{J}_{k\times\ell;M,N}|}\right) \\
&= 1 - (\frac{1}{p_1p_2} + \frac{1}{q_1q_2} - \frac{1}{p_1p_2q_1q_2}).
\end{aligned}$$

**3.** The result follows the same argument as that of Lemma 3.4.

The proof is complete. □

***Proof for Theorem 1.4*** By Lemma 2.4,

$$\begin{aligned}
h(X^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}) &= \limsup_{k,\ell\to\infty} \frac{1}{k\ell} \log \Gamma_{k\times\ell}(X^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}) \\
&= \limsup_{k,\ell\to\infty} \frac{1}{k\ell} \log \prod_{M=1}^{k} \prod_{N=1}^{\ell} b_{M,N}^{|\mathcal{K}_{k\times\ell;M,N}|} \\
&= \limsup_{k,\ell\to\infty} \frac{1}{k\ell} \sum_{M=1}^{k} \sum_{N=1}^{\ell} |\mathcal{K}_{k\times\ell;M,N}| \log b_{M,N}.
\end{aligned} \tag{15}$$

Combining Lemma 2.7 and (15), we have

$$\begin{aligned}
h(X^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}) &= \sum_{M,N=1}^{\infty} \lim_{k,\ell\to\infty} \frac{1}{k\ell} |\mathcal{K}_{k\times\ell;M,N}| \log b_{M,N} \\
&= \sum_{M,N=1}^{\infty} \lim_{k,\ell\to\infty} \frac{1}{k\ell} |\mathcal{J}_{k\times\ell;M,N}| \left[1 - (\frac{1}{p_1p_2} + \frac{1}{q_1q_2} - \frac{1}{p_1p_2q_1q_2})\right] \log b_{M,N} \\
&= \sum_{M,N=1}^{\infty} (\frac{1}{r_M} - \frac{1}{r_{M+1}})(\frac{1}{r'_N} - \frac{1}{r'_{N+1}}) \left[1 - (\frac{1}{p_1p_2} + \frac{1}{q_1q_2} - \frac{1}{p_1p_2q_1q_2})\right] \log b_{M,N} \\
&= \left[1 - (\frac{1}{p_1p_2} + \frac{1}{q_1q_2} - \frac{1}{p_1p_2q_1q_2})\right] \sum_{M,N=1}^{\infty} (\frac{1}{r_M} - \frac{1}{r_{M+1}})(\frac{1}{r'_N} - \frac{1}{r'_{N+1}}) \log b_{M,N},
\end{aligned} \tag{16}$$

which completes the proof. □

## 3 Proofs for Theorems 1.6 and 1.8

Similar to the above proofs. We gives the necessary definitions for the proof of the second type $(d, k)$-MMIS. For the step (I), we need the following definitions.

**Definition 3.1** Given $p, q \geq 2$,

1.$\bar{\mathcal{I}}_{p,q} = \{(i, j) : p \nmid i \text{ and } q \nmid j\} = \{i : p \nmid i\} \times \{j : q \nmid j\} = \mathcal{I}_p \times \mathcal{I}_q$.

2.$\bar{\mathcal{M}}_{p,q} = \{(p^\alpha, q^\beta) : \alpha, \beta \geq 0\} = \{p^\alpha : \alpha \geq 0\} \times \{q^\beta : \beta \geq 0\} = \mathcal{M}_p \times \mathcal{M}_q$.

3.$\bar{\mathcal{M}}_{p,q}(i, j) = \{(ip^\alpha, jq^\beta) : \alpha, \beta \geq 0\}$ denote the lattice $\bar{\mathcal{M}}_{p,q}$ starts at $(i, j)$.

The following lemma is analogous to Lemma 2.1.

**Lemma 3.2** *For $p, q \geq 2$,*

$$\mathbb{N}^2 = \coprod_{(i,j)\in\bar{\mathcal{I}}_{p,q}} \bar{\mathcal{M}}_{p,q}(i, j).$$

***Proof*** We first claim that for all $(i, j) \neq (i', j') \in \bar{\mathcal{I}}_{p,q}$, $\bar{\mathcal{M}}_{p,q}(i, j) \cap \bar{\mathcal{M}}_{p,q}(i', j') = \emptyset$. Suppose not, then there exist $(i, j) \neq (i', j')$ such that $\bar{\mathcal{M}}_{p,q}(i, j) \cap \bar{\mathcal{M}}_{p,q}(i', j') \neq \emptyset$. Then, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ with $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$ such that $(ip^{\alpha_1}, jq^{\beta_1}) = (i'p^{\alpha_2}, j'q^{\beta_2})$. Without loss of generality, we may assume $\alpha_1 > \alpha_2$. Therefore, $ip^{\alpha_1-\alpha_2} = i'$, and hence $p|i'$, which contradicts $(i', j') \in \bar{\mathcal{I}}_{p,q}$.

It remains to show that the equality holds. For $(i, j) \in \mathbb{N}^2$, $(i, j) = (i'p^\alpha, j'q^\beta)$, where $p \nmid i'$ and $q \nmid j'$, hence $(i, j) \in \bar{\mathcal{M}}_{p,q}(i', j')$. The converse is then clear. □

We need several more definitions for the step (II).

**Definition 3.3** For $k$, $\ell$, and $m \geq 1$,

1. $\mathcal{N}_k = \{i | 1 \leq i \leq k\}$.
2. $\mathcal{L}_{p;k}(i) = \mathcal{M}_p(i) \cap \mathcal{N}_k$ denote the subset of $\mathcal{M}_p(i)$ in the $k$ lattice, and $\mathcal{L}_{q;\ell}(j) = \mathcal{M}_q(j) \cap \mathcal{N}_\ell$ denote the subset of $\mathcal{M}_q(j)$ in the $\ell$ lattice.
3. $\bar{\mathcal{L}}_{k\times\ell}(i, j) = \mathcal{L}_{p;k}(i) \times \mathcal{L}_{q;\ell}(j)$.
4. $\bar{\mathcal{J}}_{k\times\ell;m,n} = \{(i, j) \in \mathcal{N}_{k\times\ell} : |\mathcal{L}_{p;k}(i)| = m \text{ and } |\mathcal{L}_{q;\ell}(j)| = n\}$ $= \{i \in \mathcal{N}_k : |\mathcal{L}_{p;k}(i)| = m\} \times \{j \in \mathcal{N}_\ell : |\mathcal{L}_{q;\ell}(j)| = n\}$.
5. $\bar{\mathcal{K}}_{k\times\ell;m,n} = \{(i, j) \in \bar{\mathcal{I}}_{p,q} \cap \mathcal{N}_{k\times\ell} : |\mathcal{L}_{p;k}(i)| = m \text{ and } |\mathcal{L}_{q;\ell}(j)| = n\}$ $= \{i \in \mathcal{I}_p \cap \mathcal{N}_k : |\mathcal{L}_{p;k}(i)| = m\} \times \{j \in \mathcal{I}_q \cap \mathcal{N}_\ell : |\mathcal{L}_{q;\ell}(j)| = n\}$.

We similarly compute the density limit for the independent lattices.

**Lemma 3.4** *For $k, \ell, m$, and $n \geq 1$, we have the following assertions.*

1.$|\bar{\mathcal{J}}_{k\times\ell;m,n}| = \left(\left\lfloor \frac{k}{p^{m-1}} \right\rfloor - \left\lfloor \frac{k}{p^m} \right\rfloor\right)\left(\left\lfloor \frac{\ell}{q^{n-1}} \right\rfloor - \left\lfloor \frac{\ell}{q^n} \right\rfloor\right)$.

2. $\lim\limits_{k,\ell\to\infty} \frac{|\bar{\mathcal{K}}_{k\times\ell;m,n}|}{|\bar{\mathcal{J}}_{k\times\ell;m,n}|} = (1 - \frac{1}{p})(1 - \frac{1}{q})$.

3. $\lim\limits_{k,\ell\to\infty} \frac{1}{k\ell} \sum\limits_{m=1}^{k} \sum\limits_{n=1}^{\ell} |\bar{\mathcal{K}}_{k\times\ell;m,n}| \log a_{m,n} = \sum\limits_{m,n=1}^{\infty} \lim\limits_{k,\ell\to\infty} \frac{1}{k\ell} |\bar{\mathcal{K}}_{k\times\ell;m,n}| \log a_{m,n}$.

***Proof*** **1.** Since $(i, j)$ satisfies $|\mathcal{L}_{p;k}(i)| = m$ and $|\mathcal{L}_{q;\ell}(j)| = n$ if and only if $(i, j)$ satisfies $ip^{m-1} \le k < ip^m$ and $jq^{n-1} \le \ell < jq^n$. Then $(i, j)$ satisfies $\frac{k}{p^m} < i \le \frac{k}{p^{m-1}}$ and $\frac{\ell}{q^n} < j \le \frac{\ell}{q^{n-1}}$. Therefore

$$\begin{aligned}
|\bar{\mathcal{J}}_{k\times\ell;m,n}| &= |\{i \in \mathcal{N}_k : |\mathcal{L}_{p;k}(i)| = m\}||\{j \in \mathcal{N}_\ell : |\mathcal{L}_{q;\ell}(j)| = n\}| \\
&= \left(\left\lfloor \frac{k}{p^{m-1}} \right\rfloor - \left\lfloor \frac{k}{p^m} \right\rfloor\right)\left(\left\lfloor \frac{\ell}{q^{n-1}} \right\rfloor - \left\lfloor \frac{\ell}{q^n} \right\rfloor\right).
\end{aligned}$$

**2.** The proof is similar to that for Lemma 2.2.
**3.** Define

$$\hat{K}_{k\times\ell;m,n} = \begin{cases} |\bar{\mathcal{K}}_{k\times\ell;m,n}|, & \text{if } m \le k \text{ and } n \le \ell, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\lim_{k,\ell\to\infty} \frac{1}{k\ell} \sum_{m=1}^{k} \sum_{n=1}^{\ell} |\bar{\mathcal{K}}_{k\times\ell;m,n}| \log a_{m,n} = \lim_{k,\ell\to\infty} \frac{1}{k\ell} \sum_{m,n=1}^{\infty} \hat{K}_{k\times\ell;m,n} \log a_{m,n}.$$

We claim that $\displaystyle\sum_{m,n=1}^{\infty} \frac{\hat{K}_{k\times\ell;m,n} \log a_{m,n}}{k\ell}$ converges uniformly in $k, \ell$ by Weierstrass M-test with

$$\left| \frac{\hat{K}_{k\times\ell;m,n} \log a_{m,n}}{k\ell} \right| \le \left| \frac{|\bar{\mathcal{J}}_{k\times\ell;m,n}| \log a_{m,n}}{k\ell} \right| \le \frac{1}{p^{m-1}q^{n-1}} \log a_{m,n}$$

for all $k, \ell \ge 1$ and

$$\begin{aligned}
\sum_{m,n=1}^{\infty} \frac{1}{p^{m-1}q^{n-1}} \log a_{m,n} &\le \sum_{m,n=1}^{\infty} \frac{1}{p^{m-1}q^{n-1}} \log 2^{mn} \\
&= \log 2 \sum_{m,n=1}^{\infty} \frac{mn}{p^{m-1}q^{n-1}} \\
&= \log 2 (\sum_{m=1}^{\infty} \frac{m}{p^{m-1}})(\sum_{n=1}^{\infty} \frac{n}{q^{n-1}}) < \infty.
\end{aligned}$$

Thus,

$$\lim_{k,\ell\to\infty}\frac{1}{k\ell}\sum_{m=1}^{k}\sum_{n=1}^{\ell}|\bar{\mathcal{K}}_{k\times\ell;m,n}|\log a_{m,n} = \lim_{k,\ell\to\infty}\frac{1}{k\ell}\sum_{m,n=1}^{\infty}\hat{K}_{k\times\ell;m,n}\log a_{m,n}$$
$$= \sum_{m,n=1}^{\infty}\lim_{k,\ell\to\infty}\frac{1}{k\ell}\hat{K}_{k\times\ell;m,n}\log a_{m,n}$$
$$= \sum_{m,n=1}^{\infty}\lim_{k,\ell\to\infty}\frac{1}{k\ell}|\bar{\mathcal{K}}_{k\times\ell;m,n}|\log a_{m,n}.$$

The proof is complete. □

***Proof for Theorem 1.6*** By Lemma 3.2, we have

$$\begin{aligned} h(\bar{X}_{\mathbf{p}_1}^{(2,2)}) &= \limsup_{k,\ell\to\infty}\frac{1}{k\ell}\log\Gamma_{k\times\ell}(\bar{X}_{\mathbf{p}_1}^{(2,2)}) \\ &= \limsup_{k,\ell\to\infty}\frac{1}{k\ell}\log\prod_{m=1}^{k}\prod_{n=1}^{\ell}a_{m,n}^{|\bar{\mathcal{K}}_{k\times\ell;m,n}|} \\ &= \limsup_{k,\ell\to\infty}\frac{1}{k\ell}\sum_{m=1}^{k}\sum_{n=1}^{\ell}|\bar{\mathcal{K}}_{k\times\ell;m,n}|\log a_{m,n}. \end{aligned} \tag{17}$$

Combining Lemma 3.4 and (17), we have

$$\begin{aligned} h(\bar{X}_{\mathbf{p}_1}^{(2,2)}) &= \sum_{m,n=1}^{\infty}\lim_{k,\ell\to\infty}\frac{1}{k\ell}|\bar{\mathcal{K}}_{k\times\ell;m,n}|\log a_{m,n} \\ &= \sum_{m,n=1}^{\infty}\lim_{k,\ell\to\infty}\frac{1}{k\ell}|\bar{\mathcal{J}}_{k\times\ell;m,n}|(1-\frac{1}{p})(1-\frac{1}{q})\log a_{m,n} \\ &= \sum_{m,n=1}^{\infty}\frac{1}{p^m}\frac{1}{q^n}(p-1)(q-1)(1-\frac{1}{p})(1-\frac{1}{q})\log a_{m,n} \\ &= (p-1)(q-1)(1-\frac{1}{p})(1-\frac{1}{q})\sum_{m,n=1}^{\infty}\frac{\log a_{m,n}}{p^m q^n}. \end{aligned} \tag{18}$$

The proof is complete. □

We need the following definitions to prove Theorem 1.8, parts (I) and (II).

**Definition 3.5** Let $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$ be two vectors in $\mathbb{N}^2_{\geq 2}$,

1. $\bar{\mathcal{I}}_{\mathbf{p},\mathbf{q}} = \{(i, j) : (p_1 \nmid i \text{ and } p_2 \nmid j) \text{ and } (q_1 \nmid i \text{ and } q_2 \nmid j)\}$.
2. $\bar{\mathcal{M}}_{\mathbf{p},\mathbf{q}} = \{(p_1^{\alpha_1}q_1^{\beta_1}, p_2^{\alpha_2}q_2^{\beta_2}) : \alpha_1, \beta_1, \alpha_2, \beta_2 \geq 0\}$.
3. $\bar{\mathcal{M}}_{\mathbf{p},\mathbf{q}}(i, j) = \{(ip_1^{\alpha_1}q_1^{\beta_1}, jp_2^{\alpha_2}q_2^{\beta_2}) : \alpha_1, \beta_1, \alpha_2, \beta_2 \geq 0\}$.

The partition of $\mathbb{N}^2$ can be expressed as follows.

**Lemma 3.6** *For* $\mathbf{p}, \mathbf{q} \in \mathbb{N}^2_{\geq 2}$ *with* $\gcd(p_1, q_1) = 1$ *and* $\gcd(p_2, q_2) = 1$,

$$\mathbb{N}^2 = \coprod_{(i,j)\in\bar{\mathcal{I}}_{\mathbf{p},\mathbf{q}}}\bar{\mathcal{M}}_{\mathbf{p},\mathbf{q}}(i, j).$$

***Proof*** We first claim that for all $(i, j) \neq (i', j') \in \bar{\mathcal{I}}_{\mathbf{p},\mathbf{q}}$, $\bar{\mathcal{M}}_{\mathbf{p},\mathbf{q}}(i, j) \cap \bar{\mathcal{M}}_{\mathbf{p},\mathbf{q}}(i', j') = \emptyset$. Suppose not, then there exist $(i, j) \neq (i', j') \in \bar{\mathcal{I}}_{\mathbf{p},\mathbf{q}}$ such that $\bar{\mathcal{M}}_{\mathbf{p},\mathbf{q}}(i, j) \cap \bar{\mathcal{M}}_{\mathbf{p},\mathbf{q}}(i', j') \neq \emptyset$. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_1', \alpha_2', \beta_1', \beta_2' \geq 0$ with $p_1^{\alpha_1} q_1^{\beta_1} \neq p_1^{\alpha_1'} q_1^{\beta_1'}$ or $p_2^{\alpha_2} q_2^{\beta_2} \neq p_2^{\alpha_2'} q_2^{\beta_2'}$ such that $(ip_1^{\alpha_1} q_1^{\beta_1}, jp_2^{\alpha_2} q_2^{\beta_2}) = (i' p_1^{\alpha_1'} q_1^{\beta_1'}, j' p_2^{\alpha_2'} q_2^{\beta_2'})$. Since $\gcd(p_1, q_1) = 1$ and $\gcd(p_2, q_2) = 1$, $\alpha_1 \neq \alpha_1'$ or $\beta_1 \neq \beta_1'$ or $\alpha_2 \neq \alpha_2'$ or $\beta_2' \neq \beta_2'$. Without loss of generality, we may assume $\alpha_1 > \alpha_1'$. Then by $ip_1^{\alpha_1 - \alpha_1'} q_1^{\beta_1} = i' q_1^{\beta_1'}$ and $\gcd(p_1, q_1) = 1$, $p_1 | i'$, which contradicts $(i', j') \in \mathcal{I}_{\mathbf{p},\mathbf{q}}$.

It remains to show that the equality holds. For $(i, j) \in \mathbb{N}^2$, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ with $p_1, q_1 \nmid i'$ and $p_2, q_2 \nmid j'$ such that $(i, j) = (i' p_1^{\alpha_1} q_1^{\beta_1}, j' p_2^{\alpha_2} q_2^{\beta_2})$. Hence $(i, j) \in \bar{\mathcal{M}}_{\mathbf{p},\mathbf{q}}(i', j')$ and $(i', j') \in \bar{\mathcal{I}}_{\mathbf{p},\mathbf{q}}$. The converse is then clear. $\square$

**Definition 3.7** Let $k$, $\ell$, $M$, and $N$ be integers larger than 1.

1. $\mathcal{M}_{p_1,q_1} = \{p_1^{\alpha} q_1^{\beta} : \alpha, \beta \geq 0\} = \{r_1 < r_2 < \cdots\}$ and
   $\mathcal{M}_{p_2,q_2} = \{p_2^{\alpha} q_2^{\beta} : \alpha, \beta \geq 0\} = \{r_1' < r_2' < \cdots\}$.
2. $\bar{\mathcal{J}}_{k\times\ell;M,N} = \{(i, j) \in \mathcal{N}_{k\times\ell} : ir_M \leq k < ir_{M+1} \text{ and } jr_N' \leq \ell < jr_{N+1}'\}$.
3. $\bar{\mathcal{K}}_{k\times\ell;M,N} = \{(i, j) \in \bar{\mathcal{I}}_{\mathbf{p},\mathbf{q}} \cap \mathcal{N}_{k\times\ell} : ir_M \leq k < ir_{M+1} \text{ and } jr_N' \leq \ell < jr_{N+1}'\}$.
4. For $M, N \geq 1$, we define $\bar{\mathbb{L}}_{M,N}$ as
   $\bar{\mathbb{L}}_{M,N} := \{(\alpha_1, \beta_1, \alpha_2, \beta_2) : p_1^{\alpha_1} q_1^{\beta_1} \leq r_M \text{ and } p_2^{\alpha_2} q_2^{\beta_2} \leq r_N'\}$.

We also have the limit of density for the independent lattice in the following lemma.

**Lemma 3.8** *For $k$, $\ell$, $M$, and $N \geq 1$, we have the following assertions.*

1. $|\bar{\mathcal{J}}_{k\times\ell;M,N}| = \left(\left\lfloor \dfrac{k}{r_M} \right\rfloor - \left\lfloor \dfrac{k}{r_{M+1}} \right\rfloor\right)\left(\left\lfloor \dfrac{\ell}{r_N'} \right\rfloor - \left\lfloor \dfrac{\ell}{r_{N+1}'} \right\rfloor\right)$.
2. $\lim\limits_{k,\ell\to\infty} \dfrac{|\bar{\mathcal{K}}_{k\times\ell;M,N}|}{|\bar{\mathcal{J}}_{k\times\ell;M,N}|} = (1 - \dfrac{1}{p_1})(1 - \dfrac{1}{p_2})(1 - \dfrac{1}{q_1})(1 - \dfrac{1}{q_2})$.
3. $\lim\limits_{k,\ell\to\infty} \dfrac{1}{k\ell} \sum\limits_{M=1}^{k} \sum\limits_{N=1}^{\ell} |\bar{\mathcal{K}}_{k\times\ell;M,N}| \log c_{M,N} = \sum\limits_{M,N=1}^{\infty} \lim\limits_{k,\ell\to\infty} \dfrac{1}{k\ell} |\bar{\mathcal{K}}_{k\times\ell;M,N}| \log c_{M,N}$,

where $c_{M,N}$ is the number of admissible patterns on $\bar{\mathbb{L}}_{M,N}$ lattice of 4 dimensional subshift *with forbidden set* $\mathcal{F} = \{x_{\mathbf{0}} = x_{\mathbf{e}_1} = x_{\mathbf{e}_2} = 1, x_{\mathbf{0}} = x_{\mathbf{e}_3} = x_{\mathbf{e}_4} = 1\}$.

***Proof*** The proof is similar to that of Lemma 2.7. $\square$

Similar to Definition 2.6, the lattice for $d, k \geq 2$ is defined as follows.

**Definition 3.9** For $d, k \geq 2$ and $M_1, M_2, \ldots, M_d \geq 1$, the lattice

$$\bar{\mathbb{L}}_{M_1,M_2,\ldots,M_d} := \{ \begin{array}{l}(\alpha_{1,1}, \alpha_{2,1}, \cdots, \alpha_{k-1,1}, \alpha_{1,2}, \alpha_{2,2}, \cdots, \alpha_{k-1,2}, \cdots, \alpha_{1,d}, \alpha_{2,d}, \cdots, \alpha_{k-1,d}) : \\ p_{1,\ell}^{\alpha_{1,\ell}} p_{2,\ell}^{\alpha_{2,\ell}} \cdots p_{k-1,\ell}^{\alpha_{k-1,\ell}} \leq r_{M_\ell}^{(\ell)}, 1 \leq \ell \leq d\},\end{array}$$

where

$$\mathcal{M}_{p_{1,j}, p_{2,j}, \cdots, p_{k-1,j}} = \{p_{1,j}^{\alpha_{1,j}} p_{2,j}^{\alpha_{2,j}} \cdots p_{k-1,j}^{\alpha_{k-1,j}} : \alpha_{i,j} \geq 0 \text{ for } 1 \leq i \leq k-1\} = \{r_1^{(j)} < r_2^{(j)} < \cdots\}$$

for all $j = 1, 2, \ldots, d$.

***Proof for Theorem 1.8*** By Lemma 3.6,

$$\begin{aligned} h(\bar{X}^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}) &= \limsup_{k,\ell\to\infty} \frac{1}{k\ell} \log \Gamma_{k\times\ell}(\bar{X}^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}) \\ &= \limsup_{k,\ell\to\infty} \frac{1}{k\ell} \log \prod_{M=1}^{k} \prod_{N=1}^{\ell} c_{M,N}^{|\bar{\mathcal{K}}_{k\times\ell;M,N}|} \\ &= \limsup_{k,\ell\to\infty} \frac{1}{k\ell} \sum_{M=1}^{k} \sum_{N=1}^{\ell} |\bar{\mathcal{K}}_{k\times\ell;M,N}| \log c_{M,N}. \end{aligned} \tag{19}$$

Applying Lemma 3.8 and (19), we have

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$$\begin{aligned} h(\bar{X}^{(2,3)}_{\mathbf{p}_1,\mathbf{p}_2}) &= \sum_{M,N=1}^{\infty} \lim_{k,\ell\to\infty} \frac{1}{k\ell} |\bar{\mathcal{K}}_{k\times\ell;M,N}| \log c_{M,N} \\ &= \sum_{M,N=1}^{\infty} \lim_{k,\ell\to\infty} \frac{1}{k\ell} |\bar{\mathcal{J}}_{k\times\ell;M,N}| (1-\frac{1}{p_1})(1-\frac{1}{p_2})(1-\frac{1}{q_1})(1-\frac{1}{q_2}) \log c_{M,N} \\ &= \sum_{M,N=1}^{\infty} (\frac{1}{r_M} - \frac{1}{r_{M+1}})(\frac{1}{r'_N} - \frac{1}{r'_{N+1}})(1-\frac{1}{p_1})(1-\frac{1}{p_2})(1-\frac{1}{q_1})(1-\frac{1}{q_2}) \log c_{M,N} \\ &= (1-\frac{1}{p_1})(1-\frac{1}{p_2})(1-\frac{1}{q_1})(1-\frac{1}{q_2}) \sum_{M,N=1}^{\infty} (\frac{1}{r_M} - \frac{1}{r_{M+1}})(\frac{1}{r'_N} - \frac{1}{r'_{N+1}}) \log c_{M,N}. \end{aligned}$$

The proof is complete. □

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